ALGEBRAIC GEOMETRY

NOTES BY ATTICUS WANG

Contents

1. Category theory 3
2. Sheaves 3
2.1. The espace étalé of a (pre)sheaf 3
2.2. Stalks and sheafification 3
2.3. Sheaf on a base 3
2.4. Inverse image sheaf 4
3. Affine schemes 4
3.1. Spectrum of a ring 4
3.2. Hilbert's Nullstellensatz 4
3.3. Topological properties of affine schemes 4
4. Schemes 4
4.1. Proj construction 4
4.2. Properties of schemes 4
4.3. Varieties 5
4.4. Normality and factoriality 5
4.5. Associated points 5
4.6. Weakly associated points 6
5. Morphisms of schemes 6
5.1. Morphisms to affine schemes 6
5.2. Morphisms from affine schemes 6
5.3. Functoriality of Proj 6
5.4. Veronese subring 6
5.5. The relative point of view 6
5.6. Green flags to look for in a property of morphisms 6
5.7. Finiteness conditions on morphisms 6
5.8. Elimination theory 8
5.9. Closed subschemes, and related constructions 8
5.10. Effective Cartier divisors and regular sequences 8
5.11. Fiber products 8
5.12. An interlude on closed points 9
5.13. Separated morphisms 9
5.14. Dominant rational maps between irreducible varieties 11
5.15. Ax-Grothendieck theorem 12
5.16. Proper maps 13
5.17. Chow's lemma 13
5.18. Valuative criteria 14
6. Dimension and smoothness 14
6.1. Definitions of dimension 14

Date: 18.725 and 18.726 were taught by Davesh Maulik, in fall 2022 and spring 2023 respectively.

6.2. Dimension of fibers 15
6.3. Cotangent and tangent spaces 16
6.4. Regularity and smoothness 16
6.5. Bertini's theorem 17
7. Quasicoherent sheaves 17
7.1. Basic definitions 17
7.2. f_* and f^* 18
7.3. Invertible sheaves 18
7.4. Weil divisors 18
7.5. Quasicoherent sheaf of graded module 20
7.6. Sections of line bundles 20
7.7. Ampleness 22
7.8. Projective morphism 23
7.9. Curves 24
8. Cohomology 24
8.1. Properties 24
8.2. Definition 25
8.3. Euler characteristic, Hilbert functions 25
8.4. Riemman-Roch for line bundles on a regular projective curve 26
8.5. Remarks on sheaf cohomology 27
8.6. Baby intersection theory 27
9. Curves of small genus 27
9.1. Preliminary tools 27
9.2. Genus 0 28
9.3. Hyperelliptic curves 28
9.4. Genus 1: elliptic curves 29
9.5. Genus 2 29
9.6. Genus 3 29
9.7. Genus 4 30
9.8. Genus 5 30
10. Differentials 31
10.1. Affine case 31
10.2. The cotangent sheaf 32
10.3. Smoothness 33
10.4. Invariants 34
10.5. Riemann-Hurwitz theorem 34
11. Flatness 34
11.1. Algebra 34
11.2. Geometry 35

1. Category theory

Lemma 1.1. *Consider the following cochain complex:*

$$\cdots \to C^{i-1} \xrightarrow{f^{i-1}} C^i \xrightarrow{f^i} C^{i+1} \to \dots$$

Then we have two pairs of short exact sequences:

- $0 \to \ker f^i \to C^i \to \operatorname{im} f^i \to 0$ *and* $0 \to \operatorname{im} f^{i-1} \to \ker f^i \to H^i(C^\bullet) \to 0$*;*
- $0 \to \operatorname{im} f^{i-1} \to C^i \to \operatorname{coker} f^{i-1} \to 0$ *and* $0 \to H^i(C^\bullet) \to \operatorname{coker} f^{i-1} \to \operatorname{im} f^i \to 0$*.*

Proposition 1.2 (FHHF theorem)**.** *Let $F : \mathscr{A} \to \mathscr{B}$ be a covariant functor between abelian categories, and let $C^\bullet$ be a cochain complex in $\mathscr{A}$.*

(a) If F is right exact, there is a natural morphism $FH^\bullet(C^\bullet) \to H^\bullet F(C^\bullet)$.
(b) If F is left exact, there is a natural morphism $H^\bullet F(C^\bullet) \to FH^\bullet(C^\bullet)$.
(c) If F is exact, the two morphisms are inverses of each other.

Proof. (a) Applying F on $C^i \to C^{i+1} \to \operatorname{coker} f^i \to 0$, we get a natural isomorphism $F \operatorname{coker} f^i \to \operatorname{coker} F f^i$. Applying F on $0 \to \operatorname{im} f^i \to C^{i+1} \to \operatorname{coker} f^i \to 0$, we get a natural epimorphism $F \operatorname{im} f^i \twoheadrightarrow \operatorname{im} F f^i$. Applying F on $0 \to H^i(C^\bullet) \to \operatorname{coker} f^{i-1} \to \operatorname{im} f^i \to 0$ and chasing diagrams, we get a natural map $FH^i(C^\bullet) \to H^i F(C^\bullet)$.

(b) Applying F on $0 \to \ker f^i \to C^i \to C^{i+1}$, we get a natural isomorphism $\ker F f^i \to F \ker f^i$. Applying F on $0 \to \ker f^i \to C^i \to \operatorname{im} f^i \to 0$, we get a natural monomorphism $\operatorname{im} F f^i \hookrightarrow F \operatorname{im} f^i$. Applying F on $0 \to \operatorname{im} f^{i-1} \to \ker f^i \to H^i(C^\bullet) \to 0$ and chasing diagrams, we get a natural map $H^i F(C^\bullet) \to FH^i(C^\bullet)$.

(c) Carefully trace where each element goes. ☐

Proposition 1.3 (Exactness and (co)limits)**.** *Limits commute with limits and right adjoints. In particular, right adjoints and limits are both left exact since they commute with* ker*.*

Colimits commute with colimits and left adjoints. In particular, left adjoints and colimits are both right exact since they commute with coker*.*

In Mod_A, colimits over filtered index categories are exact.

2. Sheaves

2.1. **The espace étalé of a (pre)sheaf.** Let $\mathcal{F}$ be a (pre)sheaf on X. We can construct a topological space F and a continuous $\pi : F \to X$ as follows:

- As a set, $F = \coprod_{p \in X} \mathcal{F}_p$.
- Open sets of F are generated by the following base: given an open $U \subseteq X$ and $f \in \mathcal{F}(U)$, the set $\{(p, U, f) : p \in U\}$ is open.

Then $\pi : F \to U$ is a *local homeomorphism.*

2.2. **Stalks and sheafification.**

2.3. **Sheaf on a base.** Suppose X is a topological space with $\{B_i\}$ as a base of the topology. Suppose we're given the following information:

- To each B_i, we have an associated set/abelian group/ring/module $\mathcal{F}(B_i)$;
- For each $B_i \subseteq B_j$, a restriction map $\operatorname{res}_{B_i,B_j} : \mathcal{F}(B_i) \to \mathcal{F}(B_j)$; this should be the identity when $i = j$;
- If $B_i \subseteq B_j \subseteq B_k$, then $\operatorname{res}_{B_i,B_k} = \operatorname{res}_{B_j,B_k} \circ \operatorname{res}_{B_i,B_j}$.
- If $B = \bigcup B_i$, then if $f, g \in \mathcal{F}(B)$ restricts to the same function on each $\mathcal{F}(B_i)$, then $f = g$;
- If $B = \bigcup B_i$, and $f_i \in \mathcal{F}(B_i)$ such that if for any $B_k \subseteq B_i \cap B_j$, f_i and f_j restrict to the same function on B_k, then there exists $f \in \mathcal{F}(B)$ such that f restricts on each f_i on each patch.

This is called a *sheaf on a base*. Given this information, the sheaf on any open set can be uniquely determined up to unique isomorphism.

2.4. Inverse image sheaf.

3. Affine schemes

3.1. Spectrum of a ring.

3.2. Hilbert's Nullstellensatz.

3.3. Topological properties of affine schemes.

Definition 3.3.1. A topological space X is *Noetherian* if it satisfies the d.c.c. on closed sets.

Proposition 3.3.1. *Let X be Noetherian. Then every nonempty closed set Z can be uniquely expressed as a finite union $Z = Z_1 \cup \cdots \cup Z_n$ of irreducible closed sets, none contained in any other.*

4. Schemes

4.1. Proj construction. Given a (commutative) ring A, Spec produces from it a locally ringed space $\operatorname{Spec} A$. If we take $A = k[x_1, \ldots, x_n]$, then $\operatorname{Spec} A$ is the affine n-space $\mathbb{A}^n_k$. Similarly, the Proj construction takes a $\mathbb{Z}_{\geq 0}$-graded ring S as input, and produces from this data a scheme (not necessarily affine!) $\operatorname{Proj} S$, and in the special case $S = k[x_0, \ldots, x_n]$, $\operatorname{Proj} S$ is the projective n-space $\mathbb{P}^n_k$.

Definition 4.1.1. Let S be a $\mathbb{Z}_{\geq 0}$-graded ring. The scheme $\operatorname{Proj} S$ is given by:

- As a set, the points in $\operatorname{Proj} S$ are the homogeneous prime ideals $\mathfrak{p}$ such that $S_+ \notin \mathfrak{p}$;
- As a topological space, the closed sets are given by $V(I) = \{[\mathfrak{p}] \in \operatorname{Proj} S : I \subseteq \mathfrak{p}\}$, for homogeneous ideals $I \subseteq S_+$. Equivalently, the topology is given by the base of distinguished opens $D(f) = \{[\mathfrak{p}] \in \operatorname{Proj} S : f \notin \mathfrak{p}\}$, for homogeneous $f \in S_+$.
- As a locally ringed space, the structure sheaf is given on the base by $\mathcal{O}_{\operatorname{Proj} S}(D(f)) = (S_f)_{\deg 0}$.

Definition 4.1.2. Let S be a finitely generated graded ring over A. Then a scheme of the form $\operatorname{Proj} S$ is called a *projective scheme over A*. An quasicompact open subscheme of a projective A-scheme is called a *quasiprojective A-scheme*.

4.2. Properties of schemes.

Proposition 4.2.1. *Let X be a scheme. Then the points of X correspond bijectively to irreducible closed sets of X, via the map*

$$x \mapsto \overline{\{x\}}.$$

Proof. Because the closure of an irreducible set is irreducible, this is a well-defined map. Conversely, given an irreducible closed set $T \subseteq X$, consider an affine open U such that $T \cap U \neq \varnothing$. Then $T \cap U$ is an irreducible closed set in U, so it corresponds to a unique generic point in U. For affine opens U, V both intersecting T, $U \cap V$ must also intersect T because T is irreducible. Pick an affine open $W \subseteq U \cap V$ that is distinguished in both U and V and also intersects T. Then the unique generic point corresponding to $T \cap W$ must simultaneously be the unique generic points corresponding to $T \cap U$ and $T \cap V$. In other words, there is a unique point $x \in T$ that is the unique generic point corresponding to $T \cap U$ for all affine opens U intersecting T.

We claim that $T = \overline{\{x\}}$; indeed for any closed $K \subseteq X$ containing x, and for any point $t \in T$, there is an affine open U containing t (and by default containing x too), $K \cap U$ contains x, so it must contain $T \cap U$ (the closure of $\{x\}$ in $T \cap U$). In particular, $t \in K$ as well. $\square$

Proposition 4.2.2. *Let X be a quasicompact scheme, then any point has a closed point in its closure.*

Definition 4.2.1. A scheme X is called *reduced* if all stalks are reduced rings. Equivalently, for all open U, $\mathcal{O}_X(U)$ is reduced.

Definition 4.2.2. A property P for affine open subsets of a scheme X is called *affine-local* if it satisfies:

- If an affine open $\operatorname{Spec} A$ satisfies P, then any $\operatorname{Spec} A_f$ satisfies P also.
- If $f_1, \ldots, f_n \in A$ generate the unit ideal, and all $\operatorname{Spec} A_{f_i}$ satisfy P, then $\operatorname{Spec} A$ satisfies P as well.

Lemma 4.2.3 (Affine Communication Lemma)**.** *Suppose P is an affine-local property, and $X = \bigcup_{i \in I} \operatorname{Spec} A_i$ where each $\operatorname{Spec} A_i$ satisfies property P. Then any affine open in X satisfies P.*

Properties defined in this way:

- Locally Noetherian
- Noetherian
- Locally of finite type over B
- Finite type over B

4.3. **Varieties.** An affine scheme that is reduced and of finite type over k is called an *affine k-variety.* A reduced quasiprojective k-scheme called a *projective k-variety.*

4.4. **Normality and factoriality.** A scheme X is *normal* if all of its local rings are integrally closed domains.

Because being integrally closed is a local property, $\operatorname{Spec} A$ for A integrally closed is an affine normal scheme. For a quasicompact scheme, this can also be checked at closed points only.

A scheme X is *factorial* if all of its local rings are UFDs. Since UFDs are all integrally closed, factorial schemes are normal. Factoriality is not affine-local.

4.5. **Associated points.** In the affine case, the associated points of an A-module M are primes $\mathfrak{p} \subset A$ of the form $\mathfrak{p} = \operatorname{Ann}(m)$ for some $m \in M$. (See here; also, taking $M = A/I$, we recover the usual associated points of an ideal.) They have the following properties:

Theorem 4.5.1. *Suppose A is Noetherian and $M \neq 0$ is finitely generated. Then:*

(1) $\operatorname{Ass}(M)$ *is nonempty and finite.*
(2) The natural map $M \to \prod_{\mathfrak{p} \in \operatorname{Ass}(M)} M_\mathfrak{p}$ *is injective.*
(3) $\bigcup_{\mathfrak{p} \in \operatorname{Ass}(M)}$ *is precisely the set of zerodivisors of M.*
(4) Associated primes commute with localization:

$$\operatorname{Ass}_{S^{-1}A}(S^{-1}M) = \operatorname{Ass}_A(M) \cap \operatorname{Spec} S^{-1}A.$$

In general (see here):

Definition 4.5.1. Let X be a scheme, and F a quasicoherent sheaf. A point $x \in X$ is *associated to F* if $\mathfrak{m}_x$ is an associated point of the $\mathcal{O}_{X,x}$-module F_x.

Proposition 4.5.2. *Let X be locally Noetherian, F quasicoherent. Let $U = \operatorname{Spec} A$ be an affine open, $x \in U$ corresponds to $\mathfrak{p} \subset A$, $M = \Gamma(U, F)$, then $x \in \operatorname{Ass}(F) \iff \mathfrak{p} \in \operatorname{Ass}(M)$.*

Definition 4.5.2. Let X be a scheme, F a quasicoherent sheaf. An *embedded associated point* is an associated point that is not minimal.

Proposition 4.5.3. *Let X be locally Noetherian, and F coherent (e.g. $\mathcal{O}_X$). Then the generic points of irreducible components of $\operatorname{Supp} F$ are associated points, and the rest of the associated points are embedded.*

4.6. **Weakly associated points.**

5. Morphisms of schemes

5.1. **Morphisms to affine schemes.** These have a nice characterization:

Proposition 5.1.1. *The following are equivalent:*

- *There is a morphism of schemes* $X \to \operatorname{Spec} A$*;*
- *For every open* $U \subseteq X$*,* $\mathcal{O}_X(U)$ *is an* A*-algebra;*
- *There is a ring map* $A \to \mathcal{O}_X(X)$*.*

5.2. **Morphisms from affine schemes.** Given any point $p \in X$, there is a canonical morphism $\operatorname{Spec} \mathcal{O}_{X,p} \to X$. Composing this with the map induced by $\mathcal{O}_{X,p} \to \kappa(p)$, we get a canonical $\operatorname{Spec} \kappa(p) \to X$, often written just as $p \to X$.

More generally: for a local ring $(A, \mathfrak{m})$, a scheme morphism $\pi : \operatorname{Spec} A \to X$ sending $\mathfrak{m}$ to p corresponds bijectively to local homomorphisms $\mathcal{O}_{X,p} \to A$.

Definition 5.2.1 (functor of points)**.** Let Z be a scheme, the *Z-valued points* of X (denoted $X(Z)$) are the maps $Z \to X$. (When $Z = \operatorname{Spec} A$ or $\operatorname{Spec} k$, they are the A- or k-valued points.)

If we're working with schemes over a base scheme B, then this data should also include a $Z \to B$ making $Z \to X \to B$ commute.

5.3. **Functoriality of Proj.** Suppose $\phi : S \to R$ is a map of graded rings (i.e. there exists $\mathbb{N}_+$ such that S_n maps to R_{dn} for all n). This induces a morphism of schemes

$$\phi^* : (\operatorname{Proj} R)\backslash V(\phi(S_+)) \to \operatorname{Proj} S,$$

as follows: given $f \in S_+$, there is a map of rings $S_f \to R_{\phi(f)}$, hence a map of rings $(S_f)_{\deg 0} \to (R_{\phi(f)})_{\deg 0}$, hence a morphism of affine schemes $\operatorname{Spec}(R_{\phi(f)})_{\deg 0} \to \operatorname{Spec}(S_f)_{\deg 0}$, i.e. $D(\phi(f)) \to D(f) \hookrightarrow \operatorname{Proj} S$. These glue together to form the desired morphism of schemes.

In particular, if $V(\phi(S_+))$ is empty, then we get an actual morphism $\operatorname{Proj} R \to \operatorname{Proj} S$. This is satisfied when $\operatorname{rad}(\phi(S_+)) = R_+$. (Recall from §4.1 that the radical turns out to be equal to the intersection of all homogeneous primes containing the ideal.)

5.4. **Veronese subring.**

5.5. **The relative point of view.** Instead of thinking of properties of objects, it might be better to understand them as properties of morphisms between objects. For example, given a property P about schemes, one often turns it into a property about morphisms of schemes as follows: say $\pi : X \to Y$ has P if and only if for every affine open $U \subset Y$, $\pi^{-1}(U)$ has P.

5.6. **Green flags to look for in a property of morphisms.**

(1) It is *local on the target*: for a morphism $\pi : X \to Y$ and a open cover V_i of Y, π satisfies P iff all $\pi|_{\pi^{-1}(V_i)}$ satisfy P.
(2) It is closed under composition.
(3) It is closed under base change, pullback, fibered products, etc.
(4) ...

5.7. **Finiteness conditions on morphisms.** Recall that a scheme is called *quasicompact* if it is the union of finitely many affine schemes, and a scheme is called *quasiseparated* if the intersection of any two quasicompact open subsets is quasicompact. We turn them into properties of schemes as discussed in §5.5. These are both affine-local on the target and closed under composition. Conversely, a scheme X is quasicompact (resp. quasiseparated) if the canonical $X \to \operatorname{Spec} \mathbb{Z}$ is so. Note that many schemes we commonly encounter are qcqs: in particular, all affine schemes are qcqs, and all Noetherian schemes are qcqs.

Definition 5.7.1 (affine morphisms)**.** A morphism $\pi : X \to Y$ is *affine* if the preimage of any affine open in Y is affine open in X. Affine morphisms are automatically qcqs.

Lemma 5.7.1 (qcqs lemma)**.** *If X is qcqs, $s \in \mathcal{O}_X(X)$, then the natural map $\mathcal{O}_X(X)_s \to \mathcal{O}_X(X_s)$ is an isomorphism.*

Proof. Use the qcqs property as a finite presentation. ☐

Proposition 5.7.2. *Affineness is affine-local on the target. In other words, affineness of a morphism can be checked on affine covers of the target.*

Proof. ☐

Definition 5.7.2 (finite morphisms)**.** An affine morphism $\pi : X \to Y$ is *finite* if for any affine $\operatorname{Spec} A \subset Y$, $\pi^{-1}(\operatorname{Spec} A)$ is the spectrum of a ring that is a finitely generated module over A.

Finiteness is also affine-local on the target.

Example 5.7.3. Examples of finite morphisms:

- Branched covers: consider the map $k[u] \to k[t]$ given by $u \mapsto p(t)$ for a polynomial p. Then $\operatorname{Spec} k[t] \to \operatorname{Spec} k[u]$ is a finite morphism.
- Closed embeddings: A/I is a finite A-module (generated by 1), so $\operatorname{Spec} A/I \to \operatorname{Spec} A$ is a finite morphism.
- Normalization: $k[x,y]/(y^2 - x^2 - x^3) \mapsto k[t]$ by $x \mapsto t^2 - 1$, $y \mapsto t^3 - t$ induces a morphism of schemes $\operatorname{Spec} k[t] \to \operatorname{Spec} k[x,y]/(y^2 - x^2 - x^3)$. This is a finite morphism, and it is in fact an isomorphism from $D(t^2 - 1)$ to $D(x)$.

Proposition 5.7.3 (7.3.H)**.** *If $X \to \operatorname{Spec} k$ is a finite morphism, then X is a finite union of points with the discrete topology, each point with residue field a finite extension of k.*

Proof. We must have $X = \operatorname{Spec} A$, where A is a k-algebra that is finitely generated as a module. Then A is Noetherian and any prime $\mathfrak{p} \subset A$ is maximal, so the (finitely many) irreducible components of A, which correspond to minimal primes, are all closed points. Therefore $\operatorname{Spec} A$ is finite discrete, and the residue field at each point $[\mathfrak{p}]$ is a finite extension of k. ☐

Corollary 5.7.4 (7.3.K)**.** *Finite morphisms have finite fibers.*

Definition 5.7.4 (integral morphisms)**.** A morphism $\pi : X \to Y$ is *integral* if it is affine, and for every affine open $\operatorname{Spec} B \subset Y$, $\operatorname{Spec} A = \pi^{-1}(\operatorname{Spec} B)$, $B \to A$ is an integral extension.

Because integrality is an affine-local property, a morphism being integral is affine-local on the target. Also, finite morphisms are integral, and integral morphisms are closed (they map closed sets to closed sets).

Definition 5.7.5 (finite type morphisms)**.** A morphism $\pi : X \to Y$ is *locally of finite type* if for every affine open $\operatorname{Spec} B \subset Y$, and for every $\operatorname{Spec} A \subset \pi^{-1}(\operatorname{Spec} B)$, $B \to A$ expresses A as a finitely generated B-algebra. We say π is *finite type* if it is quasicompact and locally of finite type.

Proposition 5.7.5 (7.3.P)**.** *A morphism is finite iff it is integral and of finite type.*

Definition 5.7.6 (finitely presented morphisms)**.** A morphism $\pi : X \to Y$ is *locally finitely presented* if for every affine open $\operatorname{Spec} B \subset Y$, $\pi^{-1}(\operatorname{Spec} B) = \bigcup_i \operatorname{Spec} A_i$ with each $B \to A_i$ finitely presented. We say π is finitely presented if it is locally finitely presented and qc*qs*.

It is clear that if Y is locally Noetherian, then locally of finite presentation is the same as locally of finite type, and finite presentation is the same as finite type.

Proposition 5.7.6. *Locally finitely presented-ness is affine-local on both the target and the source.*

5.8. **Elimination theory.**

Lemma 5.8.1 (Generic freeness)**.** *Let B be a Noetherian integral domain, A a finite type algebra over B, and M a finitely generated A-module. Then there exists $f \in B$ such that M_f is a free B_f-module.*

Theorem 5.8.2 (Chevalley's theorem)**.** *Let $\pi : X \to Y$ be a finite type morphism between Noetherian schemes. Then the image of any constructible set is constructible.*

Theorem 5.8.3 (Fundamental theorem of elimination theory)**.** *The map $\mathbb{P}^n_A \to \operatorname{Spec} A$ is closed, for any ring A.*

5.9. **Closed subschemes, and related constructions.**

Definition 5.9.1. A *closed embedding* $\pi : X \hookrightarrow Y$ is an affine morphism where for each $\operatorname{Spec} B \subseteq Y$ and $\operatorname{Spec} A = \pi^{-1}(\operatorname{Spec} B)$, the induced ring map $B \to A$ is surjective.

Definition 5.9.2 (equivalent to the above)**.** A *closed embedding* $\pi : X \to Y$ is a morphism such that π induces a homeomorphism of the underlying topological space of X onto a closed subset of the topological space of Y, and the induced map $\pi^\sharp : \mathcal{O}_Y \to \pi_* \mathcal{O}_X$ of sheaves on Y is surjective.

Ideal sheaf, scheme-theoretic image, intersection and union of closed subschemes

5.10. **Effective Cartier divisors and regular sequences.**

Definition 5.10.1. A *locally principal* closed subscheme $\pi : X \hookrightarrow Y$ is one for which there exists an open cover U_i of Y, such that each $\pi^{-1}(U_i) \to U_i$ is isomorphic to a closed subscheme $V(s_i) \subset U_i$, where $s_i \in \mathcal{O}_Y(U_i)$. Equivalently, we may as well take all U_i to be affine.

Definition 5.10.2. An *effective Cartier divisor* is a locally principal closed subscheme where the ideal sheaf is locally generated near every point by a non-zero divisor.

Example 5.10.3. Consider $\operatorname{Spec} A$, where $A = k[w,x,y,z]/(wz - xy)$. Let X be the open subscheme $D(y) \cup D(w)$. The closed subscheme defined by $V(z/y)$ on $D(y)$ and $V(x/w)$ on $D(w)$ is an effective Cartier divisor, but it is not generated by a single element of $\operatorname{Frac} A$.

Definition 5.10.4. Let M be an A-module. A sequence $x_1, \dots, x_r$ of elements in A is called an *M-regular sequence* if:

- For each i, x_i is not a zero divisor for $M/(x_1, \dots, x_{i-1})M$ (exists no $m \in M \backslash (x_1, \dots, x_{i-1})M$ such that $mx_i \in (x_1, \dots, x_{i-1})M$), and
- $(x_1, \dots, x_r)M \neq M$.

In particular, an A-regular sequence is just called a regular sequence.

Example 5.10.5. For any M-regular sequence $x_1, \dots, x_n$, and positive integers $a_1, \dots, a_n$, the sequence $x_1^{a_1}, \dots, x_n^{a_n}$ is a regular sequence too.

Example 5.10.6. Let $A = k[x,y,z]/(x-1)z$. Then $x, (x-1)y$ is a regular sequence, while $(x-1)y, x$ is not.

Theorem 5.10.1. *Let A be a Noetherian local ring, and M a finitely generated A-module. Then any M-regular sequence remains regular when reordered.*

Definition 5.10.7 (regular embedding)**.** Let $\pi : X \to Y$ be a locally closed embedding. Say that π is a *regular embedding* of codimension r at $x \in X$ if in $\mathcal{O}_{Y,\pi(x)}$, the ideal of X is generated by a regular sequence of length r. Say that π is a *regular embedding* if it is at all points.

5.11. **Fiber products.**

5.12. **An interlude on closed points.**

Proposition 5.12.1. *Let X be a scheme locally of finite type over a field k. If $x \in \operatorname{Spec} A \subset X$ corresponds to a maximal ideal in some affine open subscheme of X, then x is a closed point in X.*

Proof. Suppose x corresponds to $\mathfrak{m} \subset A$, then $\kappa(x) = A/\mathfrak{m}$. By the nullstellensatz, $A/\mathfrak{m}$ is a finite extension of k. Now, suppose $\operatorname{Spec} B \subset X$ is some other affine open containing x, and say x corresponds to a prime $\mathfrak{p} \subset B$. Then $\kappa(x) = \operatorname{Frac} B/\mathfrak{p}$, so in particular $k \subseteq B/\mathfrak{p} \subseteq \kappa(x)$. So $B/\mathfrak{p}$ is an integral extension of k, so it is a field as well, i.e. $\mathfrak{p}$ is maximal. So $\{x\}$ is closed in X. $\square$

Proposition 5.12.2. *Let X be a scheme locally of finite type over k. Suppose we have a morphism $\pi : \operatorname{Spec} k \to X$, then its image is a closed point.*

Proof. Let $\operatorname{Spec} A \subset X$ be an affine open subscheme. The morphism π factors through $\operatorname{Spec} A$, so we get $\phi : \operatorname{Spec} k \to \operatorname{Spec} A$. Suppose $\mathfrak{m}$ is the kernel of the corresponding map $A \to k$, and $\mathfrak{p}$ is the prime ideal corresponding to the image of π. Then we get a map of stalks $A_{\mathfrak{p}} \to k$ through which the map $A \to k$ factors. Suppose $a \notin \mathfrak{p}$, then a is invertible in $A_{\mathfrak{p}}$, so it is not in the kernel of $A \to k$, so $\mathfrak{m} \subseteq \mathfrak{p}$. Since $\mathfrak{m}$ is maximal, $\mathfrak{m} = \mathfrak{p}$, so we conclude by the previous proposition. $\square$

Proposition 5.12.3. *Let X be a scheme locally of finite type over $k = \overline{k}$. Then closed points of X are in bijection with k-points of X.*

Proof. The bijection is given by:

- Given a k-point $\operatorname{Spec} k \to X$, this maps to its image, which is a closed point in X;
- Given a closed point $x \in X$, its field of fractions is k by the nullstellensatz, so we get $\operatorname{Spec} k = \operatorname{Spec} \kappa(x) \to X$.

It suffices to verify that these two are inverses. Given a closed point $x \in X$, it is clear from definition that the image of $\operatorname{Spec} \kappa(x) \to X$ is x. On the other hand, given a k-point $\operatorname{Spec} k \to X$, it is given by $\operatorname{Spec} k \to \operatorname{Spec} A \subseteq X$, where A/k is the maximal ideal corresponding to the image x of the k-point. So $A/k = \kappa(x)$, which finishes the proof. $\square$

Corollary 5.12.4. *Let $f : X \to Y$ be a morphism between schemes over k locally of finite type. Then f maps closed points to closed points. In particular, maps between k-schemes map closed points to closed points.*

5.13. **Separated morphisms.**

Definition 5.13.1. A morphism of schemes $\pi : X \to Y$ is *separated* if the diagonal map $\Delta_\pi : X \to X \times_Y X$ is a closed embedding.

To see that this definition isn't too crazy, we notice the following.

Proposition 5.13.1. *Let $\pi : X \to Y$ be a morphism. The diagonal $\Delta_\pi : X \to X \times_Y X$ is a locally closed embedding (i.e. a closed subscheme of an open subscheme).*

Proof. Cover Y by affine opens V_i, and $\pi^{-1}(V_i)$ by affine opens U_{ij}. Then $U_{ij} \times_{V_i} U_{ij}$ is an affine open subscheme of $X \times_Y X$ by definition, and these cover the image of Δ_π. Further, it is clear that $\Delta_\pi^{-1}(U_{ij} \times_{V_i} U_{ij}) = U_{ij}$, and $\Delta|_{U_{ij}}$ is a closed embedding. $\square$

Definition 5.13.2. A *variety* over a field k is a reduced, separated, finite-type k-scheme.

Because a locally closed embedding whose image is closed is in fact a closed embedding, to check that $\pi : X \to Y$ is separated, it suffices to check that the image of Δ is closed.

Examples of separated morphisms:

- Locally closed embeddings (also called *immersions*);
- Morphisms between affine schemes;
- All quasiprojective A-schemes (with morphism to $\operatorname{Spec} A$);

- Any morphism between varieties is automatically separated and finite type (this will follow from the cancellation theorem).

Lemma 5.13.2 (Magic diagram)**.** *Let X_1, X_2, Y, Z be objects in a category where fiber products exist. Suppose we are given maps $f_1 : X_1 \to Y$, $f_2 : X_2 \to Y$, and $g : Y \to Z$. Then the following diagram is a Cartesian square:*

$$\begin{tikzcd} X_1 \times_Y X_2 \arrow[r] \arrow[d] & X_1 \times_Z X_2 \arrow[d, "f_1 \times f_2"] \\ Y \arrow[r, "\Delta"] & Y \times_Z Y. \end{tikzcd}$$

Proposition 5.13.3. *Let X be separated over a ring A. Then for $U, V \subset X$ affine opens, $U \cap V$ is an affine open as well.*

Proof. Consider the following fiber product:

$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d] & U \times_A V \arrow[d] \\ X \arrow[r, hook, "\Delta"] & X \times_A X \end{tikzcd}$$

Here, $U \cap V = U \times_X V$ is the fiber product because of the magic diagram. Now, because the bottom map is a closed embedding, so is the top map. Since $U \times_A V$ is an affine scheme, so is $U \cap V$. ☐

Proposition 5.13.4. *Separatedness is well-behaved:*

(1) affine-local on the target;
(2) stable under composition;
(3) stable under base change.

Proof. (1) This follows from the fact that $\pi : X \to Y$ is separated if and only if $\mathrm{im}(\Delta)$ is closed.

(2) Suppose $f : X \to Y, g : Y \to Z$ are separated. Consider the following commutative diagram

$$\begin{tikzcd} X \arrow[r, hook, "\Delta_f"] & X \times_Y X \arrow[r] \arrow[d] & X \times_Z X \arrow[d] \\ & Y \arrow[r, hook, "\Delta_g"] & Y \times_Z Y \end{tikzcd}$$

The square is Cartesian by the magic diagram, so the top map $X \times_Y X \to X \times_Z X$ is a closed embedding. So the composition $X \to X \times_Z X$, which can be verified to be the diagonal of $g \circ f$, is a closed embedding.

(3) Suppose

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & W \end{tikzcd}$$

is a pullback square, where $Z \to W$ is separated. It suffices to show that

$$\begin{tikzcd} X \arrow[r, "\Delta"] \arrow[d] & X \times_Y X \arrow[d] \\ Z \arrow[r, "\Delta"] & Z \times_W Z \end{tikzcd}$$

is also a pullback square, which is a straightforward diagram chase. ☐

Proposition 5.13.5. *Let $\pi : X \to Y$ be a morphism of Z-schemes, and $Y \to Z$ separated. Then its* graph $\Gamma_\pi : X \xrightarrow{(\mathrm{id}, \pi)} X \times_Z Y$ *is a closed embedding.*

Proposition 5.13.6 (Cancellation theorem). *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$, and suppose P is a property of morphisms, such that:*

- *P is stable under composition;*
- *P is stable under base change;*
- *$g \circ f$ satisfies P;*
- *$\Delta_g : Y \to Y \times_z Y$ satisfies P.*

Then f satisfies P also.

Proof. We have the following Cartesian squares:

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{\pi} & Y \\ \downarrow & & \downarrow g \\ X & \xrightarrow{g \circ f} & Z. \end{array}$$

Here, because $g \circ f$ satisfies P, so does $\pi : X \times_Z Y \to Y$. Also, we have

$$\begin{array}{ccc} X = X \times_Y Y & \xrightarrow{\Gamma} & X \times_Z Y \\ \downarrow f & & \downarrow \\ Y & \xrightarrow{\Delta_g} & Y \times_Z Y, \end{array}$$

and because Δ_g satisfies P, so does Γ. But $\pi \circ \Gamma$ is easily verified to be simply f, so f satisfies P also. $\square$

Theorem 5.13.7 (Reduced to separated theorem). *Suppose X, Y are schemes over Z, where X is reduced, and $Y \to Z$ is separated. Let $\pi, \pi' : X \to Y$ be morphisms over Z. Suppose $U \subseteq X$ is a dense open on which π and π' agree. Then $\pi = \pi'$.*

Proof. Let V be the fiber product

$$\begin{array}{ccc} V & \longrightarrow & Y \\ \downarrow & & \downarrow \Delta \\ X & \xrightarrow{(\pi;\pi')} & Y \times_Z Y. \end{array}$$

Because Δ is a closed embedding, so is $V \hookrightarrow X$. Because $\pi|_U = \pi'|_U$, we get a map $U \to V$ through the universal property of V. But U is an open subscheme of X. Because U is dense, $V = X$ as sets. Because X is reduced, $V = X$ as schemes. So $\pi = \pi'$ on all of X. $\square$

5.14. **Dominant rational maps between irreducible varieties.**

Definition 5.14.1. A *rational map* between schemes $X \dashrightarrow Y$ is a map $U \to Y$ where U is a dense open in X. Two rational maps $X \dashrightarrow Y$ are *equivalent* if $\alpha|_W = \beta|_W$ on some dense open $W \subseteq U \cap V$.

Definition 5.14.2. A morphism of schemes is *dominant* if its image is dense.

Fix a field k (algebraically closed when necessary), consider the category of irreducible varieties over k, with morphisms as dominant rational maps.

Given an irreducible variety X, because irreducible and reduced implies integral, it has a unique generic point η. The stalk at η is the *function field* $K(X)$, which is equal to the fraction field $\operatorname{Frac} A$ of any affine open $\operatorname{Spec} A \subseteq X$. Given a rational map $X \dashrightarrow Y$, this induces a field homomorphism at the stalks of the generic points.

Theorem 5.14.1. *The functor described above gives an equivalence of categories between irreducible varieties with dominant rational maps and finitely generated field L/k with inclusions of fields.*

5.15. **Ax-Grothendieck theorem.**

Theorem 5.15.1 (Ax-Grothendieck)**.** *Let X be a variety over $\mathbb{C}$, $f : X \to X$ a morphism over $\mathbb{C}$. Suppose that the map of $\mathbb{C}$-points $X(\mathbb{C}) \to X(\mathbb{C})$ is injective (as a set), then it must be surjective.*

We will define the *spreading out* of X, which is a finite type scheme over $\operatorname{Spec} R$, for some finitely generated $\mathbb{Z}$-algebra $R \subset \mathbb{C}$.

Cover X by (finitely many, since X is quasicompact) affine schemes U_i, which are of the form $\operatorname{Spec} \mathbb{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$ since X is finite type and by Hilbert's basis theorem. Because X is separated, $U_i \cap U_j$ is also affine of the above form. Even further, each $f^{-1}(U_i)$ is covered by finitely many affine opens U_{ij}, because morphisms between varieties are automatically quasicompact, and the U_{ij}'s are again of the above form. So we can take R to be the $\mathbb{Z}$-algebra generated by all coefficients of f_i appearing in U_i, $U_i \cap U_j$, and U_{ij}'s, and define $\mathcal{X}$ by glueing together $\operatorname{Spec} R[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$. The map $f : X \to X$ also spreads out to a map $F : \mathcal{X} \to \mathcal{X}$. By definition, this satisfies the following Cartesian squares:

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & \mathcal{X} \arrow[d, "F"] \\
X \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\
\operatorname{Spec} \mathbb{C} \arrow[r] & \operatorname{Spec} R.
\end{tikzcd}$$

Now, set $U = X \times_{\mathbb{C}} X \backslash \Delta(X)$, an open subscheme of $X \times_{\mathbb{C}} X$. Let W be the fiber product

$$\begin{tikzcd}
W \arrow[rr] \arrow[d] & & X \arrow[d] \\
U \arrow[r, hook] & X \times_{\mathbb{C}} X \arrow[r, "f \times_{\mathbb{C}} f"] & X \times_{\mathbb{C}} X,
\end{tikzcd}$$

and supposing $x \in X$ is a point, let Z be the fiber

$$\begin{tikzcd}
Z \arrow[r] \arrow[d] & \operatorname{Spec} \mathbb{C} \arrow[d, "x"] \\
X \arrow[r, "f"] & X.
\end{tikzcd}$$

Then $X(\mathbb{C}) \to X(\mathbb{C})$ is injective implies that $W = \varnothing$, and we wish to show surjectivity at x, i.e. $Z \neq \varnothing$.

Because spreading out behaves well with fiber products, we can similarly define the spread-out of x, U, W, Z as $\chi, \mathcal{U}, \mathcal{W}, \mathcal{Z}$.

Sketch of proof:

- Reduce the problem into showing that given $W = \varnothing$, show that $Z \neq \varnothing$.
- Spread out to get $\mathcal{X}, F, \chi, \mathcal{U}, \mathcal{W}, \mathcal{Z}$.
- $W = \varnothing$ implies $\mathcal{W} \times_R K = \varnothing$, where $K = \operatorname{Frac} R$. This implies the image of $\pi_{\mathcal{W}} : \mathcal{W} \to \operatorname{Spec} R$, which is constructible by Chevalley, does not include the generic point η_R. So $\eta_R \notin \overline{\operatorname{im} \pi_{\mathcal{W}}}$, so we may invert finitely many elements of R so that $\operatorname{im} \pi_{\mathcal{W}} = \varnothing$, i.e. $\mathcal{W} = \varnothing$.
- Let t be a closed point in $\operatorname{Spec} R$, $F_t : \mathcal{X}_t \to \mathcal{X}_t$ be the induced map. Then $\kappa(t)$ is a finite field $\mathbb{F}_q$. Because $\mathcal{W}_t(\mathbb{F}_q) = \varnothing$, the map $\mathcal{X}_t(\mathbb{F}_q) \to \mathcal{X}_t(\mathbb{F}_q)$ is injective, hence surjective. So $\mathcal{Z}_t \neq \varnothing$ for all closed points t.
- So $\pi_{\mathcal{Z}} : \mathcal{Z} \to \operatorname{Spec} R$ has image containing all closed points, which are dense in $\operatorname{Spec} R$. So the generic point η_R is contained in the image, which is constructible by Chevalley. So $\mathcal{Z} \times_R K \neq \varnothing$, which implies $Z \neq \varnothing$. This concludes the proof.

Lemma 5.15.2. *Let S be a constructible set in $\operatorname{Spec} R$, R Noetherian. If $\eta_R \notin S$, then $\eta_R \notin \overline{S}$.*

Proof. Write $S = \coprod_i (U_i \cap K_i)$ as the disjoint union of locally closed sets over a finite index set. Then $\overline{S} = \bigcup_i \overline{U_i \cap K_i}$. Suppose for contradiction $\eta_R \in \overline{U_i \cap K_i}$ for some i, then $\operatorname{Spec} R = \overline{U_i \cap K_i} \subseteq K_i$, so $K_i = \operatorname{Spec} R$ and U_i is a dense open in $\operatorname{Spec} R$, so $\eta_R \in U_i$, which implies $\eta_R \in S$, a contradiction. $\square$

Lemma 5.15.3. *Let $k \subseteq \mathbb{C}$ be a subfield, V a k-variety. Then the following are equivalent:*

- $V = \varnothing$*;*
- $V_{\mathbb{C}} := V \times_k \mathbb{C} = \varnothing$*;*
- $V_{\mathbb{C}}(\mathbb{C}) = \varnothing$*.*

5.16. **Proper maps.** Just as separatedness captures the topological concept of a Hausdorff space, properness is meant to capture the concept of compactness. Of course, quasicompactness won't do the job. Recall the topological notion:

Definition 5.16.1. A map of topological spaces is *proper* if the inverse image of any compact set is compact.

Definition 5.16.2. A *universally closed* map $f : M \to N$ of topological spaces is one such that for all $P \to N$, $f_P : P \times_N M \to P$ is a closed map.

We remark that the map from M to a point is universally closed iff M is compact.
The same definition moves over to schemes:

Definition 5.16.3. A *universally closed* morphism $f : X \to Y$ of schemes is one such that for all $Z \to Y$, $f_Z : Z \times_Y X \to Z$ is a closed morphism.

Definition 5.16.4. A morphism of schemes $\pi : X \to Y$ is *proper* if it is finite type, separated, and universally closed.

So, $X \to \operatorname{Spec} k$ being universally closed corresponds to X being "compact".

Example 5.16.5. Examples of proper morphisms:

- Closed embeddings;
- Properness is stable under composition and base change;
- $\mathbb{P}^n_A \to \operatorname{Spec} A$ is proper; as a consequence, any projective morphism $Z \hookrightarrow \mathbb{P}^n_A \to \operatorname{Spec} A$ is proper.
- In contrast, $\mathbb{A}^1_{\mathbb{C}}$ is not proper (this fits your intuition that a line is not compact). This can be seen by the following square:

$$\begin{tikzcd} \mathbb{A}^2 \arrow[r] \arrow[d] & \mathbb{A}^1 \arrow[d] \\ \mathbb{A}^1 \arrow[r] & \bullet \end{tikzcd}$$

  But the left map is not closed: $V(xy-1)$ maps to $D(x)$, which is not closed.

5.17. **Chow's lemma.** Chow's lemma says that "a proper morphism is fairly close to being a projective morphism". Note that by the fundamental theorem of elimination theory, projective morphisms are proper.

Theorem 5.17.1 (Chow)**.** *Let $f : X \to S$ be a separated, finite type morphism of Noetherian schemes. Then for there exists a diagram*

$$\begin{tikzcd} X' \arrow[r, hook, "i"] & \mathbb{P}^n_X \arrow[r, "f'"] \arrow[d, "\pi'"] & \mathbb{P}^n_S \arrow[d, "\pi"] \\ & X \arrow[r, "f"] & S \end{tikzcd}$$

where the square is Cartesian, i is a closed immersion, $f' \circ i$ is an immersion, and $\pi' \circ i$ is surjective and induces an isomorphism on a dense open set $U \subseteq X$.

In the case f is proper, f' must then be closed, so X' is a projective S-scheme that surjects onto X and is an isomorphism over a dense open of X.

5.18. **Valuative criteria.**

Theorem 5.18.1 (valuative criteria)**.** *We have the following criteria:*

- *Let $f : X \to Y$ be quasiseparated, then f is separated iff for every valuation ring V with field of fractions K, $X_Y(V) \to X_Y(K)$ is injective.*
- *Let $f : X \to Y$ be quasicompact, then f is universally closed iff for every valuation ring V with field of fractions K, $X_Y(V) \to X_Y(K)$ is surjective.*
- *Let $f : X \to Y$ be quasiseparated and finite type, then f is proper iff for every valuation ring V with field of fractions K, $X_Y(V) \to X_Y(K)$ is bijective.*

(Aside: in fact, universally closed implies quasicompact. Also, a map of schemes is a closed immersion if and only if it is a proper monomorphism.)

6. Dimension and smoothness

6.1. **Definitions of dimension.** The Krull dimension of a scheme is a purely topological construction and does not depend on the sheaf structure.

Lemma 6.1.1. *Let X be a topological space, $U \subseteq X$ open. Then there is a bijection between closed irreducible subsets of U and closed irreducible subsets of X that meet U, given by*

$$K \subseteq U \longmapsto \overline{K} \subseteq X$$
$$L \cap U \subseteq U \longleftarrow L \subseteq X.$$

Proof. First, we show that given a closed irreducible set $K \subseteq X$ that meets U, $\overline{K \cap U} = K$. Because K meets U, $K \cap U^c \neq K$, so because $K = \overline{K \cap U} \cup (K \cap U^c)$ is irreducible, $K = \overline{K \cap U}$.

Next, we show that given a closed subset $K \subset U$, $\overline{K} \cap U = K$. Clearly $K \subseteq \overline{K} \cap U$. Since K is closed in U, $K = L \cap U$ for some closed $L \subseteq X$. Then $\overline{K} \cap U \subseteq L \cap U = K \subseteq \overline{K} \cap U$, so equality holds.

Now we are ready to show the bijection. It suffices to show both maps are well-defined, since the above two paragraphs shows that the two maps are inverses of each other. Given a closed irreducible $K \subset U$, it is clear that its closure $\overline{K}$ is closed in X and meets U. To show it is irreducible, suppose $\overline{K} = C_1 \cup C_2$ for closed C_1, C_2. Then $K = \overline{K} \cap U = (C_1 \cap U) \cup (C_2 \cap U)$, so WLOG $C_1 \cap U = K$. Then $C_1 \subseteq \overline{K} = \overline{C_1 \cap U} \subseteq C_1$, so equality holds and $C_1 = \overline{K}$.

Conversely, given a closed irreducible $L \subseteq X$ that meets U, $L \cap U$ is closed in U. To show it is irreducible, suppose $L \cap U = (C_1 \cap U) \cup (C_2 \cap U)$, where $C_1, C_2 \subseteq X$ are closed. Then

$$L = \overline{L \cap U} = \overline{(C_1 \cap U) \cup (C_2 \cap U)} = \overline{C_1 \cap U} \cup \overline{C_2 \cap U},$$

so WLOG $\overline{C_1 \cap U} = L$. Then $L \cap U = \overline{C_1 \cap U} \cap U = C_1 \cap U$. This shows $L \cap U$ is irreducible, which completes the proof. □

Corollary 6.1.2. *Suppose $X = \bigcup_i U_i$ is an open cover of a topological space. Then*

$$\dim X = \sup_i \dim U_i.$$

In particular, the dimension of a scheme can be checked on any affine open cover.

Proof. Consider any sequence

$$\varnothing \neq Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n \subseteq X,$$

where Z_i are irreducible and closed. Because $Z_0 \neq \varnothing$, there exists U_i such that $Z_0 \cap U_i \neq \varnothing$. Then

$$\varnothing \neq Z_0 \cap U_i \subsetneq Z_1 \cap U_i \subsetneq \cdots \subsetneq Z_n \cap U_i \subseteq U_i$$

is also a chain of irreducible closed sets by the above lemma. This shows $\dim X \leq \sup_i \dim U_i$. Conversely, for any i and a chain of irreducible closed subsets

$$\varnothing \neq Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n \subseteq U_i$$

of U_i,

$$\varnothing \neq \overline{Z_0} \subsetneq \overline{Z_1} \subsetneq \cdots \subsetneq \overline{Z_n} \subseteq X$$

is a chain of irreducible closed sets in X, again by the above lemma. So $\dim X \geq \sup_i \dim U_i$, so equality holds. □

Definition 6.1.1. The *codimension* $\operatorname{codim}_X Y$ of an irreducible subset $Y \subseteq X$ is the supremum of lengths of increasing chains of irreducible closed subsets starting with $\overline{Y}$. The corresponding ring-theoretic notion is the *height* $\operatorname{ht} \mathfrak{p}$ of a prime ideal $\mathfrak{p}$.

Warning: Noetherian rings can be infinite-dimensional. On the other hand, Noetherian local rings must have finite dimension.

Theorem 6.1.3 (Krull's height theorem)**.** *Let A be a Noetherian ring, I a proper ideal generated by r elements, then every minimal prime of I has height at most r.*

Theorem 6.1.4 (Algebraic Hartogs's Lemma)**.** *Let A be a Noetherian integrally closed domain. Then*

$$A = \bigcap_{\operatorname{ht} \mathfrak{p} = 1} A_{\mathfrak{p}}.$$

Intuitively, this says that on a normal Noetherian scheme, a rational function that is regular outside a closed set of codimension at least 2 can be uniquely extended to a regular function on the whole scheme. Compare this with Hartogs's lemma in complex analysis.

Proof. This is trivially true when $\dim A \leq 1$. In general, suppose for contradiction $x \in \operatorname{Frac} A$ belongs to $A_{\mathfrak{p}}$ for every prime of height 1, and $x \notin A$. Let $I = \{a \in A : ax \in A\}$, then $1 \notin I$, so there exists a minimal prime $\mathfrak{q} \supseteq I$. Because $I_{\mathfrak{q}} = \{a \in A_{\mathfrak{q}} : ax \in A_{\mathfrak{q}}\}$ is not equal to $A_{\mathfrak{q}}$, we see that $\mathfrak{q}$ has height at least 2.

Localize at $\mathfrak{q}$ to assume WLOG that $(A, \mathfrak{q})$ is a local ring and $\mathfrak{q}$ is the unique prime containing I. Then $\mathfrak{q} = \operatorname{rad}(I)$, and because A is Noetherian, $\mathfrak{q}$ is finitely generated, so $I \supseteq \mathfrak{q}^n$ for some n. Take the smallest such n. Consider an element $t \in \mathfrak{q}^{n-1} \backslash I$, and let $z = xt$. Because $t \notin I$, $z = xt \notin A$, but $z\mathfrak{q} \subseteq x\mathfrak{q}^n \subseteq xI \subseteq A$.

Now, if $z\mathfrak{q} \not\subseteq \mathfrak{q}$, then $z\mathfrak{q} = A$, so $\mathfrak{q} = \frac{1}{z}A$ is a principal ideal, contradicting $\operatorname{ht} \mathfrak{q} \geq 2$. So we conclude that $z\mathfrak{q} \subseteq \mathfrak{q}$, and we have a faithful $A[z]$-action on the finitely generated A-module $\mathfrak{q}$, so z is integral over A. But A is integrally closed, so $z \in A$, a contradiction. □

6.2. **Dimension of fibers.** The main theorem here is the following:

Theorem 6.2.1. *Let X, Y be irreducible varieties, $\pi : X \to Y$ a dominant map. Suppose* $\dim X = a$, $\dim Y = b$. *Then:*

- *For any $y \in \operatorname{im} \pi$, $\dim \pi^{-1}(y) \geq a - b$.*
- *There exists a dense open $U \subset Y$, such that for any $y \in U$, $\dim \pi^{-1}(y) = a - b$.*
- *Given a point $x \in X$, define $e(x)$ to be the maximal $\dim Z$, where Z ranges among the irreducible components of $\pi^{-1}(\pi(x))$ containing x. Then $e(x)$ is an upper semi-continuous function: the sets $X_n = \{x \in X : e(x) \geq n\}$ are closed.*

6.3. Cotangent and tangent spaces.

Proposition 6.3.1. *Let X be a scheme, $f \in \mathcal{O}_x(X)$, $p \in V(f)$ a closed point, and $\overline{f}$ the image of f in $T^\vee_{X,p}$. Then*

$$T^\vee_{V(f),p} = T^\vee_{X,p}/\langle \overline{f} \rangle$$

Proposition 6.3.2 (Jacobian computes Zariski cotangent space)**.** *Let X be a finite type k-scheme, so that locally it is* $\operatorname{Spec} k[x_1, \dots, x_n]/(f_1, \dots, f_r)$. *Then for any closed point p, $T^\vee_{X,p} = \operatorname{coker} J$, where $J : k^r \to k^n$ is the linear map given by the Jacobian matrix*

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(p) & \cdots & \frac{\partial f_r}{\partial x_1}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial x_n}(p) & \cdots & \frac{\partial f_r}{\partial x_n}(p) \end{bmatrix}.$$

Proof. Translate p to the origin, and use the previous proposition repeatedly. □

Given a morphism of schemes $f : X \to Y$, mapping $p \in X$ to $q \in Y$, there is a naturally induced ring map $T^\vee_{Y,q} \to T^\vee_{X,p}$. If $\kappa(p) = \kappa(q)$, the above is a linear map, and we also get a map $T_{X,p} \to T_{Y,q}$.

6.4. Regularity and smoothness.

Proposition 6.4.1. *For a Noetherian local ring $(A, \mathfrak{m}, k)$, $\dim A \le \dim_k \mathfrak{m}/\mathfrak{m}^2$.*

Proof. By Nakayama, a set of generators of $\mathfrak{m}/\mathfrak{m}^2$ over k lifts to a set of generators of $\mathfrak{m}$, which is at least $\operatorname{ht} \mathfrak{m} = \dim A$. □

Definition 6.4.1 (regular local ring)**.** A *regular local ring* is a Noetherian local ring $(A, \mathfrak{m}, k)$ such that $\dim A = \dim_k \mathfrak{m}/\mathfrak{m}^2$.

Definition 6.4.2 (regularity)**.** A locally Noetherian scheme X is *regular* at $p \in X$ if $\mathcal{O}_{X,p}$ is a regular local ring. The word *nonsingular* is synonymous. Otherwise, we say X is *singular* at p.

X is *regular* if it is regular at all points, and it is *singular* otherwise.

Example 6.4.3. Regular local rings of dimension 0 are fields, while regular local rings of dimension 1 are DVRs.

Proposition 6.4.2 (Jacobian criterion)**.** *Suppose $X = \operatorname{Spec}[x_1, \dots, x_n]/(f_1, \dots, f_r)$ has pure dimension d. (As usual, $k = \overline{k}$). Then a k-point $p \in X$ is regular iff $\dim \operatorname{coker} J(p) = d$ at p.*

Proof. We know $\dim T^\vee_{X,p} = \dim \operatorname{coker} J(p) = d$. So it suffices to show that $\dim \mathcal{O}_{X,p} = d$. But this is clear since p is a closed point and X has pure dimension d. □

In fact, for finite-type k-schemes, it suffices to check regularity at closed points (this is a hard fact). So for such schemes, regular of pure dim d is equivalent to every irreducible component having dim d and $\dim \operatorname{coker} J(p) = d$ for all k-points p. But this still requires $k = \overline{k}$. For general k, we have an alternate notion of *smoothness over* $\operatorname{Spec} k$:

Definition 6.4.4. A scheme X/k is *smooth of dimension d over k* if there exists an affine cover by $\operatorname{Spec} k[x_1, \dots, x_n]/(f_1, \dots, f_r)$, for which the Jacobian matrix has $\dim \operatorname{coker} = d$ at all points.

Remark: k-smoothness is equivalent to the Jacobian being corank d everywhere for every affine open cover (and by any choice of generators of the ring corresponding to such an open set).

Regularity/smoothness correspond to the notion of "smoothness" in the world of manifolds. So:

schemes	manifolds
Separated	Hausdorff
Universally closed	Compact
Proper	Compact + Hausdorff
Krull dimension	Dimension
Zariski (co)tangent space	(Co)tangent space
Regular, smooth	Smooth
Singular	Singular

More or less by definition, for a finite type scheme X/k of pure dim d, where $k = \overline{k}$, X is regular at all closed points iff X is smooth over k.

Theorem 6.4.3. *Comparison between regularity and smoothness:*

(a) If k is perfect, every regular finite type k-scheme is smooth over k.
(b) Every smooth k-scheme is regular (with no hypotheses on perfection).

Example 6.4.5. Let $k = \mathbb{F}_p(t)$, L a field extension given by $L = k[x]/(x^p - t)$. Let $X = \operatorname{Spec} L$, then it is regular since L is a field. But it is not smooth of dimension 0 since the derivative of $x^p - t$ vanishes.

Theorem 6.4.4. *Regular local rings are domains, so regular implies reduced. (In fact, they are UFDs, but this is a much harder fact.)*

6.5. Bertini's theorem.

Theorem 6.5.1 (Bertini)**.** *Suppose X is a smooth subvariety of $\mathbb{P}^n_k$. Then there is a dense open $U \subseteq \mathbb{P}^{n\vee}_k$ such that for any closed point $H \in U$ (corresponding to a hyperplane in $\mathbb{P}^n_k$), H does not contain any irreducible component of X, and $H \cap X$ is k-smooth.*

7. Quasicoherent sheaves

7.1. Basic definitions.

Definition 7.1.1. Let X be a scheme. A *quasicoherent sheaf* $\mathcal{E}$ on X is an $\mathcal{O}_X$-module where there exists an affine cover $\{U_i = \operatorname{Spec} A_i \subseteq X\}$, such that $\mathcal{E}|_{U_i} \cong \widetilde{M_i}$ for A_i-modules M_i.

Proposition 7.1.1. *Let $X = \operatorname{Spec} A$, $\mathcal{E}$ a quasicoherent sheaf on X, then $\mathcal{E} \cong \widetilde{M}$ for $M = \Gamma(X, \mathcal{E})$.*

Proof. Define $\phi : \widetilde{M} \to \mathcal{E}$ on each $D(f)$ by the natural map $M_f \to \Gamma(D(f), \mathcal{E})$. Check that these are bijections using the sheaf axioms. □

Definition 7.1.2. Let X be Noetherian, then $\mathcal{E}$ is a *coherent* sheaf if there exists an affine cover $\{U_i = \operatorname{Spec} A_i \subseteq X\}$, such that $\mathcal{E}|_{U_i} \cong \widetilde{M_i}$ for *finitely generated* A_i-modules M_i.

Warning: locally free of rank r is not an affine local condition.

Proposition 7.1.2. *There is an equivalence of categories A-$\mathsf{Mod} \longleftrightarrow \mathsf{QCoh}(\operatorname{Spec} A)$.*

Corollary 7.1.3. *Exact sequences of qcoh sheaves implies exactness on* affine *opens.*

Example 7.1.3. Tensor product of qcoh sheaves: on affine opens, $(\mathcal{E}_1 \otimes \mathcal{E}_2)(U) \cong \mathcal{E}_1(U) \otimes \mathcal{E}_2(U)$. This is the same as the sheafification of the obvious presheaf tensor product.

Proposition 7.1.4. *Let $\mathcal{F}$ be a finite type qcoh sheaf on X, then its rank at a point is upper-semicontinous on X.*

7.2. f_* and f^*.

Proposition 7.2.1. *Let $f : X \to Y$ be qcqs. If $\mathcal{E} \in \mathsf{QCoh}(X)$, then $f_*\mathcal{E} \in \mathsf{QCoh}(Y)$.*

Definition 7.2.1. f^* in the affine case: for $f : \operatorname{Spec} A \to \operatorname{Spec} B$, $\mathcal{F} = \widetilde{N}$, then $f^*\mathcal{F} = \widetilde{A \otimes_B N}$.

In general, cover $f : X \to Y$ by $f|_U : U \to V$ between affine opens. Pull $\mathcal{F}$ back on each of them, and glue together by universal property. Quasicoherence is obvious.

Proposition 7.2.2. *$f^* \dashv f_*$.* □

Proposition 7.2.3. *The pullback f^* sends coherent sheaves (resp. locally free of rank r) on Y to coherent sheaves (resp. locally free of rank r) on X.*

Proposition 7.2.4 (base change map)**.**

Proposition 7.2.5 (projection formula)**.** *Let $\pi : X \to Y$ be qcqs, and $\mathcal{F}, \mathcal{G}$ QCoh sheaves on X, Y. Then there is a natural map $\pi_*\mathcal{F} \otimes \mathcal{G} \to \pi_*(\mathcal{F} \otimes \pi^*\mathcal{G})$, which is an isomorphism when either (1) $\mathcal{G}$ is locally free or (2) π is affine.*

7.3. Invertible sheaves.

Definition 7.3.1. An *invertible sheaf* on X is an $\mathcal{O}_X$-module locally free of rank 1.

Why are invertible sheaves so important?

- Use global sections of an invertible sheaf $\mathcal{L}$ as replacement for $\Gamma(X, \mathcal{O}_X)$.
- Invertible sheaves are "dual" to Weil divisors.

Invertible sheaves are preserved under $\otimes$.

Definition 7.3.2. The *dual* $\mathcal{L}^\vee$ of a qcoh sheaf $\mathcal{L}$ is defined on affine opens by

$$\Gamma(U, \mathcal{L}^\vee) := \operatorname{Hom}_{\Gamma(U, \mathcal{O}_U)}(\Gamma(U, \mathcal{L}), \Gamma(U, \mathcal{O}_U)).$$

This is also a qcoh sheaf. There is a natural pairing

$$\mathcal{L} \otimes \mathcal{L}^\vee \to \mathcal{O}_X$$

which is an isomorphism when $\mathcal{L}$ is invertible.

Definition 7.3.3. The invertible sheaves on X forms an abelian group, called the Picard group $\operatorname{Pic}(X)$. Given $f : X \to Y$, $f^* : \operatorname{Pic}(Y) \to \operatorname{Pic}(X)$ is a group homomorphism.

Example 7.3.4. Consider $X = \mathbb{P}^1$, then there is a homomorphism $\mathbb{Z} \to \operatorname{Pic}(X)$ mapping $a \mapsto \mathcal{O}(a)$. This is in fact an isomorphism.

In general, for $X = \mathbb{P}^n$, then we can similarly define $\mathcal{O}(a)$, and $\mathbb{Z} \to \operatorname{Pic}(X)$ is again an isomorphism.

7.4. Weil divisors.

Let X be a Noetherian irreducible regular scheme. (Regular local rings are UFDs, so X will be factorial.)

In topology, for a smooth compact oriented manifold M with dimension d, $H^k(M) \cong H_{d-k}(M)$. For schemes and $k = 1$, the left side is $\operatorname{Pic}(X)$, and the right side should be "codimension 1 subsets of X".

Let $p \in X$ be a codimension-1 point. Then $\mathcal{O}_{X,p}$ is a DVR. For $f \in K(X)$, we may define $v_p(f)$ by the discrete valuation.

Definition 7.4.1. A *Weil divisor* on X is a $\mathbb{Z}$-linear finite sum of irreducible codimension-1 subsets $\sum a_Y[Y]$.

For nonzero $f \in K(X)$, its principal Weil divisor

$$\operatorname{div} f = \sum_Y v_Y(f)[Y].$$

This is a finite sum.

By Hartogs's lemma 6.1.4, if $f \in K(X)^\times$ such that $v_Y(f) \geq 0$ for all Y, then $f \in \mathcal{O}_X(X)$. If $(f) = 0$, then both $f, f^{-1} \in \mathcal{O}_X(X)$, so $f \in \mathcal{O}_X(X)^\times$.

It is not hard to see that the principal divisors on $\mathbb{P}^1$ all have degree 0. In contrast, all Weil divisors of $\mathbb{A}^1$ are principal.

Definition 7.4.2. The *class group* of X is $\mathrm{Cl}(X) = \mathrm{Weil}(X)/\operatorname{Prin}(X)$.

Example 7.4.3. Let $X = \operatorname{Spec} \mathcal{O}_K$, then $\mathrm{Cl}(X) = \mathrm{Cl}_K$.

Theorem 7.4.1. *There is a natural isomorphism* $\operatorname{Pic}(X) \to \mathrm{Cl}(X)$.

Given $\mathcal{L} \in \operatorname{Pic}(X)$, and a nonzero section $s \in \Gamma(X, \mathcal{L})$, consider an irred codim 1 subset Y and its generic point p_Y. Pick an open neighborhood U of p_Y (equivalently, $U \cap Y \neq \infty$), such that $\mathcal{L}|_U \cong \mathcal{O}_U$, so that we can talk about $v_Y(s) = v_Y(s|_U)$. This is easily checked to be well-defined. So we can define

$$\operatorname{div}(s) := \sum_Y v_Y(s)[Y] \in \mathrm{Weil}(X).$$

Example 7.4.4. Consider the line bundle $\mathcal{O}(1)$ on $\mathbb{P}^1 = U_0 \cup U_1$, and the section s given by $t \in k[t]$ on U_0, and by $1 \in k[t^{-1}]$ on U_1. Then $\operatorname{div}(s) = [0]$ has degree 1.

Definition 7.4.5. A *rational section* of $\mathcal{L}$ is a section of $\mathcal{L}$ over some dense open $V \subset X$, modulo equivalence; two rational sections are the same if they agree on some smaller open.

Given any nonzero rational section s of $\mathcal{L}$, we may similarly define $\operatorname{div}(s)$: this time, s only represents a section on $U \cap V$, hence a rational function on U, to which we may still associate $v_Y(s)$. The set of $(\mathcal{L}, s)$ with $\otimes$ forms a group. What we will show is:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K(X)^\times/\mathcal{O}_X(X)^\times & \longrightarrow & (\mathcal{L}, s) & \longrightarrow & \operatorname{Pic}(X) & \longrightarrow & 0 \\
 & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\
0 & \longrightarrow & \operatorname{Prin}(X) & \longrightarrow & \mathrm{Weil}(X) & \longrightarrow & \mathrm{Cl}(X) & \longrightarrow & 0.
\end{array}$$

To show theorem 7.4.1, we need to show bijectivity of the middle vertical map.

Injectivity: Suppose $\operatorname{div}(s) = 0$ is defined on dense open V. For any irreducible codimension-1 D with generic point p, pick an affine open neighborhood $U = \operatorname{Spec} A$ of $p = [\mathfrak{p}]$, then there is an isomorphism $\mathcal{L}|_U \xrightarrow{\phi} \mathcal{O}_U$. Then the rational function that s corresponds to belongs to $A_\mathfrak{p}$. Since this holds for all height-1 $\mathfrak{p} \subset A$, $s \in A = \mathcal{O}_U(U)$. So these glue together to form a global section $s \in \mathcal{O}_X(X)$. We will show that the map $\mathcal{O}_X \to \mathcal{L}$ defined by s is an isomorphism. Indeed, locally, after composing with local trivializations $\phi : \mathcal{L}|_U \to \mathcal{O}_U$, $\phi s : \mathcal{O}_U \to \mathcal{O}_U$ still has no zeros and no poles, so it belongs to $\mathcal{O}_U(U)^\times$, i.e. is an isomorphism. Since ϕ and ϕs are both isomorphisms, so is s locally, hence globally.

Surjectivity: suppose D is a Weil divisor. Define the sheaf $\mathcal{O}(D)$ as follows: on any $U \subset X$ dense open, define

$$\Gamma(U, \mathcal{O}(D)) := \{x \in K(X)^\times : \operatorname{div}(x|_U) + D|_U \geq 0\}.$$

Define a rational section s_D of $\mathcal{O}(D)$ to be $1 \in \Gamma(U, \mathcal{O}(D)) \subseteq K(X)^\times$, where U is the complement of $\operatorname{Supp} D$. We claim that $(\mathcal{O}(D), s_D)$ is the desired preimage.

- To show that $\mathcal{O}(D)$ is a line bundle: first, we find an open cover of X, where on each open set U, $D|_U$ is principal. Suppose $S = \operatorname{Supp} D$, then $X \backslash S$ is such an open. We then construct such an open neighborhood of each $p \in S$. Consider any irreducible divisor Y where $p \in Y$. Since X is factorial, every stalk $\mathcal{O}_{X,p}$ is a UFD. Since any open neighborhood of p contains the generic point η_Y of Y, there is a natural injection $\mathcal{O}_{X,p} \to \mathcal{O}_{X,\eta_Y}$. For each affine neighborhood $U = \operatorname{Spec} A$ of $p = [\mathfrak{p}]$ and $\eta_Y = [\mathfrak{q}]$, this is the natural localization $A_\mathfrak{p} \to A_\mathfrak{q}$. The preimage of $\mathfrak{q}A_\mathfrak{q}$ under this map is a height 1 prime in $\mathcal{O}_{X,p}$, a UFD, so it is principal, say generated by $f \in \mathcal{O}_{X,p} \subseteq K(X)$. WLOG we may choose $f \in A$, then f has no poles in U, and if it has a zero at a divisor Y' containing p, say with generic point $\eta_{Y'}$, then the preimage of $\mathfrak{m}_{\eta_{Y'}}$ in $\mathcal{O}_{X,p} \to \mathcal{O}_{X,\eta_{Y'}}$ is another height 1 prime $\mathfrak{r}$ containing f. Then $\mathfrak{q} = (f) \subseteq \mathfrak{r}$, which implies $\mathfrak{q} = \mathfrak{r}$. This shows that f only has a zero of order 1 at η_Y.

Now, let

$$U' = U \cap (X \backslash \bigcup_{\substack{Z \text{ irred codim } 1 \\ p \notin Z}} Z)$$

which contains p, so it is a dense open. On U', $\operatorname{div}(f) = [Y]$.

Now, suppose $p \in Y_1, \ldots, Y_n$ where $D = \sum n_i [Y_i]$. Choose f_i so that on an open neighborhood of p, $\operatorname{div}(f_i) = [Y_i]$. Then on their intersections, which is an open neighborhood U of p,

$$\operatorname{div}|_U(\prod f_i^{n_i}) = \sum n_i [Y_i] = D|_U.$$

This shows that we can find an open cover of X where D is locally principal. Now, fix one open U in the cover, where $D = \operatorname{div}|_U(s)$. For each affine open $V \subseteq U$, there is an isomorphism $\Gamma(V, \mathcal{O}(D)) \cong \mathcal{O}_U(V)$ by sending $t \mapsto st$. This is functorial, so they glue together to form $\mathcal{O}(D)|_U \cong \mathcal{O}_U$. This shows that $\mathcal{O}(D)$ is locally free of rank 1.

- To show that $\mathcal{O}(\operatorname{div}(s)) \cong \mathcal{L}$ for $(\mathcal{L}, s)$: We claim that any open U that trivializes $\mathcal{L}$ satisfies $\mathcal{O}(\operatorname{div}(s))|_U \cong \mathcal{O}_U$. Suppose $\mathcal{L}|_U \cong \mathcal{O}_U$ takes s to a rational function on U, which we also denote by s. Then for any affine open $V = \operatorname{Spec} A \subseteq U$,

$$\begin{aligned}\Gamma(V, \mathcal{O}(\operatorname{div}(s))) &= \{t \in K^\times : \operatorname{div}_V(t) + \operatorname{div}_V(s) \geq 0\} \\ &= \{t \in K^\times : \operatorname{div}_V(st) \geq 0\} \\ &= \{t \in K^\times : st \in A\} \\ &= s^{-1}A,\end{aligned}$$

which is isomorphic to $\mathcal{O}_V(V) = A$ as A-modules by sending t to st. Furthermore, this isomorphism is clearly functorial as V ranges among affine open subsets of U, so this induces an isomorphism of sheaves $\mathcal{O}(\operatorname{div}(s))|_U \cong \mathcal{O}_U$. Composing this with $\mathcal{O}_U \cong \mathcal{L}|_U$, for sections over U, this is the bijection sending t to st. Now, the set of Us (open sets trivializing $\mathcal{L}$) forms a base of the Zariski topology on X, and the isomorphism $\Gamma(U, \mathcal{O}(\operatorname{div}(s))) \to \Gamma(U, \mathcal{L})$ is clearly functorial, so this defines an isomorphism of sheaves $\mathcal{O}(\operatorname{div}(s)) \to \mathcal{L}$.

  Suppose the canonical section "1" is a section of $\mathcal{O}(\operatorname{div}(s))$ over U. Its image is a section which, on each V in part (a) (i.e. affine opens that trivialize $\mathcal{L}$), agrees with $s|_V$. So its image is s by the sheaf axiom.

Corollary 7.4.2. *$\operatorname{Pic}(\mathbb{P}^n_k) \cong \mathbb{Z}$.*

Proof. There is an exact sequence $0 \to \mathbb{Z} \to \operatorname{Weil}(\mathbb{P}^n) \to \operatorname{Weil}(\mathbb{A}^n) \to 0$, where $\mathbb{A}^n = U_0$ is the complement of a hyperplane, and $\mathbb{Z}$ is freely generated by that hyperplane. This induces an exact $0 \to \mathbb{Z} \to \operatorname{Cl}(\mathbb{P}^n) \to \operatorname{Cl}(\mathbb{A}^n) \to 0$. But $\operatorname{Cl}(\mathbb{A}^n) = 0$. □

7.5. **Quasicoherent sheaf of graded module.**

7.6. **Sections of line bundles.** One theme we see here is that global sections of line bundles on X serve a similar purpose as functions on X.

Definition 7.6.1. Let X be a scheme, $\mathcal{L} \in \operatorname{Pic}(X)$, $s \in \Gamma(X, \mathcal{L})$, $p \in X$. The *value* of s at p, $s(p)$, is the image of s in the *fiber* $\mathcal{L}|_p := \mathcal{L}_p / \mathfrak{m}_p \mathcal{L}_p = \mathcal{L}_p \otimes_{\mathcal{O}_{X,p}} \kappa(p)$, which is naturally a 1-dim $\kappa(p)$-vector space. (In general this makes sense for any quasicoherent sheaf.)

For $s \in \Gamma(X, \mathcal{L})$, the locus of points where s does not vanish is denoted by $D(s)$. This is open.

A map $X \to \mathbb{A}^n_k$ is equivalent to choosing n global sections of $\mathcal{O}_X$. The analogous fact is:

Proposition 7.6.1. *Let X be an A-scheme, for a ring A. The following data are equivalent:*

- *A map $f : X \to \mathbb{P}^n_A$;*
- *A line bundle $\mathcal{L} \in \operatorname{Pic}(X)$, and sections $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$, such that $X = \bigcup D(s_i)$.*

When $A = k$, on k-points, this is the map $X(k) \to \mathbb{P}^n(k)$ given by $p \mapsto [s_0(p), \dots, s_n(p)]$.

Proof. ($\Longleftarrow$): Recall that affine schemes $U_i = \operatorname{Spec} A[x_{0/i}, \dots, x_{n/i}]$ cover $\mathbb{P}^n_A$. Given $(\mathcal{L}, s_0, \dots, s_n)$, we define maps $D(s_i) \to U_i$ by specifying a ring homomorphism $A[x_{0/i}, \dots, x_{n/i}] \to \Gamma(D(s_i), \mathcal{O}_X)$. Because $s_j \in \Gamma(D(s_i), \mathcal{L})$ and $s_i^{-1} \in \Gamma(D(s_i), \mathcal{L}^\vee)$, there is an element $s_j s_i^{-1} \in \Gamma(D(s_i), \mathcal{O}_X)$, which we map $x_{j/i}$ to. To check that these glue together, it suffices to show that

$$\begin{tikzcd}
A[x_{0/i}, \dots, x_{n/i}]_{x_{j/i}} \arrow[r] \arrow[d] & \Gamma(D(s_i) \cap D(s_j), \mathcal{O}_X) \arrow[d, equal] \\
A[x_{0/j}, \dots, x_{n/j}]_{x_{i/j}} \arrow[r] & \Gamma(D(s_i) \cap D(s_j), \mathcal{O}_X)
\end{tikzcd}$$

This is true because $x_{k/i} \mapsto x_{k/j} x_{i/j}^{-1} \mapsto s_k s_j^{-1} (s_i s_j^{-1})^{-1} = s_k s_i^{-1}$.

($\Longrightarrow$): Let $\mathcal{L} = f^* \mathcal{O}_{\mathbb{P}^n_A}(1)$, and $s_i = f^* x_i$ where $x_i \in A[x_0, \dots, x_n]_{\deg 0} \cong \Gamma(\mathbb{P}^n_A, \mathcal{O}(1))$. Then $D(s_i) = D(f^* x_i) = f^{-1}(D(x_i))$, so $X = \bigcup D(s_i)$. $\square$

Definition 7.6.2. Let $\mathcal{F}$ be a finite type quasicoherent sheaf on X. Say $\mathcal{F}$ is *globally generated* if for any point $p \in X$, there exists a set of $s_i \in \Gamma(X, \mathcal{F})$ such that $s_i(p)$ generate $\mathcal{L}|_p$ over $\kappa(p)$. Equivalently (Nakayama), there is a surjection of sheaves $\mathcal{O}^{\oplus I} \twoheadrightarrow \mathcal{L}$ where I is an index set.

Definition 7.6.3. Let X be a k-scheme. A finite dimensional k-subspace $W \subset \Gamma(X, \mathcal{L})$ is called a *linear series*. It is a *complete linear series* if $W \cong \Gamma(X, \mathcal{L})$ and is often written $|\mathcal{L}|$. Given a linear series W, the *base locus* is the set of points where all of W vanish. Then if W globally generates $\mathcal{L}$, we get a map $X \to \mathbb{P}_k^{\dim W - 1}$. We say $\mathcal{L}$ is *basepoint free* if is is globally generated.

Example 7.6.4. The Veronese embedding $\mathbb{P}^n \to \mathbb{P}^{\binom{n+1}{d}-1}$ can be seen as the map corresponding to picking the degree-d monomials in $\Gamma(\mathbb{P}^n, \mathcal{O}(d)) \cong k[x_0, \dots, x_n]_{\deg d}$, which globally generate $\mathcal{O}(d)$.

Example 7.6.5. All maps $\mathbb{P}^m \to \mathbb{P}^n$ are be characterized by choosing a d and $n+1$ degree-d homogeneous polynomials in $k[x_0, \dots, x_m]$ with no common zeros.

Theorem 7.6.2 (Serre's Theorem A)**.** *Suppose $S_\bullet$ is generated in degree 1, and finitely generated over $A = S_0$. Then for any finite type quasicoherent sheaf $\mathcal{F}$ on $\operatorname{Proj} S$, there exists n_0 such that for all $n > n_0$, $\mathcal{F} \otimes \mathcal{O}(n) = \mathcal{F}(n)$ is finitely globally generated.*

Theorem 7.6.3 (Curve to projective extension)**.** *Let C/k be a smooth curve (i.e. pure dimension one), Y projective over k, $p \in C$ a closed point. Then any map $f : C - p \to Y$ (uniquely) extends to C.*

Proof. Uniqueness follows from the reduced-to-separated theorem (regular local rings are reduced). To show existence, we make several reductions:

- Assume C is affine. This is because we can choose an affine neighborhood of p, and if the function is extended to that neighborhood, then it glues with f to form an extension on the whole of C.
- Assume $Y = \mathbb{P}^n_k$. This is because: suppose we have proven the theorem for $Y = \mathbb{P}^n_k$. Then we may extend $f : C - p \to Y \to \mathbb{P}^n_k$ to a map $f : C \to \mathbb{P}^n_k$. Take affine open neighborhood $\operatorname{Spec} A \subseteq C$ of p such that its image lands in $\mathbb{A}^n_k$. Then functions vanishing on $Y \cap \mathbb{A}^n_k$ pull back to functions vanishing at generic points of the irreducible components of C, hence they vanish on the entire C (by reducedness), so $\operatorname{Spec} A \to \mathbb{A}^n_k$ factors through $Y \cap \mathbb{A}^n_k$.

Now, because C is regular and p is a closed point, $\mathcal{O}_{C,p}$ is a DVR, so we can pick a uniformizer π. Pick a neighborhood V of p, such that $\pi \in \Gamma(V, \mathcal{O}_C)$. Shrink V so that $V = \operatorname{Spec} A$ is affine, π is nonvanishing on $V - p$, and the line bundle $\mathcal{L}$ induced by f is trivialized on $V - p$. Suppose $f|_{V-p} = [f_0 : f_1 : \cdots : f_n]$, $f_i \in A_\pi$ (where $V - p = \operatorname{Spec} A_\pi$). Let $m = \min v_\pi(f_i)$, then $t^{-m}g_0, \ldots, t^{-m}g_n \in A$ are $(n+1)$ functions with no common zeros, which gives a map $V \to \mathbb{P}^n_k$ extending f. This glues with f to produce an extension on the whole C. □

7.7. **Ampleness.** Ample line bundles are "positive" in certain senses, and ampleness roughly means "having many sections".

Definition 7.7.1. Let X be a proper A-scheme. An invertible sheaf $\mathcal{L}$ on X is *very ample* if there exist $n+1$ sections that globally generate $\mathcal{L}$ such that the induced map to $\mathbb{P}^n_A$ is a closed embedding.

Equivalently, $X \cong \operatorname{Proj} S_\bullet$, where $S_0 = A$ and S is generated in degree 1. Then $\mathcal{L}$ is *very ample* if $\mathcal{L} = \mathcal{O}(1)$.

Proposition 7.7.1. *If $\mathcal{L}$ is very ample, then so are $\mathcal{L}^{\otimes k}$ $(k \geq 1)$.*

Proof. Suppose $\mathcal{L} = f^*\mathcal{O}_{\mathbb{P}^n}(1)$ for $f : X \to \mathbb{P}^n$. Let $g : \mathbb{P}^n \to \mathbb{P}^N$, $N = \binom{n-1}{k} + 1$ be the Veronese embedding, so that $g^*\mathcal{O}_{\mathbb{P}^N}(1) = \mathcal{O}_{\mathbb{P}^n}(k)$. Then $(g \circ f)^*\mathcal{O}_{\mathbb{P}^N}(1) = f^*\mathcal{O}_{\mathbb{P}^n}(k) = f^*\mathcal{O}_{\mathbb{P}^n}(1)^{\otimes k} = \mathcal{L}^{\otimes_k}$, so $\mathcal{L}^{\otimes_k}$ is also pulled back from $\mathcal{O}(1)$ of a projective space, and $\mathcal{L} \hookrightarrow \mathbb{P}^n \hookrightarrow \mathbb{P}^N$ is a closed embedding. □

Lemma 7.7.2 (extending sections)**.** *Let X be qcqs, $\mathcal{L}$ a invertible sheaf, $s \in \Gamma(X, \mathcal{L})$, $\mathcal{F}$ a quasi-coherent sheaf. Then for any $t \in \Gamma(D(s), F)$, there exists $k \geq 0$, such that*

$$t \otimes s^{\otimes k} \in \Gamma(D(s), F \otimes \mathcal{L}^{\otimes k})$$

lies in the image of $\Gamma(X, F \otimes \mathcal{L}^{\otimes k})$. □

Definition 7.7.2 (ample line bundles)**.** Let X be a proper A-scheme. An invertible sheaf $\mathcal{L}$ on X is *ample* if any of the following equivalent conditions hold:

(a) $\mathcal{L}^{\otimes k}$ is very ample for some $k \geq 1$.
(a') $\mathcal{L}^{\otimes k}$ is very ample for all $k \gg 0$.
(b) For all finite type quasicoherent sheaves $\mathcal{F}$, $\mathcal{F} \otimes \mathcal{L}^{\otimes k}$ is globally generated for some $k \geq 1$.
(b') For all finite type quasicoherent sheaves $\mathcal{F}$, $\mathcal{F} \otimes \mathcal{L}^{\otimes k}$ is globally generated for all $k \gg 0$.
(c) As f varies over global sections of $\mathcal{L}^{\otimes k}$ (over all $k \geq 1$), the open sets $D(f)$ form a base of the topology on X.
(c') In the above, the affine ones already form a base.
(c") In the above, the affine ones cover X.

Proof. Clearly, (a') $\implies$ (a), (b') $\implies$ (b), and (c') $\implies$ (c), (c").

(c) $\implies$ (c'): Consider $p \in X$ and any open neighborhood U of p. WLOG U is affine and trivializes $\mathcal{L}$. Then there exists $f \in \Gamma(\mathcal{L}^{\otimes k})$ such that $D(f) \subseteq U$. This $D(f)$ is affine.

(a) $\implies$ (c): Suppose $\mathcal{L}^{\otimes k}$ is very ample. Then there is a closed immersion $i : X \hookrightarrow \mathbb{P}^n$ and $i^*(\mathcal{O}_{\mathbb{P}^n}(1)) = \mathcal{L}^{\otimes k}$. Let Z be closed in X, and p a point in the complement of Z. We wish to find a neighborhood $D(f)$ of p disjoint from Z. We can make Z into a closed subscheme. Then $Z \hookrightarrow X \hookrightarrow \mathbb{P}^n$ is a closed subscheme, so $Z \cong \operatorname{Proj} S_\bullet$ where $S = A[x_0, \ldots, x_n]/I$ for some homogeneous ideal I. Pick a homogeneous element $s \in I$, say of degree d, so that $s \in \Gamma(\mathbb{P}^n, \mathcal{O}(d))$. Then $f := i^*s \in \Gamma(X, \mathcal{L}^{\otimes kd})$ vanishes on Z, and does not vanish at p, which is what we want.

(b) $\implies$ (c): Similar to above, we wish to find a neighborhood $D(f)$ of p disjoint from Z. Pick $\mathcal{F} = \mathcal{I}_Z$ to be the ideal sheaf of Z. Then since $\mathcal{I}_Z \otimes \mathcal{L}^{\otimes k}$ is globally generated for some k, there exists $s \in \Gamma(X, \mathcal{I}_Z \otimes \mathcal{L}^{\otimes k})$ such that $s(p) \neq 0$. Since $0 \to \mathcal{I}_Z \to \mathcal{O}_X$ is an injection, tensoring with the locally free $\mathcal{L}^{\otimes k}$ gives an injection $0 \to \mathcal{I}_Z \otimes \mathcal{L}^{\otimes k} \to \mathcal{L}^{\otimes k}$. Let $f \in \Gamma(X, \mathcal{L}^{\otimes k})$ denote the image of s, and we claim that this works. For any U trivializing $\mathcal{L}$, $f|_U$ is the image of $s|_U$

under $0 \to \mathcal{I}_Z|_U \to \mathcal{O}_U$, hence vanishes on $Z \cap U$. So s vanishes on Z. Since $p \notin Z$, there exists a neighborhood U of p trivializing $\mathcal{L}$ where $\mathcal{I}_Z|_U \cong \mathcal{O}_U$. So since $s(p) \neq 0$, $f(p) \neq 0$ as well, as desired.

(c") $\Longrightarrow$ (b'): Let $X = \bigcup D(f_i)$ be the union of finitely many affine opens, where $f_i \in \Gamma(X, \mathcal{L}^{\otimes a})$. (By scaling, a can be chosen not to depend on i.) On each $D(f_i) = \operatorname{Spec} A_i$, $\mathcal{F}$ is just some finitely generated A_i-module, so it is globally generated by $s_{ij} \in \Gamma(D(f_i), \mathcal{F})$. Extend these to $\widetilde{s}_{ij} \in \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes k})$ where k can be chosen to not depend on i, j. Then $\widetilde{s}_{ij}$ generates $\mathcal{F} \otimes \mathcal{L}^{\otimes k}$ on each stalk, hence globally generates $\mathcal{F} \otimes \mathcal{L}^{\otimes k}$. In fact, this shows that $\mathcal{F} \otimes \mathcal{L}^{\otimes(k+na)}$ is globally generated for all $n \geq 0$. Arguing similarly for all residues mod a implies the desired statement.

(c") $\Longrightarrow$ (a): Let $X = \bigcup D(f_i)$ be the union of finitely many affine opens, where $f_i \in \Gamma(X, \mathcal{L}^{\otimes a})$ and $D(f_i) = \operatorname{Spec} A_i = A[a_{ij}]/I$, where $a_{ij} \in \Gamma(D(f_i), \mathcal{O}_X)$. Extend these to $\widetilde{a}_{ij} \in \Gamma(X, \mathcal{L}^{\otimes r})$. We may choose r so that $f_i, \widetilde{a}_{ij}$ are all global sections of $\mathcal{L}^{\otimes r}$. We claim that these give a closed embedding to a projective space. Since the linear series generated by f_i is already basepoint-free, this gives us a map $X \to \mathbb{P}^N_A$. We index the coordinates of $\mathbb{P}^N_A$ correspondingly with i and ij. Then it is clear that the ring homomorphisms $A[x_i, x_{ij}]/(x_k - 1) \to \Gamma(D(f_k), \mathcal{L}^{\otimes k}) = A_k$ are surjective. This shows that $X \to \mathbb{P}^N_A$ is a closed immersion.

(a), (b) $\Longrightarrow$ (a'): very ample tensor basepoint-free is very ample. □

There is another, more geometric, interpretation of ampleness.

Proposition 7.7.3 (separating points and tangent vectors). *Let X be proper over $k = \overline{k}$, $\mathcal{L}$ an invertible sheaf, and V a basepoint-free linear series giving a map $f : X \to \mathbb{P}^n$. If:*

- *for any two distinct k-points $x, y \in X$, there exists $s \in V$ with $s(x) = 0$, $s(y) \neq 0$;*
- *for any k-point x and nonzero tangent vector $\theta : \operatorname{Spec} \kappa(x)[\varepsilon] \to X$, there exists a section $s \in V$ vanishing at x such that the pullback of s along θ is nonzero,*

then $\mathcal{L}$ is very ample and f is a closed immersion.

7.8. **Projective morphism.** Recall that a morphism $X = \operatorname{Proj} S_\bullet \to \operatorname{Spec} A$, where $S_0 = A$ and $S_\bullet$ is finitely generated in degree 1, is called *projective*. We wish to define a notion of projectiveness over any base scheme.

Lemma 7.8.1. *Given a scheme Y, and the following data:*

- *for each affine open $U \subset Y$, a scheme $Z_U \to U$;*
- *for $V \subseteq U$, a map $\rho_{UV} : Z_V \subseteq Z_U$ such that $Z_V \cong Z_U \times_U V$;*
- *for $W \subset V \subset U$, $\rho_{UW} = \rho_{UV} \circ \rho_{VW}$,*

then there exists a scheme $\pi : Z \to Y$ such that $\pi^{-1}(U) = Z_U$.

Given a scheme Y, and a *graded quasicoherent sheaf of $\mathcal{O}_Y$-algebras $\mathscr{S}_\bullet = \bigoplus_{n\geq 0} \mathscr{S}_n$* such that

- $\mathscr{S}_0 = \mathcal{O}_Y$;
- $\operatorname{Sym}^\bullet \mathscr{S}_1 := \bigoplus \operatorname{Sym}^k \mathscr{S}_1 \to \mathscr{S}_\bullet$ is surjective,

we can define $\operatorname{Proj} \mathscr{S}_\bullet \to Y$ using the above gluing lemma. Also, the line bundles on each affine open glue together over $\operatorname{Proj} \mathscr{S}_\bullet$.

Example 7.8.1. Let $\mathcal{E}$ be locally free of rank r, then define $\mathscr{S}_\bullet = \operatorname{Sym}^\bullet \mathcal{E}$. Then $\operatorname{Proj} \mathscr{S}_\bullet$ is a *projective bundle* that locally looks like $U \times \mathbb{P}^{r-1}$ on affine opens trivializing $\mathcal{E}$.

Definition 7.8.2. A morphism $\pi : X \to Y$ is *projective* if $X \cong \operatorname{Proj} \mathscr{S}_\bullet$ for some $\mathscr{S}_\bullet$ as above.

Remark. Hartshorne defines projective morphisms as $X \hookrightarrow Y \times \mathbb{P}^n \to Y$, where the first map is a closed immersion.

7.9. **Curves.**

Theorem 7.9.1. *The following categories are equivalent:*

1. *Integral regular projective 1-dimensional k-varieties, and surjective k-morphisms.*
2. *Integral regular projective 1-dimensional k-varieties, and dominant k-morphisms.*
3. *Integral regular projective 1-dimensional k-varieties, and dominant rational maps.*
4. *Integral 1-dimensional k-varieties, and dominant rational maps.*
5. *The opposite category of finitely generated fields of transcendence degree 1 over k, and k-morphisms.*

8. Cohomology

8.1. **Properties.** Let $X \to \operatorname{Spec} A$ be separated. (This isn't absolutely necessary.) We will define for each $k \geq 0$ a functor $H^k(X, -) : \mathsf{QCoh}(X) \to A\text{-}\mathsf{Mod}$, such that:

- $H^0(X, -) = \Gamma(X, -)$;
- Short exact sequences of QCoh sheaves gets sent to long exact sequences of A-modules;
- Let $\pi : X \to Y$ be a morphism of schemes, and $\mathcal{F} \in \mathsf{QCoh}(X)$. Then there exist $\alpha_k : H^k(Y, \pi_* \mathcal{F}) \to H^k(X, \mathcal{F})$, which are isomorphisms when π is affine, that extend $\alpha^0 : \Gamma(Y, \pi_*\mathcal{F}) \to \Gamma(X, \mathcal{F})$. This gives, for $G \in \mathsf{QCoh}(Y)$, a composition
$$H^k(Y, G) \to H^k(Y, \pi_*\pi^* G) \to H^k(X, \pi^* G).$$
- If X is covered by n affine open charts, then $H^k(X, -) = 0$ if $k \geq n$. In particular, if X is affine, then $H^1(X, -) = 0$ (which we recall from earlier).
- $H^k(X, \bigoplus \mathcal{F}_j) = \bigoplus H^k(X, \mathcal{F}_j)$.

A preview of what's to come:

Theorem 8.1.1 (cohomologies of $\mathcal{O}(m)$)**.** *We have:*

- $H^0(\mathbb{P}^n_A, \mathcal{O}(m)) = A^{\binom{n+m}{m}}$ *if $m \geq 0$, and 0 if $m \leq 0$;*
- $H^n(\mathbb{P}^n_A, \mathcal{O}(m)) = A^{\binom{-m-1}{-m-1-n}}$ *if $-m-1 \geq n$, and 0 otherwise;*
- *All other cohomologies vanish.*

Theorem 8.1.2. *Let X be projective over A, and $\mathcal{F}$ a coherent sheaf. Then $\Gamma(X, \mathcal{F})$ is a finitely generated A-module.*

Proof. We will show in fact that $H^k(X, \mathcal{F})$ are all finitely generated over A.

Let $i : X \hookrightarrow \mathbb{P}^n_A$ be a closed embedding, then $H^k(X, \mathcal{F}) = H^k(\mathbb{P}^n_A, i_*\mathcal{F})$. So we may WLOG assume $X = \mathbb{P}^n_A$. Use descending induction on k. In the base cases $k \geq n+1$, the cohomologies all vanish.

Recall that there exists a surjection $\mathcal{O}(m)^{\oplus a} \to \mathcal{F} \to 0$. Let K be the kernel, and unwind to a long exact sequence. Suppose we want to show $H^n(X, \mathcal{F})$ is finitely generated. A segment of the long exact sequence reads:
$$\cdots \to H^n(\mathcal{O}(m)^{\oplus a}) \to H^n(\mathcal{F}) \to 0$$
and since H^n commutes with direct sums and by the explicit calculations, $H^n(\mathcal{O}(m)^{\oplus a})$ is finitely generate, so $H^n(\mathcal{F})$ is as well. Suppose now we want to show this for $n-1$. Then
$$\cdots \to H^{n-1}(\mathcal{O}(m)^{\oplus a}) \to H^{n-1}(\mathcal{F}) \to H^n(K) \to \dots,$$
and since both the left and right are finitely generated, so is the middle. □

8.2. **Definition.** Let $\mathscr{U} = \{U_i\}_{i=1}^n$ be an affine cover of X, and let $\mathcal{F}$ be a quasicoherent sheaf. Define the Čech complex

$$C^k_{\mathscr{U}}(X, \mathcal{F}) := \prod_{\substack{|I|=k+1 \\ I=\{i_0,\dots,i_k\}\subseteq[n]}} \Gamma(U_{i_0} \cap \cdots \cap U_{i_k}, \mathcal{F})$$

with obvious differentials

$$d : C^k_{\mathscr{U}}(X, \mathcal{F}) \to C^{k+1}_{\mathscr{U}}(X, \mathcal{F})$$

by alternatingly summing over the restriction maps. A short exact sequence of QCoh sheaves induces a short exact sequence of Čech complexes (this is where it is crucial that we're working with QCoh sheaves), which then induces the long exact sequence. It is then obvious that if X is covered by n affine open charts, then H^k vanishes for $k \geq n$.

Theorem 8.2.1. *Let X be quasicompact and separated. The Čech cohomology is independent of the (finite) affine cover $\mathscr{U}$.*

Proof. The proof proceeds in several steps.

Step 1: it suffices to show that the Čech complexes of $\{U_i\}_{i=1}^n$ and $\{U_i\}_{i=1}^{n+1}$ are quasi-isomorphic.

Step 2: The kernel of the surjection $C_{\{U_i\}_{i=1}^{n+1}}(X, \mathcal{F}) \twoheadrightarrow C_{\{U_i\}_{i=1}^n}(X, \mathcal{F})$ is the chain complex whose k-th term is the product over all $I \subseteq [n+1]$ containing $n+1$, $|I| = k+1$. The goal is then to show that this is exact. But this is exactly the augmented Čech complex $C_{\{U_i \cap U_{n+1}\}_{i=1}^n}(U_{n+1}, \mathcal{F})$. So it suffices to show that for affine schemes X, the Čech cohomology vanishes except at degree 0.

Step 3: Suppose that X is affine and $\{U_i\}$ cover X, and suppose U_n already is X. Then the augmented Čech complex of X surjects onto the augmented Čech complex of $U_1 \cap \cdots \cap U_{n-1}$, and the kernel is the Čech complex for U_n, which is just the Čech complex for X shifted by one. So by the cohomology long exact sequence, the cohomology of the middle row vanishes.

Step 4: In general, suppose X is affine and $\{U_i\}$ cover X. Then there is an affine cover $D(f_j)$ where each $D(f_j)$ lies inside some $\{U_i\}$. Then the Čech complex localized at each f_i is exact, so the original complex is exact as well. $\square$

More consequences of cohomology:

Proposition 8.2.2. *Pushforwards of coherent sheaves by projective morphisms (of locally Noetherian schemes) is coherent.*

Proposition 8.2.3. *Suppose Y is locally Noetherian. Then a morphism $\pi : X \to Y$ is projective and affine iff it is finite.*

Proposition 8.2.4. *Suppose Y is Noetherian. Then a morphism $\pi : X \to Y$ is projective and has finite fiber iff it is finite.*

Proposition 8.2.5 (fiber dimension of projective morphism is upper-semicontinuous)**.** *Let $\pi : X \to Y$ be projective, and let Y be locallly Noetherian. Then the set $\{q \in Y : \dim \pi^{-1}(q) \geq k\}$ is Zariski-closed.*

Theorem 8.2.6 (Serre vanishing)**.** *Let $\mathcal{F}$ be coherent on a projective X/A. Then for all $m \gg 0$, $H^i(X, \mathcal{F}(m)) = 0$ for all $i > 0$.*

8.3. **Euler characteristic, Hilbert functions.** We work with a projective k-scheme X, and $\mathcal{F} \in \mathsf{Coh}(X)$. The *Euler characteristic*

$$\chi(\mathcal{F}) := \sum_{i \geq 0} \dim_k H^i(X, \mathcal{F}).$$

For example, for $X = \mathbb{P}^n$, $\mathcal{F} = \mathcal{O}(m)$, then

$$\chi(\mathcal{O}(m)) = \frac{1}{n!}(m+1)(m+2)\dots(m+n)$$

for *all* m, n. A general heuristic is that χ is better behaved than individual cohomology groups, and we study the individual cohomologies by proving vanishing theorems.

Proposition 8.3.1. *Let* $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ *be an exact sequence of coherent sheaves. Then* $\chi(\mathcal{G}) = \chi(\mathcal{F}) + \chi(\mathcal{H})$.

Let $i : X \hookrightarrow \mathbb{P}^N_k$ be a fixed embedding. Then by definition, $\mathcal{O}_X(1) = i^*\mathcal{O}_{\mathbb{P}^N}(1)$.

Definition 8.3.1. The *Hilbert function* of $\mathcal{F}$ is defined by

$$h_{\mathcal{F}}(m) = \dim_k H^0(X, \mathcal{F}(m)) = \dim_k H^0(X, \mathcal{F} \otimes \mathcal{O}_X(1)^{\otimes m}).$$

Example 8.3.2. Let $\mathcal{F} = \mathcal{I}_X$ be the ideal sheaf of X. Then we have an exact sequence

$$0 \to \mathcal{I}_X \to \mathcal{O}_{\mathbb{P}^N} \to i_*\mathcal{O}_X \to 0.$$

Tensoring with $\mathcal{O}_{\mathbb{P}^N}(m)$, we get

$$0 \to \mathcal{I}_X(m) \to \mathcal{O}_{\mathbb{P}^N}(m) \to (i_*\mathcal{O}_X)(m) \to 0.$$

By the projection formula, $(i_*\mathcal{O}_X)(m) = i_*(\mathcal{O}_X(m))$. Taking $\Gamma(\mathbb{P}^n, -)$ gives us

$$0 \to H^0(\mathbb{P}^N, \mathcal{I}_X(m)) \to H^0(\mathbb{P}^N, \mathcal{O}(m)) \to H^0(X, \mathcal{O}_X(m))$$

where the last map is just restriction to X. So in other words, $H^0(\mathbb{P}^N, \mathcal{I}_X(m))$ should be interpreted as the degree-m homogeneous polynomials in $x_0, \dots, x_N$ that vanish on X. In particular, it depends on the way X is embedded into $\mathbb{P}^N$.

Theorem 8.3.2. *The function* $t \mapsto \chi(\mathcal{F}(t))$ *is a polynomial in* $\mathbb{Q}[t]$ *whose degree is* $\dim \operatorname{Supp} \mathcal{F}$.

Hence, by Serre vanishing, for $m \gg 0$, the Hilbert function is a polynomial, called the *Hilbert polynomial* $p_{\mathcal{F}}(m)$. In particular, the Hilbert polynomial $p_X(m)$ of $\mathcal{O}_X$ is a polynomial of degree $\dim X$.

Proof. (TODO) □

Example 8.3.3. Let $X = V(f)$ be a degree-d hypersurface. Then

$$p_X(m) = p_{\mathbb{P}^n}(m) - p_{\mathbb{P}^n}(m-d) = \frac{1}{n!}((m+1)\dots(m+n) - (m+1-d)\dots(m+n-d)).$$

In particular, its leading term is $\frac{d}{n!}m^{n-1}$.

Remark. In general, for a closed subscheme $X \hookrightarrow \mathbb{P}^n$, its *degree* is defined as the positive integer a such that the leading coefficient of $p_X(t)$ is $\frac{a}{n!}$. Another piece of information is the constant term $p_X(0) = \chi(X, \mathcal{O}_X)$. This is one minus the arithmetic genus.

8.4. **Riemman-Roch for line bundles on a regular projective curve.** Let C be a regular projective curve over k (not necessarily alg. closed), D a Weil divisor. Recall that if $D = \sum a_p[p]$, then $\deg D = \sum a_p \deg p$.

Theorem 8.4.1. *We have* $\deg D = \chi(C, \mathcal{O}(D)) - \chi(C, \mathcal{O}_C)$.

Definition 8.4.1. For a line bundle $\mathcal{L}$ on C, define its degree $\deg \mathcal{L} = \chi(C, \mathcal{O}(D)) - \chi(C, \mathcal{O}_C)$.

Definition 8.4.2. For a scheme X, the *arithmetic genus* is defined to be $g = 1 - \chi(X, \mathcal{O}_X)$. When X is a integral projective curve over an algebraically closed field, it is true that $h^0(X, \mathcal{O}_X) = 1$, so $h^1(X, \mathcal{O}_X) = g$.

8.5. Remarks on sheaf cohomology.

Theorem 8.5.1 (Künneth formula)**.** *Let X, Y projective schemes over k, $\mathcal{F} \in \mathsf{QCoh}(X)$, $\mathcal{G} \in \mathsf{QCoh}(Y)$. Define $\mathcal{F} \boxtimes \mathcal{G} = \pi_1^*\mathcal{F} \otimes \pi_2^*\mathcal{G}$, where π_1, π_2 are projection maps from $X \times Y$. Then*

$$H^m(X \times Y, \mathcal{F} \boxtimes \mathcal{G}) = \bigoplus_{p+q=m} H^p(X, \mathcal{F}) \otimes_k H^q(Y, \mathcal{G}).$$

Theorem 8.5.2 (cup product)**.** *There is a ...*

8.6. Baby intersection theory.

Definition 8.6.1. Let X be a smooth projective scheme over k. Given a line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_n$

9. Curves of small genus

We use the machinery of cohomology of line bundles to study curves of small genus.

Definition 9.1. In this section, a *curve* C is a projective, geometrically integral, geometrically regular, dimension-1 scheme over a field k.

9.1. Preliminary tools.

Definition 9.1.1 (degree of a finite morphism at a point)**.** Let $\pi : X \to Y$ be a finite morphism. Then $\pi_*\mathcal{O}_X$ is a finite type quasicoherent sheaf, so we may consider the rank d of $f_*\mathcal{O}_X$ at a point $y \in Y$. We call d the *degree* of π at y. Equivalently, the degree is $d = \dim_{\kappa(y)} \Gamma(\mathcal{O}_{\pi^{-1}(y)}, \pi^{-1}(y))$ (just unwind the definition).

Remark. The degree of π is upper-semicontinuous on Y.

Lemma 9.1.1. *Let $\pi : X \to Y$ be a finite morphism of Noetherian schemes, whose degree at every point of Y is either 0 or 1. Then π is a closed embedding.*

Theorem 9.1.2 (separating points and tangent vectors)**.** *Let k be algebraically closed. Let $\pi : X \to Y$ be a projective morphism of finite-type k-schemes that is injective on closed points and injective on tangent vectors at closed points. Then π is a closed embedding.*

Proof. Since closed embeddings are affine-local on the target, we may WLOG $Y = \operatorname{Spec} B$. Since π is projective, its fiber dimension is upper-semicontinuous on Y, so $\{y \in Y : \dim \pi^{-1}(y) \geq 1\}$ is closed. If it is nonempty, then it contains a closed point, which contradicts with injectivity. So the fibers are finite type and dimension 0 over the Spec of a field, hence finite. So π is projective with finite fibers, hence finite (Theorem 8.2.4).

Now, for any *closed* point $y \in Y$, we claim that the degree of π at y is at most 1. Suppose $\pi^{-1}(y)$ is nonempty, then it contains 1 point x that is finite over $\operatorname{Spec} k$, so it has to be $\operatorname{Spec} A$, where A is a finite k-algebra with one prime ideal $\mathfrak{m}$. Then k must be the residue field. Suppose for contradiction that $\dim_k A \neq 1$, then $A_\mathfrak{m} \neq k$. But $A_\mathfrak{m} = \mathcal{O}_{\pi^{-1}(y),x} = \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} k$, so $\mathfrak{m}_y\mathcal{O}_{X,x} \neq \mathfrak{m}_x$. So $\mathfrak{m}_y/\mathfrak{m}_y^2 \to \mathfrak{m}_x/\mathfrak{m}_x^2$ is not surjective as maps of k-vector spaces, which contradicts π being injective on tangent vectors, i.e. $(\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee \to (\mathfrak{m}_y/\mathfrak{m}_y^2)^\vee$ being injective. So we conclude that the degree of π at closed points is at most 1. But since the degree of π is upper-semicontinuous, its degree at all points is at most 1. Hence we are done by the previous lemma. $\square$

Lemma 9.1.3. *Suppose $\mathcal{L}$ is a degree $2g-2$ line bundle, then $h^0(C, \mathcal{L}) = g-1$ or g, with $h^0(C, \mathcal{L}) = g$ iff $\mathcal{L} = \omega_C$.*

Theorem 9.1.4. *Let k be algebraically closed. Suppose $\mathcal{L}$ is a line bundle on a curve C, and let $g = h^1(X, \mathcal{O}_X)$ be the arithmetic genus of C.*

- *If $\deg \mathcal{L} \geq 2g$, then $\mathcal{L}$ is basepoint-free.*
- *If $\deg \mathcal{L} \geq 2g+1$, then $\mathcal{L}$ is very ample (in fact, any basis of $\Gamma(C, \mathcal{L})$ gives a closed embedding $C \hookrightarrow \mathbb{P}_k^{\deg \mathcal{L} - g}$).*

9.2. Genus 0.

Example 9.2.1. The curve $x^2+y^2+z^2=0$ in $\mathbb{P}^2_{\mathbb{R}}$ has genus 0, and is not isomorphic to $\mathbb{P}^1_{\mathbb{R}}$.

Proposition 9.2.1. *Any genus 0 curve C with a k-point is isomorphic to $\mathbb{P}^1_k$.*

Proposition 9.2.2. *All genus 0 curves can be described as conics in $\mathbb{P}^2_k$.*

Proposition 9.2.3. *Suppose C is a curve not isomorphic to $\mathbb{P}^1_k$, then any degree 1 line bundle $\mathcal{L}$ has $h^0(C,\mathcal{L})<2$. As a consequence, for degree-1 points p,q on C, $\mathcal{O}(p)\cong\mathcal{O}(q)$ iff $p=q$.*

9.3. Hyperelliptic curves.

Assume k algebraically closed with characteristic not 2.

Definition 9.3.1. A genus g curve C is *hyperelliptic* if it admits a double cover (i.e. degree 2 finite morphism) $\pi : C\to\mathbb{P}^1_k$ (which we may as well fix).

Then the preimage of any closed point consists of either 1 or 2 points.

Theorem 9.3.1 (hyperelliptic Riemann-Hurwitz)**.** *Let C be a hyperelliptic curve with double cover $\pi : C\to\mathbb{P}^1_k$. Then π has $2g+2$ branch points (closed points $p\in\mathbb{P}^1_k$ where $\pi^{-1}(p)$ is a single point).*

Proposition 9.3.2. *Let $p_1,\dots,p_r$ be distinct closed points in $\mathbb{P}^1_k$. If r is even, then there is precisely one double cover branched at those points. If r is odd, then there are none.*

Proof. Suppose 0 and ∞ are distinct from $p_1,\dots,p_r$. Then all branch points are in $\mathbb{A}^1$. Any double cover $C'\to\mathbb{A}^1$ gives rise to a quadratic field extension $K/k(x)$, which must be Galois. Find $y\in K$ such that the nontrivial element σ in the Galois group maps $y\mapsto -y$. Then $y^2\in k(x)$, so we can replace y by an appropriate $k(x)$-multiple such that y^2 is a polynomial, monic with no repeated factors, say $y^2=f(x)=x^N+a_{N-1}x^{N-1}+\cdots+a_0$. This is a curve C'_0 in $\mathbb{A}^2$, and by the Jacobian criterion, this curve is regular. Thus C'_0 and C' are both normalizations of $\mathbb{A}^1$ in $k[x](y)$, hence isomorphic. Because the branch points are $p_1,\dots,p_r$, we conclude that $f(x)=(x-p_1)\dots(x-p_r)$.

In the projective situation, we simply do the same for $k[u]$, $u=x^{-1}$, which gives rise to the curve C'' defined by $z^2=(u-\frac{1}{p_1})\dots(u-\frac{1}{p_r})$. So the double cover $C\to\mathbb{P}^1$ has to be glued using C' and C''. Thus, in $K(C)$, we must have $z^2=u^rf(1/u)=f(x)/x^r=y^2/x^r$. If r is even, then there is a unique way to glue, i.e. identifying $z=y/x^{r/2}$. If r is odd, x does not have a square root in $k(x)[y]/(y^2-f(x))$, so there is no way to glue C' and C'' together compatibly. □

Proof of hyperelliptic Riemann-Hurwitz. We now have an explicit description of $\pi : C\to\mathbb{P}^1_k$, in terms of covering it by two affine opens. Writing down the Čech complex then easily tells us that $g=h^1(C,\mathcal{O}_C)=\frac{r}{2}-1$, as desired. □

Proposition 9.3.3. *Suppose $g\geq 2$. If $\mathcal{L}$ corresponds to a hyperelliptic cover $C\to\mathbb{P}^1$, then $\mathcal{L}^{\otimes(g-1)}\cong\omega_C$.*

Proof. Compose the hyperelliptic map with the $(g-1)$-th Veronese embedding

$$C\to\mathbb{P}^1\to\mathbb{P}^{g-1}$$

then the pullback of $\mathcal{O}_{\mathbb{P}^{g-1}}(1)$ along this composition is $\mathcal{L}^{\otimes(g-1)}$. The pullback $H^0(\mathbb{P}^{g-1},\mathcal{O}(1))\to H^0(C,\mathcal{L}^{\otimes(g-1)})$ is injective: if a hyperplane $s\in H^0(\mathbb{P}^{g-1},\mathcal{O}(1))$ is pulled back to 0, then s vanishes on all of the image of C, so the image of C (a rational normal curve) is contained in a hyperplane, which is impossible. So $\mathcal{L}^{\otimes(g-1)}$ is a degree $2g-2$ line bundle that has at least g linearly independent sections, so it is equal to ω_C. □

Proposition 9.3.4. *Any curve of genus at least 2 admits at most one hyperelliptic cover.*

Proof. The hyperelliptic map, if it exists, can be reconstructed from the canonical linear series given by ω_C. □

Proposition 9.3.5. *A curve C of genus at least 1 is hyperelliptic iff it has a degree 2 line bundle $\mathcal{L}$ with $h^0(C, \mathcal{L}) = 2$.*

Proof. Suppose $\mathcal{L}$ is a degree 2 line bundle with $h^0(C, \mathcal{L}) \geq 2$. We claim $h^0(C, \mathcal{L}) = 2$. Suppose otherwise. Consider a closed point p, and the exact sequence $0 \to \mathcal{O}(-p) \to \mathcal{O}_C \to \mathcal{O}|_p \to 0$. Tensoring with $\mathcal{L}$ gives $0 \to \mathcal{L}(-p) \to \mathcal{L} \to \mathcal{L}|_p \to 0$. Writing down the long exact sequence gives $h^0(C, \mathcal{L}(-p)) + 1 \geq h^0(C, \mathcal{L}) \geq 3$, so $h^0(C, \mathcal{L}(-p)) \geq 2$. But $\mathcal{L}(-p)$ has degree 1, so this contradicts with Proposition 9.2.3. So $h^0(C, \mathcal{L}) = 2$. Let s_1, s_2 be linearly independent sections, we claim that this is basepoint-free. Suppose $\operatorname{div}(s_1) = p + q_1$, $\operatorname{div}(s_2) = p + q_2$. Then $\mathcal{O}(q_1) = \mathcal{L}(-p) = \mathcal{O}(q_2)$, which implies $q_1 = q_2$, so s_1/s_2 has no zeros and no poles and therefore constant, which contradicts them being linearly independent.

Now, return to the original problem. Suppose C is hyperelliptic, then the pullback of $\mathcal{O}_{\mathbb{P}^1}(1)$ is a degree 2 line bundle with at least 2 sections, so by our discussion above it has exactly 2 sections. Conversely, suppose $\mathcal{L}$ is a degree 2 bundle with 2 sections, then it is basepoint-free and thus gives a map to $\mathbb{P}^1$, which has degree 2. $\square$

9.4. **Genus 1: elliptic curves.**

9.5. **Genus 2.** We claim that in this case all curves are hyperelliptic. Let C be a curve of genus $g = 2$. Then ω_C has degree $2g - 2 = 2$, and has 2 sections. By Proposition 9.3.5, it is basepoint-free and gives a double cover to $\mathbb{P}^1$. Conversely, any double cover gives a degree 2 line bundle with 2 sections, which must be ω_C.

9.6. **Genus 3.**

Proposition 9.6.1 (canonical embedding)**.** *Let k be algebraically closed. Suppose C is not hyperelliptic, then ω_C gives a closed embedding $C \hookrightarrow \mathbb{P}^{g-1}$.*

Proof. To show ω_C is basepoint-free, it suffices to show that given any closed point p,

$$h^0(C, \omega_C(-p)) = h^0(C, \omega_C) - 1.$$

By Riemann-Roch: $h^0(C, \omega_C(-p)) - h^0(C, \mathcal{O}(p)) = \deg \omega_C(-p) - g + 1 = 2g - 3 - g + 1 = g - 2$. But $h^0(C, \mathcal{O}(p)) = 1$ by Proposition 9.2.3, so indeed $h^0(C, \omega_C(-p)) = g - 1 = h^0(C, \omega_C) - 1$.

Now, to show ω_C is very ample, it suffices to show that given any closed points p, q (not necessarily different),

$$h^0(C, \omega_C(-p - q)) = h^0(C, \omega_C) - 2.$$

By Riemann-Roch: $h^0(C, \omega_C(-p - q)) - h^0(C, \mathcal{O}(p + q)) = \deg \omega_C(-p - q) - g + 1 = 2g - 4 - g + 1 = g - 3$. Because C is not hyperelliptic, then the degree 2 line bundle $\mathcal{O}(p + q)$ must have $h^0(C, \mathcal{O}(p + q)) = 1$. So $h^0(C, \omega_C(-p - q)) = g - 2 = h^0(C, \omega_C) - 2$ as desired. $\square$

Specializing to the genus 3 case, the canonical embedding gives an embedding $C \hookrightarrow \mathbb{P}^2$ as a degree 4 curve. Conversely, I claim that every quartic curve in $\mathbb{P}^2$ is canonically embedded. The curve has genus $1 - p_C(0) = 1 - \binom{2}{2} + \binom{-2}{2} = 3$. The embedding is given by a line bundle of degree 4 with at least 3 sections, so it has to be ω_C. In conclusion, there is a bijection between genus 3 non-hyperelliptic curves and quartics in $\mathbb{P}^2$ (up to $\mathrm{PGL}_3(k)$).

Example 9.6.1. The *Klein quartic* $x^3y + y^3z + z^3x = 0$ has 168 automorphisms.

Definition 9.6.2. A curve admitting a degree 3 cover of $\mathbb{P}^1$ is called *trigonal*.

Proposition 9.6.2. *Every non-hyperelliptic genus 3 curve is trigonal.*

9.7. **Genus 4.** The canonical embedding i maps a genus 4 curve C as a sextic curve in $\mathbb{P}^3$. We claim that this is in bijection with regular complete intersections of a quadric surface and a cubic surface.

By Riemann-Roch,

$$h^0(C, i^*\mathcal{O}(2)) = h^0(C, \omega_C^{\otimes 2}) = \deg \omega_C^{\otimes 2} - g + 1 = 12 - 4 + 1 = 9,$$

while $h^0(\mathbb{P}^3, \mathcal{O}(2)) = \binom{5}{2} = 10$, so the pullback

$$H^0(\mathbb{P}^3, \mathcal{O}(2)) \to H^0(C, i^*\mathcal{O}(2))$$

has a nontrivial kernel. The kernel (which is $H^0(\mathbb{P}^3, \mathcal{I}_{C/\mathbb{P}^3} \otimes \mathcal{O}(2))$ from the closed subscheme exact sequence) is a quadric surface that contains C.

Now, this quadric surface Q is given by some quadratic form which can be represented by a matrix. We may as well diagonalize it (assuming $\operatorname{char} k \neq 2$). Its rank determines the shape of Q:

- rank 1: double plane
- rank 2: two planes
- rank 3: cone
- rank 4: regular quadric.

The first two cases cannot happen, i.e. C does not lie in a hyperplane, because $H^0(\mathbb{P}^3, \mathcal{O}(1)) \to H^0(C, \omega_C)$ is injective. So we conclude that Q is irreducible.

In addition, we claim that C cannot lie in two distinct quadric surfaces. Otherwise, by Bezout, their intersection has degree $2 \times 2 = 4 < 6$, but C is contained in this intersection, hence must have a larger degree.

So we ask, does C lie in a cubic surface? Repeating the same calculation, we see that

$$\dim \ker(H^0(\mathbb{P}^3, \mathcal{O}(3)) \to H^0(C, i^*\mathcal{O}(3))) \geq 5.$$

Since we require the cubic surface to not contain Q, a 4-dimensional subspace is forbidden, so there exists at least one cubic surface K not containing Q. Now, K and Q share no components, so $K \cap Q$ is a complete intersection, containing C as a closed subscheme. By Bezout's theorem, $K \cap Q$ has degree 6. By a calculation on Hilbert polynomials, $K \cap Q$ has genus $1 - \binom{3}{3} + \binom{3-2}{3} + \binom{3-3}{3} - \binom{3-5}{3} = 4$. Since the genus and degree completely determine the Hilbert polynomial (which has degree 1), we conclude that $C = K \cap Q$.

Conversely, any regular complete intersection of a quadric Q and a cubic K is a curve C of genus 4 and degree 6. Then C does not lie inside a hyperplane, because otherwise (say it lies inside H), then $H \cap Q$ is a degree 2 curve containing C, a degree 6 curve, which is impossible. Thus, $\mathcal{O}_C(1)$ has degree 6 and at least 4 sections, so it must be equal to ω_C. This means that C is canonically embedded.

9.8. **Genus 5.** We can mimic the genus 4 case: the dualizing sheaf ω_C has degree $2g - 2 = 8$ and $g = 5$ sections, so it canonically embeds C as a degree 8 curve in $\mathbb{P}^4$. By Riemann-Roch,

$$h^0(C, \omega_C^{\otimes 2}) = \deg \omega_C^{\otimes 2} - g + 1 = 16 - 5 + 1 = 12,$$

while $h^0(\mathbb{P}^4, \mathcal{O}(2)) = \binom{4+2}{4} = 15$. So

$$\dim \ker(H^0(\mathbb{P}^4, \mathcal{O}(2)) \to H^0(C, \omega_C^{\otimes 2})) \geq 3.$$

Then there exist 3 linearly independent quadrics containing C. (However, we will see later that not all genus 5 curves are canonically embedded as the complete intersection of 3 quadrics; the exceptional ones are precisely the trigonal curves.)

Conversely, suppose C is the regular complete intersection of 3 quadrics. Then its genus is given by the inclusion-exclusion formula:

$$g = 1 - \binom{4}{4} + 3 \cdot \binom{4-2}{4} - 3 \cdot \binom{4-4}{4} + \binom{4-6}{4} = 5.$$

Also, C has degree $2^3 = 8$ by Bezout's theorem. To show it is canonically embedded, it suffices to show $\mathcal{O}_C(1)$ has at least 5 sections, i.e. it does not lie in a plane. Suppose it does, then C is a closed subscheme of the complete intersection of two quadrics and a plane, which is a curve of degree $2^2 = 4$. But C has degree $8 > 4$, so it cannot be contained in a curve of degree 4, a contradiction. So $\mathcal{O}_C(1)$ has degree 8 and at least 5 sections, so it must be isomorphic to ω_C, as desired.

Unfortunately, this stops working for genus $g \geq 6$:

Proposition 9.8.1. *Any canonical genus g curve, where $g \geq 6$, is not a complete intersection.*

10. Differentials

In this section we take another familiar object in differential geometry (differential forms) and transport it to schemes.

As motivation, consider the case where U is an open set in $\mathbb{R}^n$. Then we have a map $d : C^\infty(U) \to \Omega^1(U)$, mapping $f \mapsto \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$, which satisfies $d(fg) = f(dg) + (df)g$. On any smooth manifold M, we have the same construction on every coordinate patch, which glue together. More generally, for a smooth map $M \to N$, we have the notion of a sheaf of *relative* differential forms.

The corresponding algebraic version is the "cotangent sheaf".

10.1. **Affine case.** We start from the simplest (affine) case.

Definition 10.1.1. Let $i : B \to A$ be a map of rings. The *module of derivations* is an A-module M, and a map of abelian groups $d : A \to M$ (not a map of A-modules!) such that:

- $i(B) \subset \ker d$;
- $d(aa') = a(da') + (da)a'$.

Note that then d is a map of B-modules.

Definition 10.1.2. The module of Kähler differentials $(\Omega_{A/B}, d)$ is the universal such module: given any module of derivation (M, d'), there exists a unique map of A-modules $p : \Omega_{A/B} \to M$, such that $p \circ d = d'$. It is constructed as

$$\Omega_{A/B} = \left(\bigoplus_{a \in A} A da \right) \Big/ \langle d(i(b)), d(a + a') - d(a) - d(a'), d(aa') - ad(a') - d(a)a' \rangle.$$

Note that if A is a finitely generated algebra over B by $a_1, \ldots, a_n$, then $\Omega_{A/B}$ is a finitely generated module over A by $da_1, \ldots, da_n$. It is even finitely presented when A is.

Example 10.1.3. Let $A = B[x_1, \ldots, x_n]$, then $\Omega_{A/B} = \bigoplus_{i=1}^n A dx_i$, with $df(x_1, \ldots, x_n) = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$.

Example 10.1.4. Let $A = B[x, y]/(f(x, y))$. Then

$$\Omega_{A/B} = \frac{A dx \oplus A dy}{(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy)}.$$

Say $A = k[x, y]/(xy)$, then $\Omega_{A/k} = (A dx \oplus A dy)/(y dx + x dy)$. Its rank at all points $(x, y) \neq (0, 0)$ is 1, but the rank jumps to 2 at $(0, 0)$. This already indicates that $\Omega_{A/B}$ captures smoothness information.

Lemma 10.1.1. *Let $T \subset B$, $S \subset A$ be multiplicatively closed sets such that $i(T) \subset S$, then*

$$\Omega_{S^{-1}A/T^{-1}B} = S^{-1}\Omega_{A/B}.$$

Lemma 10.1.2. *Let $C \to B \to A$ be maps of rings. Then*

$$A \otimes \Omega_{B/C} \to \Omega_{A/C} \to \Omega_{A/B} \to 0$$

is exact, where the first map is given by $a \otimes db \mapsto ad(i(b))$ and the second map is $da \mapsto da$.

Remark. In manifolds: let $M \xrightarrow{\pi} N \to \{*\}$ be smooth, then $0 \to T_{M/N} \to T_M \to \pi^* T_N \to 0$ is exact. Dualize to get a similar expression as in lemma 10.1.2.

Remark. If the maps of $\operatorname{Spec} A \to \operatorname{Spec} B \to \operatorname{Spec} C$ are smooth, then the sequence in lemma 10.1.2 is short exact (the leftmost map is injective).

Lemma 10.1.3. *Let $C \to B \to A$ be maps of rings, where $A = B/I$. Then we can continue the exact sequence to the left:*

$$I/I^2 \xrightarrow{\delta} A \otimes \Omega_{B/C} \to \Omega_{A/C} \to 0$$

where I/I^2 is a $B/I = A$-module, and $\delta : i \mapsto 1 \otimes di$; note that it is well-defined because $\delta(ii') = i \otimes di' + i' \otimes di \in I \otimes \Omega_{B/C}$, hence is zero.

Remark. In the differential-geometric picture: $M \to N$ is an embedded submanifold, and we have an exact sequence

$$0 \to T_M \to T_N|_M \to \text{normal bundle} \to 0.$$

So I/I^2 is called the *conormal sheaf* of $\operatorname{Spec} A \hookrightarrow \operatorname{Spec} B$.

In general, there is a way to extend the right exact sequence into a long exact sequence, analogous to sheaf cohomology. The difficulty is that we're starting with a sequence of rings, which is not an abelian category. This is called André–Quillen homology.

10.2. **The cotangent sheaf.** Let $\pi : X \to Y$ be a morphism of schemes. Define the cotangent sheaf $\Omega_{X/Y}$, a sheaf of $\mathcal{O}_X$-modules, by gluing together on affine opens. The tangent sheaf $T_{X/Y} = \Omega_{X/Y}^\vee$.

There is another way to define this. In the affine case, let $B \to A$ be a ring map. Consider

$$I = \ker(A \otimes_B A \xrightarrow{a \otimes a' \mapsto aa'} A),$$

which is generated by tensors of the form $a \otimes 1 - 1 \otimes a$. Then one can show that I/I^2, which is an $A = A \otimes_B A/I$-module, is just $\Omega_{A/B}$, with $d : A \to I/I^2$ sending $a \mapsto (a \otimes 1 - 1 \otimes a) \pmod{I^2}$. We can use this to directly define $\Omega_{X/Y}$, as the sheaf $\mathcal{I}/\mathcal{I}^2$ where $\mathcal{I}$ is the ideal sheaf of $\Delta : X \to X \times_Y X$.

The analogous versions of lemmata 10.1.2 and 10.1.3 are then:

Lemma 10.2.1. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be maps of schemes, then we have an exact sequence*

$$f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0.$$

Lemma 10.2.2. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be maps of schemes, where f is a closed immersion. then we have an exact sequence*

$$\mathcal{I}/\mathcal{I}^2 \to f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to 0,$$

where $\mathcal{I}$ is the ideal sheaf of f, and $\mathcal{I}/\mathcal{I}^2$ is the conormal sheaf.

The following proposition justifies the importance of Ω.

Proposition 10.2.3. *Suppose X is a scheme over k, and $p \in X(k)$. Then*

$$i_p^*\Omega_{X/k} = T_p^\vee = \mathfrak{m}/\mathfrak{m}^2$$

is the Zariski cotangent space at p.

Proof. When $X = \operatorname{Spec} A$, a k-point is a maximal ideal $\mathfrak{m} \subset A$ with $A/\mathfrak{m} = k$. So it suffices to show $\Omega_{A/k} \otimes_A k \cong \mathfrak{m}/\mathfrak{m}^2$. Taking the dual, we have to show

$$\operatorname{Hom}(\mathfrak{m}/\mathfrak{m}^2, k) \cong \operatorname{Hom}(\Omega_{A/k} \otimes_A k, k).$$

The RHS is $\operatorname{Hom}(\Omega_{A/k}, k)$ by tensor-hom adjunction, and by the universal property this is just k-derivations $d : A \to k$. This necessarily kills k and $\mathfrak{m}^2$, so induces a map $\mathfrak{m}/\mathfrak{m}^2 \to k$. Conversely, any map $\mathfrak{m}/\mathfrak{m}^2 \to k$ extends to a k-derivation $d : A \to k$. $\square$

Example 10.2.1. Let $X = \mathbb{P}^1_k$, and consider $\Omega_{\mathbb{P}^1/k}$, which is a line bundle. In fact, by taking an affine chart $\mathbb{A}^1 = \operatorname{Spec} k[x]$ and a rational section dx of the line bundle, because

$$dx = d(1/x^{-1}) = \frac{1}{(x^{-1})^2} d(x^{-1}),$$

we conclude that $\Omega_{\mathbb{P}^1/k} \cong \mathcal{O}(-2) \cong \omega_{\mathbb{P}^1}$. In fact, this is true for all smooth projective curves.

Example 10.2.2. In the case $X = \mathbb{P}^n_k$, we have a map

$$\mathcal{O}_X \xrightarrow{x_0,\dots,x_n} \mathcal{O}(1)^{\oplus(n+1)}$$

and dualizing it we get the *Euler sequence*

$$(*) \qquad 0 \to \Omega_{\mathbb{P}^n/k} \to \mathcal{O}(-1)^{\oplus(n+1)} \xrightarrow{x_0,\dots,x_n} \mathcal{O}_X \to 0.$$

(Intuition: $\mathbb{C}^\times \to \mathbb{C}^{n+1}\backslash 0 \xrightarrow{\pi} \mathbb{P}^n$, which gives $0 \to \langle \sum x_i \frac{\partial}{\partial x_i} \rangle \to \bigoplus \mathbb{C}\frac{\partial}{\partial x_i} \to \pi^* T_{\mathbb{P}^n} \to 0$.)

Proof. Write $(*)$ as a map of graded modules: let $S = k[x_0, \dots, x_n]$, then $S(-1)$ shifts the indexing toward the left by 1. Then let M be the kernel

$$(**) \qquad 0 \to M \to S(-1)^{\oplus(n+1)} \to S \to 0$$

where the latter map is given by $e_i \mapsto x_i$ (e_i is the generator of each copy of $S(-1)$, which has degree 1). To calculate $\widetilde{M}$ on each $D(x_i)$, we localize $(**)$ at x_i and take the degree 0 component. It is a free k-vector space spanned by $\frac{1}{x_i}(e_j - \frac{x_j}{x_i} e_i)$, and we take each of these to $d(x_{j/i})$, which are free generators of the sections of $\Omega_{\mathbb{P}^n/k}$ over $D(x_i)$. It suffices then to check that these isomorphisms glue together to show that $\Omega_{\mathbb{P}^n/k} \cong \widetilde{M}$. □

The *canonical bundle* $K_{\mathbb{P}^n/k} := \bigwedge^n \Omega_{\mathbb{P}^n/k}$ can then be calculated as $\mathcal{O}(-n-1)$, which is just the sheaf $\omega_{\mathbb{P}^n/k}$ appearing in Serre duality. This will be true for all smooth projective varieties.

10.3. **Smoothness.** Recall the definition of smoothness over a field: $X \to \operatorname{Spec} k$ is smooth of dimension d if it can be covered with affine charts $\operatorname{Spec} k[x_1, \dots, x_n]/(f_1, \dots, f_r)$ where the Jacobian matrix has corank d at all points. We now make an equivalent, cleaner definition:

Definition 10.3.1. Let X be a k-scheme, then X is *smooth of dimension* d if X is locally of finite type, of pure dimension d, and $\Omega_{X/k}$ is locally free of rank d.

Theorem 10.3.1 (conormal exact sequence for smooth varieties)**.** *Let $i : X \hookrightarrow Y$ be smooth k-varieties of dimension d, e. Then*

$$0 \to \mathcal{I}/\mathcal{I}^2 \to i^*\Omega_Y \to \Omega_X \to 0$$

is exact, and $\mathcal{I}/\mathcal{I}^2$ is locally free of rank $e - d$. (Recall this is not usually left exact.) Conversely, if Y is smooth, $\mathcal{I}/\mathcal{I}^2$ is locally free, and the above sequence is exact, then X is smooth.

The normal sheaf is $N_{X/Y} = (\mathcal{I}/\mathcal{I}^2)^\vee$. When $X \hookrightarrow Y$ is a (Weil) divisor, the conormal sheaf $\mathcal{I}/\mathcal{I}^2$ is denoted by $\mathcal{O}_X(-X)$, by which we really mean $\mathcal{O}_Y(-X)|_X$.

Proposition 10.3.2 (adjunction formula)**.** *Let $i : X \hookrightarrow Y$ be a divisor, then*

$$\omega_X = i^*(\omega_Y \otimes \mathcal{O}_Y(X)).$$

Proof. By the conormal exact sequence, we see that

$$i^* K_Y = i^*(\bigwedge^{\dim Y} \Omega_Y) = \bigwedge^{1}(\mathcal{I}/\mathcal{I}^2) \otimes \bigwedge^{\dim Y - 1} \Omega_X = i^*\mathcal{O}_Y(-X) \otimes K_X,$$

and we know $K_X = \omega_X$ for smooth projective varieties. □

10.4. **Invariants.** We can use $\Omega_{X/k}$ and $\bigwedge^q \Omega_{X/k} = \Omega^q_{X/k}$ to define invariants, such as the *Hodge numbers*

$$h^p(X, \Omega^q_{X/k}).$$

What's interesting is that for $p = 0$ we get *birational invariants*:

Theorem 10.4.1. *Let X, Y be smooth projective varieties that are birationally isomorphic. Then $h^0(X, \Omega^q_{X/k}) = h^0(Y, \Omega^q_{Y/k})$.*

This works not just for $\bigwedge^q$, but for any covariant tensor operation.

Definition 10.4.1 (plurigenera)**.** The rth plurigenus of a smooth projective k-variety X is $h^0(X, K_X^{\otimes r})$.

Definition 10.4.2 (Kodaira dimension)**.** By asymptotic Riemann-Roch, $h^0(X, K_X^{\otimes r})$ is eventually polynomial in r. The Kodaira dimension $\kappa(X)$ is the degree of this polynomial (defined to be -1 if the polynomial is identically zero).

10.5. **Riemann-Hurwitz theorem.**

Theorem 10.5.1. *Let $\pi : X \to Y$ be a finite separable morphism of regular projective curves, of pure degree n. Then*

$$2g(X) - 2 = n(2g(Y) - 2) + \deg R,$$

where R is the ramification divisor.

As an application, we may count the number of tangent lines from a point $p \in \mathbb{P}^2$ to a degree d plane curve $C \subset \mathbb{P}^2$. (The answer is $d^2 - d$.)

11. Flatness

The idea is to capture "nice families of schemes".

11.1. **Algebra.**

Definition 11.1.1. Let X be a scheme, $\mathcal{F} \in \mathsf{QCoh}(X)$, then $\mathcal{F}$ is *flat* if $\mathcal{F}_x$ is flat over $\mathcal{O}_{X,x}$ (or equivalently, affine locally instead of stalkwise).

Let $f : X \to Y$ be a morphism, then it is *flat* if for $x \in X$, $y = f(x) \in Y$, $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is flat.

Example 11.1.2. Closed embeddings in general will not be flat. For example, $\operatorname{Spec} k \xrightarrow{0} \mathbb{A}^1$ is not flat, because k is not a flat $k[x]$ module: take $k[x] \xrightarrow{[x]} k[x]$.

Lemma 11.1.1. *Let $0 \to N_1 \to N_2 \to N_3 \to 0$ be a short exact sequence of A-modules, where N_3 is flat. Then for any A-module M,*

$$0 \to N_1 \otimes M \to N_2 \otimes M \to N_3 \otimes M \to 0$$

is exact.

Proof. $\operatorname{Tor}_1(N_3, M) = 0$. □

Geometrically: suppose $0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$ are QCoh on Y, where $\mathcal{E}_3$ is flat. Then for any morphism $f : X \to Y$, pulling back to $0 \to f^*\mathcal{E}_1 \to f^*\mathcal{E}_2 \to f^*\mathcal{E}_3 \to 0$ is also exact.

Lemma 11.1.2. *Suppose $0 \to M_1 \to M_2 \to M_3 \to 0$, then:*

- *If M_2, M_3 are flat, so is M_1;*
- *If M_1, M_3 are flat, so is M_2.*

Lemma 11.1.3. *Suppose $0 \to M_1 \to \cdots \to M_n \to 0$ is an exact complex. If $M_2, \ldots, M_n$ are flat, then so is M_1.*

Lemma 11.1.4. *Suppose $0 \to M_1 \to \cdots \to M_n \to 0$ is an exact complex, where all M_i are flat. The for any N, $0 \to M_1 \otimes N \to \cdots \to M_n \otimes N \to 0$ is exact.*

Proposition 11.1.5. *Let $(A, \mathfrak{m}, k)$ be a local Noetherian ring. Then any finitely generated, flat A-module is free.*

Proof. By Nakayama, we can pick lifts of generators of $M \otimes_A k$ to get

$$0 \to K \to A^{\oplus r} \to M \to 0.$$

Since M is flat, tensoring with k gives an exact sequence

$$0 \to K \otimes_A k \to k^{\oplus r} \to M \otimes k \to 0,$$

but $k^{\oplus r} \cong M \otimes k$, so $K \otimes_A k = 0$, so $K = 0$ by Nakayama. □

Theorem 11.1.6. *Suppose for any finitely generated ideal $I \subset A$, $\mathrm{Tor}_1(M, A/I) = 0$. Then M is flat.*

Corollary 11.1.7. *Let A be a PID. Then M is flat iff M is torsion-free.*

Corollary 11.1.8. *Let $\pi : X \to C$ be dominant, where X is integral and C is a regular curve. Then π is flat.*

Proof. It suffices to check that $\mathcal{O}_{X,x}$ are torsion free. But since π is dominant, this is automatically true. □

Example 11.1.3. The resolution of a node is not flat.

11.2. **Geometry.** Assume all schemes are locally Noetherian or something.

Theorem 11.2.1. *Let $f : X \to Y$ be a flat morphism. Given $x \in X$, $y = f(x) \in Y$, then*

$$\dim_x(X_y) = \dim_x X - \dim_y Y.$$

Here $\dim_x$ means local dimension, i.e. dimension of the local ring at x.

Proof. Use induction on $\dim_y Y$. We may replace Y by $\operatorname{Spec} \mathcal{O}_{Y,y}$ and X by $X \times_Y \operatorname{Spec} \mathcal{O}_{Y,y}$. The base case $\dim Y = 0$ is easy, since then $X_y = X$ (as topological spaces) and $\dim X_y = \dim X - 0$. In general, suppose $\dim Y = n$. Pick $t \in \mathfrak{m}_Y \subset \mathcal{O}_{Y,y}$ a non-zero-divisor. By flatness (which is just torsion-free over local ring), the image of t under $f^\# : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is also a non-zero-divisor. By Krull's principal ideal theorem, every irreducible component of $V(t) \hookrightarrow Y$ has codimension 1, and so is every irreducible component of $V(f^\#(t)) \hookrightarrow X$. We are then done by inductive hypothesis. □

Theorem 11.2.2. *Let $X \to Y$ be projective, $\mathcal{F} \in \mathsf{Coh}(X)$ flat over Y. Then the map*

$$y \mapsto \chi(\mathcal{F}|_{X_y})$$

is locally constant.

Remark: conversely, let $X \hookrightarrow \mathbb{P}^n \times Y \to Y$, $\mathcal{F} \in \mathsf{Coh}(X)$, and Y is reduced. If the Hilbert polynomials $p_{\mathcal{F}|_{X_y}}(t)$ are independent of the choice of $y \in Y$, then $\mathcal{F}$ is flat over Y.

Proof. Reduce to $Y = \operatorname{Spec} A$. It is enough to show that $\chi(\mathcal{F}(m)|_{X_y})$ is independent of Y for m sufficiently large. By Serre vanishing, pick m large enough so that $H^k(X, \mathcal{F}(m)) = 0$ for $k \geq 1$. Then the augmented Čech complex associated to $\mathcal{F}(m)$

$$0 \to H^0(X, \mathcal{F}(m)) \to C^0(X, \mathcal{F}(m)) \to \cdots \to C^n(X, \mathcal{F}(m)) \to 0$$

is a long exact sequence, and since $\mathcal{F}$ is flat over Y, all but the first term are flat over A. Then so is $H^0(X, \mathcal{F}(m))$. Since it is flat and finitely presented, it is projective (locally free).

Now, to restrict it to X_y, it is enough to tensor this Čech complex with $\kappa(y)$, which by flatness gives another exact complex. We then conclude that $H^k(X_y, \mathcal{F}(m)|_{X_y}) = 0$ for $k \geq 1$, and is equal to $H^0(X, \mathcal{F}(m)) \otimes_A \kappa(y)$ for $k = 0$. This number is dependent of y since $H^0(X, \mathcal{F}(m))$ is locally free. $\square$

Corollary 11.2.3. *Let $X \hookrightarrow \mathbb{P}^n \times Y \to Y$ be flat, and Y is connected. Then the Hilbert polynomials $p_{X_y}(t)$ are independent of the choice of $y \in Y$.*

Corollary 11.2.4. *Let $C \times Y \to Y$ be a flat morphism, where C is a projective curve and Y is connected. Let $\mathcal{L}$ be a line bundle on $C \times Y$. Then $\deg \mathcal{L}|_{C \times \{y\}}$ is independent of $y \in Y$.*

Suppose we have a family of schemes, parametrized by one parameter $t \neq 0$. We would like to define a limit at $t = 0$. In other words, if we have a scheme lying over, say, $\mathbb{A}^1 - \{0\}$, we would like to uniquely extend it to be over $\mathbb{A}^1$.

Theorem 11.2.5 (uniqueness of flat limits)**.** *Let A be a DVR, $K = \operatorname{Frac} A$. Let η be the generic point of $\operatorname{Spec} A$. Let X be a Noetherian scheme over $\operatorname{Spec} A$. Given a closed subscheme $Z_\eta \hookrightarrow X_\eta$ of the generic fiber, consider its scheme-theoretic closure $Z = \overline{Z_\eta} \hookrightarrow X$. Then this is the unique closed subscheme $Z \hookrightarrow X$ that is flat over $\operatorname{Spec} A$, and restricts to Z_η on X_η.*