MEEG 829 - Project
Inverted Pendulum on a Cart

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Objective: The goal of this (individual) project is to analyze a particular physical system, the so-called "broom balancing problem", from a linearization point of view. That is, you will design a controller for a nonlinear system using linear feedback design methods. This will give you a chance to use some of what you learned in MEEG 621 "Linear Systems," plus maybe a few other techniques as well. More importantly, it will illustrate a very common design technique for nonlinear systems: linearize, then hope for the best! We will learn more analytical methods towards the end of the term. To complete the project, you may have to recall or read a few things from MEEG 621.

1 Inverted Pendulum on a Cart Model

Consider the inverted pendulum positioning system described in Figure 1. Assuming that friction is negligible, but not assuming that the mass of the pendulum is small in comparison to the mass of the cart, the dynamics of the system is

$$\begin{aligned}\frac{d^2\phi}{dt^2} &= \frac{(M+m)mgLsin\phi - (mL\dot{\phi})^2\sin\phi\cos\phi - (mL\cos\phi)\mu}{(M+m)(J+mL^2)-(mL)^2(\cos\phi)^2} \\ \frac{d^2 s}{dt^2} &= \frac{(J+mL^2)mL\dot{\phi}^2\sin\phi - (mL)^2 g\sin\phi\cos\phi + (J+mL^2)\mu}{(M+m)(J+mL^2)-(mL)^2\cos^2\phi}\end{aligned} \tag{1}$$

where, here, the dots represent differentiation with respect to *physical* time t. You may wish to do the

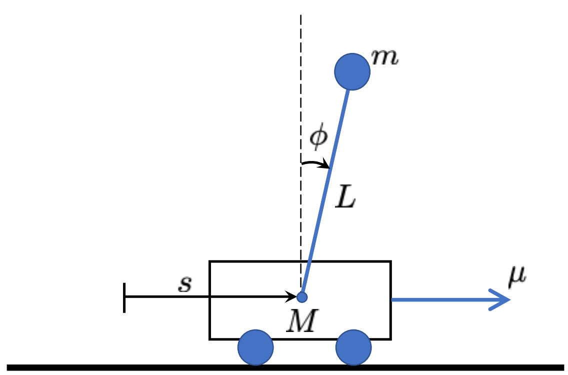

Figure 1: Inverted pendulum on a cart.

tedious derivation for your own pleasure. The method of Lagrange can be used to get the model in a much easier and quicker way (you could this symbolically in MATLAB and verify the above equations. But none of this is NOT required, just assume that they are right and free of typos.

Next, we "non-dimensionalize" the model. The concept of dimensionless variables is very important in practice because it often leads to models that are numerically better conditioned, and, also, it often leads to fewer parameters (as is the case in our example below). The technique is typically taught in courses like Thermodynamics and Fluid Mechanics. In what follows, we introduce the dimensionless variables:

$$\bar{s} = \left(\frac{M}{m}+1\right)\frac{s}{L}, \quad \bar{\mu} = \frac{1}{(M+m)g}\mu, \quad d = 1+c, \quad \text{and} \quad \bar{t} = \frac{t}{T}, \tag{2}$$

where

$$T^2 = \left(\frac{J}{mL^2}+\frac{M}{m+M}\right)\frac{L}{g}, \quad b = \left(\frac{M}{m}+1\right)\left(\frac{J}{mL^2}+1\right), \quad \text{and} \quad c = \frac{m^2L^2}{J(m+M)+mML^2}.$$

In terms of the dimensionless variables (2), the dynamics (1) of the inverted pendulum take the form

$$\begin{aligned} \frac{d^2\phi}{d\bar{t}^2} &= -\frac{c\dot{\phi}^2\sin\phi\cos\phi}{1+c\sin^2\phi}+\frac{\sin\phi}{1+c\sin^2\phi}-\frac{\cos\phi}{1+c\sin^2\phi}\,\bar{\mu} \\ \frac{d^2\bar{s}}{d\bar{t}^2} &= \frac{d\dot{\phi}^2\sin\phi}{1+c\sin^2\phi}-\frac{\sin\phi\cos\phi}{1+c\sin^2\phi}+\frac{b}{1+c\sin^2\phi}\,\bar{\mu}, \end{aligned} \tag{3}$$

where, here the dots represent differentiation with respect to time the *normalized* time $\bar{t}$. You may wish to verify that the denominator in the model never vanishes, so that the equations are well-defined.

2 Project Problems

Problem 1 [10 points]: Verify that the equation for ϕ in (3) indeed follows from (1) after substitution of the given dimensionless quantities and proper application of the chain rule. Doing the same for s is similar and is not required here.

Problem 2 [10 points]: Construct a state space representation of (3) and linearize the equations about the upright equilibrium point. Choose the states as $x_1 = \phi$, $x_2 = \dot{\phi}$, $x_3 = \bar{s}$, $x_4 = \dot{\bar{s}}$.

For the rest of the project, the following data is assumed:

$$M = 25 \text{ Kg}, \quad m = 20 \text{ Kg}, \quad L = 9.81 \text{ m}, \quad g = 9.81 \text{m/s}^2, \quad \text{and} \quad J = \frac{1}{3}mL^2$$

while s is in meters and ϕ is in radians. Linearizing (3) about the origin should yield the linearized system:

$$\dot{x} = Ax + B\bar{\mu}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ -1 \\ 0 \\ b \end{bmatrix}$$

Problem 3 [5 points]: Verify that the linearized system is completely controllable and then determine a state variable feedback control law

$$\bar{\mu} = -kx + v \tag{4}$$

that will place the closed-loop eigenvalues at $(-3, -2, -0.7 \pm j0.2)$. Use the "place" command in MATLAB.

Problem 4 [5 points]: In Problem 3, you calculated a control law (4). Now apply this control law to both the linearized system and the nonlinear system using $v = 0$. This gives you two closed-loop systems:

$$\dot{x} = (A - Bk)x \tag{5}$$

and

$$\dot{x} = f(x, -kx). \tag{6}$$

Verify that

$$A - Bk = \frac{\partial}{\partial x} \left(f(x, -kx)\right)\Big|_{x=0}, \tag{7}$$

that is, show that (5) is the linearization of (5) i.e. of the nonlinear system in closed loop with the linear controller. What can you conclude about the stability properties of the origin for the nonlinear closed-loop system? See Theorem 4.7 on page 139 of Khalil. This theorem is really a justification for the existence of linear control theory. See also Definition 4.1 on page 112.

Problem 5 [10 points]: Simulate both closed-loop systems for several initial conditions. Use your imagination when selecting initial conditions; for example, the cart at rest but not at the origin, and the bar leaning to one side, but also at rest; cart moving to one side with some velocity, and the bar initially at rest and straight up, etc. Compare the response of the linear model to that of the nonlinear model. With $x_2(0) = 0$, $x_3(0) = 0$ and $x_4(0) = 0$, explore how large you can make $x_1(0)$ (the initial angle) before the nonlinear model goes unstable.

Problem 6 [10 points]: Suppose now that you can only measure the outputs

$$y_1 = \phi \quad \text{and} \quad y_2 = \bar{s}.$$

Verify that the system is completely observable. Then, design an observer

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + B\bar{\mu} + L(y - \hat{y}) \\ \hat{y} &= c\hat{x} \end{aligned} \tag{8}$$

that will allow you to implement the control law obtained in Problem 3. Place the eigenvalues of the observer matrix $A - LC$ so that they are three times faster than those of the closed-loop system $A - Bk$ designed in Problem 3 (see the location of the eigenvalues in Problem 3).

Problem 7 [15 points]: Consider the dynamic (= state feedback plus the observer) controller by augmenting the open-loop system and the observer (8) and applying the control law

$$\bar{\mu} = -k\hat{x}. \tag{9}$$

Note that now the input $\bar{\mu}$ is proportional to the *estimated* value of the state $\hat{x}$ as opposed to the state x. Simulate this dynamic controller on both the linearized and the nonlinear system models for a variety of initial conditions, and plot the outputs $y_1(t)$ and $y_2(t)$. How does the response with the dynamic controller ($u = -k\hat{x}$ with $\hat{x}$ being the estimated state) compare to the response with the static controller ($u = -kx$ with x being the estimated state)? You should see that being able to measure states instead of estimating them is really useful! Always take the initial conditions of the observer to be $[0,0,0,0]$. With $x_2(0) = 0$, $x_3(0) = 0$ and $x_4(0) = 0$, explore how large you can make $x_1(0)$ (the initial angle) before the nonlinear model goes unstable. It will be smaller than in Problem 5.

Problem 8 [10 points]: Explore the following design option for the observer. Let your linear observer from Problem 6 be as in (8). You recognize the observer as being a copy of the linearized model plus the "output injection" term $L(y - \hat{y})$, which corrects for errors in the estimate of the output. For the nonlinear plant, try implementing the observer as

$$\begin{aligned} \dot{\hat{x}} &= f(\hat{x}, \bar{\mu}) + L(y - \hat{y}) \\ \hat{y} &= c\hat{x} \end{aligned} \tag{10}$$

which you recognize as being a copy of the nonlinear model plus linear output injection. Then implement the controller (9) and see if this observer provides better estimates and hence better performance. Why should the origin of the closed-loop nonlinear system still be asymptotically stable? Hint: linearize the closed-loop system about the origin.

3 Instructions for the report

I assume that you will have many pages of simulations, computations, etc. While I want most of this included in your report, I wish to receive an organized package that is easy to read and follow. I will deduct points for poorly presented work. In fact, my assumption is that everyone is capable of completing this project correctly. The surest way to loose points, therefore, is to make your presentation hard to read.
Use the following structure for your report:

Page 1: Use the required title page (see below).

Page 2: Contents page. List Problems one through eight and include page numbers.

Page 3: Copy the following statement providing your answers:

1. Using linear state variable feedback and with the nonlinear model initialized at $x_2(0) = 0$, $x_3(0) = 0$ and $x_4(0) = 0$, the largest I could make $x_1(0)$ before the nonlinear model went unstable was (units of radians). The linear state variable feedback that I used was ______ (give the feedback gains).

2. Using linear state variable feedback plus an observer, with the observer initialized at the origin and $x_2(0) = 0$, $x_3(0) = 0$ and $x_4(0) = 0$, the largest I could make $x_1(0)$ before the nonlinear model went unstable was ______ (units of radians). The observer that I used to obtain the best response was linear/nonlinear (specify which one) and the observer gain was ______ (give the observer gain matrix). Put all documentation in Problem 7 or 8.

Page 4: Begin answers to problems one through eight.

A few more instructions:

- Start each problem on a separate page; start the page with a statement of the problem being worked.
- DO NOT group all simulations, MATLAB printouts or hand computations at the end as one giant appendix. Instead, group them with the problem that is being worked. That is, all simulations for problem five will follow problem five, those for problem seven will be with problem seven, etc.
- Only include the most important of your printouts. Use good judgment on this. Remember, I have to read a bunch of these. What will make yours stand out will be its clarity. More is not always better!
- Include your MATLAB code, even the code used to generate your plots (your code should be given separately for each problem, and remember that at the end, you should upload a **single** pdf file)
- Put labels or numbers on your plots. You can number them by hand.
- You do NOT have to type set your solutions. Handwriting is perfectly fine. I really do not care.
- Scan all your documents and upload your report as a **single** pdf file on Canvas.

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Due date: May 15, 2020

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LAST NAME (PRINTED)	FIRST NAME (PRINTED)

This should be the cover page of your project