# MEEG 829
# Inverted Pendulum on a Cart Project

**Prof. Poulakakis**

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| **KARAKASIS** | **CHRYSOSTOMOS** |
|---|---|
| LAST NAME (PRINTED) | FIRST NAME (PRINTED) |

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1. Using linear state variable feedback and with the nonlinear model initialized at $x_2(0) = 0$, $x_3(0) = 0$ and $x_4(0) = 0$, the largest I could make $x_1(0)$ before the nonlinear model went unstable was **0.658** (units of radians). The linear state variable feedback that I used was:

$$\mathbf{k}=\begin{bmatrix}-19.3000 - 22.9750 - 1.5900 - 5.5250\end{bmatrix}.$$

2. Using linear state variable feedback plus an observer, with the observer initialized at the origin and $x_2(0) = 0$, $x_3(0) = 0$ and $x_4(0) = 0$, the largest I could make $x_1(0)$ before the nonlinear model went unstable was **0.340** (units of radians). The observer that I used to obtain the best response was **nonlinear** and the observer gain was:

$$\mathbf{L}=\begin{bmatrix} 9.2929 & 0.3630 \\ 8.9077 & -3.1719 \\ 3.9579 & 9.9071 \\ 22.2846 & 23.2335 \end{bmatrix}.$$

# 1 Problem 1

Problem 1

$$\frac{d^2\phi}{d\bar{t}^2} = \frac{d^2\phi}{dt^2} T^2 \overset{①}{=} \frac{(M+m)mgL\sin\phi \cdot T^2}{(M+m)(J+mL^2)-(mL)^2(\cos\phi)^2} - \frac{\left(mL\frac{d\phi}{dt}\right)^2 \sin\phi\cos\phi \, T^2}{(M+m)(J+mL^2)-(mL)^2(\cos\phi)^2}$$

$$- \frac{(mL\cos\phi)\mu}{(M+m)(J+mL^2)-(mL)^2(\cos\phi)^2} \cdot T^2$$

We will analyze each term independently:

- $$\frac{(M+m)mgL\sin\phi \, T^2}{(M+m)(J+mL^2)-(mL)^2(\cos\phi)^2} = \frac{(M+m)mgL\sin\phi}{[(M+m)(J+mL^2)-(mL)^2(\cos\phi)^2]} \cdot \left(\frac{J}{mL^2}+\frac{M}{m+M}\right) \cdot \frac{L}{g}$$

$$= \frac{(M+m)\, mL^2\sin\phi}{[(M+m)(J+mL^2)-(mL)^2(\cos\phi)^2]} \cdot \left(\frac{J}{mL^2}+\frac{M}{m+M}\right)$$

$$= \frac{J(M+m)\sin\phi + mL^2M\sin\phi}{(M+m)(J+mL^2)-(mL)^2(\cos\phi)^2}$$

$$= \frac{\sin\phi\,(J(M+m)+mL^2M)}{(M+m)(J+mL^2)-(mL)^2(1-(\sin\phi)^2)} \qquad \left((\cos\phi)^2 = 1-(\sin\phi)^2\right)$$

$$= \frac{\sin\phi\,[J(M+m)+mL^2M]}{(M+m)(J+mL^2)-(mL)^2+(mL)^2(\sin\phi)^2}$$

$$= \frac{\sin\phi}{\frac{(M+m)(J+mL^2)-(mL)^2}{[J(M+m)+mL^2M]} + \frac{(mL)^2(\sin\phi)^2}{[J(M+m)+mL^2M]}}$$

$$= \frac{\sin\phi}{\frac{(M+m)J+MmL^2+(mL)^2-(mL)^2}{[J(M+m)+mL^2M]} + \frac{(mL)^2(\sin\phi)^2}{[J(M+m)+mL^2M]}} = \boxed{\frac{\sin\phi}{1+c(\sin\phi)^2}} \checkmark$$

Figure 1: "Handwritten Solution for Problem 1 - Page 1"

$$\bullet \quad \frac{-\left(mL\frac{d\phi}{dt}\right)^2 \sin\phi\cos\phi\, T^2}{(M+m)(J+mL^2)-(mL)^2(\cos\phi)^2} \qquad \text{where } \frac{d\phi}{dt} = \frac{d\phi}{d\tilde{t}}\frac{d\tilde{t}}{dt} = \dot{\phi}\,\frac{1}{T}$$

$$= \frac{-\left(mL\,\dot{\phi}\,\frac{1}{T}\right)^2 \sin\phi\cos\phi\, T^2}{(M+m)(J+mL^2)-(mL)^2(\cos\phi)^2} = \frac{-(mL)^2\,\dot{\phi}^2\,\frac{1}{T^2}\sin\phi\cos\phi\, T^2}{(M+m)(J+mL^2)-(mL)^2(\cos\phi)^2}$$

$$= \frac{-(mL)^2\,\dot{\phi}^2\sin\phi\cos\phi}{(M+m)(J+mL^2)-(mL)^2(1-(\sin\phi)^2)} = \frac{-(mL)^2\,\dot{\phi}^2\sin\phi\cos\phi}{(M+m)(J+mL^2)-(mL)^2+(mL)^2(\sin\phi)^2}$$

$$= \frac{-(mL)^2\,\dot{\phi}^2\sin\phi\cos\phi}{J(M+m)+mML^2+(mL)^2(\sin\phi)^2} = \frac{-\frac{(mL)^2}{[J(M+m)+mML^2]}\,\dot{\phi}^2\sin\phi\cos\phi}{1+\frac{(mL)^2(\sin\phi)^2}{[J(M+m)+mML^2]}}$$

$$= \boxed{\frac{-c\,\dot{\phi}^2\sin\phi\cos\phi}{1+c(\sin\phi)^2}}$$

$$\bullet \quad \frac{-(mL\cos\phi)\,\mu\, T^2}{(M+m)(J+mL^2)-(mL)^2(\cos\phi)^2} = \frac{-(mL\cos\phi)\,\mu\, T^2}{J(M+m)+mML^2+(mL)^2-(mL)^2(1-(\sin\phi)^2)}$$

$$= \frac{-(mL\cos\phi)\,\mu\, T^2}{J(M+m)+mML^2+(mL)^2(\sin\phi)^2} = \frac{-\cos\phi\,\frac{mL\,\mu\, T^2}{[J(M+m)+mML^2]}}{1+c(\sin\phi)^2}$$

$$= \frac{-\cos\phi}{1+c(\sin\phi)^2}\cdot\frac{mL\,\mu}{[J(M+m)+mML^2]}\cdot\left(\frac{J}{mL^2}+\frac{M}{m+M}\right)\frac{L}{g}$$

$$= \frac{-\cos\phi}{1+c(\sin\phi)^2}\cdot\frac{\mu\,\frac{1}{g}}{[J(M+m)+mML^2]}\cdot\left(J+\frac{mL^2M}{m+M}\right)$$

$$= \frac{-\cos\phi}{1+c(\sin\phi)^2}\cdot\frac{\mu\,\frac{1}{g}}{[J(M+m)+mML^2]}\cdot\left(\frac{J(m+M)+mL^2M}{m+M}\right) = \boxed{\frac{-\cos\phi}{1+c(\sin\phi)^2}\,\bar{\mu}}$$

Figure 2: "Handwritten Solution for Problem 1 - Page 2"

## 2 Problem 2

Problem 2

$$x_1 = \phi \Rightarrow \dot{x}_1 = \dot{\phi} = x_2 \qquad (3.1)$$

$$x_2 = \dot{\phi} \Rightarrow \dot{x}_2 = \ddot{\phi} = \frac{d^2\phi}{dt^2} \overset{(3)}{=} -\frac{c\dot{\phi}^2 \sin\phi\cos\phi}{1+c\sin^2\phi} + \frac{\sin\phi}{1+c\sin^2\phi} - \frac{\cos\phi}{1+c\sin^2\phi}\bar{\mu}$$

$$x_3 = \bar{s} \Rightarrow \dot{x}_3 = \dot{\bar{s}} = x_4 \qquad (3.3)$$

$$x_4 = \dot{\bar{s}} \Rightarrow \dot{x}_4 = \ddot{\bar{s}} = \frac{d^2\bar{s}}{dt^2} \overset{(3)}{=} \frac{d\cdot\dot{\phi}^2\sin\phi}{1+c\sin^2\phi} - \frac{\sin\phi\cos\phi}{1+c\sin^2\phi} + \frac{b\,\bar{\mu}}{1+c\sin^2\phi}$$

$$(3.2) \Rightarrow \dot{x}_2 = -\frac{c\,x_2^2\sin(x_1)\cos(x_1)}{1+c\sin^2(x_1)} + \frac{\sin(x_1)}{1+c\sin^2(x_1)} - \frac{\cos(x_1)}{1+c\sin^2(x_1)}\bar{\mu} \qquad (3.2)$$

$$(3.4) \Rightarrow \dot{x}_4 = \frac{d\cdot x_2^2\sin(x_1)}{1+c\sin^2(x_1)} - \frac{\sin(x_1)\cos(x_1)}{1+c\sin^2(x_1)} + \frac{b\,\bar{\mu}}{1+c\sin^2(x_1)} \qquad (3.4)$$

Now, we find the equilibrium points $\Rightarrow \dot{x}_1 = \dot{x}_2 = \dot{x}_3 = \dot{x}_4 = 0$

$\dot{x}_1 = 0 \Rightarrow \boxed{x_2 = 0}$ (3.5)

$\dot{x}_2 = 0 \overset{(3.5)}{\Rightarrow} \sin(x_1) - \cos(x_1)\bar{\mu} = 0 \Rightarrow \sin(x_1) = \cos(x_1)\bar{\mu}$ (3.6)

$\dot{x}_3 = 0 \Rightarrow \boxed{x_4 = 0}$ (3.7)

$\dot{x}_4 = 0 \overset{(3.5)}{\Rightarrow} -\sin(x_1)\cos(x_1) + b\bar{\mu} = 0 \Rightarrow \sin(x_1)\cos(x_1) = b\bar{\mu}$ (3.8)

$$\overset{(3.6)}{\underset{(3.8)}{\Rightarrow}} \begin{cases} \tan(x_1) = \bar{\mu} \\ \cos^2(x_1)\bar{\mu} = b\bar{\mu} \end{cases} \rightarrow \begin{cases} \tan(x_1) = \bar{\mu} \\ \bar{\mu}(\cos^2(x_1) - b) = 0 \end{cases} \rightarrow \begin{cases} \tan(x_1) = \bar{\mu} \\ \bar{\mu} = 0 \text{ or } \cos^2(x_1) = b \end{cases}$$

We can evaluate $b$, given the values of $M, m, L, g$

$$b = \left(\frac{25}{20}+1\right)\left(\frac{\frac{1}{3}mL^2}{mL^2}+1\right) = \left(\frac{5}{4}+1\right)\left(\frac{1}{3}+1\right) = \frac{9}{4}\cdot\frac{4}{3} = 3 > 1$$

Hence, it is impossible that $\cos^2(x_1) = b \Rightarrow \boxed{\bar{\mu} = 0} \overset{(3.6)}{\Rightarrow} \sin(x_1) = 0$

Therefore, $\boxed{x_1 = k\pi}$, $k = 0, \pm1, \pm2, \dots$

Figure 3: "Handwritten Solution for Problem 2 - Page 1"

Therefore, we have shown that $x = (k\pi, 0, x_3, 0)$, $k = 0, \pm 1, \pm 2, \ldots$ and $x_3 \in \mathbb{R}$ are equilibrium points, when the input $\bar{\mu} = 0$.

The linearization of the system about the origin will yield

$$\dot{x} = Ax + B\bar{\mu}$$

where $A = \left.\dfrac{\partial f}{\partial x}\right|_{\substack{x=0\\ \bar{\mu}=0}}$ and $B = \left.\dfrac{\partial f}{\partial \bar{\mu}}\right|_{\substack{x=0\\ \bar{\mu}=0}}$

$$\frac{\partial f}{\partial \bar{\mu}} = \begin{bmatrix} \partial f_1/\partial \bar{\mu} \\ \partial f_2/\partial \bar{\mu} \\ \partial f_3/\partial \bar{\mu} \\ \partial f_4/\partial \bar{\mu} \end{bmatrix} = \begin{bmatrix} 0 \\ -\dfrac{\cos\phi}{(1+c\sin^2\phi)} \\ 0 \\ \dfrac{b}{(1+c\sin^2\phi)} \end{bmatrix} \Rightarrow B = \left.\frac{\partial f}{\partial \mu}\right|_{\substack{x=0\\ \bar{\mu}=0}} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ b \end{bmatrix}$$

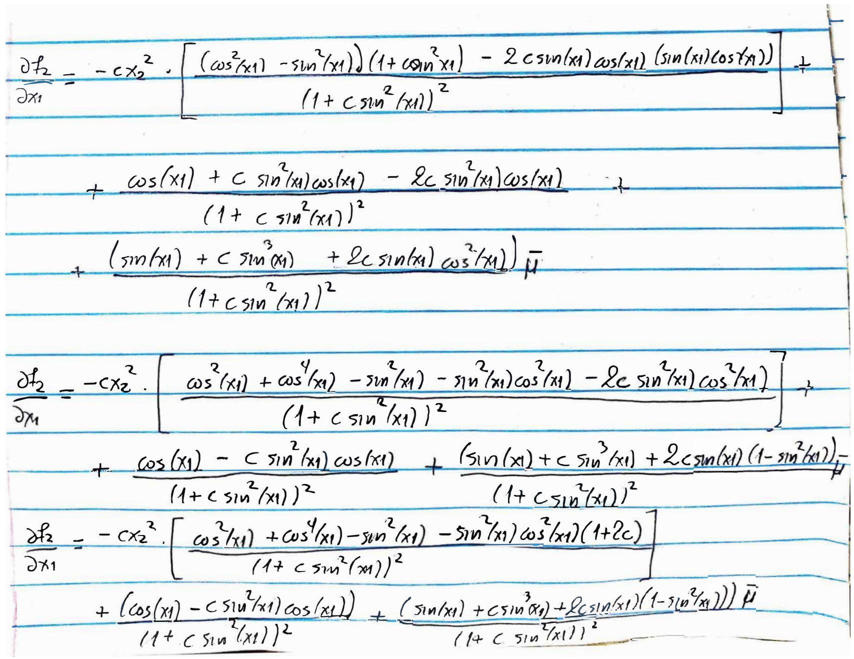
$$\frac{\partial f}{\partial x} = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 & \partial f_1/\partial x_3 & \partial f_1/\partial x_4 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 & \partial f_2/\partial x_3 & \partial f_2/\partial x_4 \\ \partial f_3/\partial x_1 & \partial f_3/\partial x_2 & \partial f_3/\partial x_2 & \partial f_3/\partial x_4 \\ \partial f_4/\partial x_1 & \partial f_4/\partial x_2 & \partial f_4/\partial x_3 & \partial f_4/\partial x_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{\partial f_4}{\partial x_1} & \dfrac{\partial f_4}{\partial x_2} & 0 & 0 \end{bmatrix}$$

$$\frac{\partial f_2}{\partial x_1} = -cx_2^2 \cdot \frac{\dfrac{\partial(\sin x_1 \cos x_1)}{\partial x_1}(1+c\sin^2 x_1) - \dfrac{\partial(1+c\sin^2 x_1)}{\partial x_1}(\sin x_1 \cos x_1)}{(1+c\sin^2(x_1))^2} +$$

$$+ \frac{\cos(x_1)(1+c\sin^2 x_1) - \sin(x_1)(2c\sin(x_1)\cos(x_1))}{(1+c\sin^2 x_1)^2} -$$

$$- \left[\frac{(-\sin(x_1))(1+c\sin^2 x_1) - \cos(x_1)(2c\sin(x_1)\cos(x_1))}{(1+c\sin^2(x_1))}\bar{\mu}\right]$$

Figure 4: "Handwritten Solution for Problem 2 - Page 2"



$$\frac{\partial f_2}{\partial x_1} = -c x_2^2 \cdot \left[ \frac{(\cos^2(x_1) - \sin^2(x_1))(1 + c\sin^2 x_1) - 2c\sin(x_1)\cos(x_1)(\sin(x_1)\cos(x_1))}{(1 + c\sin^2(x_1))^2} \right] +$$

$$+ \frac{\cos(x_1) + c\sin^2(x_1)\cos(x_1) - 2c\sin^2(x_1)\cos(x_1)}{(1 + c\sin^2(x_1))^2} +$$

$$+ \frac{(\sin(x_1) + c\sin^3(x_1) + 2c\sin(x_1)\cos^2(x_1))\bar{\mu}}{(1 + c\sin^2(x_1))^2}$$

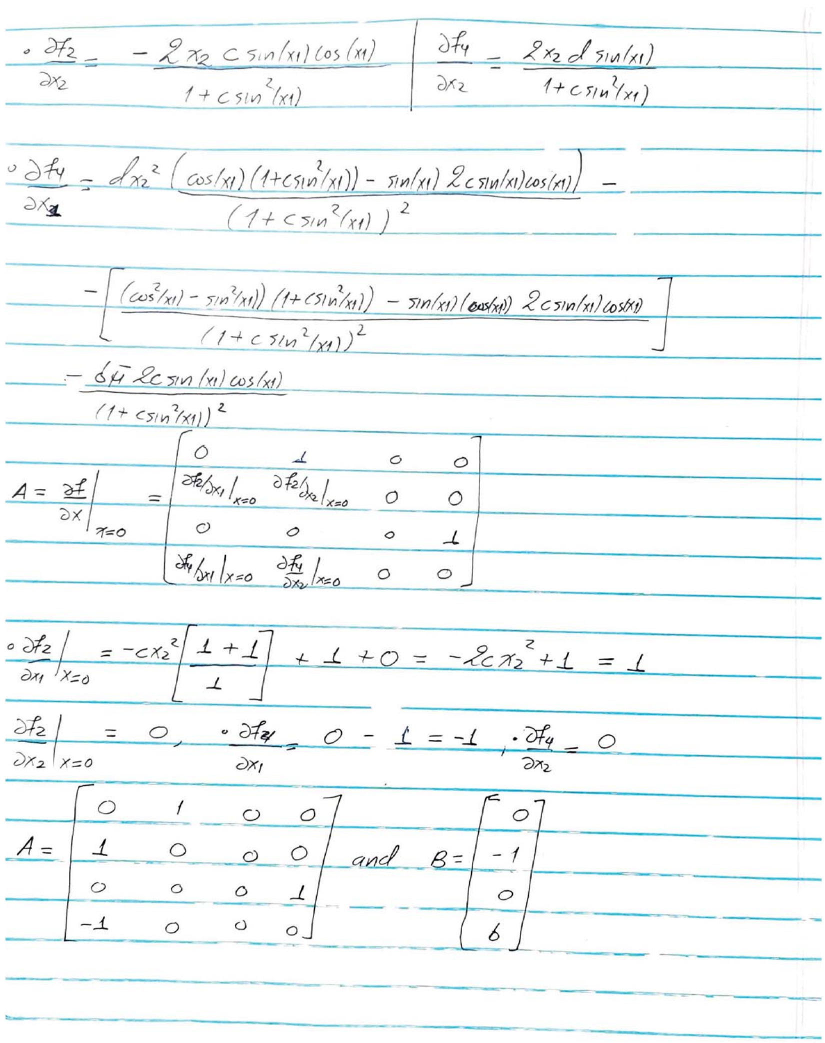
$$\frac{\partial f_2}{\partial x_1} = -c x_2^2 \cdot \left[ \frac{\cos^2(x_1) + \cos^4(x_1) - \sin^2(x_1) - \sin^2(x_1)\cos^2(x_1) - 2c\sin^2(x_1)\cos^2(x_1)}{(1 + c\sin^2(x_1))^2} \right] +$$

$$+ \frac{\cos(x_1) - c\sin^2(x_1)\cos(x_1)}{(1 + c\sin^2(x_1))^2} + \frac{(\sin(x_1) + c\sin^3(x_1) + 2c\sin(x_1)(1 - \sin^2(x_1)))\bar{\mu}}{(1 + c\sin^2(x_1))^2}$$

$$\frac{\partial f_2}{\partial x_1} = -c x_2^2 \cdot \left[ \frac{\cos^2(x_1) + \cos^4(x_1) - \sin^2(x_1) - \sin^2(x_1)\cos^2(x_1)(1+2c)}{(1 + c\sin^2(x_1))^2} \right]$$

$$+ \frac{(\cos(x_1) - c\sin^2(x_1)\cos(x_1))}{(1 + c\sin^2(x_1))^2} + \frac{(\sin(x_1) + c\sin^3(x_1) + 2c\sin(x_1)(1 - \sin^2(x_1)))\bar{\mu}}{(1 + c\sin^2(x_1))^2}$$

Figure 5: "Handwritten Solution for Problem 2 - Page 3"



- $\dfrac{\partial f_2}{\partial x_2} = \dfrac{-2 x_2 c \sin(x_1)\cos(x_1)}{1 + c\sin^2(x_1)}$ $\quad\Big|\quad$ $\dfrac{\partial f_4}{\partial x_2} = \dfrac{2 x_2 d \sin(x_1)}{1 + c\sin^2(x_1)}$

- $$\frac{\partial f_4}{\partial x_1} = d x_2^2 \frac{(\cos(x_1)(1 + c\sin^2(x_1)) - \sin(x_1)\, 2c\sin(x_1)\cos(x_1))}{(1 + c\sin^2(x_1))^2} -$$

$$- \left[ \frac{(\cos^2(x_1) - \sin^2(x_1))(1 + c\sin^2(x_1)) - \sin(x_1)(\cos(x_1))\, 2c\sin(x_1)\cos(x_1)}{(1 + c\sin^2(x_1))^2} \right]$$

$$- \frac{b\bar{\mu}\, 2c\sin(x_1)\cos(x_1)}{(1 + c\sin^2(x_1))^2}$$

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \partial f_2/\partial x_1|_{x=0} & \partial f_2/\partial x_2|_{x=0} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \partial f_4/\partial x_1|_{x=0} & \frac{\partial f_4}{\partial x_2}\big|_{x=0} & 0 & 0 \end{bmatrix}$$

- $\left.\dfrac{\partial f_2}{\partial x_1}\right|_{x=0} = -c x_2^2 \left[\dfrac{1+1}{1}\right] + 1 + 0 = -2c x_2^2 + 1 = 1$

$\left.\dfrac{\partial f_2}{\partial x_2}\right|_{x=0} = 0$, $\quad \cdot\, \dfrac{\partial f_4}{\partial x_1} = 0 - 1 = -1$, $\quad \cdot\, \dfrac{\partial f_4}{\partial x_2} = 0$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ -1 \\ 0 \\ b \end{bmatrix}$$

Figure 6: "Handwritten Solution for Problem 2 - Page 4"

## 3 Problem 3

Problem 3

- Verify that $\dot{x} = Ax + B\bar{\mu}$ is completely controllable.

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 6 \end{bmatrix}, \quad AB = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 6 \\ 0 \end{bmatrix}, \quad A \cdot AB = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$$A \cdot A^2B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathcal{C} = \begin{bmatrix} 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 \\ 0 & 6 & 0 & 1 \\ 6 & 0 & 1 & 0 \end{bmatrix} \overset{\delta=3}{=} \begin{bmatrix} 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 \\ 0 & 3 & 0 & 1 \\ 3 & 0 & 1 & 0 \end{bmatrix}$$

It can be easily observed that $\mathcal{C}$ has four linearly independent columns and hence rank $\mathcal{C} = 4 = n$.
Therefore, the linearized system is completely controllable.

Now, we consider the state variable feedback control law

$$\bar{\mu} = -kx + v \implies \dot{x} = Ax + B(-kx + v)$$

$$\dot{x} = (A - Bk)x + Bv$$

Using the "place" command in MATLAB, we calculated that

$$K = [-19.3 \quad -22.975 \quad -1.59 \quad -5.525]$$

will place the closed-loop eigenvalues at $(-3, -2, -0.7 \pm j0.2)$.

Figure 7: "Handwritten Solution for Problem 3 - Page 1"

## 3.1 Matlab Code for Problem 3

```
%%%%%% ---------------------------------------------------------------- %%%%%%
%Problem 3
M = 25;                                                     %Kg
m = 20;                                                     %Kg
L = 9.81;                                                   %m
g = 9.81;                                                   %m/s^2
J = m*(L^2)/3;

b = ((M/m)+1)*(1+J/(m*(L^2)));                              %dimensionless variable
c = (m^2)*(L^2)/(J*(m+M)+m*M*(L^2));                        %dimensionless variable
d = c+1;                                                    %dimensionless variable

%The linearization matrices A and B were computed by hand in the report
A=[0 1 0 0; 1 0 0 0; 0 0 0 1; -1 0 0 0 ];
B=[0; -1; 0; b];

%The desired eigenvalues for the state variable feedback control law
p1=-3;
p2=-2;
p3=-0.7-1i*0.2;
p4=-0.7+1i*0.2;
p=[p1 p2 p3 p4];

%Controllability Matrix
contr=[B A*B A*A*B A*A*A*B];
%Verification of the Controllability Test
rank(contr);

%Calculation of the gain matrix for the state feedback controller
K=place(A,B,p);
%%%%%% ---------------------------------------------------------------- %%%%%%
```

# 4 Problem 4

Problem 4

Here we apply the control law $\bar{\mu} = -kx + v$, where $v = 0$

$\boxed{\dot{x} = (A - Bk)x \quad (5)}$ $\boxed{\dot{x} = f(x, -kx) \quad ⑥}$

(linearized system) (nonlinear system)

First, we will compute $f(x, -kx)$

where $x = [x_1\ x_2\ x_3\ x_4]^T$ and $\bar{\mu} = -kx$

$$\bar{\mu} = -[k_1\ k_2\ k_3\ k_4][x_1\ x_2\ x_3\ x_4]^T$$

$$\bar{U} = -(k_1 x_1 + k_2 x_2 + k_3 x_3 + k_4 x_4)$$

Hence, $\dot{x}_1 = x_2$

$$\dot{x}_2 = \frac{-c x_2^2 \sin(x_1)\cos(x_1)}{1 + c\sin^2(x_1)} + \frac{\sin(x_1)}{1 + c\sin^2(x_1)} - \frac{\cos(x_1)(-k_1x_1 - k_2x_2 - k_3x_3 - k_4x_4)}{1 + c\sin^2(x_1)}$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = \frac{d x_2^2 \sin(x_1)}{1 + c\sin^2(x_1)} - \frac{\sin(x_1)\cos(x_1)}{1 + c\sin^2(x_1)} + \frac{b(-k_1x_1 - k_2x_2 - k_3x_3 - k_4x_4)}{1 + c\sin^2(x_1)}$$

Now,

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x}(x, -kx) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 & \partial f_2/\partial x_3 & \partial f_2/\partial x_4 \\ 0 & 0 & 0 & 1 \\ \partial f_4/\partial x_1 & \partial f_4/\partial x_2 & \partial f_4/\partial x_3 & \partial f_4/\partial x_4 \end{bmatrix}$$

Using the results we had previously, we have:

- $$\frac{\partial f_2}{\partial x_1} = -c x_2^2 \left[\frac{\cos^2(x_1) + \cos^4(x_1) - \sin^2(x_1) - \sin^2(x_1)\cos^2(x_1)(1+2c)}{(1 + c\sin^2(x_1))^2}\right] +$$

$$+ \frac{(\cos(x_1) - c\sin^2(x_1)\cos(x_1))}{(1 + c\sin^2(x_1))^2} + \frac{(\sin(x_1) + c\sin^3(x_1) + 2c\sin(x_1)(1 - \sin^2(x_1)))}{(1 + c\sin^2(x_1))^2}(-kx)$$

$$- \frac{\cos(x_1)}{1 + c\sin^2(x_1)}(-k_1), \quad \text{where } k_1 = \frac{\partial(-kx)}{\partial x_1}$$

Figure 8: "Handwritten Solution for Problem 4 - Page 1"

$$\frac{\partial f_2}{\partial x_2} = \frac{-2x_2\, c \sin(x_1)\cos(x_1)}{1+c\sin^2(x_1)} - \frac{\cos(x_1)}{(1+c\sin^2(x_1))}(-k_2)$$

$$\frac{\partial f_2}{\partial x_3} = \frac{-\cos(x_1)}{(1+c\sin^2(x_1))}(-k_3) \qquad \frac{\partial f_2}{\partial x_4} = \frac{-\cos(x_1)}{(1+c\sin^2(x_1))}(-k_4)$$

$$\frac{\partial f_4}{\partial x_1} = \frac{d x_2^2\left(\cos(x_1)(1+c\sin^2(x_1)) - \sin(x_1)2c\sin(x_1)\cos(x_1)\right)}{(1+c\sin^2(x_1))^2} - \left[\frac{(\cos^2(x_1)-\sin^2(x_1))(1+c\sin^2(x_1)) - \sin^2(x_1)\cos^2(x_1)2c}{(1+c\sin^2(x_1))^2}\right]$$
$$- \frac{b2c\sin(x_1)\cos(x_1)}{(1+c\sin^2(x_1))^2}(-k_1x_1-k_2x_2-k_3x_3-k_4x_4) + \frac{b}{(1+c\sin^2(x_1))}(-k_1)$$

$$\frac{\partial f_4}{\partial x_2} = \frac{2x_2\, d\sin(x_1)}{1+c\sin^2(x_1)} + \frac{b}{(1+c\sin^2(x_1))}(-k_2), \quad \frac{\partial f_4}{\partial x_3} = \frac{b(-k_3)}{(1+c\sin^2(x_1))}, \quad \frac{\partial f_4}{\partial x_4} = \frac{b(-k_4)}{(1+c\sin^2(x_1))}$$

Hence $$\left.\frac{\partial}{\partial x}(f(x,-kx))\right|_{x=0} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1+k_1 & k_2 & k_3 & k_4 \\ 0 & 0 & 0 & 1 \\ -1-bk_1 & -bk_2 & -bk_3 & -bk_4 \end{bmatrix}$$

$$A - Bk = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \\ 0 \\ b \end{bmatrix}\begin{bmatrix} k_1 & k_2 & k_3 & k_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1+k_1 & k_2 & k_3 & k_4 \\ 0 & 0 & 0 & 1 \\ -1-bk_1 & -bk_2 & -bk_3 & -bk_4 \end{bmatrix}$$

Therefore, we verify that $A - Bk = \left.\frac{\partial}{\partial x}(f(x,-kx))\right|_{x=0}$

Figure 9: "Handwritten Solution for Problem 4 - Page 2"

As a result, we have shown that $\dot{x}=(A-Bk)x$ is the linearization of $\dot{x}=f(x,-kx)$ and hence using Theorem 4.7 of Khalil, we can study the stability properties of the origin for the nonlinear closed-loop system, by finding the eigenvalues of $(A-Bk)$.

However, we already know the eigenvalues, since the applied control law was designed so that the closed loop eigenvalues will be placed at $(-3,-2,-0.7\pm j0.2)$.

Hence, we can observe that $Re(\lambda_i)<0 \ \forall \lambda_i$ of $(A-Bk)$ meaning that the origin is asymptotically stable.

Figure 10: "Handwritten Solution for Problem 4 - Page 3"

# 5 Problem 5

Problem 5

In this problem, we will simulate both closed-loop systems for several initial conditions.

Cases of initial conditions

1) The cart at rest but not at the origin, and the bar leaning to one side, but also at rest.

$x_1 = \phi \neq 0$, $x_2 = \dot{\phi} = 0$, $x_3 = s \neq 0$, $x_4 = \dot{s} = 0$

In fact, we picked $x_1(0) = \frac{\pi}{11}$ and $x_3(0) = 0.25$

2) The cart moving to one side with some velocity, and the bar initially at rest and straight up.

$x_1 = \phi = 0$, $x_2 = \dot{\phi} = 0$, $x_3 = s \neq 0$, $x_4 = \dot{s} \neq 0$

In fact, we picked $x_3(0) = 1.25$ and $x_4(0) = 1$

3) The cart moving to one side with some velocity and the bar leaning to one side with some angular velocity.

$x_1 = \phi \neq 0$, $x_2 = \dot{\phi} \neq 0$, $x_3 = s \neq 0$, $x_4 = \dot{s} \neq 0$

In fact, we picked $x_1(0) = \frac{\pi}{11}$, $x_2(0) = \frac{\pi}{40}$, $x_3(0) = 0.2$, $x_4(0) = 0.1$

4) The cart moving with negative velocity and the bar leaning to the negative side, but not at rest.

$x_1 = \phi < 0$, $x_2 = \dot{\phi} > 0$, $x_3 = s > 0$, $x_4 = \dot{s} < 0$

In fact, we picked $x_1(0) = -\frac{\pi}{11}$, $x_2(0) = \frac{\pi}{40}$, $x_3(0) = 0.2$, $x_4(0) = -0.1$

Figure 11: "Handwritten Solution for Problem 5 - Page 1"

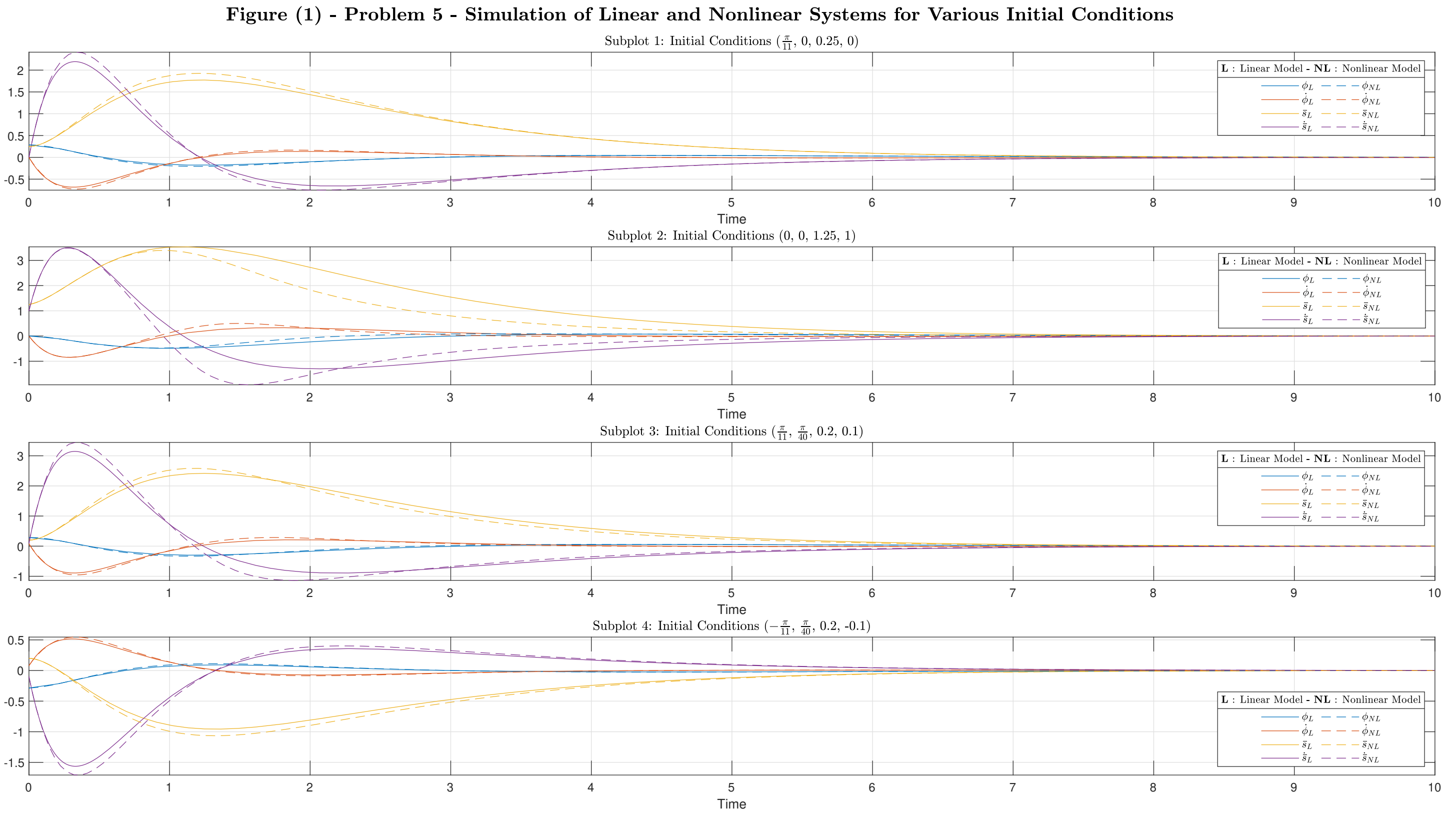



Figure 12: ”Figure 1 of Problem 5”

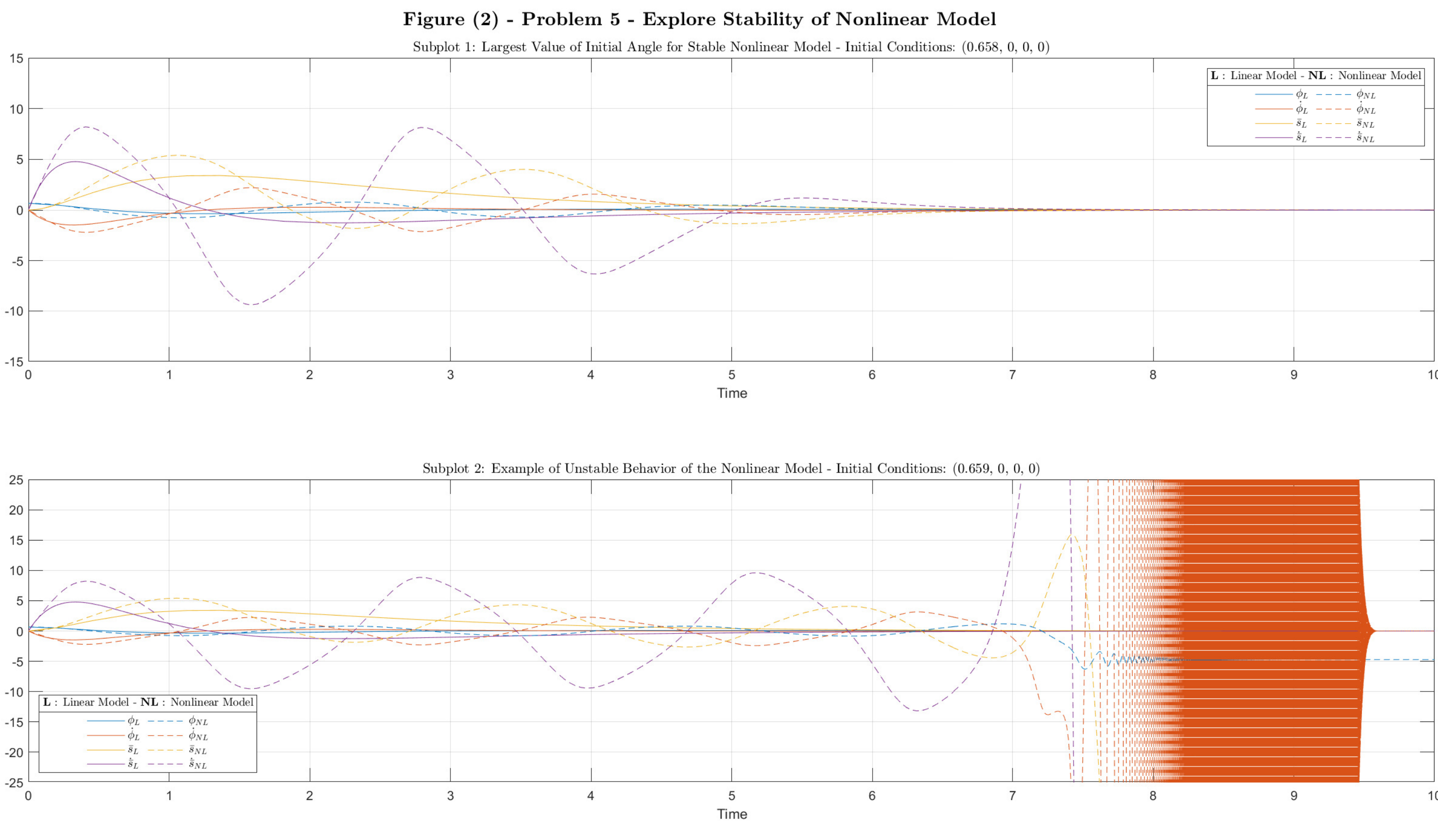



Figure 13: ”Figure 2 of Problem 5”

▷ Comparison of the response of the linear model to that of the nonlinear model.

In general, we can observe that both responses are similar for all four sets of initial conditions. Moreover, the responses of the nonlinear model present larger overshoot in most cases (Subplots 1, 3, 4), while parameters such as rise time and settling time appear to be similar.

▷ Comments about the plots

Overall, the responses are similar for all 4 I.C.s, where we observe behaviors that seem logical, when we are imagining how we would balance a broom. The fourth case (Subplot 4) was selected to examine the behavior of the system for negative I.C.s and we observe a mirroring behavior of that in Subplot 3. Furthermore, we should note that besides the figures in the report, we experimented with different I.C.s that required the use of a numerical solver ("ode45"). In many cases, the code would run forever and hence only I.C.s that resulted in finite execution times and stable behaviors were selected.

▷ With $x_2(0)=0, x_3(0)=0$ and $x_4(0)=0$, the largest value for the initial angle $x_1(0)$ before the nonlinear model goes unstable is 0.658 rad. This was illustrated in Figure 2, where Subplot 1 depicts the stable behavior for $x_1(0)=0.658$ rad, and Subplot 2 depicts the unstable behavior for $x_1(0)=0.659$ rad.

Figure 14: "Handwritten Solution for Problem 5 - Page 2"

## 5.1 Matlab Code for Problem 5

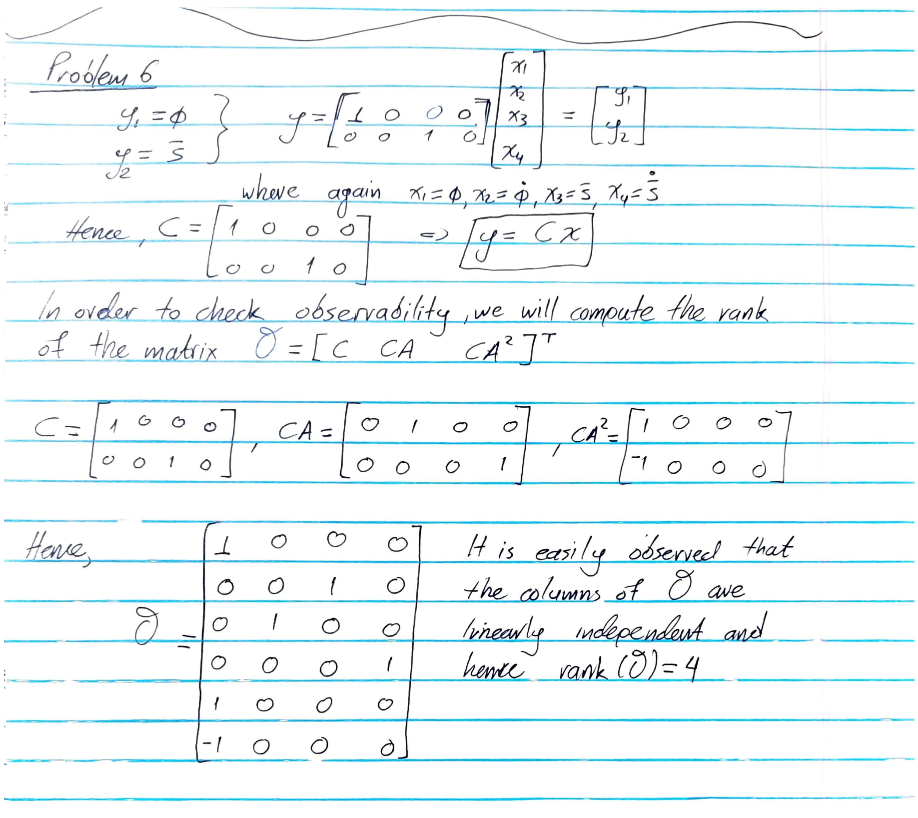
```
%%%%%% -------------------------------------------------------------- %%%%%%

%Problem 5
syms x(t) [4 1] real

%Control Law
mu = -real(K)*x;

%Linear Model Closed-Loop System
x_dot_linear = (A-B*K)*x;

%Nonlinear Model Closed-Loop System
x1_dot_nl = x2;
x2_dot_nl = (-c*(x2^2)*sin(x1)*cos(x1)/(1+c*(sin(x1))^2)) + (sin(x1)/(1+c*(sin(x1))^2)) - ...
    (cos(x1)*mu/(1+c*(sin(x1))^2));
x3_dot_nl = x4;
x4_dot_nl = (d*(x2^2)*sin(x1)/(1+c*(sin(x1))^2)) - (sin(x1)*cos(x1)/(1+c*(sin(x1))^2)) + ...
    (b*mu/(1+c*(sin(x1))^2));
x_dot_nonlinear=[x1_dot_nl; x2_dot_nl; x3_dot_nl; x4_dot_nl];

%Differential Equations for Linear Model
odes = diff(x) == x_dot_linear;
%Differential Equations for Nonlinear Model
odes_nl= diff(x) == x_dot_nonlinear;

figure(1)
sgtitle('$\textbf{Figure (1) - Problem 5 - Simulation of Linear and Nonlinear Systems for Various Initial ...
    Conditions}$','Interpreter','latex')
subplot(4,1,1);
%The cart at rest but not at the origin and the bar leaning to one side,
%but also at rest
init_conds=[pi/11; 0; 0.25; 0];
cond1 = x1(0) == init_conds(1);
cond2 = x2(0) == init_conds(2);
cond3 = x3(0) == init_conds(3);
cond4 = x4(0) == init_conds(4);
conds = [cond1; cond2; cond3; cond4];

%Solution of the ODE for the Linear Model
[x1Sol, x2Sol, x3Sol, x4Sol] = dsolve(odes,conds);

%Solution of the ODE for the Nonlinear Model
M=odeFunction(x_dot_nonlinear,x);
[t,sol] = ode45(M,[0 10],init_conds);

%Plotting of the Responses for both Close-Loop Systems
fplot(x1Sol,[0 10],'Color',[0 0.4470 0.7410])
hold on
fplot(x2Sol,[0 10],'Color',[0.8500 0.3250 0.0980])
hold on
fplot(x3Sol,[0 10],'Color',[0.9290 0.6940 0.1250])
hold on
fplot(x4Sol,[0 10],'Color',[0.4940 0.1840 0.5560])
hold on;
plot(t,sol(:,1),'m--','Color',[0 0.4470 0.7410])
hold on;
plot(t,sol(:,2),'y--','Color',[0.8500 0.3250 0.0980])
hold on;
plot(t,sol(:,3),'--','Color',[0.9290 0.6940 0.1250])
hold on;
plot(t,sol(:,4),'--','Color',[0.4940 0.1840 0.5560])
grid on
title('Subplot 1: Initial Conditions ($\frac{\pi}{11}$, 0, 0.25, 0)','Interpreter','latex')
xlabel('Time')
lgd=legend('$\phi_{L}$','$\dot{\phi}_{L}$','$\bar{s}_{L}$','$\dot{\bar{s}}_{L}$','$\phi_{NL}$','$\dot{\phi}_{NL}$','$
title(lgd,'$\textbf{L}$ : Linear Model - $\textbf{NL}$ : Nonlinear Model','Interpreter','latex')

subplot(4,1,2);
%The cart moving to one side with some velocity and the bar initially at
%rest and straight up
init_conds=[0; 0; 1.25; 1];
cond1 = x1(0) == init_conds(1);
```

```
cond2 = x2(0) == init_conds(2);
cond3 = x3(0) == init_conds(3);
cond4 = x4(0) == init_conds(4);
conds = [cond1; cond2; cond3; cond4];

%Solution of the ODE for the Linear Model
[x1Sol, x2Sol, x3Sol, x4Sol] = dsolve(odes,conds);

%Solution of the ODE for the Nonlinear Model
M=odeFunction(x_dot_nonlinear,x);
[t,sol] = ode45(M,[0 10],init_conds);

%Plotting of the Responses for both Close-Loop Systems
fplot(x1Sol,[0 10],'Color',[0 0.4470 0.7410])
hold on
fplot(x2Sol,[0 10],'Color',[0.8500 0.3250 0.0980])
hold on
fplot(x3Sol,[0 10],'Color',[0.9290 0.6940 0.1250])
hold on
fplot(x4Sol,[0 10],'Color',[0.4940 0.1840 0.5560])
hold on;
plot(t,sol(:,1),'m--','Color',[0 0.4470 0.7410])
hold on;
plot(t,sol(:,2),'y--','Color',[0.8500 0.3250 0.0980])
hold on;
plot(t,sol(:,3),'--','Color',[0.9290 0.6940 0.1250])
hold on;
plot(t,sol(:,4),'--','Color',[0.4940 0.1840 0.5560])
grid on
title('Subplot 2: Initial Conditions (0, 0, 1.25, 1)','Interpreter','latex')
xlabel('Time')
lgd=legend('$\phi_{L}$','$\dot{\phi}_{L}$','$\bar{s}_{L}$','$\dot{\bar{s}}_{L}$','$\phi_{NL}$','$\dot{\phi}_{NL}$','$
title(lgd,'$\textbf{L}$ : Linear Model - $\textbf{NL}$ : Nonlinear Model','Interpreter','latex')

subplot(4,1,3);
%The cart moving to one side with some velocity and the bar leaning to one
%side not at rest
init_conds=[pi/11; pi/40; 0.2; 0.1];
cond1 = x1(0) == init_conds(1);
cond2 = x2(0) == init_conds(2);
cond3 = x3(0) == init_conds(3);
cond4 = x4(0) == init_conds(4);
conds = [cond1; cond2; cond3; cond4];

%Solution of the ODE for the Linear Model
[x1Sol, x2Sol, x3Sol, x4Sol] = dsolve(odes,conds);

%Solution of the ODE for the Noninear Model
M=odeFunction(x_dot_nonlinear,x);
[t,sol] = ode45(M,[0 10],init_conds);

%Plotting of the Responses for both Close-Loop Systems
fplot(x1Sol,[0 10],'Color',[0 0.4470 0.7410])
hold on
fplot(x2Sol,[0 10],'Color',[0.8500 0.3250 0.0980])
hold on
fplot(x3Sol,[0 10],'Color',[0.9290 0.6940 0.1250])
hold on
fplot(x4Sol,[0 10],'Color',[0.4940 0.1840 0.5560])
hold on;
plot(t,sol(:,1),'m--','Color',[0 0.4470 0.7410])
hold on;
plot(t,sol(:,2),'y--','Color',[0.8500 0.3250 0.0980])
hold on;
plot(t,sol(:,3),'--','Color',[0.9290 0.6940 0.1250])
hold on;
plot(t,sol(:,4),'--','Color',[0.4940 0.1840 0.5560])
grid on
title('Subplot 3: Initial Conditions ($\frac{\pi}{11}$, $\frac{\pi}{40}$, 0.2, 0.1)','Interpreter','latex')
xlabel('Time')
lgd=legend('$\phi_{L}$','$\dot{\phi}_{L}$','$\bar{s}_{L}$','$\dot{\bar{s}}_{L}$','$\phi_{NL}$','$\dot{\phi}_{NL}$','$
title(lgd,'$\textbf{L}$ : Linear Model - $\textbf{NL}$ : Nonlinear Model','Interpreter','latex')

subplot(4,1,4);
%The cart moving to one side with some negative velocity and the bar leaning to the ...
```

```
    negative
%side not at rest
init_conds=[-pi/11; pi/40; 0.2; -0.1];
cond1 = x1(0) == init_conds(1);
cond2 = x2(0) == init_conds(2);
cond3 = x3(0) == init_conds(3);
cond4 = x4(0) == init_conds(4);
conds = [cond1; cond2; cond3; cond4];

%Solution of the ODE for the Linear Model
[x1Sol, x2Sol, x3Sol, x4Sol] = dsolve(odes,conds);

%Solution of the ODE for the Nonlinear Model
M=odeFunction(x_dot_nonlinear,x);
[t,sol] = ode45(M,[0 10],init_conds);

%Plotting of the Responses for both Close-Loop Systems
fplot(x1Sol,[0 10],'Color',[0 0.4470 0.7410])
hold on
fplot(x2Sol,[0 10],'Color',[0.8500 0.3250 0.0980])
hold on
fplot(x3Sol,[0 10],'Color',[0.9290 0.6940 0.1250])
hold on
fplot(x4Sol,[0 10],'Color',[0.4940 0.1840 0.5560])
hold on;
plot(t,sol(:,1),'m--','Color',[0 0.4470 0.7410])
hold on;
plot(t,sol(:,2),'y--','Color',[0.8500 0.3250 0.0980])
hold on;
plot(t,sol(:,3),'--','Color',[0.9290 0.6940 0.1250])
hold on;
plot(t,sol(:,4),'--','Color',[0.4940 0.1840 0.5560])
grid on
title('Subplot 4: Initial Conditions ($-\frac{\pi}{11}$, $\frac{\pi}{40}$, 0.2, -0.1)','Interpreter','latex')
xlabel('Time')
lgd=legend('$\phi_{L}$','$\dot{\phi}_{L}$','$\bar{s}_{L}$','$\dot{\bar{s}}_{L}$','$\phi_{NL}$','$\dot{\phi}_{NL}$','$
title(lgd,'$\textbf{L}$ : Linear Model - $\textbf{NL}$ : Nonlinear Model','Interpreter','latex')
%%%%%% ------------------------------------------------------------ %%%%%%

%%%%%% ------------------------------------------------------------ %%%%%%
%Problem 5 - Connection between Stability and the initial angle x1(0)

figure(2)
sgtitle('\textbf{Figure (2) - Problem 5 - Explore Stability of Nonlinear Model}','Interpreter','latex')
subplot(2,1,1);
init_conds=[0.658; 0; 0; 0];
cond1 = x1(0) == init_conds(1);
cond2 = x2(0) == init_conds(2);
cond3 = x3(0) == init_conds(3);
cond4 = x4(0) == init_conds(4);
conds = [cond1; cond2; cond3; cond4];

%Solution of the ODE for the Linear Model
[x1Sol, x2Sol, x3Sol, x4Sol] = dsolve(odes,conds);

%Solution of the ODE for the Nonlinear Model
M=odeFunction(x_dot_nonlinear,x);
[t,sol] = ode45(M,[0 10],init_conds);

%Plotting of the Responses for both Close-Loop Systems
fplot(x1Sol,[0 10],'Color',[0 0.4470 0.7410])
hold on
fplot(x2Sol,[0 10],'Color',[0.8500 0.3250 0.0980])
hold on
fplot(x3Sol,[0 10],'Color',[0.9290 0.6940 0.1250])
hold on
fplot(x4Sol,[0 10],'Color',[0.4940 0.1840 0.5560])
hold on;
plot(t,sol(:,1),'m--','Color',[0 0.4470 0.7410])
hold on;
plot(t,sol(:,2),'y--','Color',[0.8500 0.3250 0.0980])
hold on;
plot(t,sol(:,3),'--','Color',[0.9290 0.6940 0.1250])
hold on;
plot(t,sol(:,4),'--','Color',[0.4940 0.1840 0.5560])
grid on
```

```
title('Subplot 1: Largest Value of Initial Angle for Stable Nonlinear Model - Initial Conditions: (0.658, ...
    0, 0, 0)','Interpreter','latex')
xlabel('Time')
lgd=legend('$\phi_{L}$','$\dot{\phi}_{L}$','$\bar{s}_{L}$','$\dot{\bar{s}}_{L}$','$\phi_{NL}$','$\dot{\phi}_{NL}$','$
title(lgd,'$\textbf{L}$ : Linear Model - $\textbf{NL}$ : Nonlinear Model','Interpreter','latex')
ylim([-15 15]);

subplot(2,1,2);
init_conds=[0.659; 0; 0; 0];
cond1 = x1(0) == init_conds(1);
cond2 = x2(0) == init_conds(2);
cond3 = x3(0) == init_conds(3);
cond4 = x4(0) == init_conds(4);
conds = [cond1; cond2; cond3; cond4];

%Solution of the ODE for the Linear Model
[x1Sol, x2Sol, x3Sol, x4Sol] = dsolve(odes,conds);

%Solution of the ODE for the Nonlinear Model
M=odeFunction(x_dot_nonlinear,x);
[t,sol] = ode45(M,[0 10],init_conds);

%Plotting of the Responses for both Close-Loop Systems
fplot(x1Sol,[0 10],'Color',[0 0.4470 0.7410])
hold on
fplot(x2Sol,[0 10],'Color',[0.8500 0.3250 0.0980])
hold on
fplot(x3Sol,[0 10],'Color',[0.9290 0.6940 0.1250])
hold on
fplot(x4Sol,[0 10],'Color',[0.4940 0.1840 0.5560])
hold on;
plot(t,sol(:,1),'m--','Color',[0 0.4470 0.7410])
hold on;
plot(t,sol(:,2),'y--','Color',[0.8500 0.3250 0.0980])
hold on;
plot(t,sol(:,3),'--','Color',[0.9290 0.6940 0.1250])
hold on;
plot(t,sol(:,4),'--','Color',[0.4940 0.1840 0.5560])
grid on
title('Subplot 2: Example of Unstable Behavior of the Nonlinear Model - Initial Conditions: (0.659, 0, 0, ...
    0)','Interpreter','latex')
xlabel('Time')
lgd=legend('$\phi_{L}$','$\dot{\phi}_{L}$','$\bar{s}_{L}$','$\dot{\bar{s}}_{L}$','$\phi_{NL}$','$\dot{\phi}_{NL}$','$
title(lgd,'$\textbf{L}$ : Linear Model - $\textbf{NL}$ : Nonlinear Model','Interpreter','latex')
ylim([-25 25]);
%%%%%% ---------------------------------------------------------------- %%%%%%

%%%%%% ---------------------------------------------------------------- %%%%%%
```

# 6 Problem 6



Problem 6

$$\left.\begin{aligned} y_1 &= \phi \\ y_2 &= \bar{s} \end{aligned}\right\} \quad y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

where again $x_1 = \phi,\ x_2 = \dot{\phi},\ x_3 = \bar{s},\ x_4 = \dot{\bar{s}}$

Hence, $C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \Rightarrow \boxed{y = Cx}$

In order to check observability, we will compute the rank of the matrix $\mathcal{O} = [C \quad CA \quad CA^2]^T$

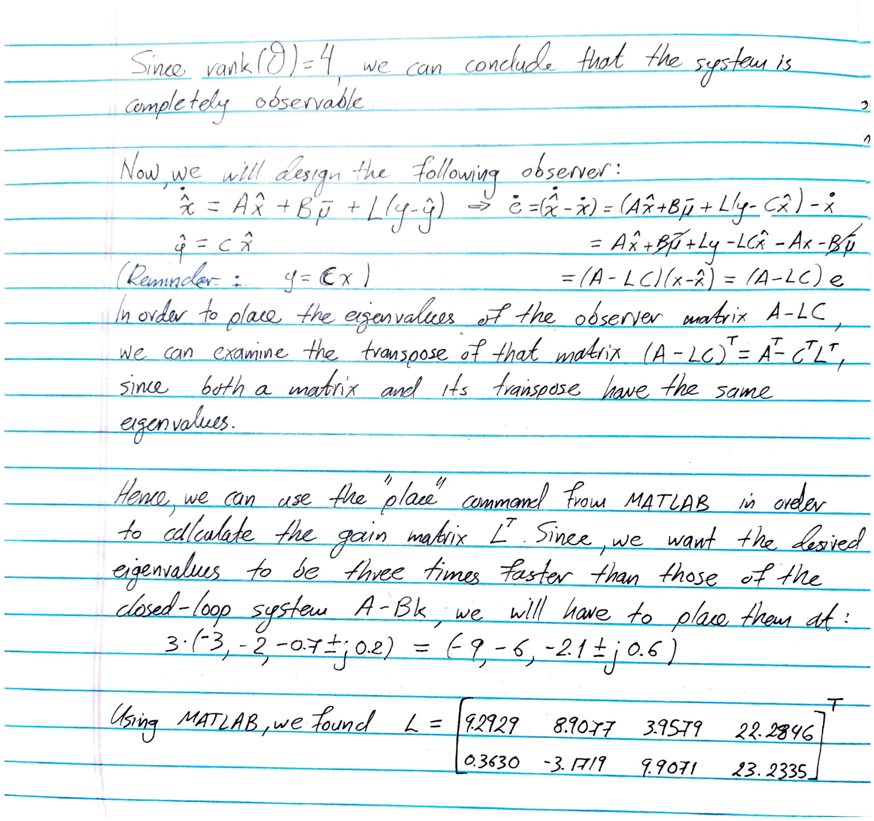
$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad CA = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad CA^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

Hence,

$$\mathcal{O} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

It is easily observed that the columns of $\mathcal{O}$ are linearly independent and hence $\operatorname{rank}(\mathcal{O}) = 4$

Figure 15: "Handwritten Solution for Problem 6 - Page 1"



Since $\operatorname{rank}(\mathcal{O}) = 4$, we can conclude that the system is completely observable

Now, we will design the following observer:

$$\dot{\hat{x}} = A\hat{x} + B\bar{u} + L(y - \hat{y}) \Rightarrow \dot{e} = (\dot{\hat{x}} - \dot{x}) = (A\hat{x} + B\bar{u} + L(y - C\hat{x})) - \dot{x}$$

$$\hat{y} = C\hat{x} \qquad = A\hat{x} + \cancel{B\bar{u}} + Ly - LC\hat{x} - Ax - \cancel{B\bar{u}}$$

(Reminder: $y = Cx$) $\qquad = (A - LC)(x - \hat{x}) = (A - LC)e$

In order to place the eigenvalues of the observer matrix $A - LC$, we can examine the transpose of that matrix $(A - LC)^T = A^T - C^T L^T$, since both a matrix and its transpose have the same eigenvalues.

Hence, we can use the "place" command from MATLAB in order to calculate the gain matrix $L^T$. Since, we want the desired eigenvalues to be three times faster than those of the closed-loop system $A - Bk$, we will have to place them at:

$$3 \cdot (-3, -2, -0.7 \pm j0.2) = (-9, -6, -2.1 \pm j0.6)$$

Using MATLAB, we found $L = \begin{bmatrix} 9.2929 & 8.9077 & 3.9579 & 22.2846 \\ 0.3630 & -3.1719 & 9.9071 & 23.2335 \end{bmatrix}^T$

Figure 16: "Handwritten Solution for Problem 6 - Page 2"

### 6.1 Matlab Code for Problem 6

```

%We derive the C matrix from the analysis in the report
C=[1 0 0 0; 0 0 1 0];
%Observability Matrix
O=[C; C*A; C*A*A];
rank(O);

%The desired eigenvalues for the observer matrix A-LC
% p1=-3;
% p2=-2;
% p3=-0.7-1i*0.2;
% p4=-0.7+1i*0.2;
pobs=3*[p1 p2 p3 p4];

%Calculation of the gain matrix for the state feedback controller
Ltran=place(A',C',pobs);
L=Ltran';
%%%%%% -------------------------------------------------------------- %%%%%%

%%%%%% -------------------------------------------------------------- %%%%%%
```

# 7 Problem 7

Problem 7

We consider the dynamic controller by augmenting the open-loop system and the observer (8), while applying the following control law:

$$\bar{\mu} = -k\hat{x} \quad (9)$$

Part a

$$\begin{aligned} \dot{x} &= Ax + B\bar{\mu} \\ y &= Cx \end{aligned} \quad \overset{(9)}{\Longrightarrow} \quad \begin{aligned} \dot{x} &= Ax - Bk\hat{x} \text{ (input)} \\ y &= Cx \text{ (output)} \end{aligned}$$

$$\left.\begin{aligned} \dot{\hat{x}} &= A\hat{x} + B\bar{\mu} + L(y-\hat{y}) \\ \hat{y} &= C\hat{x} \end{aligned}\right\} (8) \quad \overset{(9)}{\Longrightarrow} \quad \begin{aligned} \dot{\hat{x}} &= A\hat{x} - Bk\hat{x} + Ly - LC\hat{x} \\ \dot{\hat{x}} &= (A-Bk-LC)\underset{\text{output}}{\hat{x}} + L\underset{\text{input}}{y} \end{aligned}$$

Therefore, the dynamic controller for the linearized model is:

$$\left.\begin{aligned} \dot{x} &= Ax - Bk\hat{x} \\ y &= Cx \\ \dot{\hat{x}} &= (A-Bk-LC)\hat{x} + Ly \end{aligned}\right\} \quad \boxed{\begin{aligned} \dot{x} &= Ax - Bk\hat{x} \\ \dot{\hat{x}} &= (A-Bk-LC)\hat{x} + LCx \end{aligned}}$$

Specifically, for the required simulation, we integrated both states $x$ and $\hat{x}$ into an augmented system and solved the corresponding differential equations, using again the numerical solver "ode45". Subsequently, we plotted the outputs $y_1(t) = x_1(t) = \varphi$ and $y_2(t) = x_3(t) = \bar{s}$.

Likewise, for the nonlinear model, we will have:

$$\boxed{\begin{aligned} \dot{x} &= f(x, -k\hat{x}) \\ \dot{\hat{x}} &= (A-Bk-LC)\hat{x} + LCx \end{aligned}}$$

For the simulations, the same Initial Conditions as in Problem 5 were used, in order to compare the performance of each controller.

Figure 17: "Handwritten Solution for Problem 7 - Page 1"

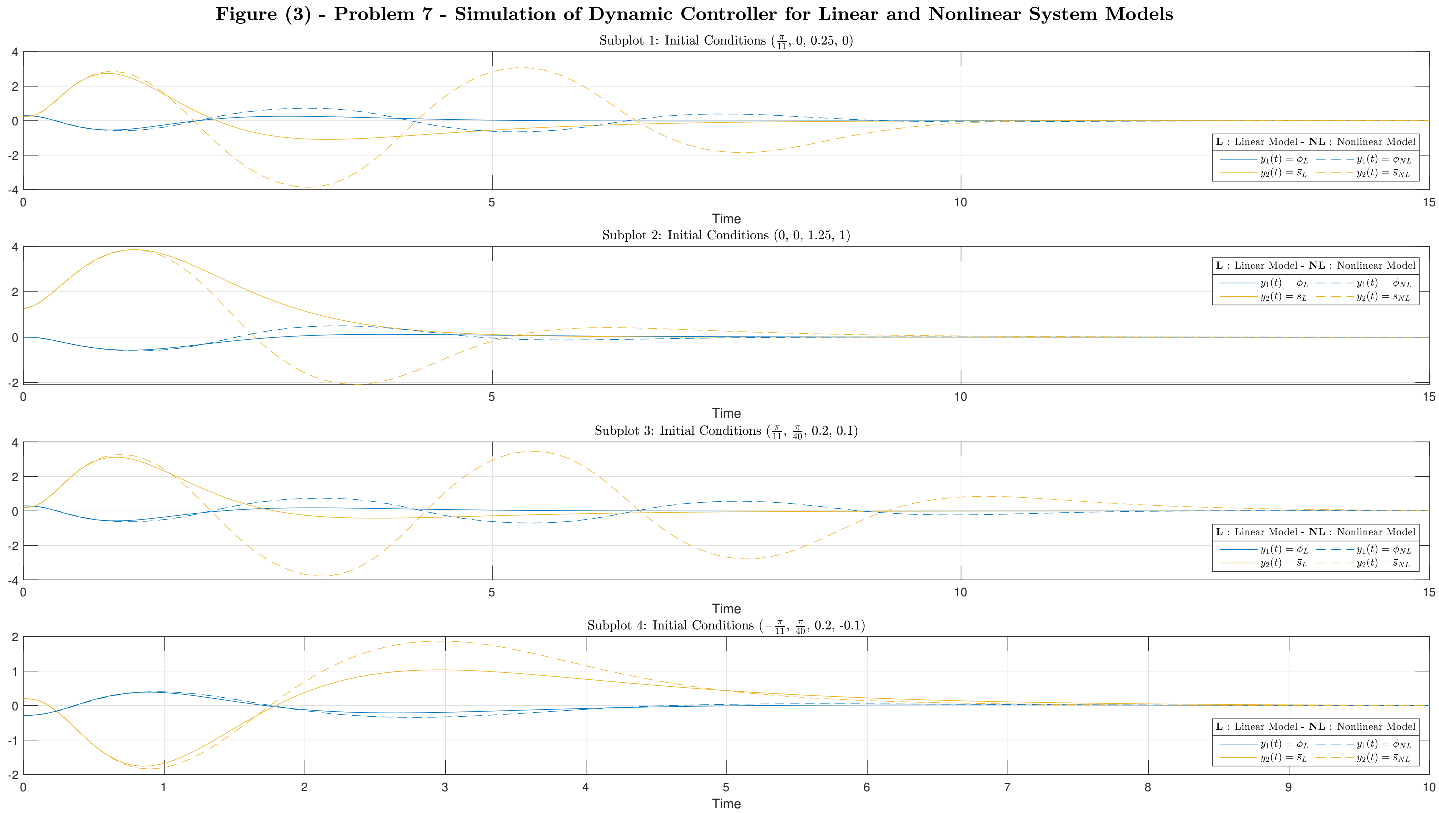



Figure 18: "Figure 3 of Problem 7"

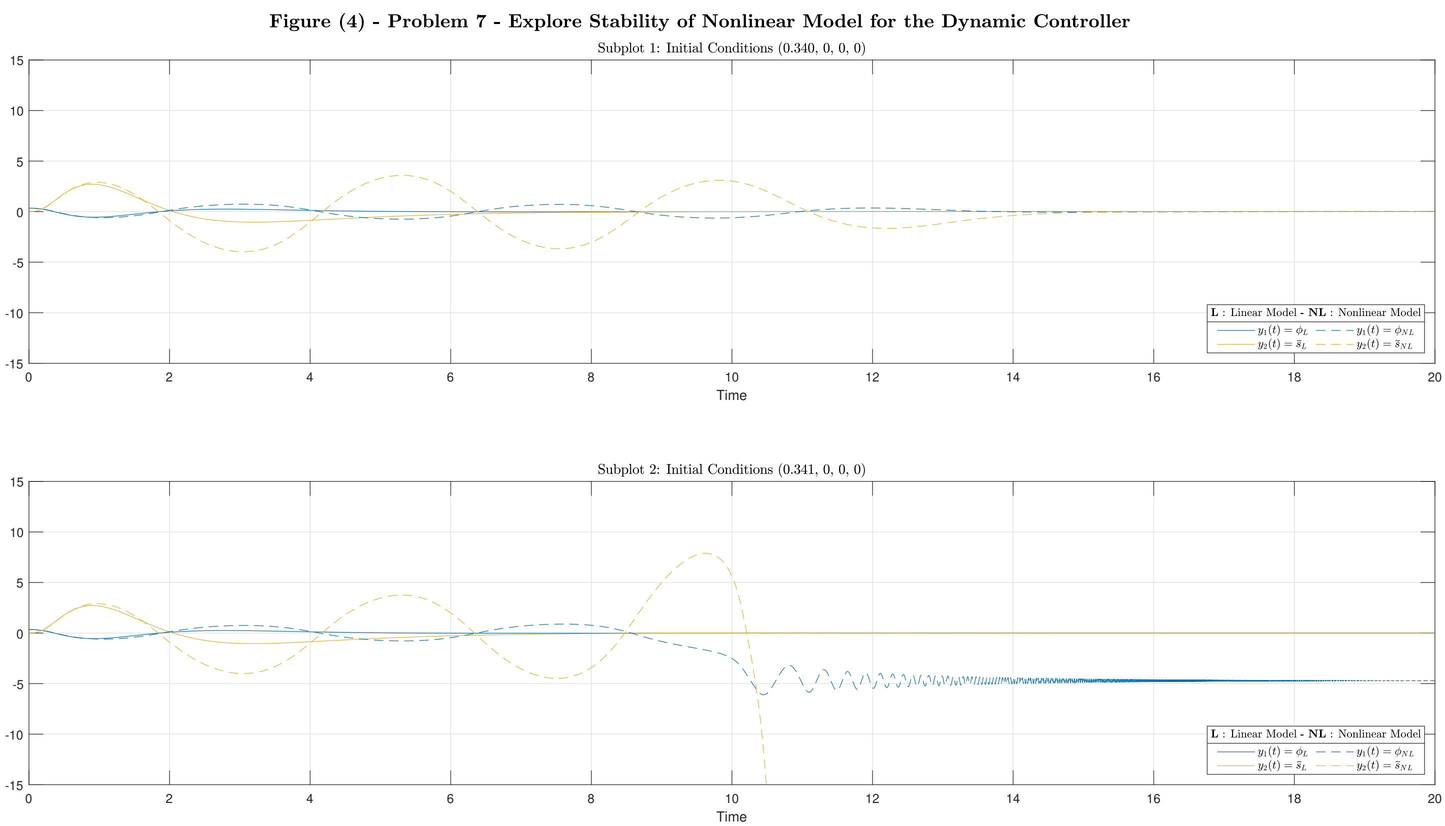



Figure 19: "Figure 4 of Problem 7"

The simulations for Problem 7 are presented in Figure 3.

▷ By comparing the plots with the corresponding ones in Figure 1, we can observe that for the same I.Cs, both the linear and nonlinear model dynamic controllers perform worse than the static controllers. In general, there are more oscillations, larger peaks, increased settling times, while the difference between the "linear and nonlinear model" controllers is more evident. Therefore, we can verify that being able to measure states instead of estimating is quite useful.

▷ With $x_2(0) = 0$, $x_3(0) = 0$ and $x_4(0) = 0$, we explored how large we can make $x_1(0)$ before the nonlinear model goes unstable. As we can observe in Figure 4, the largest "stable" value for $x_1(0)$ is 0.340 rad. In Subplot 1, we can observe the stable behavior and in Subplot 2, we can observe the unstable behavior (finite escape time) for $x_1(0) = 0.341$ rad.
Compared to Problem 5, we can observe that this value is smaller.

Figure 20: "Handwritten Solution for Problem 7 - Page 2"

## 7.1 Matlab Code for Problem 7

```
syms x_full(t) [8 1] real
x_full_14=[x_full1;x_full2;x_full3;x_full4];
x_full_58=[x_full5;x_full6;x_full7;x_full8];

%Linear Model
x_dot_full_14 = A*x_full_14 -B*K*x_full_58;
x_dot_full_58 = (A-B*K-L*C)*x_full_58+L*C*x_full_14;
x_dot_full=[x_dot_full_14;x_dot_full_58];

% Nonlinear Model
%Appliead Control Law
mu = -real(K)*x_full_58;
x_dot_full_nl_1 = x_full2;
x_dot_full_nl_2 = (-c*(x_full2^2)*sin(x_full1)*cos(x_full1)/(1+c*(sin(x_full1))^2)) + ...
    (sin(x_full1)/(1+c*(sin(x_full1))^2)) - (cos(x_full1)*mu/(1+c*(sin(x_full1))^2));
x_dot_full_nl_3 = x_full4;
x_dot_full_nl_4 = (d*(x_full2^2)*sin(x_full1)/(1+c*(sin(x_full1))^2)) - ...
    (sin(x_full1)*cos(x_full1)/(1+c*(sin(x_full1))^2)) + (b*mu/(1+c*(sin(x_full1))^2));
x_dot_full_nl_58 = (A-B*K-L*C)*x_full_58+L*C*x_full_14;
x_dot_full_nonlinear=[x_dot_full_nl_1; x_dot_full_nl_2; x_dot_full_nl_3; x_dot_full_nl_4; x_dot_full_nl_58];

figure(3)
sgtitle('$\textbf{Figure (3) - Problem 7 - Simulation of Dynamic Controller for Linear and Nonlinear ...
    System Models}$','Interpreter','latex')

subplot(4,1,1);
%The cart at rest but not at the origin and the bar leaning to one side,
%but also at rest
init_conds=[pi/11; 0; 0.25; 0];
%Always take the initial conditions of the observer to be [0,0,0,0]
init_conds=[init_conds;0;0;0;0];
%Solution of Differential Equations for Linear Model
M=odeFunction(x_dot_full,x_full);
[t_linear,sol_linear] = ode45(M,[0 15],init_conds);
%Solution of Differential Equations for Nonlinear Model
M=odeFunction(x_dot_full_nonlinear,x_full);
[t_nonlinear,sol_nonlinear] = ode45(M,[0 15],init_conds);
%Plotting ouputs y1(t) and y2(t)
%y1(t)=phi=x1=x_full1(t)
plot(t_linear,sol_linear(:,1),'','Color',[0 0.4470 0.7410])
hold on;
%y2(t)=s bar=x3=x_full3(t)
plot(t_linear,sol_linear(:,3),'','Color',[0.9290 0.6940 0.1250])
hold on;
plot(t_nonlinear,sol_nonlinear(:,1),'--','Color',[0 0.4470 0.7410])
hold on;
%y2(t)=s bar=x3=x_full3(t)
plot(t_nonlinear,sol_nonlinear(:,3),'--','Color',[0.9290 0.6940 0.1250])
hold on;
grid on
title('Subplot 1: Initial Conditions ($\frac{\pi}{11}$, 0, 0.25, 0)','Interpreter','latex')
xlabel('Time')
lgd=legend('$y_{1}(t)=\phi_{L}$','$y_{2}(t)=\bar{s}_{L}$','$y_{1}(t)=\phi_{NL}$','$y_{2}(t)=\bar{s}_{NL}$','Location'
title(lgd,'$\textbf{L}$ : Linear Model - $\textbf{NL}$ : Nonlinear Model','Interpreter','latex')

subplot(4,1,2);
%The cart moving to one side with some velocity and the bar initially at
%rest and straight up
init_conds=[0; 0; 1.25; 1];
%Always take the initial conditions of the observer to be [0,0,0,0]
init_conds=[init_conds;0;0;0;0];
%Solution of Differential Equations for Linear Model
M=odeFunction(x_dot_full,x_full);
[t_linear,sol_linear] = ode45(M,[0 15],init_conds);
%Solution of Differential Equations for Nonlinear Model
M=odeFunction(x_dot_full_nonlinear,x_full);
[t_nonlinear,sol_nonlinear] = ode45(M,[0 15],init_conds);
%Plotting ouputs y1(t) and y2(t)
%y1(t)=phi=x1=x_full1(t)
plot(t_linear,sol_linear(:,1),'','Color',[0 0.4470 0.7410])
hold on;
%y2(t)=s bar=x3=x_full3(t)
```

```
plot(t_linear,sol_linear(:,3),'','Color',[0.9290 0.6940 0.1250])
hold on;
plot(t_nonlinear,sol_nonlinear(:,1),'--','Color',[0 0.4470 0.7410])
hold on;
%y2(t)=s bar=x3=x_full3(t)
plot(t_nonlinear,sol_nonlinear(:,3),'--','Color',[0.9290 0.6940 0.1250])
hold on;
grid on
title('Subplot 2: Initial Conditions (0, 0, 1.25, 1)','Interpreter','latex')
xlabel('Time')
lgd=legend('$y_{1}(t)=\phi_{L}$','$y_{2}(t)=\bar{s}_{L}$','$y_{1}(t)=\phi_{NL}$','$y_{2}(t)=\bar{s}_{NL}$','Location'
title(lgd,'$\textbf{L}$ : Linear Model - $\textbf{NL}$ : Nonlinear Model','Interpreter','latex')

subplot(4,1,3);
%The cart moving to one side with some velocity and the bar leaning to one
%side not at rest
init_conds=[pi/11; pi/40; 0.2; 0.1];
%Always take the initial conditions of the observer to be [0,0,0,0]
init_conds=[init_conds;0;0;0;0];
%Solution of Differential Equations for Linear Model
M=odeFunction(x_dot_full,x_full);
[t_linear,sol_linear] = ode45(M,[0 15],init_conds);
%Solution of Differential Equations for Nonlinear Model
M=odeFunction(x_dot_full_nonlinear,x_full);
[t_nonlinear,sol_nonlinear] = ode45(M,[0 15],init_conds);
%Plotting ouputs y1(t) and y2(t)
%y1(t)=phi=x1=x_full1(t)
plot(t_linear,sol_linear(:,1),'','Color',[0 0.4470 0.7410])
hold on;
%y2(t)=s bar=x3=x_full3(t)
plot(t_linear,sol_linear(:,3),'','Color',[0.9290 0.6940 0.1250])
hold on;
plot(t_nonlinear,sol_nonlinear(:,1),'--','Color',[0 0.4470 0.7410])
hold on;
%y2(t)=s bar=x3=x_full3(t)
plot(t_nonlinear,sol_nonlinear(:,3),'--','Color',[0.9290 0.6940 0.1250])
hold on;
grid on
title('Subplot 3: Initial Conditions ($\frac{\pi}{11}$, $\frac{\pi}{40}$, 0.2, 0.1)','Interpreter','latex')
xlabel('Time')
lgd=legend('$y_{1}(t)=\phi_{L}$','$y_{2}(t)=\bar{s}_{L}$','$y_{1}(t)=\phi_{NL}$','$y_{2}(t)=\bar{s}_{NL}$','Location'
title(lgd,'$\textbf{L}$ : Linear Model - $\textbf{NL}$ : Nonlinear Model','Interpreter','latex')

subplot(4,1,4);
%The cart moving to one side with some negative velocity and the bar leaning to the ...
    negative
%side not at rest
init_conds=[-pi/11; pi/40; 0.2; -0.1];
%Always take the initial conditions of the observer to be [0,0,0,0]
init_conds=[init_conds;0;0;0;0];
%Solution of Differential Equations for Linear Model
M=odeFunction(x_dot_full,x_full);
[t_linear,sol_linear] = ode45(M,[0 10],init_conds);
%Solution of Differential Equations for Nonlinear Model
M=odeFunction(x_dot_full_nonlinear,x_full);
[t_nonlinear,sol_nonlinear] = ode45(M,[0 10],init_conds);
%Plotting ouputs y1(t) and y2(t)
%y1(t)=phi=x1=x_full1(t)
plot(t_linear,sol_linear(:,1),'','Color',[0 0.4470 0.7410])
hold on;
%y2(t)=s bar=x3=x_full3(t)
plot(t_linear,sol_linear(:,3),'','Color',[0.9290 0.6940 0.1250])
hold on;
plot(t_nonlinear,sol_nonlinear(:,1),'--','Color',[0 0.4470 0.7410])
hold on;
%y2(t)=s bar=x3=x_full3(t)
plot(t_nonlinear,sol_nonlinear(:,3),'--','Color',[0.9290 0.6940 0.1250])
hold on;
grid on
title('Subplot 4: Initial Conditions ($-\frac{\pi}{11}$, $\frac{\pi}{40}$, 0.2, -0.1)','Interpreter','latex')
xlabel('Time')
lgd=legend('$y_{1}(t)=\phi_{L}$','$y_{2}(t)=\bar{s}_{L}$','$y_{1}(t)=\phi_{NL}$','$y_{2}(t)=\bar{s}_{NL}$','Location'
title(lgd,'$\textbf{L}$ : Linear Model - $\textbf{NL}$ : Nonlinear Model','Interpreter','latex')

%%%%%% ------------------------------------------------------------ %%%%%%
```

```
%Problem 7 - Connection between Stability of Dynamic Controller and the initial angle x1(0)
figure(4)
sgtitle('\textbf{Figure (4) - Problem 7 - Explore Stability of Nonlinear Model for the Dynamic ...
    Controller}','Interpreter','latex')

subplot(2,1,1);
%I.C.s with the Largest Value of x1(0) with Stable Behavior
init_conds=[0.340; 0; 0; 0];
%Always take the initial conditions of the observer to be [0,0,0,0]
init_conds=[init_conds;0;0;0;0];
%Solution of Differential Equations for Linear Model
M=odeFunction(x_dot_full,x_full);
[t_linear,sol_linear] = ode45(M,[0 20],init_conds);
%Solution of Differential Equations for Nonlinear Model
M=odeFunction(x_dot_full_nonlinear,x_full);
[t_nonlinear,sol_nonlinear] = ode45(M,[0 20],init_conds);
%Plotting ouputs y1(t) and y2(t)
%y1(t)=phi=x1=x_full1(t)
plot(t_linear,sol_linear(:,1),'','Color',[0 0.4470 0.7410])
hold on;
%y2(t)=s bar=x3=x_full3(t)
plot(t_linear,sol_linear(:,3),'','Color',[0.9290 0.6940 0.1250])
hold on;
plot(t_nonlinear,sol_nonlinear(:,1),'--','Color',[0 0.4470 0.7410])
hold on;
%y2(t)=s bar=x3=x_full3(t)
plot(t_nonlinear,sol_nonlinear(:,3),'--','Color',[0.9290 0.6940 0.1250])
hold on;
ylim([-15 15]);
grid on
title('Subplot 1: Initial Conditions (0.340, 0, 0, 0)','Interpreter','latex')
xlabel('Time')
lgd=legend('$y_{1}(t)=\phi_{L}$','$y_{2}(t)=\bar{s}_{L}$','$y_{1}(t)=\phi_{NL}$','$y_{2}(t)=\bar{s}_{NL}$','Location'
title(lgd,'$\textbf{L}$ : Linear Model - $\textbf{NL}$ : Nonlinear Model','Interpreter','latex')

subplot(2,1,2);
%I.C.s with the first Value of x1(0) with Unstable Behavior
init_conds=[0.341; 0; 0; 0];
%Always take the initial conditions of the observer to be [0,0,0,0]
init_conds=[init_conds;0;0;0;0];
%Solution of Differential Equations for Linear Model
M=odeFunction(x_dot_full,x_full);
[t_linear,sol_linear] = ode45(M,[0 20],init_conds);
%Solution of Differential Equations for Nonlinear Model
M=odeFunction(x_dot_full_nonlinear,x_full);
[t_nonlinear,sol_nonlinear] = ode45(M,[0 20],init_conds);
%Plotting ouputs y1(t) and y2(t)
%y1(t)=phi=x1=x_full1(t)
plot(t_linear,sol_linear(:,1),'','Color',[0 0.4470 0.7410])
hold on;
%y2(t)=s bar=x3=x_full3(t)
plot(t_linear,sol_linear(:,3),'','Color',[0.9290 0.6940 0.1250])
hold on;
plot(t_nonlinear,sol_nonlinear(:,1),'--','Color',[0 0.4470 0.7410])
hold on;
%y2(t)=s bar=x3=x_full3(t)
plot(t_nonlinear,sol_nonlinear(:,3),'--','Color',[0.9290 0.6940 0.1250])
hold on;
ylim([-15 15]);
grid on
title('Subplot 2: Initial Conditions (0.341, 0, 0, 0)','Interpreter','latex')
xlabel('Time')
lgd=legend('$y_{1}(t)=\phi_{L}$','$y_{2}(t)=\bar{s}_{L}$','$y_{1}(t)=\phi_{NL}$','$y_{2}(t)=\bar{s}_{NL}$','Location'
title(lgd,'$\textbf{L}$ : Linear Model - $\textbf{NL}$ : Nonlinear Model','Interpreter','latex')
%%%%%% ------------------------------------------------------------------------ %%%%%%

%%%%%% ------------------------------------------------------------------------ %%%%%%
```

# 8 Problem 8

Problem 8

In the previous problem, both the linear and nonlinear showed the same observer. In this case, we will use a different observer for the nonlinear model.

$$\left.\begin{aligned} \dot{x} &= f(x, -k\hat{x}) \\ \dot{\hat{x}} &= f(\hat{x}, -k\hat{x}) + L(y-\hat{y}) \\ \hat{y} &= C\hat{x} \\ y &= Cx \end{aligned}\right\} \quad \boxed{\begin{aligned} \dot{x} &= f(x, -k\hat{x}) \\ \dot{\hat{x}} &= f(\hat{x}, -k\hat{x}) + LC(x-\hat{x}) \end{aligned}}$$

For the linear model, the same equation as in Problem 7 apply:

$$\boxed{\begin{aligned} \dot{x} &= Ax - Bk\hat{x} \\ \dot{\hat{x}} &= (A - Bk - LC)\hat{x} + LCx \end{aligned}}$$

In this case, like the last problem, we integrate the state $x$ and the observer $\hat{x}$ in the augmented state $x_{augm} = \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$. Hence, we can examine the properties of the closed-loop nonlinear system, by linearizing the system about the origin.

$$\frac{\partial f}{\partial x_{augm}} = \begin{bmatrix}
\partial f_1/\partial x_1 & \partial f_1/\partial x_2 & \partial f_1/\partial x_3 & \partial f_1/\partial x_4 & \partial f_1/\partial \hat{x}_1 & \partial f_1/\partial \hat{x}_2 & \partial f_1/\partial \hat{x}_3 & \partial f_1/\partial \hat{x}_4 \\
\partial f_2/\partial x_1 & \partial f_2/\partial x_2 & \partial f_2/\partial x_3 & \partial f_2/\partial x_4 & \partial f_2/\partial \hat{x}_1 & \partial f_2/\partial \hat{x}_2 & \partial f_2/\partial \hat{x}_3 & \partial f_2/\partial \hat{x}_4 \\
\partial f_3/\partial x_1 & \partial f_3/\partial x_2 & \partial f_3/\partial x_3 & \partial f_3/\partial x_4 & \partial f_3/\partial \hat{x}_1 & \partial f_3/\partial \hat{x}_2 & \partial f_3/\partial \hat{x}_3 & \partial f_3/\partial \hat{x}_4 \\
\partial f_4/\partial x_1 & \partial f_4/\partial x_2 & \partial f_4/\partial x_3 & \partial f_4/\partial x_4 & \partial f_4/\partial \hat{x}_1 & \partial f_4/\partial \hat{x}_2 & \partial f_4/\partial \hat{x}_3 & \partial f_4/\partial \hat{x}_4 \\
\partial f_5/\partial x_1 & \partial f_5/\partial x_2 & \partial f_5/\partial x_3 & \partial f_5/\partial x_4 & \partial f_5/\partial \hat{x}_1 & \partial f_5/\partial \hat{x}_2 & \partial f_5/\partial \hat{x}_3 & \partial f_5/\partial \hat{x}_4 \\
\partial f_6/\partial x_1 & \partial f_6/\partial x_2 & \partial f_6/\partial x_3 & \partial f_6/\partial x_4 & \partial f_6/\partial \hat{x}_1 & \partial f_6/\partial \hat{x}_2 & \partial f_6/\partial \hat{x}_3 & \partial f_6/\partial \hat{x}_4 \\
\partial f_7/\partial x_1 & \partial f_7/\partial x_2 & \partial f_7/\partial x_3 & \partial f_7/\partial x_4 & \partial f_7/\partial \hat{x}_1 & \partial f_7/\partial \hat{x}_2 & \partial f_7/\partial \hat{x}_3 & \partial f_7/\partial \hat{x}_4 \\
\partial f_8/\partial x_1 & \partial f_8/\partial x_2 & \partial f_8/\partial x_3 & \partial f_8/\partial x_4 & \partial f_8/\partial \hat{x}_1 & \partial f_8/\partial \hat{x}_2 & \partial f_8/\partial \hat{x}_3 & \partial f_8/\partial \hat{x}_4
\end{bmatrix}$$

where $x = [x_1\ x_2\ x_3\ x_4]^T$, $\hat{x} = [\hat{x}_1\ \hat{x}_2\ \hat{x}_3\ \hat{x}_4]^T$,

Figure 21: "Handwritten Solution for Problem 8 - Page 1"

Now $f(x,-k\hat{x}) = [f_1\ f_2\ f_3\ f_4]^T$ and $(f(\hat{x},-k\hat{x}) + LC(x-\hat{x})) = [f_5\ f_6\ f_7\ f_8]^T$

- $f_1 = \dot{x}_1 = x_2$
- $f_2 = \dot{x}_2 = \dfrac{-c\cdot x_2^2 \sin(x_1)\cos(x_1)}{1+c\sin^2(x_1)} + \dfrac{\sin(x_1)}{1+c\sin^2(x_1)} - \dfrac{\cos(x_1)}{(1+c\sin^2(x_1))}(-k_1\hat{x}_1 - k_2\hat{x}_2 - k_3\hat{x}_3 - k_4\hat{x}_4)$
- $f_3 = \dot{x}_3 = x_4$
- $f_4 = \dot{x}_4 = \dfrac{d x_2^2 \sin(x_1)}{1+c\sin^2(x_1)} - \dfrac{\sin(x_1)\cos(x_1)}{1+c\sin^2(x_1)} + \dfrac{b(-k_1\hat{x}_1 - k_2\hat{x}_2 - k_3\hat{x}_3 - k_4\hat{x}_4)}{1+c\sin^2(x_1)}$

$$LC(x-\hat{x}) = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \\ L_{31} & L_{32} \\ L_{41} & L_{42} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1-\hat{x}_1 \\ x_2-\hat{x}_2 \\ x_3-\hat{x}_3 \\ x_4-\hat{x}_4 \end{bmatrix} = \begin{bmatrix} L_{11} & 0 & L_{12} & 0 \\ L_{21} & 0 & L_{22} & 0 \\ L_{31} & 0 & L_{32} & 0 \\ L_{41} & 0 & L_{42} & 0 \end{bmatrix} \begin{bmatrix} x_1-\hat{x}_1 \\ x_2-\hat{x}_2 \\ x_3-\hat{x}_3 \\ x_4-\hat{x}_4 \end{bmatrix}$$

$$LC(x-\hat{x}) = \begin{bmatrix} L_{11}(x_1-\hat{x}_1) + L_{12}(x_3-\hat{x}_3) \\ L_{21}(x_1-\hat{x}_1) + L_{22}(x_3-\hat{x}_3) \\ L_{31}(x_1-\hat{x}_1) + L_{32}(x_3-\hat{x}_3) \\ L_{41}(x_1-\hat{x}_1) + L_{42}(x_3-\hat{x}_3) \end{bmatrix}$$

Hence,
- $f_5 = \hat{x}_2 + L_{11}(x_1-\hat{x}_1) + L_{12}(x_3-\hat{x}_3)$
- $f_6 = \dfrac{-c\hat{x}_2^2 \sin(\hat{x}_1)\cos(\hat{x}_1)}{1+c\sin^2(\hat{x}_1)} + \dfrac{\sin(\hat{x}_1)}{1+c\sin^2(\hat{x}_1)} - \dfrac{\cos(\hat{x}_1)}{(1+c\sin^2(\hat{x}_1))}(-k_1\hat{x}_1 - k_2\hat{x}_2 - k_3\hat{x}_3 - k_4\hat{x}_4) + {}$
  $+ L_{21}(x_1-\hat{x}_1) + L_{22}(x_3-\hat{x}_3)$
- $f_7 = \hat{x}_4 + L_{31}(x_1-\hat{x}_1) + L_{32}(x_3-\hat{x}_3)$
- $f_8 = \dfrac{d\hat{x}_2^2 \sin(\hat{x}_1)}{1+c\sin^2(\hat{x}_1)} - \dfrac{\sin(\hat{x}_1)\cos(\hat{x}_1)}{1+c\sin^2(\hat{x}_1)} + \dfrac{b(-k_1\hat{x}_1 - k_2\hat{x}_2 - k_3\hat{x}_3 - k_4\hat{x}_4)}{(1+c\sin^2(\hat{x}_1))} + L_{41}(x_1-\hat{x}_1) + L_{42}(x_3-\hat{x}_3)$

Figure 22: "Handwritten Solution for Problem 8 - Page 2"

Now, we will compute all derivatives for the Jacobian.

$$\triangleright\ \frac{\partial f_1}{\partial x_1} = 0, \quad \frac{\partial f_1}{\partial x_2} = 1, \quad \frac{\partial f_1}{\partial x_3} = \frac{\partial f_1}{\partial x_4} = \frac{\partial f_1}{\partial \hat{x}_1} = \frac{\partial f_1}{\partial \hat{x}_2} = \frac{\partial f_1}{\partial \hat{x}_3} = \frac{\partial f_1}{\partial \hat{x}_4} = 0$$

$$\triangleright\ \frac{\partial f_2}{\partial x_1} = -c x_2^2 \left[ \frac{\cos^2(x_1) + \cos^4(x_1) - \sin^2(x_1) - \sin^2(x_1)\cos^2(x_1)(1+2c)}{(1+c\sin^2(x_1))^2} \right] + \frac{(\cos(x_1) - c\sin^2(x_1)\cos(x_1))}{(1+c\sin^2(x_1))^2} +$$

$$+ \frac{(\sin(x_1) + c\sin^2(x_1) + 2c\sin(x_1)(1-\sin^2(x_1)))}{(1+c\sin^2(x_1))^2}(-k_1\hat{x}_1 - k_2\hat{x}_2 - k_3\hat{x}_3 - k_4\hat{x}_4)$$

$$\triangleright\ \frac{\partial f_2}{\partial x_2} = \frac{-2x_2 c\sin(x_1)\cos(x_1)}{1+c\sin^2(x_1)}, \quad \frac{\partial f_2}{\partial x_3} = \frac{\partial f_2}{\partial x_4} = 0, \quad \frac{\partial f_2}{\partial \hat{x}_1} = \frac{\cos(x_1)}{(1+c\sin^2(x_1))} k_1$$

$$\triangleright\ \frac{\partial f_2}{\partial \hat{x}_2} = \frac{\cos(x_1)}{(1+c\sin^2(x_1))} k_2, \quad \frac{\partial f_2}{\partial \hat{x}_3} = \frac{\cos(x_1)}{(1+c\sin^2(x_1))} k_3, \quad \frac{\partial f_2}{\partial \hat{x}_4} = \frac{\cos(x_1)}{(1+c\sin^2(x_1))} k_4$$

$$\triangleright\ \frac{\partial f_3}{\partial x_1} = \frac{\partial f_3}{\partial x_2} = \frac{\partial f_3}{\partial x_3} = 0, \quad \frac{\partial f_3}{\partial x_4} = 1, \quad \frac{\partial f_3}{\partial \hat{x}_1} = \frac{\partial f_3}{\partial \hat{x}_2} = \frac{\partial f_3}{\partial \hat{x}_3} = \frac{\partial f_3}{\partial \hat{x}_4} = 0$$

$$\triangleright\ \frac{\partial f_4}{\partial x_1} = \frac{d x_2^2 \left[\cos(x_1)(1+c\sin^2(x_1)) - \sin(x_1)\, 2c\sin(x_1)\cos(x_1)\right]}{(1+c\sin^2(x_1))^2} -$$

$$- \left[ \frac{(\cos^2(x_1) - \sin^2(x_1))(1+c\sin^2(x_1)) - \sin^2(x_1)\cos^2(x_1)\, 2c}{(1+c\sin^2(x_1))^2} \right] -$$

$$- \frac{b\, 2c\sin(x_1)\cos(x_1)}{(1+c\sin^2(x_1))^2}(-k_1\hat{x}_1 - k_2\hat{x}_2 - k_3\hat{x}_3 - k_4\hat{x}_4)$$

$$\triangleright\ \frac{\partial f_4}{\partial x_2} = \frac{2x_2 d\sin(x_1)}{1+c\sin^2(x_1)}, \quad \frac{\partial f_4}{\partial x_3} = \frac{\partial f_4}{\partial x_4} = 0, \quad \frac{\partial f_4}{\partial \hat{x}_1} = \frac{-b k_1}{1+c\sin^2(x_1)}, \quad \frac{\partial f_4}{\partial \hat{x}_2} = \frac{-b k_2}{(1+c\sin^2(x_1))}$$

$$\triangleright\ \frac{\partial f_4}{\partial \hat{x}_3} = \frac{-b k_3}{1+c\sin^2(x_1)}, \quad \frac{\partial f_4}{\partial \hat{x}_4} = \frac{-b k_4}{1+c\sin^2(x_1)}$$

Figure 23: "Handwritten Solution for Problem 8 - Page 3"

▷ $\dfrac{\partial f_5}{\partial x_1} = L_{11}, \quad \dfrac{\partial f_5}{\partial x_2} = 0, \quad \dfrac{\partial f_5}{\partial x_3} = L_{12}, \quad \dfrac{\partial f_5}{\partial x_4} = 0, \quad \dfrac{\partial f_5}{\partial \hat{x}_1} = -L_{11}, \quad \dfrac{\partial f_5}{\partial \hat{x}_2} = 1$

$\dfrac{\partial f_5}{\partial \hat{x}_3} = -L_{12}, \quad \dfrac{\partial f_5}{\partial \hat{x}_4} = 0$

▷ $\dfrac{\partial f_6}{\partial x_1} = L_{21}, \quad \dfrac{\partial f_6}{\partial x_2} = 0, \quad \dfrac{\partial f_6}{\partial x_3} = L_{22}, \quad \dfrac{\partial f_6}{\partial x_4} = 0,$

$$\frac{\partial f_6}{\partial \hat{x}_1} = -c\hat{x}_2^2 \left[ \frac{\cos^2(\hat{x}_1) + \cos^4(\hat{x}_1) - \sin^2(\hat{x}_1) - \sin^2(\hat{x}_1)\cos^2(\hat{x}_1)(1+2c)}{(1+c\sin^2(\hat{x}_1))^2} \right] +$$

$$+ \frac{(\cos(\hat{x}_1) - c\sin^2(\hat{x}_1)\cos(\hat{x}_1))}{(1+c\sin^2(\hat{x}_1))^2} + \frac{(\sin(\hat{x}_1) + c\sin^3(\hat{x}_1) + 2c\sin(\hat{x}_1)(1-\sin^2(\hat{x}_1)))}{(1+c\sin^2(\hat{x}_1))^2}(-k_1\hat{x}_1 - k_2\hat{x}_2 - k_3\hat{x}_3 - k_4\hat{x}_4)$$

$$- \frac{\cos(\hat{x}_1)}{1+c\sin^2(\hat{x}_1)}(-k_1) - L_{21}$$

▷ $\dfrac{\partial f_6}{\partial \hat{x}_2} = \dfrac{-2\hat{x}_2 c\sin(\hat{x}_1)\cos(\hat{x}_1)}{1+c\sin^2(\hat{x}_1)} - \dfrac{\cos(\hat{x}_1)}{(1+c\sin^2(\hat{x}_1))}(-k_2), \quad \dfrac{\partial f_6}{\partial \hat{x}_3} = \dfrac{+\cos(\hat{x}_1)}{(1+c\sin^2(\hat{x}_1))}k_3 - L_{22}$

$\dfrac{\partial f_6}{\partial \hat{x}_4} = \dfrac{\cos(\hat{x}_1)}{(1+c\sin^2(\hat{x}_1))}k_4$, ▷ $\dfrac{\partial f_7}{\partial x_1} = L_{31}, \quad \dfrac{\partial f_7}{\partial x_2} = 0, \quad \dfrac{\partial f_7}{\partial x_3} = L_{32}, \quad \dfrac{\partial f_7}{\partial x_4} = 0$

▷ $\dfrac{\partial f_7}{\partial \hat{x}_1} = -L_{31}, \quad \dfrac{\partial f_7}{\partial \hat{x}_2} = 0, \quad \dfrac{\partial f_7}{\partial \hat{x}_3} = -L_{32}, \quad \dfrac{\partial f_7}{\partial \hat{x}_4} = 1$

▷ $\dfrac{\partial f_8}{\partial x_1} = L_{41}, \quad \dfrac{\partial f_8}{\partial x_2} = 0, \quad \dfrac{\partial f_8}{\partial x_3} = L_{42}, \quad \dfrac{\partial f_8}{\partial x_4} = 0,$

$$\frac{\partial f_8}{\partial \hat{x}_1} = \frac{d\hat{x}_2^2\left[\cos(\hat{x}_1)(1+c\sin^2(\hat{x}_1)) - \sin(\hat{x}_1)2c\sin(\hat{x}_1)\cos(\hat{x}_1)\right]}{(1+c\sin^2(\hat{x}_1))^2} -$$

$$- \left[ \frac{(\cos^2(\hat{x}_1) - \sin^2(\hat{x}_1))(1+c\sin^2(\hat{x}_1)) - \sin^2(\hat{x}_1)\cos^2(\hat{x}_1)2c}{(1+c\sin^2(\hat{x}_1))^2} \right] - \frac{b2c\sin(\hat{x}_1)\cos(\hat{x}_1)}{(1+c\sin^2(\hat{x}_1))^2}\left[-k_1\hat{x}_1 - k_2\hat{x}_2 - k_3\hat{x}_3 - k_4\hat{x}_4\right]$$

$$+ \frac{b(-k_1)}{(1+c\sin^2(\hat{x}_1))} - L_{41}$$

Figure 24: "Handwritten Solution for Problem 8 - Page 4"

$$\frac{\partial f_8}{\partial \hat{x}_2} = \frac{2\hat{x}_2 d \sin(\hat{x}_1)}{1+c\sin^2(\hat{x}_1)} + \frac{b(-k_2)}{(1+c\sin^2(\hat{x}_1))}, \quad \frac{\partial f_8}{\partial \hat{x}_3} = +\frac{b(-k_3)}{(1+c\sin^2(\hat{x}_1))} - L_{42}$$

$$\frac{\partial f_8}{\partial \hat{x}_4} = \frac{b(-k_4)}{(1+c\sin^2(\hat{x}_1))}$$

Since we want to linearize about the origin $(x=0, \hat{x}=0)$; we need to calculate the Jacobian when $\boxed{x=\hat{x}=0.}$

▷ $\partial f_1/\partial x_1 = 0$, $\partial f_1/\partial x_2 = 1$, $\partial f_1/\partial x_3 = \ldots = \partial f_1/\partial \hat{x}_4 = 0$

▷ $\partial f_2/\partial x_1 = 0 + 1 + 0 = 1$, $\partial f_2/\partial x_2 = 0$, $\partial f_2/\partial x_3 = \partial f_2/\partial x_4 = 0$, $\partial f_2/\partial \hat{x}_1 = k_1$

$\circ$ $\partial f_2/\partial \hat{x}_2 = k_2$, $\partial f_2/\partial \hat{x}_3 = k_3$, $\partial f_2/\partial \hat{x}_4 = k_4$,

▷ $\partial f_3/\partial x_1 = \partial f_3/\partial x_2 = \partial f_3/\partial x_3 = 0$, $\partial f_3/\partial x_4 = 1$, $\partial f_3/\partial \hat{x}_1 = \partial f_3/\partial \hat{x}_2 = \partial f_3/\partial \hat{x}_3 = \partial f_3/\partial \hat{x}_4 = 0$

▷ $\partial f_4/\partial x_1 = 0 - 1 = -1$, $\partial f_4/\partial x_2 = 0$, $\partial f_4/\partial x_3 = \partial f_4/\partial x_4 = 0$, $\frac{\partial f_4}{\partial \hat{x}_1} = -bk_1$

$\frac{\partial f_4}{\partial \hat{x}_2} = -bk_2$, $\frac{\partial f_4}{\partial \hat{x}_3} = -bk_3$, $\frac{\partial f_4}{\partial \hat{x}_4} = -bk_4$

▷ $\frac{\partial f_5}{\partial x_1} = L_{11}$, $\frac{\partial f_5}{\partial x_2} = 0$, $\partial f_5/\partial x_3 = L_{12}$, $\partial f_5/\partial x_4 = 0$, $\partial f_5/\partial \hat{x}_1 = -L_{11}$, $\partial f_5/\partial \hat{x}_2 = 1$

$\partial f_5/\partial \hat{x}_3 = -L_{12}$, $\partial f_5/\partial \hat{x}_4 = 0$

▷ $\partial f_6/\partial x_1 = L_{21}$, $\partial f_6/\partial x_2 = 0$, $\partial f_6/\partial x_3 = L_{22}$, $\partial f_6/\partial x_4 = 0$, $\partial f_6/\partial \hat{x}_1 = 0 + 1 + 0 + k_1 - L_{21}$

$\partial f_6/\partial \hat{x}_1 = 1 + k_1 - L_{21}$, $\partial f_6/\partial \hat{x}_2 = k_2$, $\partial f_6/\partial \hat{x}_3 = k_3 - L_{22}$, $\partial f_6/\partial \hat{x}_4 = k_4$

▷ $\partial f_7/\partial x_1 = L_{31}$, $\partial f_7/\partial x_2 = 0$, $\partial f_7/\partial x_3 = L_{32}$, $\partial f_7/\partial x_4 = 0$, $\partial f_7/\partial \hat{x}_1 = -L_{31}$, $\partial f_7/\partial \hat{x}_2 = 0$

$\partial f_7/\partial \hat{x}_3 = -L_{32}$, $\partial f_7/\partial \hat{x}_4 = 1$

▷ $\partial f_8/\partial x_1 = L_{41}$, $\partial f_8/\partial x_2 = 0$, $\partial f_8/\partial x_3 = L_{42}$, $\partial f_8/\partial x_4 = 0$, $\partial f_8/\partial \hat{x}_1 = -1 - bk_1 - L_{41}$

$\partial f_8/\partial \hat{x}_2 = -bk_2$, $\partial f_8/\partial \hat{x}_3 = -bk_3 - L_{42}$, $\partial f_8/\partial \hat{x}_4 = -bk_4$

Figure 25: "Handwritten Solution for Problem 8 - Page 5"

Therefore, we have:

$$A_{augm} = \frac{\partial f}{\partial x_{augm}}\Bigg|_{\substack{x=0 \\ \hat{x}=0}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & k_1 & k_2 & k_3 & k_4 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -\delta k_1 & -\delta k_2 & -\delta k_3 & -\delta k_4 \\ L_{11} & 0 & L_{12} & 0 & -L_{11} & 1 & -L_{12} & 0 \\ L_{21} & 0 & L_{22} & 0 & 1+k_1-L_{21} & k_2 & k_3-L_{22} & k_4 \\ L_{31} & 0 & L_{32} & 0 & -L_{31} & 0 & -L_{32} & 1 \\ L_{41} & 0 & L_{42} & 0 & -1-\delta k_1-L_{41} & -\delta k_2 & -\delta k_3-L_{42} & -\delta k_4 \end{bmatrix}$$

Using Matlab, we calculated the eigenvalues of the matrix in order to examine the properties of the eq. points at the origin. Specifically these are the respective eigenvalues: $\lambda_1 = -9$, $\lambda_2 = -6$, $\lambda_3 = -0.7 + j0.2$, $\lambda_4 = -0.7 - j0.2$, $\lambda_5 = -2.1 + j0.6$, $\lambda_6 = -2.1 - j0.6$, $\lambda_7 = -2$, $\lambda_8 = -3$. We can see that $Re(\lambda_i(A_{augm})) < 0 \ \forall \lambda_i$, meaning that the origin is still asymptotically stable.

In Figure 5, we plot the responses of the new controllers both linear and nonlinear, for the same I.C. as in Problem 5 and 7. Compared to Figure 3, we can observe that this observer indeed provides better estimates and hence better performance. Namely, less oscillations are observed, smaller peaks and settling times and closer behavior between linear and nonlinear models.

Figure 26: "Handwritten Solution for Problem 8 - Page 6"

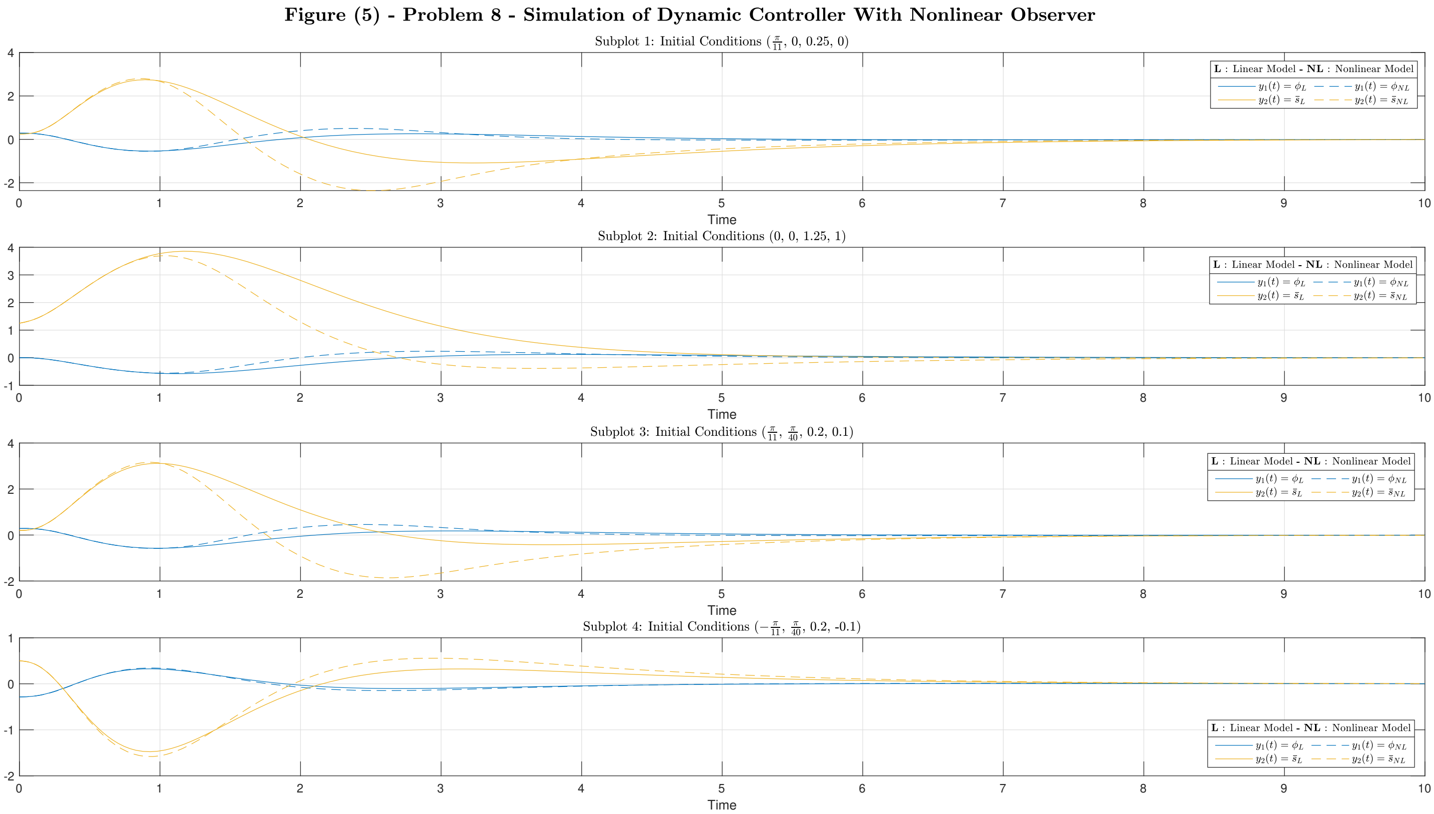



Figure 27: "Figure 5 of Problem 8"

## 8.1 Matlab Code for Problem 8

```
syms x_full(t) [8 1] real
x_full_14=[x_full1;x_full2;x_full3;x_full4];
x_full_58=[x_full5;x_full6;x_full7;x_full8];

%Linear Model - Same as in Problem 7
x_dot_full_14 = A*x_full_14 -B*K*x_full_58;
%Linear Observer
x_dot_full_58 = (A-B*K-L*C)*x_full_58+L*C*x_full_14;
x_dot_full=[x_dot_full_14;x_dot_full_58];

% Nonlinear Model - Same as in Problem 7
mu = -real(K)*x_full_58;
x_dot_full_nl_1 = x_full2;
x_dot_full_nl_2 = (-c*(x_full2^2)*sin(x_full1)*cos(x_full1)/(1+c*(sin(x_full1))^2)) + ...
    (sin(x_full1)/(1+c*(sin(x_full1))^2)) - (cos(x_full1)*mu/(1+c*(sin(x_full1))^2));
x_dot_full_nl_3 = x_full4;
x_dot_full_nl_4 = (d*(x_full2^2)*sin(x_full1)/(1+c*(sin(x_full1))^2)) - ...
    (sin(x_full1)*cos(x_full1)/(1+c*(sin(x_full1))^2)) + (b*mu/(1+c*(sin(x_full1))^2));
%Nonlinear Observer
x_dot_full_nl_5 = x_full6;
x_dot_full_nl_6 = (-c*(x_full6^2)*sin(x_full5)*cos(x_full5)/(1+c*(sin(x_full5))^2)) + ...
    (sin(x_full5)/(1+c*(sin(x_full5))^2)) - (cos(x_full5)*mu/(1+c*(sin(x_full5))^2));
x_dot_full_nl_7 = x_full8;
x_dot_full_nl_8 = (d*(x_full6^2)*sin(x_full5)/(1+c*(sin(x_full5))^2)) - ...
    (sin(x_full5)*cos(x_full5)/(1+c*(sin(x_full5))^2)) + (b*mu/(1+c*(sin(x_full5))^2));
x_dot_full_nl_58 = [x_dot_full_nl_5; x_dot_full_nl_6; x_dot_full_nl_7; x_dot_full_nl_8] ...
    +L*C*(x_full_14-x_full_58);
x_dot_full_nonlinear=[x_dot_full_nl_1; x_dot_full_nl_2; x_dot_full_nl_3; x_dot_full_nl_4; x_dot_full_nl_58];

figure(5)
sgtitle('$\textbf{Figure (5) - Problem 8 - Simulation of Dynamic Controller With Nonlinear ...
    Observer}$','Interpreter','latex')

subplot(4,1,1);
%The cart at rest but not at the origin and the bar leaning to one side,
%but also at rest
init_conds=[pi/11; 0; 0.25; 0];
%Always take the initial conditions of the observer to be [0,0,0,0]
init_conds=[init_conds;0;0;0;0];
%Solution of Differential Equations for Linear Model
M=odeFunction(x_dot_full,x_full);
[t_linear,sol_linear] = ode45(M,[0 10],init_conds);
%Solution of Differential Equations for Nonlinear Model
M=odeFunction(x_dot_full_nonlinear,x_full);
% [t_nonlinear,sol_nonlinear] = ode45(M,[0 5],init_conds);
[t_nonlinear,sol_nonlinear] = ode45(M,[0 10],init_conds);

%Plotting ouputs y1(t) and y2(t)
%y1(t)=phi=x1=x_full1(t)
plot(t_linear,sol_linear(:,1),'','Color',[0 0.4470 0.7410])
hold on;
%y2(t)=s bar=x3=x_full3(t)
plot(t_linear,sol_linear(:,3),'','Color',[0.9290 0.6940 0.1250])
hold on;
plot(t_nonlinear,sol_nonlinear(:,1),'--','Color',[0 0.4470 0.7410])
hold on;
%y2(t)=s bar=x3=x_full3(t)
plot(t_nonlinear,sol_nonlinear(:,3),'--','Color',[0.9290 0.6940 0.1250])
hold on;
% ylim([-15 15]);
grid on
title('Subplot 1: Initial Conditions ($\frac{\pi}{11}$, 0, 0.25, 0)','Interpreter','latex')
xlabel('Time')
lgd=legend('$y_{1}(t)=\phi_{L}$','$y_{2}(t)=\bar{s}_{L}$','$y_{1}(t)=\phi_{NL}$','$y_{2}(t)=\bar{s}_{NL}$','Location'
title(lgd,'$\textbf{L}$ : Linear Model - $\textbf{NL}$ : Nonlinear Model','Interpreter','latex')

subplot(4,1,2);
%The cart moving to one side with some velocity and the bar initially at
%rest and straight up
init_conds=[0; 0; 1.25; 1];
%Always take the initial conditions of the observer to be [0,0,0,0]
init_conds=[init_conds;0;0;0;0];
```

```
%Solution of Differential Equations for Linear Model
M=odeFunction(x_dot_full,x_full);
[t_linear,sol_linear] = ode45(M,[0 10],init_conds);
%Solution of Differential Equations for Nonlinear Model
M=odeFunction(x_dot_full_nonlinear,x_full);
[t_nonlinear,sol_nonlinear] = ode45(M,[0 10],init_conds);
%Plotting ouputs y1(t) and y2(t)
%y1(t)=phi=x1=x_full1(t)
plot(t_linear,sol_linear(:,1),'','Color',[0 0.4470 0.7410])
hold on;
%y2(t)=s bar=x3=x_full3(t)
plot(t_linear,sol_linear(:,3),'','Color',[0.9290 0.6940 0.1250])
hold on;
plot(t_nonlinear,sol_nonlinear(:,1),'--','Color',[0 0.4470 0.7410])
hold on;
%y2(t)=s bar=x3=x_full3(t)
plot(t_nonlinear,sol_nonlinear(:,3),'--','Color',[0.9290 0.6940 0.1250])
hold on;
% ylim([-15 15]);
grid on
title('Subplot 2: Initial Conditions (0, 0, 1.25, 1)','Interpreter','latex')
xlabel('Time')
lgd=legend('$y_{1}(t)=\phi_{L}$','$y_{2}(t)=\bar{s}_{L}$','$y_{1}(t)=\phi_{NL}$','$y_{2}(t)=\bar{s}_{NL}$','Location'
title(lgd,'$\textbf{L}$ : Linear Model - $\textbf{NL}$ : Nonlinear Model','Interpreter','latex')

subplot(4,1,3);
%The cart moving to one side with some velocity and the bar leaning to one
%side not at rest
init_conds=[pi/11; pi/40; 0.2; 0.1];
%Always take the initial conditions of the observer to be [0,0,0,0]
init_conds=[init_conds;0;0;0;0];
%Solution of Differential Equations for Linear Model
M=odeFunction(x_dot_full,x_full);
[t_linear,sol_linear] = ode45(M,[0 10],init_conds);
%Solution of Differential Equations for Nonlinear Model
M=odeFunction(x_dot_full_nonlinear,x_full);
[t_nonlinear,sol_nonlinear] = ode45(M,[0 10],init_conds);
%Plotting ouputs y1(t) and y2(t)
%y1(t)=phi=x1=x_full1(t)
plot(t_linear,sol_linear(:,1),'','Color',[0 0.4470 0.7410])
hold on;
%y2(t)=s bar=x3=x_full3(t)
plot(t_linear,sol_linear(:,3),'','Color',[0.9290 0.6940 0.1250])
hold on;
plot(t_nonlinear,sol_nonlinear(:,1),'--','Color',[0 0.4470 0.7410])
hold on;
%y2(t)=s bar=x3=x_full3(t)
plot(t_nonlinear,sol_nonlinear(:,3),'--','Color',[0.9290 0.6940 0.1250])
hold on;
% ylim([-15 15]);
grid on
title('Subplot 3: Initial Conditions ($\frac{\pi}{11}$, $\frac{\pi}{40}$, 0.2, 0.1)','Interpreter','latex')
xlabel('Time')
lgd=legend('$y_{1}(t)=\phi_{L}$','$y_{2}(t)=\bar{s}_{L}$','$y_{1}(t)=\phi_{NL}$','$y_{2}(t)=\bar{s}_{NL}$','Location'
title(lgd,'$\textbf{L}$ : Linear Model - $\textbf{NL}$ : Nonlinear Model','Interpreter','latex')

subplot(4,1,4);
%The cart moving to one side with some negative velocity and the bar leaning to the ...
    negative
%side not at rest
init_conds=[-pi/11; pi/55; 0.5; -0.25];
%Always take the initial conditions of the observer to be [0,0,0,0]
init_conds=[init_conds;0;0;0;0];
%Solution of Differential Equations for Linear Model
M=odeFunction(x_dot_full,x_full);
[t_linear,sol_linear] = ode45(M,[0 10],init_conds);
%Solution of Differential Equations for Nonlinear Model
M=odeFunction(x_dot_full_nonlinear,x_full);
[t_nonlinear,sol_nonlinear] = ode45(M,[0 10],init_conds);
%Plotting ouputs y1(t) and y2(t)
%y1(t)=phi=x1=x_full1(t)
plot(t_linear,sol_linear(:,1),'','Color',[0 0.4470 0.7410])
hold on;
%y2(t)=s bar=x3=x_full3(t)
plot(t_linear,sol_linear(:,3),'','Color',[0.9290 0.6940 0.1250])
```

```
hold on;
plot(t_nonlinear,sol_nonlinear(:,1),'--','Color',[0 0.4470 0.7410])
hold on;
%y2(t)=s bar=x3=x_full3(t)
plot(t_nonlinear,sol_nonlinear(:,3),'--','Color',[0.9290 0.6940 0.1250])
hold on;
% ylim([-15 15]);
grid on
title('Subplot 4: Initial Conditions ($-\frac{\pi}{11}$, $\frac{\pi}{40}$, 0.2, -0.1)','Interpreter','latex')
xlabel('Time')
lgd=legend('$y_{1}(t)=\phi_{L}$','$y_{2}(t)=\bar{s}_{L}$','$y_{1}(t)=\phi_{NL}$','$y_{2}(t)=\bar{s}_{NL}$','Location'
title(lgd,'$\textbf{L}$ : Linear Model - $\textbf{NL}$ : Nonlinear Model','Interpreter','latex')

%%%%%% ---------------------------------------------------------------- %%%%%%
%Problem 8 - Linearization about the Origin
A_augm=[0 1 0 0 0 0 0 0;
        1 0 0 0 K(1) K(2) K(3) K(4);
        0 0 0 1 0 0 0 0;
        -1 0 0 0 -b*K(1) -b*K(2) -b*K(3) -b*K(4);
        L(1,1) 0 L(1,2) 0 -L(1,1) 1 -L(1,2) 0;
        L(2,1) 0 L(2,2) 0 1+K(1)-L(2,1) K(2) K(3)-L(2,2) K(4);
        L(3,1) 0 L(3,2) 0 -L(3,1) 0 -L(3,2) 1;
        L(4,1) 0 L(4,2) 0 -1-b*K(1)-L(4,1) -b*K(2) -b*K(3)-L(4,2) -b*K(4)]
eig(A_augm)
%%%%%% ---------------------------------------------------------------- %%%%%%
```