

**MEEG 829**  
**Inverted Pendulum on a Cart Project**

**Prof. Poulakakis**

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**KARAKASIS**

LAST NAME (PRINTED)

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**CHRYSOSTOMOS**

FIRST NAME (PRINTED)

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1. Using linear state variable feedback and with the nonlinear model initialized at  $x_2(0) = 0$ ,  $x_3(0) = 0$  and  $x_4(0) = 0$ , the largest I could make  $x_1(0)$  before the nonlinear model went unstable was **0.658** (units of radians). The linear state variable feedback that I used was:

$$\mathbf{k} = [-19.3000 - 22.9750 - 1.5900 - 5.5250].$$

2. Using linear state variable feedback plus an observer, with the observer initialized at the origin and  $x_2(0) = 0$ ,  $x_3(0) = 0$  and  $x_4(0) = 0$ , the largest I could make  $x_1(0)$  before the nonlinear model went unstable was **0.340** (units of radians). The observer that I used to obtain the best response was **nonlinear** and the observer gain was:

$$\mathbf{L} = \begin{bmatrix} 9.2929 & 0.3630 \\ 8.9077 & -3.1719 \\ 3.9579 & 9.9071 \\ 22.2846 & 23.2335 \end{bmatrix}.$$

# 1 Problem 1

## Problem 1

$$\frac{d^2\phi}{dt^2} = \frac{d^2\phi}{dt^2} + T^2 \stackrel{(1)}{=} \frac{(M+m)mgL\sin\phi \cdot T^2}{(M+m)(J+ml^2) - (ml)^2(\cos\phi)^2} - \frac{(ml)\frac{d\phi}{dt})^2 \sin\phi \cos\phi}{(M+m)(J+ml^2) - (ml)^2(\cos\phi)^2} T^2$$

$$- \frac{(ml\cos\phi)\mu}{(M+m)(J+ml^2) - (ml)^2(\cos\phi)^2} T^2$$

We will analyze each term independently:

$$\begin{aligned} & \frac{(M+m)mgL\sin\phi T^2}{(M+m)(J+ml^2) - (ml)^2(\cos\phi)^2} = \frac{(M+m)mgL\sin\phi}{[(M+m)(J+ml^2) - (ml)^2(\cos\phi)^2]} \cdot \frac{\left(\frac{J}{ml^2} + \frac{M}{m+M}\right) \cdot L}{g} \\ &= \frac{(M+m)mL^2\sin\phi}{[(M+m)(J+ml^2) - (ml)^2(\cos\phi)^2]} \cdot \left(\frac{J}{ml^2} + \frac{M}{m+M}\right) \\ &= \frac{J(M+m)\sin\phi + ml^2M\sin\phi}{(M+m)(J+ml^2) - (ml)^2(\cos\phi)^2} \\ &= \frac{\sin\phi (J(M+m) + ml^2M)}{(M+m)(J+ml^2) - (ml)^2(1 - \sin\phi)^2} \quad (\cos\phi)^2 = 1 - (\sin\phi)^2 \\ &= \frac{\sin\phi [J(M+m) + ml^2M]}{(M+m)(J+ml^2) - (ml)^2 + (ml)^2(\sin\phi)^2} \\ &= \frac{\sin\phi}{\frac{(M+m)(J+ml^2) - (ml)^2}{[J(M+m) + ml^2M]} + \frac{(ml)^2(\sin\phi)^2}{[J(M+m) + ml^2M]}} \\ &= \frac{\sin\phi}{\frac{(M+m)J + Mml^2 + ml^2k^2 - (ml)^2}{[J(M+m) + ml^2M]} + \frac{(ml)^2(\sin\phi)^2}{[J(M+m) + ml^2M]}} = \boxed{\frac{\sin\phi}{L + C(\sin\phi)^2}} \quad \checkmark \end{aligned}$$

Figure 1: "Handwritten Solution for Problem 1 - Page 1"

$$\bullet - \frac{(mL \frac{d\phi}{dt})^2 \sin\phi \cos\phi T^2}{(M+m)(J+ml^2) - (ml)^2(\cos\phi)^2}$$
 whence  $\frac{d\phi}{dt} = \frac{d\phi}{dt} \frac{dt}{dt} = \dot{\phi} \frac{1}{T}$

$$= - \frac{(ml \dot{\phi} \frac{1}{T})^2 \sin\phi \cos\phi T^2}{(M+m)(J+ml^2) - (ml)^2(\cos\phi)^2} = - \frac{(ml)^2 \dot{\phi}^2 \frac{1}{T^2} \sin\phi \cos\phi T^2}{(M+m)(J+ml^2) - (ml)^2(\cos\phi)^2}$$

$$= - \frac{(ml)^2 \dot{\phi}^2 \sin\phi \cos\phi}{(M+m)(J+ml^2) - (ml)^2(1-\sin\phi)^2} = - \frac{(ml)^2 \dot{\phi}^2 \sin\phi \cos\phi}{(M+m)(J+ml^2) - (ml)^2 + (ml)^2(\sin\phi)^2}$$

$$= - \frac{(ml)^2 \dot{\phi}^2 \sin\phi \cos\phi}{J(M+m) + ml^2 + (ml)^2(\sin\phi)^2} = - \frac{(ml)^2 \dot{\phi}^2 \sin\phi \cos\phi}{[J(M+m) + ml^2] + 1 + (ml)^2(\sin\phi)^2}$$

$$= \boxed{- \frac{c \dot{\phi}^2 \sin\phi \cos\phi}{1 + c(\sin\phi)^2}}$$
  

$$\bullet - \frac{(ml \cos\phi) \mu T^2}{(M+m)(J+ml^2) - (ml)^2(\cos\phi)^2} = - \frac{(ml \cos\phi) \mu T^2}{J(M+m) + ml^2 + ml^2 - (ml)^2(1-\sin\phi)^2}$$

$$= - \frac{(ml \cos\phi) \mu T^2}{J(M+m) + ml^2 + ml^2 + (ml)^2(\sin\phi)^2} = - \frac{ml \mu T^2}{\cos\phi [J(M+m) + ml^2]}$$

$$= - \frac{\cos\phi}{1 + c(\sin\phi)^2} \cdot \frac{ml \mu}{[J(M+m) + ml^2]} \cdot \left( \frac{J}{ml^2} + \frac{M}{m+M} \right) \frac{L}{g}$$

$$= - \frac{\cos\phi}{1 + c(\sin\phi)^2} \cdot \frac{\mu}{[J(M+m) + ml^2]} \cdot \frac{1}{g} \cdot \left( J + \frac{ml^2 M}{m+M} \right)$$

$$= - \frac{\cos\phi}{1 + c(\sin\phi)^2} \cdot \frac{\mu}{[\bar{J}(M+m) + ml^2]} \cdot \frac{1}{g} \cdot \left( \frac{J(M+m) + ml^2 M}{m+M} \right) = \boxed{- \frac{\cos\phi \bar{\mu}}{1 + c(\sin\phi)^2}}$$

Figure 2: "Handwritten Solution for Problem 1 - Page 2"

## 2 Problem 2

Problem 2

$$x_1 = \phi \Rightarrow \ddot{x}_1 = \dot{\phi} = x_2 \quad (3.1)$$

$$x_2 = \dot{\phi} \Rightarrow \ddot{x}_2 = \ddot{\phi} = \frac{d^2\phi}{dt^2} = -\frac{c\phi \sin \phi \cos \phi}{1+c \sin^2 \phi} + \frac{\sin \phi}{1+c \sin^2 \phi} - \frac{\cos \phi}{1+c \sin^2 \phi} \bar{\mu}$$

$$x_3 = \bar{s} \Rightarrow \ddot{x}_3 = \frac{\ddot{s}}{\bar{s}} = x_4 \quad (3.3)$$

$$x_4 = \bar{s} \Rightarrow \ddot{x}_4 = \ddot{\bar{s}} = \frac{d^2\bar{s}}{dt^2} = \frac{d\dot{\phi}^2 \bar{s}}{1+c \sin^2 \phi} - \frac{\sin \phi \cos \phi}{1+c \sin^2 \phi} + \frac{b}{1+c \sin^2 \phi} \bar{\mu}$$

$$(3.2) \Rightarrow \ddot{x}_2 = -\frac{c x_2^2 \sin(x_1) \cos(x_1)}{1+c \sin^2(x_1)} + \frac{\sin(x_1)}{1+c \sin^2(x_1)} - \frac{\cos(x_1)}{1+c \sin^2(x_1)} \bar{\mu} \quad (3.2)$$

$$(3.4) \Rightarrow \ddot{x}_4 = \frac{d \cdot x_2^2 \sin(x_1)}{1+c \sin^2(x_1)} - \frac{\sin(x_1) \cos(x_1)}{1+c \sin^2(x_1)} + \frac{b}{1+c \sin^2(x_1)} \bar{\mu} \quad (3.4)$$

Now, we find the equilibrium points  $\Rightarrow \ddot{x}_1 = \ddot{x}_2 = \ddot{x}_3 = \ddot{x}_4 = 0$

$$\ddot{x}_1 = 0 \Rightarrow \boxed{x_2 = 0} \quad (3.5)$$

$$\ddot{x}_2 = 0 \stackrel{(3.5)}{\Rightarrow} \sin(x_1) - \cos(x_1) \bar{\mu} = 0 \Rightarrow \sin(x_1) = \cos(x_1) \bar{\mu} \quad (3.6)$$

$$\ddot{x}_3 = 0 \Rightarrow \boxed{x_4 = 0} \quad (3.7)$$

$$\ddot{x}_4 = 0 \stackrel{(3.5)}{\Rightarrow} -\sin(x_1) \cos(x_1) + b \bar{\mu} = 0 \Rightarrow \sin(x_1) \cos(x_1) = b \bar{\mu} \quad (3.8)$$

$$\begin{aligned} (3.6) \tan(x_1) &= \bar{\mu} \\ (3.8) \cos^2(x_1) \bar{\mu} &= b \bar{\mu} \end{aligned} \rightarrow \left\{ \begin{array}{l} \tan(x_1) = \bar{\mu} \\ \bar{\mu} (\cos^2(x_1) - b) = 0 \end{array} \right\} \tan(x_1) = \bar{\mu} \dots ;$$

$\bar{\mu} = 0 \text{ or } \cos^2(x_1) = b$

We can evaluate  $b$ , given the values of  $M, m, L, g$

$$b = \left( \frac{25}{20} + 1 \right) \left( \frac{1}{3} \frac{mL^2}{mL^2} + 1 \right) = \left( \frac{5}{4} + 1 \right) \left( \frac{1}{3} + 1 \right) = \frac{9}{4} \frac{4}{3} = 3 > 1$$

Hence, it is impossible that  $\cos^2(x_1) = b \Rightarrow \boxed{\bar{\mu} = 0} \stackrel{(3.8)}{\Rightarrow} \sin(x_1) = 0$

Therefore,  $\boxed{x_1 = K\pi}$ ,  $K = 0, \pm 1, \pm 2, \dots$

Figure 3: "Handwritten Solution for Problem 2 - Page 1"

Therefore, we have shown that  $x = (kn, 0, x_3, 0)$ ,  $k=0, \pm 1, \pm 2, \dots$  and  $x_3 \in \mathbb{R}$  are equilibrium points, when the input  $\bar{\mu} = 0$ .

The linearization of the system about the origin will yield

$$\ddot{x} = Ax + B\bar{\mu}$$

where  $A = \frac{\partial f}{\partial x} \Big|_{\substack{x=0 \\ \bar{\mu}=0}}$  and  $B = \frac{\partial f}{\partial \bar{\mu}} \Big|_{\substack{x=0 \\ \bar{\mu}=0}}$

$$\frac{\partial f}{\partial \bar{\mu}} = \begin{bmatrix} \frac{\partial f_1}{\partial \bar{\mu}} \\ \frac{\partial f_2}{\partial \bar{\mu}} \\ \frac{\partial f_3}{\partial \bar{\mu}} \\ \frac{\partial f_4}{\partial \bar{\mu}} \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{\cos \phi}{(1+c \sin^2 \phi)} \\ 0 \\ 0 \end{bmatrix} \Rightarrow B = \frac{\partial f}{\partial \bar{\mu}} \Big|_{\substack{x=0 \\ \bar{\mu}=0}} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

$$(1+c \sin^2 \phi)$$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \frac{\partial f_1}{\partial x_4} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \frac{\partial f_2}{\partial x_4} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} & \frac{\partial f_3}{\partial x_4} \\ \frac{\partial f_4}{\partial x_1} & \frac{\partial f_4}{\partial x_2} & \frac{\partial f_4}{\partial x_3} & \frac{\partial f_4}{\partial x_4} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{\partial f_4}{\partial x_1} & \frac{\partial f_4}{\partial x_2} & 0 & 0 \end{bmatrix}$$

$$\frac{\partial f_2}{\partial x_1} = -cx_2^2 \cdot \frac{\partial (\sin x_1 \cos x_1)}{\partial x_1} (1+c \sin^2 x_1) - \frac{\partial (1+c \sin^2 x_1)}{\partial x_1} (\sin x_1 \cos x_1) +$$

$$+ \frac{\cos(x_1)(1+c \sin^2 x_1) - \sin(x_1)(2c \sin(x_1) \cos(x_1))}{(1+c \sin^2 x_1)^2} -$$

$$-\left[ \frac{(-\sin(x_1))(1+c \sin^2 x_1) - \cos(x_1)(2c \sin(x_1) \cos(x_1))}{(1+c \sin^2 x_1)} \bar{\mu} \right]$$

Figure 4: "Handwritten Solution for Problem 2 - Page 2"

$$\begin{aligned}
 \frac{\partial f_2}{\partial x_1} &= -cx_2^2 \cdot \left[ \frac{(\cos^2(x_1) - \sin^2(x_1))(1 + \cos^2(x_1)) - 2c \sin(x_1) \cos(x_1)(\sin(x_1) \cos(x_1))}{(1 + c \sin^2(x_1))^2} \right] + \\
 &\quad + \frac{\cos(x_1) + c \sin^2(x_1) \cos(x_1) - 2c \sin^2(x_1) \cos(x_1)}{(1 + c \sin^2(x_1))^2} + \\
 &\quad + \frac{(\sin(x_1) + c \sin^3(x_1) + 2c \sin(x_1) \cos^2(x_1)) \bar{\mu}}{(1 + c \sin^2(x_1))^2} \\
 \frac{\partial f_2}{\partial x_1} &= -cx_2^2 \cdot \left[ \frac{\cos^2(x_1) + \cos^4(x_1) - \sin^2(x_1) - \sin^3(x_1) \cos^3(x_1) - 2c \sin^3(x_1) \cos^3(x_1)}{(1 + c \sin^2(x_1))^2} \right] + \\
 &\quad + \frac{\cos(x_1) - c \sin^2(x_1) \cos(x_1)}{(1 + c \sin^2(x_1))^2} + \frac{(\sin(x_1) + c \sin^3(x_1) + 2c \sin(x_1)(1 - \sin^2(x_1))) \bar{\mu}}{(1 + c \sin^2(x_1))^2} \\
 \frac{\partial f_2}{\partial x_1} &= -cx_2^2 \cdot \left[ \frac{\cos^3(x_1) + \cos^4(x_1) - \sin^2(x_1) - \sin^3(x_1) \cos^3(x_1)(1+2c)}{(1 + c \sin^2(x_1))^2} \right] \\
 &\quad + \frac{(\cos(x_1) - c \sin^2(x_1) \cos(x_1))}{(1 + c \sin^2(x_1))^2} + \frac{(\sin(x_1) + c \sin^3(x_1) + 2c \sin(x_1)(1 - \sin^2(x_1))) \bar{\mu}}{(1 + c \sin^2(x_1))^2}
 \end{aligned}$$

Figure 5: "Handwritten Solution for Problem 2 - Page 3"

$$\begin{aligned}
 \circ \frac{\partial f_2}{\partial x_2} &= -\frac{2x_2 c \sin(x_1) \cos(x_1)}{1 + c \sin^2(x_1)} \quad \left| \quad \frac{\partial f_4}{\partial x_2} = \frac{2x_2 d \sin(x_1)}{1 + c \sin^2(x_1)} \right. \\
 \circ \frac{\partial f_4}{\partial x_2} &= dx_2^2 \left( \cos(x_1)(1 + c \sin^2(x_1)) - \sin(x_1) 2c \sin(x_1) \cos(x_1) \right) - \\
 &\quad - \left[ \frac{(\cos^3(x_1) - \sin^2(x_1))(1 + c \sin^2(x_1)) - \sin(x_1)(\cos(x_1)) 2c \sin(x_1) \cos(x_1)}{(1 + c \sin^2(x_1))^2} \right] \\
 &\quad - \frac{6\bar{\mu} 2c \sin(x_1) \cos(x_1)}{(1 + c \sin^2(x_1))^2} \\
 A = \frac{\partial f}{\partial x} \Big|_{x=0} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{\partial f_2}{\partial x_1} \Big|_{x=0} & \frac{\partial f_2}{\partial x_2} \Big|_{x=0} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{\partial f_4}{\partial x_1} \Big|_{x=0} & \frac{\partial f_4}{\partial x_2} \Big|_{x=0} & 0 & 0 \end{bmatrix} \\
 \circ \frac{\partial f_2}{\partial x_1} \Big|_{x=0} &= -cx_2^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + 1 + 0 = -cx_2^2 + 1 = 1 \\
 \circ \frac{\partial f_2}{\partial x_2} \Big|_{x=0} &= 0, \quad \circ \frac{\partial f_4}{\partial x_1} \Big|_{x=0} = 0 - \frac{1}{1} = -1, \quad \circ \frac{\partial f_4}{\partial x_2} \Big|_{x=0} = 0 \\
 A &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 6 \end{bmatrix}
 \end{aligned}$$

Figure 6: "Handwritten Solution for Problem 2 - Page 4"

### 3 Problem 3

Problem 3

• Verify that  $\dot{x} = Ax + B\bar{\mu}$  is completely controllable.

$$\mathcal{C} = [B \ AB \ A^2B \ A^3B]$$

$$B = \begin{bmatrix} 0 \\ -1 \\ 0 \\ b \end{bmatrix}, \quad AB = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 0 \\ b \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ b \\ 0 \end{bmatrix}, \quad A \cdot AB = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ b \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$$A \cdot A^2B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 0 \\ b \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ b \\ 0 \end{bmatrix}$$

$$\mathcal{C} = \begin{bmatrix} 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 \\ 0 & b & 0 & 1 \\ b & 0 & 1 & 0 \end{bmatrix} \xrightarrow{S=3} \begin{bmatrix} 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 \\ 0 & 3 & 0 & 1 \\ 3 & 0 & 1 & 0 \end{bmatrix}$$

It can be easily observed that  $\mathcal{C}$  has four linearly independent columns and hence  $\text{rank } \mathcal{C} = 4 = n$ .

Therefore, the linearized system is completely controllable.

Now, we consider the state variable feedback control law

$$\bar{\mu} = -kx + v \Rightarrow \dot{x} = Ax + B(-kx + v)$$

$$\dot{x} = (A - Bk)x + BV$$

Using the "place" command in MATLAB, we calculated that

$$K = [-19.3 \quad -22.975 \quad -1.59 \quad -5.525]$$

will place the closed-loop eigenvalues at  $(-3, -2, -0.7 \pm j0.2)$ .

Figure 7: "Handwritten Solution for Problem 3 - Page 1"

### 3.1 Matlab Code for Problem 3

```
1 %%%%%%
2 %Problem 3
3 M = 25;                                %Kg
4 m = 20;                                 %Kg
5 L = 9.81;                               %m
6 g = 9.81;                               %m/s^2
7 J = m*(L^2)/3;                          %dimensionless variable
8
9 b = ((M/m)+1)*(1+J/(m*(L^2)));        %dimensionless variable
10 c = (m^2)*(L^2)/(J*(m+M)+m*M*(L^2)); %dimensionless variable
11 d = c+1;                               %dimensionless variable
12
13 %The linearization matrices A and B were computed by hand in the report
14 A=[0 1 0 0; 1 0 0 0; 0 0 0 1; -1 0 0 0];
15 B=[0; -1; 0; b];
16
17 %The desired eigenvalues for the state variable feedback control law
18 p1=-3;
19 p2=-2;
20 p3=-0.7-1i*0.2;
21 p4=-0.7+1i*0.2;
22 p=[p1 p2 p3 p4];
23
24 %Controllability Matrix
25 contr=[B A*B A*A*B A*A*A*B];
26 %Verification of the Controllability Test
27 rank(contr);
28
29 %Calculation of the gain matrix for the state feedback controller
30 K=place(A,B,p);
31 %%%% %%
```

#### 4 Problem 4

##### Problem 4

Here, we apply the control law  $\bar{\mu} = -kx + v$ , where  $v=0$

$$\dot{x} = (A - Bk)x \quad (5)$$

(linearized system)

$$\ddot{x} = f(x, -kx) \quad (6)$$

(nonlinear system)

First, we will compute  $f(x, -kx)$

where  $x = [x_1 \ x_2 \ x_3 \ x_4]^T$  and  $\bar{\mu} = -kx$

$$\bar{\mu} = [k_1 \ k_2 \ k_3 \ k_4] [x_1 \ x_2 \ x_3 \ x_4]^T$$

$$\bar{\mu} = -(k_1 x_1 + k_2 x_2 + k_3 x_3 + k_4 x_4)$$

Hence,  $\dot{x}_1 = x_2$

$$\dot{x}_2 = -\frac{cx_2^2 \sin(x_1) \cos(x_1)}{1+c \sin^2(x_1)} + \frac{\sin(x_1)}{1+c \sin^2(x_1)} - \frac{\cos(x_1)(-k_1 x_1 - k_2 x_2 - k_3 x_3 - k_4 x_4)}{1+c \sin^2(x_1)}$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = \frac{dx_2^2 \sin(x_1)}{1+c \sin^2(x_1)} - \frac{\sin(x_1) \cos(x_1)}{1+c \sin^2(x_1)} + \frac{6(-k_1 x_1 - k_2 x_2 - k_3 x_3 - k_4 x_4)}{1+c \sin^2(x_1)}$$

Now,

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x}(-kx) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \frac{\partial f_2}{\partial x_4} \\ 0 & 0 & 0 & 1 \\ \frac{\partial f_4}{\partial x_1} & \frac{\partial f_4}{\partial x_2} & \frac{\partial f_4}{\partial x_3} & \frac{\partial f_4}{\partial x_4} \end{bmatrix}$$

Using the results we had previously, we have:

$$\begin{aligned} \frac{\partial f_2}{\partial x_1} &= -cx_2^2 \left[ \frac{\cos^2(x_1) + \cos^4(x_1) - \sin^2(x_1) - \sin^2(x_1)\cos^2(x_1)(1+2c)}{(1+c \sin^2(x_1))^2} \right] + \\ &+ \frac{(\cos(x_1) - c \sin^2(x_1) \cos(x_1))}{(1+c \sin^2(x_1))^2} + \frac{(\sin(x_1) + c \sin^3(x_1) + 2c \sin(x_1)(1-\sin^2(x_1)))(-kx)}{(1+c \sin^2(x_1))^2} \\ &- \frac{\cos(x_1) (-k_1)}{1+c \sin^2(x_1)}, \text{ where } k_1 = \frac{\partial f_2}{\partial x_1} \end{aligned}$$

Figure 8: "Handwritten Solution for Problem 4 - Page 1"

$$\frac{\partial f_2}{\partial x_2} = \frac{-2x_2 c \sin(x_1) \cos(x_1)}{1 + c \sin^2(x_1)} - \frac{\cos(x_1)}{(1 + c \sin^2(x_1))} (-k_2)$$

$$\frac{\partial f_2}{\partial x_3} = -\frac{\cos(x_1)}{(1 + c \sin^2(x_1))} (-k_3) \quad \frac{\partial f_2}{\partial x_4} = -\frac{\cos(x_1)}{(1 + c \sin^2(x_1))} (-k_4)$$

$$\begin{aligned} \frac{\partial f_4}{\partial x_1} &= \frac{d x_2^2 (\cos(x_1)(1+c \sin^2(x_1)) - \sin(x_1) 2c \sin(x_1) \cos(x_1))}{(1+c \sin^2(x_1))^2} - \\ &- \left[ \frac{(\cos^2(x_1) - \sin^2(x_1))(1+c \sin^2(x_1)) - \sin^2(x_1) \cos^2(x_1) 2c}{(1+c \sin^2(x_1))^2} \right] \\ &- \frac{62c \sin(x_1) \cos(x_1)}{(1+c \sin^2(x_1))^2} (-k_1 x_1 - k_2 x_2 - k_3 x_3 - k_4 x_4) + \frac{6}{(1+c \sin^2(x_1))} (-k_1) \end{aligned}$$

$$\frac{\partial f_4}{\partial x_2} = \frac{2x_2 d \sin(x_1)}{1 + c \sin^2(x_1)} + \frac{6}{(1 + c \sin^2(x_1))} (-k_2), \quad \frac{\partial f_4}{\partial x_3} = \frac{6}{(1 + c \sin^2(x_1))} (-k_3), \quad \frac{\partial f_4}{\partial x_4} = \frac{6}{(1 + c \sin^2(x_1))} (-k_4)$$

Hence  $\frac{\partial (f(x, -kx))}{\partial x} \Big|_{x=0} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1+k_1 & k_2 & k_3 & k_4 \\ 0 & 0 & 0 & 1 \\ -1-bk_1 & -bk_2 & -bk_3 & -bk_4 \end{bmatrix}$

$$A - Bk = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \\ 0 \\ b \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 4k_1 & k_2 & k_3 & k_4 \\ 0 & 0 & 0 & 1 \\ -1-bk_1 & -bk_2 & -bk_3 & -bk_4 \end{bmatrix}$$

Therefore, we verify that  $A - Bk = \frac{\partial (f(x, -kx))}{\partial x} \Big|_{x=0}$

Figure 9: "Handwritten Solution for Problem 4 - Page 2"

As a result, we have shown that  $\ddot{x} = (A - Bk)x$  is the linearization of  $\dot{x} = f(x, -kx)$  and hence using Theorem 4.7 of Khalil, we can study the stability properties of the origin for the nonlinear closed-loop systems, by finding the eigenvalues of  $(A - Bk)$ .

However, we already know the eigenvalues, since the applied control law was designed so that the closed-loop eigenvalues will be placed at  $(-3, -2, -0.7 \pm j0.2)$ .

Hence, we can observe that  $\text{Re}(\lambda_i) < 0 \forall \lambda_i$  of  $(A - Bk)$  meaning that the origin is asymptotically stable.

Figure 10: "Handwritten Solution for Problem 4 - Page 3"

## 5 Problem 5

### Problem 5

In this problem, we will simulate both closed-loop systems for several initial conditions.

#### Cases of initial conditions

- 1) The cart at rest but not at the origin, and the bar leaning to one side, but also at rest.

$$x_1 = \phi \neq 0, x_2 = \dot{\phi} = 0, x_3 = \bar{s} \neq 0, x_4 = \dot{\bar{s}} = 0$$

In fact, we picked  $x_1(0) = \frac{\pi}{11}$  and  $x_3(0) = 0.25$

- 2) The cart moving to one side with some velocity, and the bar initially at rest and straight up.

$$x_1 = \phi = 0, x_2 = \dot{\phi} \neq 0, x_3 = \bar{s} \neq 0, x_4 = \dot{\bar{s}} \neq 0$$

In fact, we picked  $x_3(0) = 1.25$  and  $x_4(0) = 1$

- 3) The cart moving to one side with some velocity and the bar leaning to one side with some angular velocity.

$$x_1 = \phi \neq 0, x_2 = \dot{\phi} \neq 0, x_3 = \bar{s} \neq 0, x_4 = \dot{\bar{s}} \neq 0$$

In fact, we picked  $x_1(0) = \frac{\pi}{11}, x_2(0) = \frac{\pi}{40}, x_3(0) = 0.2, x_4(0) = 0.1$

- 4) The cart moving with negative velocity and the bar leaning to the negative side, but not at rest.

$$x_1 = \phi < 0, x_2 = \dot{\phi} > 0, x_3 = \bar{s} > 0, x_4 = \dot{\bar{s}} < 0$$

In fact, we picked  $x_1(0) = -\frac{\pi}{11}, x_2(0) = \frac{\pi}{40}, x_3(0) = 0.2, x_4(0) = -0.1$

Figure 11: "Handwritten Solution for Problem 5 - Page 1"

**Figure (1) - Problem 5 - Simulation of Linear and Nonlinear Systems for Various Initial Conditions**

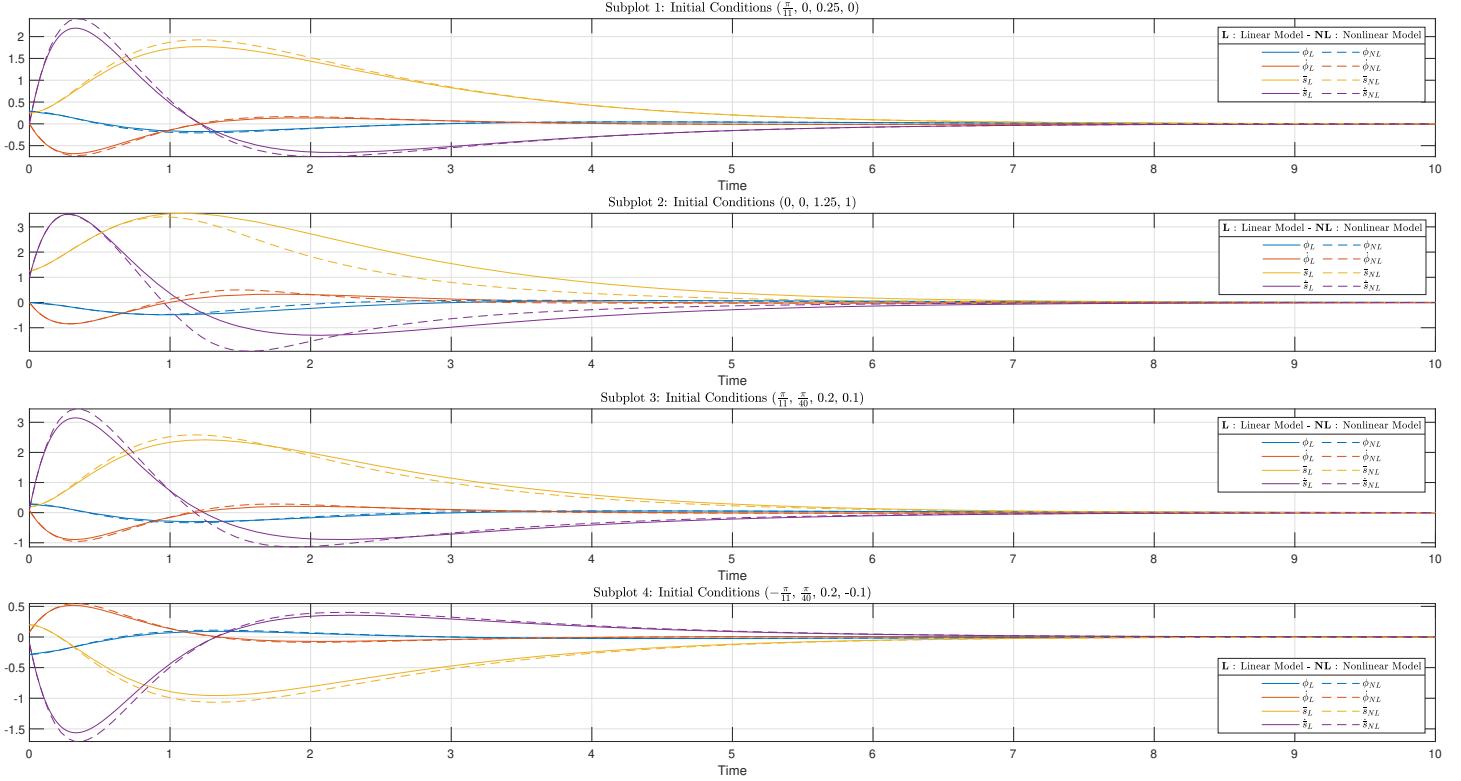


Figure 12: "Figure 1 of Problem 5"

**Figure (2) - Problem 5 - Explore Stability of Nonlinear Model**

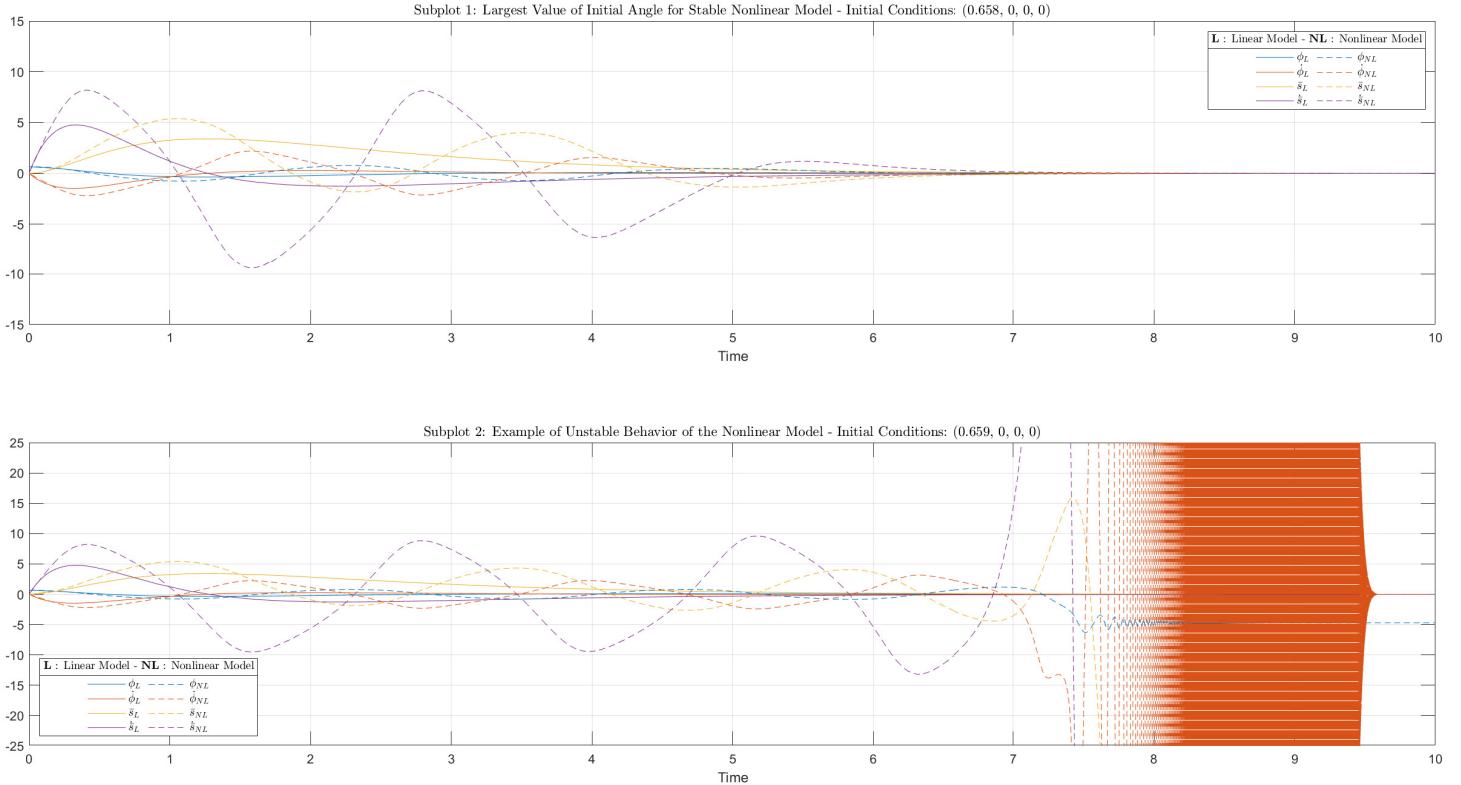


Figure 13: "Figure 2 of Problem 5"

▷ Comparison of the response of the linear model to that of the nonlinear model.

In general, we can observe that both responses are similar for all four sets of initial conditions. Moreover, the responses of the nonlinear model present larger overshoot in most cases (Subplots 1, 3, 4), while parameters such as rise time and settling time appear to be similar.

▷ Comments about the plots

Overall, the responses are similar for all 4 I.C.s, where we observe behaviors that seem logical, when we are imagining how we would balance a broom. The fourth case (Subplot 4) was selected to examine the behavior of the system for negative I.C.s and we observe a mirroring behavior of that in Subplot 3. Furthermore, we should note that besides the figures in the report, we experimented with different I.C.s that required the use of a numerical solver ("ode45"). In many cases, the code would run forever and hence only I.C.s that resulted in finite execution times and stable behaviors were selected.

▷ With  $\pi_2(0)=0, \pi_3(0)=0$  and  $\pi_4(0)=0$ , the largest value for the initial angle  $\pi_1(0)$  before the nonlinear model goes unstable is. 0.658 rad. This was illustrated in Figure 2, where Subplot 1 depicts the stable behavior for  $\pi_1(0)=0.058$  rad, and Subplot 2 depicts the unstable behavior for  $\pi_1(0)=0.659$  rad.

Figure 14: "Handwritten Solution for Problem 5 - Page 2"

## 5.1 Matlab Code for Problem 5

```

1 %%%%%%
2
3 %Problem 5
4 syms x(t) [4 1] real
5
6 %Control Law
7 mu = -real(K)*x;
8
9 %Linear Model Closed-Loop System
10 x_dot_linear = (A-B*K)*x;
11
12 %Nonlinear Model Closed-Loop System
13 x1_dot_nl = x2;
14 x2_dot_nl = (-c*(x^2)*sin(x1)*cos(x1)/(1+c*(sin(x1))^2)) + (sin(x1)/(1+c*(sin(x1))^2)) - ...
    (cos(x1)*mu/(1+c*(sin(x1))^2));
15 x3_dot_nl = x4;
16 x4_dot_nl = (d*(x^2)*sin(x1)/(1+c*(sin(x1))^2)) - (sin(x1)*cos(x1)/(1+c*(sin(x1))^2)) + ...
    (b*mu/(1+c*(sin(x1))^2));
17 x_dot_nonlinear=[x1_dot_nl; x2_dot_nl; x3_dot_nl; x4_dot_nl];
18
19 %Differential Equations for Linear Model
20 odes = diff(x) == x_dot_linear;
21 %Differential Equations for Nonlinear Model
22 odes_nl= diff(x) == x_dot_nonlinear;
23
24 figure(1)
25 sgtitle('$\text{Figure (1) - Problem 5 - Simulation of Linear and Nonlinear Systems for Various Initial ...}$', 'Interpreter', 'latex')
26 subplot(4,1,1);
27 %The cart at rest but not at the origin and the bar leaning to one side,
28 %but also at rest
29 initconds=[pi/11; 0; 0.25; 0];
30 cond1 = x1(0) == initconds(1);
31 cond2 = x2(0) == initconds(2);
32 cond3 = x3(0) == initconds(3);
33 cond4 = x4(0) == initconds(4);
34 cond = [cond1; cond2; cond3; cond4];
35
36 %Solution of the ODE for the Linear Model
37 [x1Sol, x2Sol, x3Sol, x4Sol] = dsolve(odes,cond);
38
39 %Solution of the ODE for the Nonlinear Model
40 M=odeFunction(x_dot_nonlinear,x);
41 [t,sol] = ode45(M,[0 10],initconds);
42
43 %Plotting of the Responses for both Close-Loop Systems
44 fplot(x1Sol,[0 10], 'Color',[0 0.4470 0.7410])
45 hold on
46 fplot(x2Sol,[0 10], 'Color',[0.8500 0.3250 0.0980])
47 hold on
48 fplot(x3Sol,[0 10], 'Color',[0.9290 0.6940 0.1250])
49 hold on
50 fplot(x4Sol,[0 10], 'Color',[0.4940 0.1840 0.5560])
51 hold on;
52 plot(t,sol(:,1), 'm--', 'Color',[0 0.4470 0.7410])
53 hold on;
54 plot(t,sol(:,2), 'y--', 'Color',[0.8500 0.3250 0.0980])
55 hold on;
56 plot(t,sol(:,3), 'r--', 'Color',[0.9290 0.6940 0.1250])
57 hold on;
58 plot(t,sol(:,4), 'g--', 'Color',[0.4940 0.1840 0.5560])
59 grid on
60 title('Subplot 1: Initial Conditions ($\frac{\pi}{11}, 0, 0.25, 0)', 'Interpreter', 'latex')
61 xlabel('Time')
62 lgd=legend('$\phi_L$', '$\dot{\phi}_L$', '$\bar{s}_L$', '$\dot{\bar{s}}_L$', '$\phi_{NL}$', '$\dot{\phi}_{NL}$', '$\bar{s}_{NL}$', '$\dot{\bar{s}}_{NL}$')
63 title(lgd, '$\text{L} : \text{Linear Model} - \text{NL} : \text{Nonlinear Model}', 'Interpreter', 'latex')
64
65 subplot(4,1,2);
66 %The cart moving to one side with some velocity and the bar initially at
67 %rest and straight up
68 initconds=[0; 0; 1.25; 1];
69 cond1 = x1(0) == initconds(1);

```

```

70 cond2 = x2(0) == initconds(2);
71 cond3 = x3(0) == initconds(3);
72 cond4 = x4(0) == initconds(4);
73 conds = [cond1; cond2; cond3; cond4];
74
75 %Solution of the ODE for the Linear Model
76 [x1Sol, x2Sol, x3Sol, x4Sol] = dsolve(odes,conds);
77
78 %Solution of the ODE for the Nonlinear Model
79 M=odefunction(x.dot_nonlinear,x);
80 [t,sol] = ode45(M,[0 10],initconds);
81
82 %Plotting of the Responses for both Close-Loop Systems
83 fplot(x1Sol,[0 10],'Color',[0 0.4470 0.7410])
84 hold on
85 fplot(x2Sol,[0 10],'Color',[0.8500 0.3250 0.0980])
86 hold on
87 fplot(x3Sol,[0 10],'Color',[0.9290 0.6940 0.1250])
88 hold on
89 fplot(x4Sol,[0 10],'Color',[0.4940 0.1840 0.5560])
90 hold on;
91 plot(t,sol(:,1),'m—','Color',[0 0.4470 0.7410])
92 hold on;
93 plot(t,sol(:,2),'y—','Color',[0.8500 0.3250 0.0980])
94 hold on;
95 plot(t,sol(:,3),'—','Color',[0.9290 0.6940 0.1250])
96 hold on;
97 plot(t,sol(:,4),'—','Color',[0.4940 0.1840 0.5560])
98 grid on
99 title('Subplot 2: Initial Conditions (0, 0, 1.25, 1)', 'Interpreter', 'latex')
100 xlabel('Time')
101 lgd=legend('$\phi_{L}$', '$\dot{\phi}_{L}$', '$\bar{s}_{L}$', '$\dot{\bar{s}}_{L}$', '$\phi_{NL}$', '$\dot{\phi}_{NL}$', '$\textbf{L}$ : Linear Model - $\textbf{NL}$ : Nonlinear Model', 'Interpreter', 'latex')
102
103
104 subplot(4,1,3);
105 %The cart moving to one side with some velocity and the bar leaning to one
106 %side not at rest
107 initconds=[pi/11; pi/40; 0.2; 0.1];
108 cond1 = x1(0) == initconds(1);
109 cond2 = x2(0) == initconds(2);
110 cond3 = x3(0) == initconds(3);
111 cond4 = x4(0) == initconds(4);
112 conds = [cond1; cond2; cond3; cond4];
113
114 %Solution of the ODE for the Linear Model
115 [x1Sol, x2Sol, x3Sol, x4Sol] = dsolve(odes,conds);
116
117 %Solution of the ODE for the Nonlinear Model
118 M=odefunction(x.dot_nonlinear,x);
119 [t,sol] = ode45(M,[0 10],initconds);
120
121 %Plotting of the Responses for both Close-Loop Systems
122 fplot(x1Sol,[0 10],'Color',[0 0.4470 0.7410])
123 hold on
124 fplot(x2Sol,[0 10],'Color',[0.8500 0.3250 0.0980])
125 hold on
126 fplot(x3Sol,[0 10],'Color',[0.9290 0.6940 0.1250])
127 hold on
128 fplot(x4Sol,[0 10],'Color',[0.4940 0.1840 0.5560])
129 hold on;
130 plot(t,sol(:,1),'m—','Color',[0 0.4470 0.7410])
131 hold on;
132 plot(t,sol(:,2),'y—','Color',[0.8500 0.3250 0.0980])
133 hold on;
134 plot(t,sol(:,3),'—','Color',[0.9290 0.6940 0.1250])
135 hold on;
136 plot(t,sol(:,4),'—','Color',[0.4940 0.1840 0.5560])
137 grid on
138 title('Subplot 3: Initial Conditions ($\frac{\pi}{11}, \frac{\pi}{40}, 0.2, 0.1)$', 'Interpreter', 'latex')
139 xlabel('Time')
140 lgd=legend('$\phi_{L}$', '$\dot{\phi}_{L}$', '$\bar{s}_{L}$', '$\dot{\bar{s}}_{L}$', '$\phi_{NL}$', '$\dot{\phi}_{NL}$', '$\textbf{L}$ : Linear Model - $\textbf{NL}$ : Nonlinear Model', 'Interpreter', 'latex')
141
142
143 subplot(4,1,4);
144 %The cart moving to one side with some negative velocity and the bar leaning to the ...

```

```

    negative
145 %side not at rest
146 initconds=[-pi/11; pi/40; 0.2; -0.1];
147 cond1 = x1(0) == initconds(1);
148 cond2 = x2(0) == initconds(2);
149 cond3 = x3(0) == initconds(3);
150 cond4 = x4(0) == initconds(4);
151 conds = [cond1; cond2; cond3; cond4];
152
153 %Solution of the ODE for the Linear Model
154 [x1Sol, x2Sol, x3Sol, x4Sol] = dsolve(odes,conds);
155
156 %Solution of the ODE for the Nonlinear Model
157 M=odeFunction(x_dot_nonlinear,x);
158 [t,sol] = ode45(M,[0 10],initconds);
159
160 %Plotting of the Responses for both Close-Loop Systems
161 fplot(x1Sol,[0 10], 'Color',[0 0.4470 0.7410])
162 hold on
163 fplot(x2Sol,[0 10], 'Color',[0.8500 0.3250 0.0980])
164 hold on
165 fplot(x3Sol,[0 10], 'Color',[0.9290 0.6940 0.1250])
166 hold on
167 fplot(x4Sol,[0 10], 'Color',[0.4940 0.1840 0.5560])
168 hold on;
169 plot(t,sol(:,1),'m—','Color',[0 0.4470 0.7410])
170 hold on;
171 plot(t,sol(:,2),'y—','Color',[0.8500 0.3250 0.0980])
172 hold on;
173 plot(t,sol(:,3),'—','Color',[0.9290 0.6940 0.1250])
174 hold on;
175 plot(t,sol(:,4),'—','Color',[0.4940 0.1840 0.5560])
176 grid on
177 title('Subplot 4: Initial Conditions ($-\frac{\pi}{11}$, $\frac{\pi}{40}$, 0.2, -0.1)', 'Interpreter', 'latex')
178 xlabel('Time')
179 lgd=legend('$\phi_{L}$', '$\dot{\phi}_{L}$', '$\bar{s}_{L}$', '$\dot{\bar{s}}_{L}$', '$\phi_{NL}$', '$\dot{\phi}_{NL}$', '$\textbf{L}$ : Linear Model — $\textbf{NL}$ : Nonlinear Model', 'Interpreter', 'latex')
180 title(lgd, '$\textbf{L}$ : Linear Model — $\textbf{NL}$ : Nonlinear Model', 'Interpreter', 'latex')
181 %%%%%%
182 %%%%%%
183 %%%%%%
184 %Problem 5 – Connection between Stability and the initial angle x1(0)
185
186 figure(2)
187 sgtitle('Figure (2) – Problem 5 – Explore Stability of Nonlinear Model', 'Interpreter', 'latex')
188 subplot(2,1,1);
189 initconds=[0.658; 0; 0; 0];
190 cond1 = x1(0) == initconds(1);
191 cond2 = x2(0) == initconds(2);
192 cond3 = x3(0) == initconds(3);
193 cond4 = x4(0) == initconds(4);
194 conds = [cond1; cond2; cond3; cond4];
195
196 %Solution of the ODE for the Linear Model
197 [x1Sol, x2Sol, x3Sol, x4Sol] = dsolve(odes,conds);
198
199 %Solution of the ODE for the Nonlinear Model
200 M=odeFunction(x_dot_nonlinear,x);
201 [t,sol] = ode45(M,[0 10],initconds);
202
203 %Plotting of the Responses for both Close-Loop Systems
204 fplot(x1Sol,[0 10], 'Color',[0 0.4470 0.7410])
205 hold on
206 fplot(x2Sol,[0 10], 'Color',[0.8500 0.3250 0.0980])
207 hold on
208 fplot(x3Sol,[0 10], 'Color',[0.9290 0.6940 0.1250])
209 hold on
210 fplot(x4Sol,[0 10], 'Color',[0.4940 0.1840 0.5560])
211 hold on;
212 plot(t,sol(:,1),'m—','Color',[0 0.4470 0.7410])
213 hold on;
214 plot(t,sol(:,2),'y—','Color',[0.8500 0.3250 0.0980])
215 hold on;
216 plot(t,sol(:,3),'—','Color',[0.9290 0.6940 0.1250])
217 hold on;
218 plot(t,sol(:,4),'—','Color',[0.4940 0.1840 0.5560])
219 grid on

```

```

220 title('Subplot 1: Largest Value of Initial Angle for Stable Nonlinear Model - Initial Conditions: (0.658, ...  

221     0, 0, 0)', 'Interpreter', 'latex')  

222 xlabel('Time')  

223 lgd=legend('$\phi_{L}$', '$\dot{\phi}_{L}$', '$\bar{s}_{L}$', '$\dot{\bar{s}}_{L}$', '$\phi_{NL}$', '$\dot{\phi}_{NL}$', '$\textbf{L}$ : Linear Model - $\textbf{NL}$ : Nonlinear Model', 'Interpreter', 'latex')  

224 title(lgd, '$\textbf{L}$ : Linear Model - $\textbf{NL}$ : Nonlinear Model', 'Interpreter', 'latex')  

225 ylim([-15 15]);  

226 subplot(2,1,2);  

227 initconds=[0.659; 0; 0; 0];  

228 cond1 = x1(0) == initconds(1);  

229 cond2 = x2(0) == initconds(2);  

230 cond3 = x3(0) == initconds(3);  

231 cond4 = x4(0) == initconds(4);  

232 conds = [cond1; cond2; cond3; cond4];  

233  

234 %Solution of the ODE for the Linear Model  

235 [x1Sol, x2Sol, x3Sol, x4Sol] = dsolve(odes,conds);  

236  

237 %Solution of the ODE for the Nonlinear Model  

238 M=odeFunction(x_dot_nonlinear,x);  

239 [t,sol] = ode45(M,[0 10],initconds);  

240  

241 %Plotting of the Responses for both Close-Loop Systems  

242 fplot(x1Sol,[0 10], 'Color',[0 0.4470 0.7410])  

243 hold on  

244 fplot(x2Sol,[0 10], 'Color',[0.8500 0.3250 0.0980])  

245 hold on  

246 fplot(x3Sol,[0 10], 'Color',[0.9290 0.6940 0.1250])  

247 hold on  

248 fplot(x4Sol,[0 10], 'Color',[0.4940 0.1840 0.5560])  

249 hold on;  

250 plot(t,sol(:,1), 'm—', 'Color',[0 0.4470 0.7410])  

251 hold on;  

252 plot(t,sol(:,2), 'y—', 'Color',[0.8500 0.3250 0.0980])  

253 hold on;  

254 plot(t,sol(:,3), '—', 'Color',[0.9290 0.6940 0.1250])  

255 hold on;  

256 plot(t,sol(:,4), '—', 'Color',[0.4940 0.1840 0.5560])  

257 grid on  

258 title('Subplot 2: Example of Unstable Behavior of the Nonlinear Model - Initial Conditions: (0.659, 0, 0, ...  

259     0)', 'Interpreter', 'latex')  

260 xlabel('Time')  

261 lgd=legend('$\phi_{L}$', '$\dot{\phi}_{L}$', '$\bar{s}_{L}$', '$\dot{\bar{s}}_{L}$', '$\phi_{NL}$', '$\dot{\phi}_{NL}$', '$\textbf{L}$ : Linear Model - $\textbf{NL}$ : Nonlinear Model', 'Interpreter', 'latex')  

262 title(lgd, '$\textbf{L}$ : Linear Model - $\textbf{NL}$ : Nonlinear Model', 'Interpreter', 'latex')  

263 ylim([-25 25]);  

264 %%%%%% —————— %%%%%%  

265 %%%%%% —————— %%%%%%

```

## 6 Problem 6

Problem 6

$$\left. \begin{array}{l} y_1 = \phi \\ y_2 = \bar{s} \end{array} \right\} \quad \dot{y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

where again  $x_1 = \phi, x_2 = \dot{\phi}, x_3 = \bar{s}, x_4 = \dot{\bar{s}}$

Hence,  $C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \Rightarrow \boxed{y = Cx}$

In order to check observability, we will compute the rank of the matrix  $\mathcal{O} = [C \quad CA \quad CA^2]^T$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad CA = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad CA^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

Hence,

$$\mathcal{O} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

It is easily observed that the columns of  $\mathcal{O}$  are linearly independent and hence  $\text{rank}(\mathcal{O}) = 4$

Figure 15: "Handwritten Solution for Problem 6 - Page 1"

Since  $\text{rank}(\mathcal{O}) = 4$ , we can conclude that the system is completely observable.

Now, we will design the following observer:

$$\begin{aligned} \hat{x} &= Ax + B\bar{y} + L(y - \hat{y}) \Rightarrow \dot{\hat{x}} = (\dot{x} - \hat{x}) = (A\hat{x} + B\bar{y} + L(y - \hat{y}) - \dot{x}) \\ \hat{y} &= C\hat{x} \\ (\text{Reminder: } y = Cx) \quad &= A\hat{x} + B\bar{y} + Ly - LC\hat{x} - Ax - B\bar{y} \\ &= (A - LC)(x - \hat{x}) = (A - LC)e \end{aligned}$$

In order to place the eigenvalues of the observer matrix  $A - LC$ , we can examine the transpose of that matrix  $(A - LC)^T = A^T - C^T L^T$ , since both a matrix and its transpose have the same eigenvalues.

Hence, we can use the "place" command from MATLAB in order to calculate the gain matrix  $L^T$ . Since, we want the desired eigenvalues to be three times faster than those of the closed-loop system  $A - Bk$ , we will have to place them at:

$$3 \cdot (-3, -2, -0.7 \pm j0.2) = (-9, -6, -2.1 \pm j0.6)$$

Using MATLAB, we found  $L = \begin{bmatrix} 9.2929 & 8.9077 & 3.9579 & 22.2846 \\ 0.3630 & -3.1719 & 9.9071 & 23.2335 \end{bmatrix}^T$

Figure 16: "Handwritten Solution for Problem 6 - Page 2"

### 6.1 Matlab Code for Problem 6

```

1 %We derive the C matrix from the analysis in the report
2 C=[1 0 0 0; 0 0 1 0];
3 %Observability Matrix
4 O=[C; C*A; C*A*A];
5 rank(O);
6
7 %The desired eigenvalues for the observer matrix A-LC
8 % p1=-3;
9 % p2=-2;
10 % p3=-0.7-1i*0.2;
11 % p4=-0.7+1i*0.2;
12 pobs=3*[p1 p2 p3 p4];
13
14 %Calculation of the gain matrix for the state feedback controller
15 Ltran=place(A',C',pobs);
16 L=Ltran';
17 %%%
18 %%%
19 %%%
20 %%%

```

## 7 Problem 7

### Problem 7

We consider the dynamic controller by augmenting the open-loop system and the observer (8), while applying the following control law:

$$\begin{array}{l} \bar{\mu} = -k\hat{x} \quad (9) \\ \text{From } \dot{x} = Ax + B\bar{\mu} \quad | \quad (9) \Rightarrow \dot{x} = Ax - Bk\hat{x} \quad (\text{input}) \\ y = Cx \quad | \quad y = Cx \quad (\text{output}) \\ \dot{\hat{x}} = A\hat{x} + B\bar{\mu} + L(y - \hat{y}) \quad | \quad (8) \Rightarrow \dot{\hat{x}} = A\hat{x} - Bk\hat{x} + Ly - LC\hat{x} \\ \hat{y} = C\hat{x} \quad | \quad \dot{\hat{x}} = (A - Bk - LC)\hat{x} + Ly \quad | \quad \text{input} \end{array}$$

Therefore, the dynamic controller for the linearized model is:

$$\left. \begin{array}{l} \dot{x} = Ax - Bk\hat{x} \\ y = Cx \\ \dot{\hat{x}} = (A - Bk - LC)\hat{x} + Ly \end{array} \right\} \quad \boxed{\begin{array}{l} \dot{x} = Ax - Bk\hat{x} \\ \dot{\hat{x}} = (A - Bk - LC)\hat{x} + LCx \end{array}}$$

Specifically for the required simulation, we integrated both states  $x$  and  $\hat{x}$  into an augmented system and solved the corresponding differential equations, using again the numerical solver "ode45". Subsequently, we plotted the outputs  $y_1(t) = x_1(t) = p$  and  $y_2(t) = x_2(t) = \bar{s}$ .

Likewise, for the nonlinear model, we will have:

$$\boxed{\begin{array}{l} \dot{x} = f(x, -k\hat{x}) \\ \dot{\hat{x}} = (A - Bk - LC)\hat{x} + LCx \end{array}}$$

For the simulations, the same initial conditions as in Problem 5 were used, in order to compare the performance of each controller.

Figure 17: "Handwritten Solution for Problem 7 - Page 1"

**Figure (3) - Problem 7 - Simulation of Dynamic Controller for Linear and Nonlinear System Models**

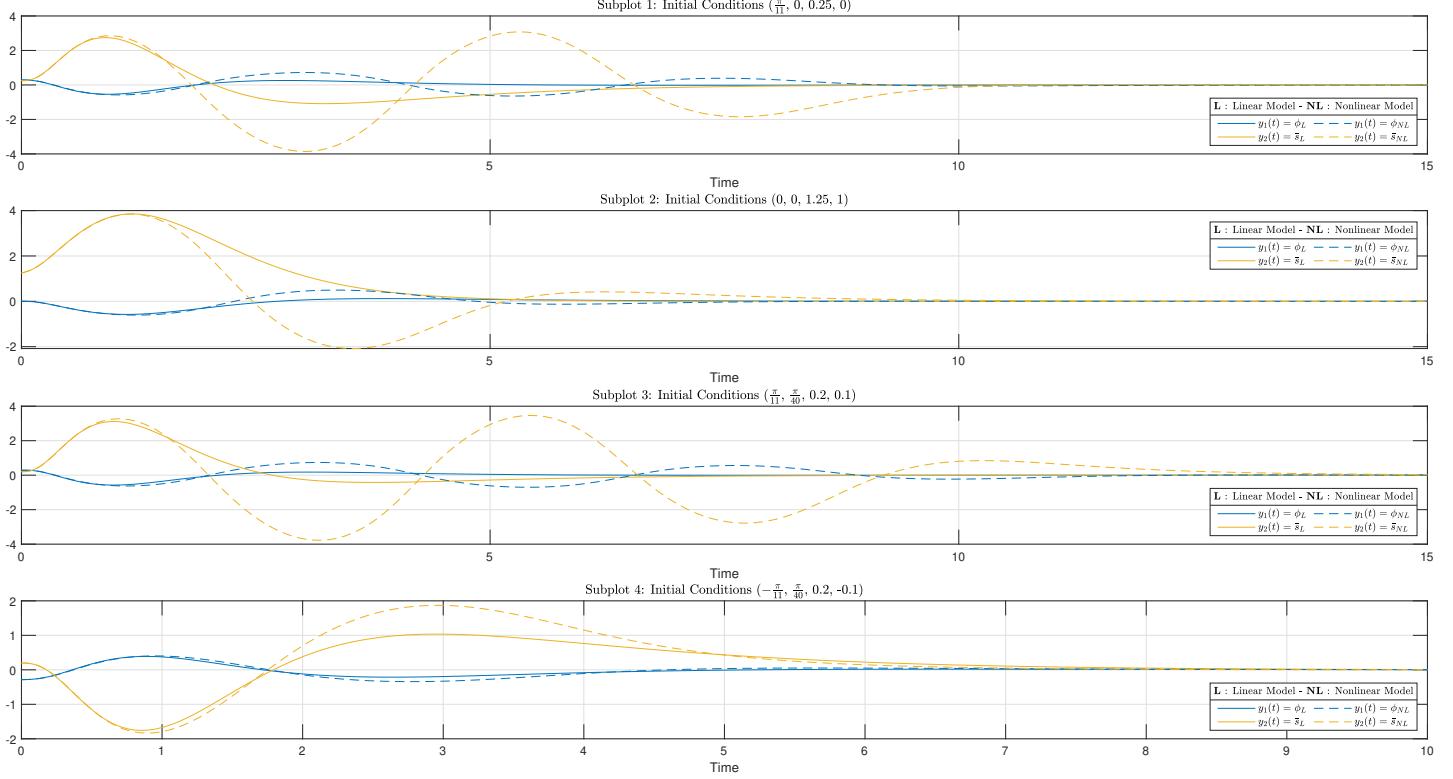


Figure 18: "Figure 3 of Problem 7"

**Figure (4) - Problem 7 - Explore Stability of Nonlinear Model for the Dynamic Controller**

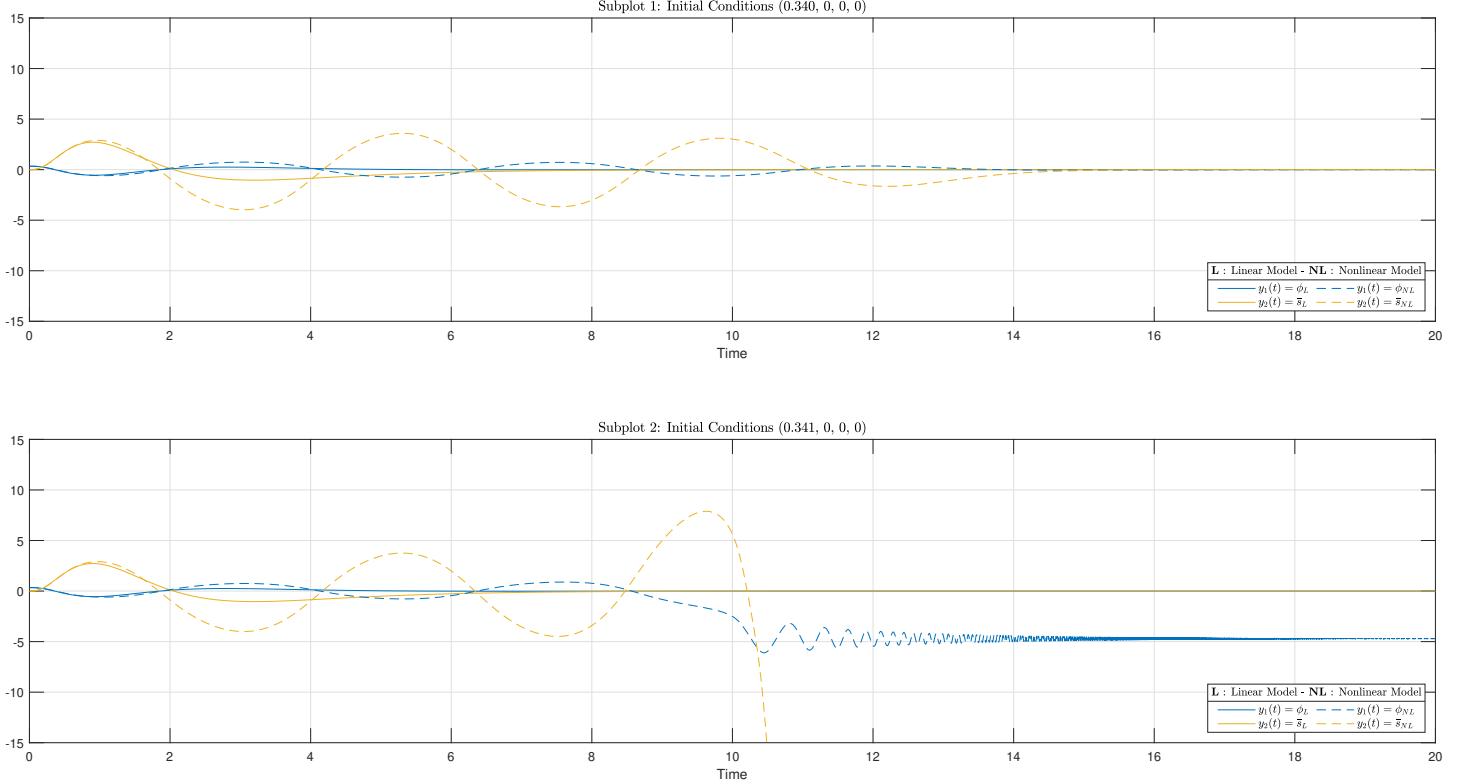


Figure 19: "Figure 4 of Problem 7"

The simulations for Problem 7 are presented in Figure 3.

► By comparing the plots with the corresponding ones in Figure 1, we can observe that for the same I.C.s, both the linear and nonlinear model dynamic controllers perform worse than the static controllers. In general, there are more oscillations, larger peaks, increased settling times, while the difference between the linear and nonlinear model's controllers is more evident.

Therefore, we can verify that being able to measure states instead of estimating is quite useful.

► With  $\pi_2(0)=0$ ,  $\pi_3(0)=0$  and  $\pi_4(0)=0$ , we explored how large we can make  $\pi_1(0)$  before the nonlinear model goes unstable. As we can observe in Figure 4, the largest "stable" value for  $\pi_1(0)$  is 0.340 rad. In Subplot 1, we can observe the stable behavior and in Subplot 2, we can observe the unstable behavior (finite escape time) for  $\pi_1(0)=0.341$  rad.

Compared to Problem 5, we can observe that this value is smaller.

Figure 20: "Handwritten Solution for Problem 7 - Page 2"

## 7.1 Matlab Code for Problem 7

```

1  syms x_full(t) [8 1] real
2  x_full_14=[x_full1;x_full2;x_full3;x_full4];
3  x_full_58=[x_full5;x_full6;x_full7;x_full8];
4
5  %Linear Model
6  x_dot_full_14 = A*x_full_14 -B*K*x_full_58;
7  x_dot_full_58 = (A-B*K-L*C)*x_full_58+L*C*x_full_14;
8  x_dot_full=[x_dot_full_14;x_dot_full_58];
9
10 % Nonlinear Model
11 %Appliead Control Law
12 mu = -real(K)*x_full_58;
13 x_dot_full_nl_1 = x_full2;
14 x_dot_full_nl_2 = (-c*(x_full2^2)*sin(x_full1)*cos(x_full1)/(1+c*(sin(x_full1))^2)) + ...
    (sin(x_full1)/(1+c*(sin(x_full1))^2)) - (cos(x_full1)*mu/(1+c*(sin(x_full1))^2));
15 x_dot_full_nl_3 = x_full4;
16 x_dot_full_nl_4 = (d*(x_full2^2)*sin(x_full1)/(1+c*(sin(x_full1))^2)) - ...
    (sin(x_full1)*cos(x_full1)/(1+c*(sin(x_full1))^2)) + (b*mu/(1+c*(sin(x_full1))^2));
17 x_dot_full_nl_58 = (A-B*K-L*C)*x_full_58+L*C*x_full_14;
18 x_dot_full_nonlinear=[x_dot_full_nl_1; x_dot_full_nl_2; x_dot_full_nl_3; x_dot_full_nl_4; x_dot_full_nl_58];
19
20 figure(3)
21 sgtitle('$\text{Figure (3) - Problem 7 - Simulation of Dynamic Controller for Linear and Nonlinear ...}$','Interpreter','latex')
22
23 subplot(4,1,1);
24 %The cart at rest but not at the origin and the bar leaning to one side,
25 %but also at rest
26 initconds=[pi/11; 0; 0.25; 0];
27 %Always take the initial conditions of the observer to be [0,0,0,0]
28 initconds=[initconds;0;0;0];
29 %Solution of Differential Equations for Linear Model
30 M=odeFunction(x_dot_full,x_full);
31 [t_linear,sol_linear] = ode45(M,[0 15],initconds);
32 %Solution of Differential Equations for Nonlinear Model
33 M=odeFunction(x_dot_full_nonlinear,x_full);
34 [t_nonlinear,sol_nonlinear] = ode45(M,[0 15],initconds);
35 %Plotting ouputs y1(t) and y2(t)
36 %y1(t)=phi=x1=x_full1(t)
37 plot(t_linear,sol_linear(:,1),'', 'Color',[0 0.4470 0.7410])
38 hold on;
39 %y2(t)=s bar=x3=x_full3(t)
40 plot(t_linear,sol_linear(:,3),'', 'Color',[0.9290 0.6940 0.1250])
41 hold on;
42 plot(t_nonlinear,sol_nonlinear(:,1),'—', 'Color',[0 0.4470 0.7410])
43 hold on;
44 %y2(t)=s bar=x3=x_full3(t)
45 plot(t_nonlinear,sol_nonlinear(:,3),'—', 'Color',[0.9290 0.6940 0.1250])
46 hold on;
47 grid on
48 title('Subplot 1: Initial Conditions ($\frac{\pi}{11}, 0, 0.25, 0)$','Interpreter','latex')
49 xlabel('Time')
50 lgd=legend('$y_1(t)=\phi$','$y_2(t)=\bar{s}$','$y_1(t)=\phi$','$y_2(t)=\bar{s}$','Location')
51 title(lgd,'$\text{L : Linear Model - NL : Nonlinear Model}$','Interpreter','latex')
52
53 subplot(4,1,2);
54 %The cart moving to one side with some velocity and the bar initially at
55 %rest and straight up
56 initconds=[0; 0; 1.25; 1];
57 %Always take the initial conditions of the observer to be [0,0,0,0]
58 initconds=[initconds;0;0;0];
59 %Solution of Differential Equations for Linear Model
60 M=odeFunction(x_dot_full,x_full);
61 [t_linear,sol_linear] = ode45(M,[0 15],initconds);
62 %Solution of Differential Equations for Nonlinear Model
63 M=odeFunction(x_dot_full_nonlinear,x_full);
64 [t_nonlinear,sol_nonlinear] = ode45(M,[0 15],initconds);
65 %Plotting ouputs y1(t) and y2(t)
66 %y1(t)=phi=x1=x_full1(t)
67 plot(t_linear,sol_linear(:,1),'', 'Color',[0 0.4470 0.7410])
68 hold on;
69 %y2(t)=s bar=x3=x_full3(t)

```

```

70 plot(t_linear,sol_linear(:,3),'', 'Color',[0.9290 0.6940 0.1250])
71 hold on;
72 plot(t_nonlinear,sol_nonlinear(:,1),'—', 'Color',[0 0.4470 0.7410])
73 hold on;
74 %y2(t)=s bar=x3=x_full3(t)
75 plot(t_nonlinear,sol_nonlinear(:,3),'—', 'Color',[0.9290 0.6940 0.1250])
76 hold on;
77 grid on
78 title('Subplot 2: Initial Conditions (0, 0, 1.25, 1)', 'Interpreter','latex')
79 xlabel('Time')
80 lgd=legend('$y_{\{1\}}(t)=\phi_{\{L\}}$','$y_{\{2\}}(t)=\bar{s}_{\{L\}}$','$y_{\{1\}}(t)=\phi_{\{NL\}}$','$y_{\{2\}}(t)=\bar{s}_{\{NL\}}$', 'Location'
81 title(lgd,'$\textbf{L} : Linear Model - \textbf{NL} : Nonlinear Model', 'Interpreter','latex')
82
83 subplot(4,1,3);
84 %The cart moving to one side with some velocity and the bar leaning to one
85 %side not at rest
86 initconds=[pi/11; pi/40; 0.2; 0.1];
87 %Always take the initial conditions of the observer to be [0,0,0,0]
88 initconds=[initconds;0;0;0];
89 %Solution of Differential Equations for Linear Model
90 M=odeFunction(x_dot_full,x_full);
91 [t_linear,sol_linear] = ode45(M,[0 15],initconds);
92 %Solution of Differential Equations for Nonlinear Model
93 M=odeFunction(x_dot_full.nonlinear,x_full);
94 [t_nonlinear,sol_nonlinear] = ode45(M,[0 15],initconds);
95 %Plotting ouputs y1(t) and y2(t)
96 %y1(t)=phi=x1=x_full1(t)
97 plot(t_linear,sol_linear(:,1),'', 'Color',[0 0.4470 0.7410])
98 hold on;
99 %y2(t)=s bar=x3=x_full3(t)
100 plot(t_linear,sol_linear(:,3),'', 'Color',[0.9290 0.6940 0.1250])
101 hold on;
102 plot(t_nonlinear,sol_nonlinear(:,1),'—', 'Color',[0 0.4470 0.7410])
103 hold on;
104 %y2(t)=s bar=x3=x_full3(t)
105 plot(t_nonlinear,sol_nonlinear(:,3),'—', 'Color',[0.9290 0.6940 0.1250])
106 hold on;
107 grid on
108 title('Subplot 3: Initial Conditions ($\frac{\pi}{11}, \frac{\pi}{40}, 0.2, 0.1)', 'Interpreter','latex')
109 xlabel('Time')
110 lgd=legend('$y_{\{1\}}(t)=\phi_{\{L\}}$','$y_{\{2\}}(t)=\bar{s}_{\{L\}}$','$y_{\{1\}}(t)=\phi_{\{NL\}}$','$y_{\{2\}}(t)=\bar{s}_{\{NL\}}$', 'Location'
111 title(lgd,'$\textbf{L} : Linear Model - \textbf{NL} : Nonlinear Model', 'Interpreter','latex')
112
113 subplot(4,1,4);
114 %The cart moving to one side with some negative velocity and the bar leaning to the ...
115
116 %negative
117 %side not at rest
118 initconds=[-pi/11; pi/40; 0.2; -0.1];
119 %Always take the initial conditions of the observer to be [0,0,0,0]
120 initconds=[initconds;0;0;0];
121 %Solution of Differential Equations for Linear Model
122 M=odeFunction(x_dot_full,x_full);
123 [t_linear,sol_linear] = ode45(M,[0 10],initconds);
124 %Solution of Differential Equations for Nonlinear Model
125 M=odeFunction(x_dot_full.nonlinear,x_full);
126 [t_nonlinear,sol_nonlinear] = ode45(M,[0 10],initconds);
127 %Plotting ouputs y1(t) and y2(t)
128 %y1(t)=phi=x1=x_full1(t)
129 plot(t_linear,sol_linear(:,1),'', 'Color',[0 0.4470 0.7410])
130 hold on;
131 %y2(t)=s bar=x3=x_full3(t)
132 plot(t_linear,sol_linear(:,3),'', 'Color',[0.9290 0.6940 0.1250])
133 hold on;
134 %y2(t)=s bar=x3=x_full3(t)
135 plot(t_nonlinear,sol_nonlinear(:,3),'—', 'Color',[0.9290 0.6940 0.1250])
136 hold on;
137 grid on
138 title('Subplot 4: Initial Conditions ($-\frac{\pi}{11}, \frac{\pi}{40}, 0.2, -0.1)', 'Interpreter','latex')
139 xlabel('Time')
140 lgd=legend('$y_{\{1\}}(t)=\phi_{\{L\}}$','$y_{\{2\}}(t)=\bar{s}_{\{L\}}$','$y_{\{1\}}(t)=\phi_{\{NL\}}$','$y_{\{2\}}(t)=\bar{s}_{\{NL\}}$', 'Location'
141 title(lgd,'$\textbf{L} : Linear Model - \textbf{NL} : Nonlinear Model', 'Interpreter','latex')
142
143 %%%%%%

```

```

144
145 %Problem 7 – Connection between Stability of Dynamic Controller and the initial angle x1(0)
146 figure(4)
147 sgttitle('Figure (4) – Problem 7 – Explore Stability of Nonlinear Model for the Dynamic ...
    Controller','Interpreter','latex')
148
149 subplot(2,1,1);
150 %I.C.s with the Largest Value of x1(0) with Stable Behavior
151 initconds=[0.340; 0; 0; 0];
152 %Always take the initial conditions of the observer to be [0,0,0,0]
153 initconds=[initconds;0;0;0;0];
154 %Solution of Differential Equations for Linear Model
155 M=odeFunction(x_dot_full,x_full);
156 [t_linear,sol_linear] = ode45(M,[0 20],initconds);
157 %Solution of Differential Equations for Nonlinear Model
158 M=odeFunction(x_dot_full_nonlinear,x_full);
159 [t_nonlinear,sol_nonlinear] = ode45(M,[0 20],initconds);
160 %Plotting ouputs y1(t) and y2(t)
161 %y1(t)=phi=x1=x_full1(t)
162 plot(t_linear,sol_linear(:,1),'', 'Color',[0 0.4470 0.7410])
163 hold on;
164 %y2(t)=s bar=x3=x_full3(t)
165 plot(t_linear,sol_linear(:,3),'', 'Color',[0.9290 0.6940 0.1250])
166 hold on;
167 plot(t_nonlinear,sol_nonlinear(:,1),'—', 'Color',[0 0.4470 0.7410])
168 hold on;
169 %y2(t)=s bar=x3=x_full3(t)
170 plot(t_nonlinear,sol_nonlinear(:,3),'—', 'Color',[0.9290 0.6940 0.1250])
171 hold on;
172 ylim([-15 15]);
173 grid on
174 title('Subplot 1: Initial Conditions (0.340, 0, 0, 0)', 'Interpreter','latex')
175 xlabel('Time')
176 lgd=legend('$y_{\{1\}}(t)=\phi_{\{L\}}$','$y_{\{2\}}(t)=\bar{s}_{\{L\}}$','$y_{\{1\}}(t)=\phi_{\{NL\}}$','$y_{\{2\}}(t)=\bar{s}_{\{NL\}}$', 'Location')
177 title(lgd,'$L$ : Linear Model – $NL$ : Nonlinear Model', 'Interpreter','latex')
178
179 subplot(2,1,2);
180 %I.C.s with the first Value of x1(0) with Unstable Behavior
181 initconds=[0.341; 0; 0; 0];
182 %Always take the initial conditions of the observer to be [0,0,0,0]
183 initconds=[initconds;0;0;0;0];
184 %Solution of Differential Equations for Linear Model
185 M=odeFunction(x_dot_full,x_full);
186 [t_linear,sol_linear] = ode45(M,[0 20],initconds);
187 %Solution of Differential Equations for Nonlinear Model
188 M=odeFunction(x_dot_full_nonlinear,x_full);
189 [t_nonlinear,sol_nonlinear] = ode45(M,[0 20],initconds);
190 %Plotting ouputs y1(t) and y2(t)
191 %y1(t)=phi=x1=x_full1(t)
192 plot(t_linear,sol_linear(:,1),'', 'Color',[0 0.4470 0.7410])
193 hold on;
194 %y2(t)=s bar=x3=x_full3(t)
195 plot(t_linear,sol_linear(:,3),'', 'Color',[0.9290 0.6940 0.1250])
196 hold on;
197 plot(t_nonlinear,sol_nonlinear(:,1),'—', 'Color',[0 0.4470 0.7410])
198 hold on;
199 %y2(t)=s bar=x3=x_full3(t)
200 plot(t_nonlinear,sol_nonlinear(:,3),'—', 'Color',[0.9290 0.6940 0.1250])
201 hold on;
202 ylim([-15 15]);
203 grid on
204 title('Subplot 2: Initial Conditions (0.341, 0, 0, 0)', 'Interpreter','latex')
205 xlabel('Time')
206 lgd=legend('$y_{\{1\}}(t)=\phi_{\{L\}}$','$y_{\{2\}}(t)=\bar{s}_{\{L\}}$','$y_{\{1\}}(t)=\phi_{\{NL\}}$','$y_{\{2\}}(t)=\bar{s}_{\{NL\}}$', 'Location')
207 title(lgd,'$L$ : Linear Model – $NL$ : Nonlinear Model', 'Interpreter','latex')
208 %%%%%%
209 %%%%%%
210 %%%%%%

```

## 8 Problem 8

### Problem 8

In the previous problem, both the linear and nonlinear shared the same observer. In this case, we will use a different observer for the nonlinear model.

$$\left. \begin{array}{l} \dot{x} = f(x - k\hat{x}) \\ \dot{\hat{x}} = f(\hat{x}, -k\hat{x}) + L(y - \hat{y}) \\ \hat{y} = C\hat{x} \\ y = Cx \end{array} \right\} \quad \boxed{\begin{array}{l} \ddot{x} = f(x, -k\hat{x}) \\ \ddot{\hat{x}} = f(\hat{x}, -k\hat{x}) + LC(x - \hat{x}) \end{array}}$$

For the linear model, the same equation as in Problem 7 apply:

$$\boxed{\begin{array}{l} \dot{x} = Ax - Bk\hat{x} \\ \dot{\hat{x}} = (A - Bk - LC)\hat{x} + LCx \end{array}}$$

In this case, like the last problem, we integrate the state  $x$  and the observer  $\hat{x}$  in the augmented state  $\vec{x}_{\text{augm}} = \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$ . Hence, we can examine the properties of the closed-loop nonlinear system, by linearizing the system about the origin.

$$\vec{x}_{\text{augm}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \frac{\partial f_1}{\partial x_4} & \frac{\partial f_1}{\partial \hat{x}_1} & \frac{\partial f_1}{\partial \hat{x}_2} & \frac{\partial f_1}{\partial \hat{x}_3} & \frac{\partial f_1}{\partial \hat{x}_4} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \frac{\partial f_2}{\partial x_4} & \frac{\partial f_2}{\partial \hat{x}_1} & \frac{\partial f_2}{\partial \hat{x}_2} & \frac{\partial f_2}{\partial \hat{x}_3} & \frac{\partial f_2}{\partial \hat{x}_4} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} & \frac{\partial f_3}{\partial x_4} & \frac{\partial f_3}{\partial \hat{x}_1} & \frac{\partial f_3}{\partial \hat{x}_2} & \frac{\partial f_3}{\partial \hat{x}_3} & \frac{\partial f_3}{\partial \hat{x}_4} \\ \frac{\partial f_4}{\partial x_1} & \frac{\partial f_4}{\partial x_2} & \frac{\partial f_4}{\partial x_3} & \frac{\partial f_4}{\partial x_4} & \frac{\partial f_4}{\partial \hat{x}_1} & \frac{\partial f_4}{\partial \hat{x}_2} & \frac{\partial f_4}{\partial \hat{x}_3} & \frac{\partial f_4}{\partial \hat{x}_4} \\ \frac{\partial f_5}{\partial x_1} & \frac{\partial f_5}{\partial x_2} & \frac{\partial f_5}{\partial x_3} & \frac{\partial f_5}{\partial x_4} & \frac{\partial f_5}{\partial \hat{x}_1} & \frac{\partial f_5}{\partial \hat{x}_2} & \frac{\partial f_5}{\partial \hat{x}_3} & \frac{\partial f_5}{\partial \hat{x}_4} \\ \frac{\partial f_6}{\partial x_1} & \frac{\partial f_6}{\partial x_2} & \frac{\partial f_6}{\partial x_3} & \frac{\partial f_6}{\partial x_4} & \frac{\partial f_6}{\partial \hat{x}_1} & \frac{\partial f_6}{\partial \hat{x}_2} & \frac{\partial f_6}{\partial \hat{x}_3} & \frac{\partial f_6}{\partial \hat{x}_4} \\ \frac{\partial f_7}{\partial x_1} & \frac{\partial f_7}{\partial x_2} & \frac{\partial f_7}{\partial x_3} & \frac{\partial f_7}{\partial x_4} & \frac{\partial f_7}{\partial \hat{x}_1} & \frac{\partial f_7}{\partial \hat{x}_2} & \frac{\partial f_7}{\partial \hat{x}_3} & \frac{\partial f_7}{\partial \hat{x}_4} \\ \frac{\partial f_8}{\partial x_1} & \frac{\partial f_8}{\partial x_2} & \frac{\partial f_8}{\partial x_3} & \frac{\partial f_8}{\partial x_4} & \frac{\partial f_8}{\partial \hat{x}_1} & \frac{\partial f_8}{\partial \hat{x}_2} & \frac{\partial f_8}{\partial \hat{x}_3} & \frac{\partial f_8}{\partial \hat{x}_4} \end{bmatrix}$$

where  $x = [x_1 \ x_2 \ x_3 \ x_4]^T$ ,  $\hat{x} = [\hat{x}_1 \ \hat{x}_2 \ \hat{x}_3 \ \hat{x}_4]^T$ ,

Figure 21: "Handwritten Solution for Problem 8 - Page 1"

$$\text{Now } \dot{\mathbf{f}}(x - k\hat{x}) = [f_1 \ f_2 \ f_3 \ f_4]^T \text{ and } (\dot{\mathbf{f}}(\hat{x} - k\hat{x}) + LC(x - \hat{x})) = [f_5 \ f_6 \ f_7 \ f_8]^T$$

$$\circ \dot{f}_1 = \dot{x}_1 = x_2$$

$$\circ \dot{f}_2 = \dot{x}_2 = -\frac{c \cdot x_2^2 \sin(x_1) \cos(x_1)}{1 + c \sin^2(x_1)} + \frac{\sin(x_1)}{1 + c \sin^2(x_1)} - \frac{\cos(x_1)}{(1 + c \sin^2(x_1))} (-k_1 \hat{x}_1 - k_2 \hat{x}_2 - k_3 \hat{x}_3 - k_4 \hat{x}_4)$$

$$\circ \dot{f}_3 = \dot{x}_3 = x_4$$

$$\circ \dot{f}_4 = \dot{x}_4 = \frac{d x_2^2 \sin(x_1)}{1 + c \sin^2(x_1)} - \frac{\sin(x_1) \cos(x_1)}{1 + c \sin^2(x_1)} + \frac{b(-k_1 \hat{x}_1 - k_2 \hat{x}_2 - k_3 \hat{x}_3 - k_4 \hat{x}_4)}{1 + c \sin^2(x_1)}$$

$$LC(x - \hat{x}) = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \\ L_{31} & L_{32} \\ L_{41} & L_{42} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 - \hat{x}_1 \\ x_2 - \hat{x}_2 \\ x_3 - \hat{x}_3 \\ x_4 - \hat{x}_4 \end{bmatrix} = \begin{bmatrix} L_{11} & 0 & L_{12} & 0 \\ L_{21} & 0 & L_{22} & 0 \\ L_{31} & 0 & L_{32} & 0 \\ L_{41} & 0 & L_{42} & 0 \end{bmatrix} \begin{bmatrix} x_1 - \hat{x}_1 \\ x_2 - \hat{x}_2 \\ x_3 - \hat{x}_3 \\ x_4 - \hat{x}_4 \end{bmatrix}$$

$$LC(x - \hat{x}) = \begin{bmatrix} L_{11}(x_1 - \hat{x}_1) + L_{12}(x_3 - \hat{x}_3) \\ L_{21}(x_1 - \hat{x}_1) + L_{22}(x_3 - \hat{x}_3) \\ L_{31}(x_1 - \hat{x}_1) + L_{32}(x_3 - \hat{x}_3) \\ L_{41}(x_1 - \hat{x}_1) + L_{42}(x_3 - \hat{x}_3) \end{bmatrix}$$

$$\text{Hence, } \circ \dot{f}_5 = \dot{\hat{x}}_2 + L_{11}(x_1 - \hat{x}_1) + L_{12}(x_3 - \hat{x}_3)$$

$$\circ \dot{f}_6 = -\frac{c \cdot \hat{x}_2^2 \sin(\hat{x}_1) \cos(\hat{x}_1)}{1 + c \sin^2(\hat{x}_1)} + \frac{\sin(\hat{x}_1)}{1 + c \sin^2(\hat{x}_1)} - \frac{\cos(\hat{x}_1)}{(1 + c \sin^2(\hat{x}_1))} (-k_1 \hat{x}_1 - k_2 \hat{x}_2 - k_3 \hat{x}_3 - k_4 \hat{x}_4) + L_{21}(x_1 - \hat{x}_1) + L_{22}(x_3 - \hat{x}_3)$$

$$\circ \dot{f}_7 = \dot{\hat{x}}_4 + L_{31}(x_1 - \hat{x}_1) + L_{32}(x_3 - \hat{x}_3)$$

$$\circ \dot{f}_8 = \frac{d \hat{x}_2^2 \sin(\hat{x}_1)}{1 + c \sin^2(\hat{x}_1)} - \frac{\sin(\hat{x}_1) \cos(\hat{x}_1)}{1 + c \sin^2(\hat{x}_1)} + \frac{b(-k_1 \hat{x}_1 - k_2 \hat{x}_2 - k_3 \hat{x}_3 - k_4 \hat{x}_4)}{(1 + c \sin^2(\hat{x}_1))} + L_{41}(x_1 - \hat{x}_1) + L_{42}(x_3 - \hat{x}_3)$$

Figure 22: "Handwritten Solution for Problem 8 - Page 2"

Now, we will compute all derivatives for the Jacobian.

$$\triangleright \frac{\partial f_1}{\partial x_1} = 0, \frac{\partial f_1}{\partial x_2} = 1, \frac{\partial f_1}{\partial x_3} = \frac{\partial f_1}{\partial x_4} = \frac{\partial f_1}{\partial \hat{x}_1} = \frac{\partial f_1}{\partial \hat{x}_2} = \frac{\partial f_1}{\partial \hat{x}_3} = \frac{\partial f_1}{\partial \hat{x}_4} = 0$$

$$\triangleright \frac{\partial f_2}{\partial x_1} = -cx_2^2 \left[ \frac{\cos^2(x_1) + \cos^4(x_1) - \sin^2(x_1) - \sin^4(x_1) \cos^2(x_1)(1+2c)}{(1+c\sin^2(x_1))^2} \right] + \frac{(\cos(x_1) - c\sin^2(x_1)\cos(x_1))}{(1+c\sin^2(x_1))^2} + \\ + \frac{(\sin(x_1) + c\sin^2(x_1) + 2c\sin(x_1)(1-\sin^2(x_1)))(-k_1\hat{x}_1 - k_2\hat{x}_2 - k_3\hat{x}_3 - k_4\hat{x}_4)}{(1+c\sin^2(x_1))^2}$$

$$\triangleright \frac{\partial f_2}{\partial x_2} = \frac{-2x_2 c\sin(x_1) \cos(x_1)}{1+c\sin^2(x_1)}, \frac{\partial f_2}{\partial x_3} = \frac{\partial f_2}{\partial x_4} = 0, \frac{\partial f_2}{\partial \hat{x}_1} = \frac{\cos(x_1)}{(1+c\sin^2(x_1))} k_1$$

$$\triangleright \frac{\partial f_2}{\partial \hat{x}_2} = \frac{\cos(x_1)}{(1+c\sin^2(x_1))} k_2, \frac{\partial f_2}{\partial \hat{x}_3} = \frac{\cos(x_1)}{(1+c\sin^2(x_1))} k_3, \frac{\partial f_2}{\partial \hat{x}_4} = \frac{\cos(x_1)}{(1+c\sin^2(x_1))} k_4$$

$$\triangleright \frac{\partial f_3}{\partial x_1} = \frac{\partial f_3}{\partial x_2} = \frac{\partial f_3}{\partial x_3} = 0, \frac{\partial f_3}{\partial x_4} = 1, \frac{\partial f_3}{\partial \hat{x}_1} = \frac{\partial f_3}{\partial \hat{x}_2} = \frac{\partial f_3}{\partial \hat{x}_3} = \frac{\partial f_3}{\partial \hat{x}_4} = 0$$

$$\triangleright \frac{\partial f_4}{\partial x_1} = \frac{d\hat{x}_2^2}{dx_1} \left[ \cos(x_1)(1+c\sin^2(x_1)) - \sin(x_1)2c\sin(x_1)\cos(x_1) \right] - \\ - \left[ \frac{(\cos^2(x_1) - \sin^2(x_1))(1+c\sin^2(x_1)) - \sin^2(x_1)\cos^2(x_1)2c}{(1+c\sin^2(x_1))^2} \right] - \\ - \frac{62c\sin(x_1)\cos(x_1)(-k_1\hat{x}_1 - k_2\hat{x}_2 - k_3\hat{x}_3 - k_4\hat{x}_4)}{(1+c\sin^2(x_1))^2}$$

$$\triangleright \frac{\partial f_4}{\partial x_2} = \frac{2x_2 d\sin(x_1)}{1+c\sin^2(x_1)}, \frac{\partial f_4}{\partial x_3} = \frac{\partial f_4}{\partial x_4} = 0, \frac{\partial f_4}{\partial \hat{x}_1} = \frac{-6k_1}{1+c\sin^2(x_1)}, \frac{\partial f_4}{\partial \hat{x}_2} = \frac{-6k_2}{1+c\sin^2(x_1)}$$

$$\triangleright \frac{\partial f_4}{\partial \hat{x}_3} = \frac{-6k_3}{1+c\sin^2(x_1)}, \frac{\partial f_4}{\partial \hat{x}_4} = \frac{-6k_4}{1+c\sin^2(x_1)}$$

Figure 23: "Handwritten Solution for Problem 8 - Page 3"

$$\triangleright \frac{\partial f_5}{\partial x_1} = L_{11}, \frac{\partial f_5}{\partial x_2} = 0, \frac{\partial f_5}{\partial x_3} = L_{12}, \frac{\partial f_5}{\partial x_4} = 0, \frac{\partial f_5}{\partial \dot{x}_1} = -L_{11}, \frac{\partial f_5}{\partial \dot{x}_2} = L$$

$$\frac{\partial f_5}{\partial \dot{x}_3} = -L_{12}, \frac{\partial f_5}{\partial \dot{x}_4} = 0$$

$$\triangleright \frac{\partial f_6}{\partial x_1} = L_{21}, \frac{\partial f_6}{\partial x_2} = 0, \frac{\partial f_6}{\partial x_3} = L_{22}, \frac{\partial f_6}{\partial x_4} = 0,$$

$$\frac{\partial f_6}{\partial \dot{x}_1} = -c \dot{x}_2^2 \left[ \frac{\cos^2(\dot{x}_1) + \cos^4(\dot{x}_1) - \sin^2(\dot{x}_1) - \sin^3(\dot{x}_1) \cos^2(\dot{x}_1) / (1+2c)}{(1+c \sin^2(\dot{x}_1))^2} \right] +$$

$$+ \frac{(\cos(\dot{x}_1) - c \sin^2(\dot{x}_1) \cos(\dot{x}_1))}{(1+c \sin^2(\dot{x}_1))^2} + \frac{(\sin(\dot{x}_1) + c \sin^3(\dot{x}_1) + 2c \sin(\dot{x}_1) (1-\sin^2(\dot{x}_1))) (-k_1 \dot{x}_1 - k_2 \dot{x}_2 - k_3 \dot{x}_3 - k_4 \dot{x}_4)}{(1+c \sin^2(\dot{x}_1))^2}$$

$$- \frac{\cos(\dot{x}_1) (-k_1)}{1+c \sin^2(\dot{x}_1)} - L_{21}$$

$$\triangleright \frac{\partial f_6}{\partial \dot{x}_2} = -2 \dot{x}_2 c \sin(\dot{x}_1) \cos(\dot{x}_1) - \frac{\cos(\dot{x}_1) (-k_2)}{1+c \sin^2(\dot{x}_1)}, \frac{\partial f_6}{\partial \dot{x}_3} = \frac{+\cos(\dot{x}_1)}{(1+c \sin^2(\dot{x}_1))} k_3 - L_{22}$$

$$\frac{\partial f_6}{\partial \dot{x}_4} = \frac{\cos(\dot{x}_1)}{(1+c \sin^2(\dot{x}_1))} k_4, \triangleright \frac{\partial f_7}{\partial x_1} = L_{31}, \frac{\partial f_7}{\partial x_2} = 0, \frac{\partial f_7}{\partial x_3} = L_{32}, \frac{\partial f_7}{\partial x_4} = 0$$

$$\triangleright \frac{\partial f_7}{\partial \dot{x}_1} = -L_{31}, \frac{\partial f_7}{\partial \dot{x}_2} = 0, \frac{\partial f_7}{\partial \dot{x}_3} = -L_{32}, \frac{\partial f_7}{\partial \dot{x}_4} = 1$$

$$\triangleright \frac{\partial f_8}{\partial x_1} = L_{41}, \frac{\partial f_8}{\partial x_2} = 0, \frac{\partial f_8}{\partial x_3} = L_{42}, \frac{\partial f_8}{\partial x_4} = 0,$$

$$\frac{\partial f_8}{\partial \dot{x}_1} = d \dot{x}_2^2 \left[ \cos(\dot{x}_1) / (1+c \sin^2(\dot{x}_1)) - \sin(\dot{x}_1) 2c \sin(\dot{x}_1) \cos(\dot{x}_1) \right] -$$

$$- \left[ \frac{(\cos^3(\dot{x}_1) - \sin^2(\dot{x}_1)) / (1+c \sin^2(\dot{x}_1)) - \sin^2(\dot{x}_1) \cos^2(\dot{x}_1) 2c}{(1+c \sin^2(\dot{x}_1))^2} \right] - 6c \sin(\dot{x}_1) \cos(\dot{x}_1) \left[ -k_1 \dot{x}_1 - k_2 \dot{x}_2 - k_3 \dot{x}_3 - k_4 \dot{x}_4 \right] / (1+c \sin^2(\dot{x}_1))^2$$

$$+ \frac{6(-k_1)}{(1+c \sin^2(\dot{x}_1))} - L_{41}$$

Figure 24: "Handwritten Solution for Problem 8 - Page 4"

$$\frac{\partial f_8}{\partial \hat{x}_2} = \frac{2\hat{x}_2 \sin(\hat{x}_1)}{1 + \cos^2(\hat{x}_1)} + \frac{b(-k_2)}{(1 + \cos^2(\hat{x}_1))}, \quad \frac{\partial f_8}{\partial \hat{x}_3} = \frac{b(-k_3)}{(1 + \cos^2(\hat{x}_1))} - L_{42}$$

$$\frac{\partial f_8}{\partial \hat{x}_4} = \frac{b(-k_4)}{(1 + \cos^2(\hat{x}_1))}$$

Since we want to linearize about the origin ( $\hat{x}=0$ ,  $\hat{x}=0$ ), we need to calculate the Jacobian when  $\boxed{\hat{x}=\hat{x}=0}$ .

$$\triangleright \frac{\partial f_1}{\partial x_1} = 0, \frac{\partial f_1}{\partial x_2} = 1, \frac{\partial f_1}{\partial x_3} = \dots = \frac{\partial f_1}{\partial x_4} = 0$$

$$\triangleright \frac{\partial f_2}{\partial x_1} = 0 + 1 + 0 = 1, \frac{\partial f_2}{\partial x_2} = 0, \frac{\partial f_2}{\partial x_3} = \frac{\partial f_2}{\partial x_4} = 0, \frac{\partial f_2}{\partial \hat{x}_1} = k_1$$

$$\circ \frac{\partial f_2}{\partial \hat{x}_2} = k_2, \frac{\partial f_2}{\partial \hat{x}_3} = k_3, \frac{\partial f_2}{\partial \hat{x}_4} = k_4$$

$$\triangleright \frac{\partial f_3}{\partial x_1} = \frac{\partial f_3}{\partial x_2} = \frac{\partial f_3}{\partial x_3} = 0, \frac{\partial f_3}{\partial x_4} = 1, \frac{\partial f_3}{\partial \hat{x}_1} = \frac{\partial f_3}{\partial \hat{x}_2} = \frac{\partial f_3}{\partial \hat{x}_3} = \frac{\partial f_3}{\partial \hat{x}_4} = 0$$

$$\triangleright \frac{\partial f_4}{\partial x_1} = 0 - 1 = -1, \frac{\partial f_4}{\partial x_2} = 0, \frac{\partial f_4}{\partial x_3} = \frac{\partial f_4}{\partial x_4} = 0, \frac{\partial f_4}{\partial \hat{x}_1} = -bk_1$$

$$\frac{\partial f_4}{\partial \hat{x}_2} = -bk_2, \frac{\partial f_4}{\partial \hat{x}_3} = -bk_3, \frac{\partial f_4}{\partial \hat{x}_4} = -bk_4$$

$$\triangleright \frac{\partial f_5}{\partial x_1} = L_{11}, \frac{\partial f_5}{\partial x_2} = 0, \frac{\partial f_5}{\partial x_3} = L_{12}, \frac{\partial f_5}{\partial x_4} = 0, \frac{\partial f_5}{\partial \hat{x}_1} = -L_{11}, \frac{\partial f_5}{\partial \hat{x}_2} = 1$$

$$\frac{\partial f_5}{\partial \hat{x}_3} = -L_{12}, \frac{\partial f_5}{\partial \hat{x}_4} = 0$$

$$\triangleright \frac{\partial f_6}{\partial x_1} = L_{21}, \frac{\partial f_6}{\partial x_2} = 0, \frac{\partial f_6}{\partial x_3} = L_{22}, \frac{\partial f_6}{\partial x_4} = 0, \frac{\partial f_6}{\partial \hat{x}_1} = 0 + 1 + 0 + k_1 - L_{21}$$

$$\frac{\partial f_6}{\partial \hat{x}_2} = 1 + k_1 - L_{21}, \frac{\partial f_6}{\partial \hat{x}_3} = k_2, \frac{\partial f_6}{\partial \hat{x}_4} = k_3 - L_{22}, \frac{\partial f_6}{\partial \hat{x}_4} = k_4$$

$$\triangleright \frac{\partial f_7}{\partial x_1} = L_{31}, \frac{\partial f_7}{\partial x_2} = 0, \frac{\partial f_7}{\partial x_3} = L_{32}, \frac{\partial f_7}{\partial x_4} = 0, \frac{\partial f_7}{\partial \hat{x}_1} = -L_{31}, \frac{\partial f_7}{\partial \hat{x}_2} = 0$$

$$\frac{\partial f_7}{\partial \hat{x}_3} = -L_{32}, \frac{\partial f_7}{\partial \hat{x}_4} = 1$$

$$\triangleright \frac{\partial f_8}{\partial x_1} = L_{41}, \frac{\partial f_8}{\partial x_2} = 0, \frac{\partial f_8}{\partial x_3} = L_{42}, \frac{\partial f_8}{\partial x_4} = 0, \frac{\partial f_8}{\partial \hat{x}_1} = -1 - bk_1 - L_{41}$$

$$\frac{\partial f_8}{\partial \hat{x}_2} = -bk_2, \frac{\partial f_8}{\partial \hat{x}_3} = -bk_3 - L_{42}, \frac{\partial f_8}{\partial \hat{x}_4} = -bk_4$$

Figure 25: "Handwritten Solution for Problem 8 - Page 5"

Therefore, we have:

$$A = \frac{\partial f}{\partial x} \Big|_{\substack{x=0 \\ \dot{x}=0}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & k_1 & k_2 & k_3 & k_4 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -bk_1 & -bk_2 & -bk_3 & -bk_4 \\ L_{11} & 0 & L_{12} & 0 & -L_{11} & 1 & -L_{12} & 0 \\ L_{21} & 0 & L_{22} & 0 & 1+k_1-L_{21} & k_2 & k_3-L_{22} & k_4 \\ L_{31} & 0 & L_{32} & 0 & -L_{31} & 0 & -L_{32} & 1 \\ L_{41} & 0 & L_{42} & 0 & -1-bk_1-L_{41} & -bk_2 & -bk_3-L_{42} & -bk_4 \end{bmatrix}$$

Using Matlab, we calculated the eigenvalues of the matrix in order to examine the properties of the eq points at the origin. Specifically these are the respective eigenvalues:

$$\lambda_1 = -9, \lambda_2 = -6, \lambda_3 = -0.7 + j0.2, \lambda_4 = -0.7 - j0.2, \lambda_5 = -21 + j0.6, \lambda_6 = -21 - j0.6$$

$\lambda_7 = -2, \lambda_8 = -3$ . We can see that  $\text{Re}(\lambda_i(A_{\text{augm}})) < 0 \forall \lambda_i$ , meaning that the origin is still asymptotically stable.

In Figure 5, we plot the responses of the new controllers both linear and nonlinear, for the same I.C. as in Problem 5 and 7. Compared to Figure 3, we can observe that this observer indeed provides better estimates and hence better performance. Namely, less oscillations are observed, smaller peaks and settling times and closer behavior between linear and nonlinear models.

Figure 26: "Handwritten Solution for Problem 8 - Page 6"

**Figure (5) - Problem 8 - Simulation of Dynamic Controller With Nonlinear Observer**

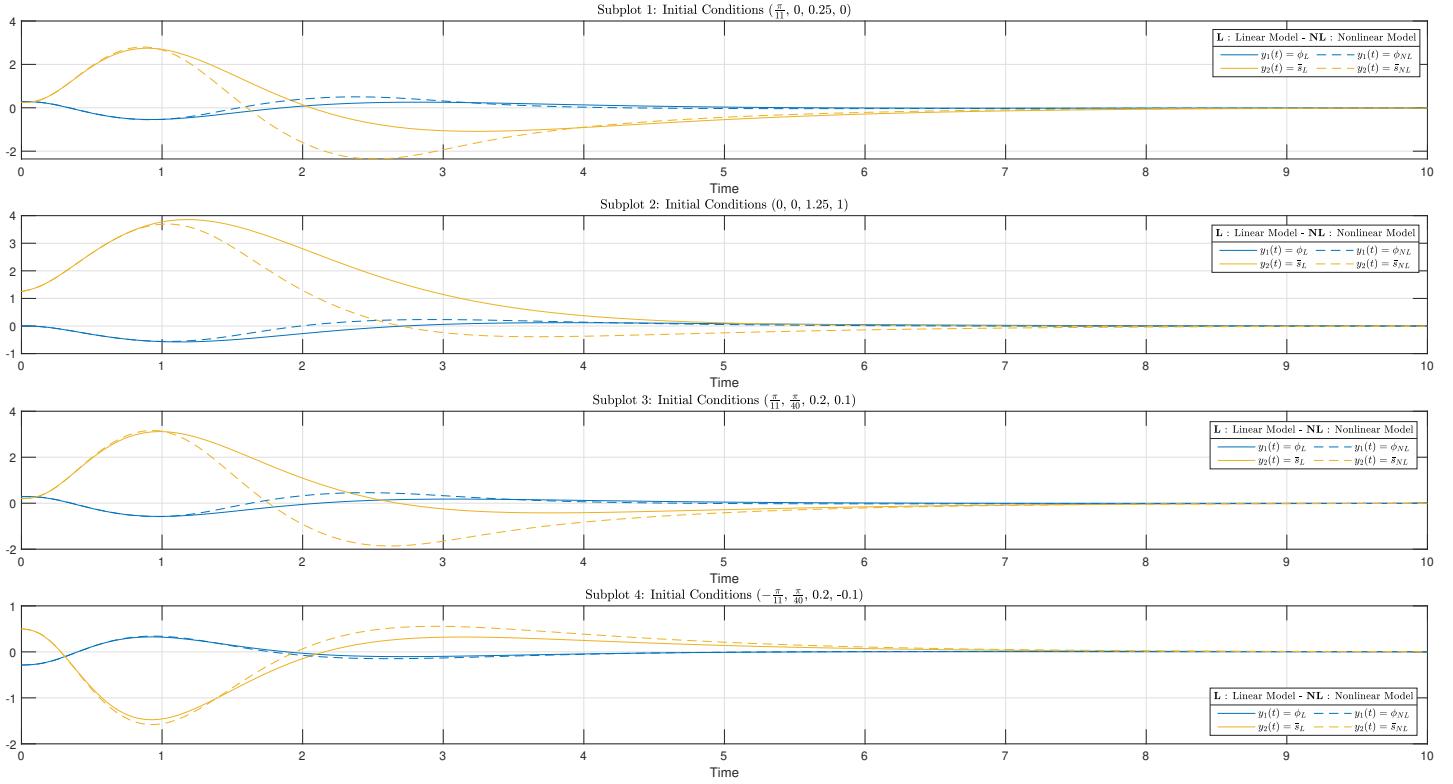


Figure 27: "Figure 5 of Problem 8"

## 8.1 Matlab Code for Problem 8

```

1  syms x_full(t) [8 1] real
2  x_full_14=[x_full1;x_full2;x_full3;x_full4];
3  x_full_58=[x_full5;x_full6;x_full7;x_full8];
4
5  %Linear Model - Same as in Problem 7
6  x_dot_full_14 = A*x_full_14 -B*K*x_full_58;
7  %Linear Observer
8  x_dot_full_58 = (A-B*K-L*C)*x_full_58+L*C*x_full_14;
9  x_dot_full=[x_dot_full_14;x_dot_full_58];
10
11 % Nonlinear Model - Same as in Problem 7
12 mu = -real(K)*x_full_58;
13 x_dot_full_nl_1 = x_full2;
14 x_dot_full_nl_2 = (-c*(x_full2^2)*sin(x_full1)*cos(x_full1)/(1+c*(sin(x_full1))^2)) + ...
    (sin(x_full1)/(1+c*(sin(x_full1))^2)) - (cos(x_full1)*mu/(1+c*(sin(x_full1))^2));
15 x_dot_full_nl_3 = x_full4;
16 x_dot_full_nl_4 = (d*(x_full2^2)*sin(x_full1)/(1+c*(sin(x_full1))^2)) - ...
    (sin(x_full1)*cos(x_full1)/(1+c*(sin(x_full1))^2)) + (b*mu/(1+c*(sin(x_full1))^2));
17 %Nonlinear Observer
18 x_dot_full_nl_5 = x_full6;
19 x_dot_full_nl_6 = (-c*(x_full6^2)*sin(x_full5)*cos(x_full5)/(1+c*(sin(x_full5))^2)) + ...
    (sin(x_full5)/(1+c*(sin(x_full5))^2)) - (cos(x_full5)*mu/(1+c*(sin(x_full5))^2));
20 x_dot_full_nl_7 = x_full8;
21 x_dot_full_nl_8 = (d*(x_full6^2)*sin(x_full5)/(1+c*(sin(x_full5))^2)) - ...
    (sin(x_full5)*cos(x_full5)/(1+c*(sin(x_full5))^2)) + (b*mu/(1+c*(sin(x_full5))^2));
22 x_dot_full_nl_58 = [x_dot_full_nl_5; x_dot_full_nl_6; x_dot_full_nl_7; x_dot_full_nl_8] ...
    +L*C*(x_full_14-x_full_58);
23 x_dot_full_nonlinear=[x_dot_full_nl_1; x_dot_full_nl_2; x_dot_full_nl_3; x_dot_full_nl_4; x_dot_full_nl_58];
24
25 figure(5)
26 sgttitle('Figure (5) - Problem 8 - Simulation of Dynamic Controller With Nonlinear ...')
    'Interpreter','latex')
27
28 subplot(4,1,1);
29 %The cart at rest but not at the origin and the bar leaning to one side,
30 %but also at rest
31 initconds=[pi/11; 0; 0.25; 0];
32 %Always take the initial conditions of the observer to be [0,0,0,0]
33 initconds=[initconds;0;0;0;0];
34 %Solution of Differential Equations for Linear Model
35 M=odeFunction(x_dot_full,x_full);
36 [t_linear,sol_linear] = ode45(M,[0 10],initconds);
37 %Solution of Differential Equations for Nonlinear Model
38 M=odeFunction(x_dot_full_nonlinear,x_full);
39 % [t_nonlinear,sol_nonlinear] = ode45(M,[0 5],initconds);
40 [t_nonlinear,sol_nonlinear] = ode45(M,[0 10],initconds);
41
42 %Plotting ouputs y1(t) and y2(t)
43 %y1(t)=phi=x1=x_full1(t)
44 plot(t_linear,sol_linear(:,1),'', 'Color',[0 0.4470 0.7410])
45 hold on;
46 %y2(t)=s_bar=x3=x_full3(t)
47 plot(t_linear,sol_linear(:,3),'', 'Color',[0.9290 0.6940 0.1250])
48 hold on;
49 plot(t_nonlinear,sol_nonlinear(:,1),'—', 'Color',[0 0.4470 0.7410])
50 hold on;
51 %y2(t)=s_bar=x3=x_full3(t)
52 plot(t_nonlinear,sol_nonlinear(:,3),'—', 'Color',[0.9290 0.6940 0.1250])
53 hold on;
54 % ylim([-15 15]);
55 grid on
56 title('Subplot 1: Initial Conditions ($\frac{\pi}{11}, 0, 0.25, 0)', 'Interpreter','latex')
57 xlabel('Time')
58 lgd=legend('$y_{\{1\}}(t)=\phi_{\{L\}}$', '$y_{\{2\}}(t)=\bar{s}_{\{L\}}$', '$y_{\{1\}}(t)=\phi_{\{NL\}}$', '$y_{\{2\}}(t)=\bar{s}_{\{NL\}}$', 'Location')
59 title(lgd,'$\\textbf{L} : Linear Model - $\\textbf{NL} : Nonlinear Model', 'Interpreter','latex')
60
61 subplot(4,1,2);
62 %The cart moving to one side with some velocity and the bar initially at
63 %rest and straight up
64 initconds=[0; 0; 1.25; 1];
65 %Always take the initial conditions of the observer to be [0,0,0,0]
66 initconds=[initconds;0;0;0;0];

```

```

67 %Solution of Differential Equations for Linear Model
68 M=odeFunction(x_dot_full,x_full);
69 [t_linear,sol_linear] = ode45(M,[0 10],initconds);
70 %Solution of Differential Equations for Nonlinear Model
71 M=odeFunction(x_dot_full_nonlinear,x_full);
72 [t_nonlinear,sol_nonlinear] = ode45(M,[0 10],initconds);
73 %Plotting ouputs y1(t) and y2(t)
74 %y1(t)=phi=x1=x_full1(t)
75 plot(t_linear,sol_linear(:,1),'', 'Color',[0 0.4470 0.7410])
76 hold on;
77 %y2(t)=s bar=x3=x_full3(t)
78 plot(t_linear,sol_linear(:,3),'', 'Color',[0.9290 0.6940 0.1250])
79 hold on;
80 plot(t_nonlinear,sol_nonlinear(:,1),'—', 'Color',[0 0.4470 0.7410])
81 hold on;
82 %y2(t)=s bar=x3=x_full3(t)
83 plot(t_nonlinear,sol_nonlinear(:,3),'—', 'Color',[0.9290 0.6940 0.1250])
84 hold on;
85 % ylim([-15 15]);
86 grid on
87 title('Subplot 2: Initial Conditions (0, 0, 1.25, 1)', 'Interpreter', 'latex')
88 xlabel('Time')
89 lgd=legend('$y_1(t)=\phi_{L}', '$y_2(t)=\bar{s}', '$y_1(t)=\phi_{NL}', '$y_2(t)=\bar{s}_{NL}', 'Location'
90 title(lgd,'$\\textbf{L} : Linear Model - $\\textbf{NL} : Nonlinear Model', 'Interpreter', 'latex')
91
92 subplot(4,1,3);
93 %The cart moving to one side with some velocity and the bar leaning to one
94 %side not at rest
95 initconds=[pi/11; pi/40; 0.2; 0.1];
96 %Always take the initial conditions of the observer to be [0,0,0,0]
97 initconds=[initconds;0;0;0];
98 %Solution of Differential Equations for Linear Model
99 M=odeFunction(x_dot_full,x_full);
100 [t_linear,sol_linear] = ode45(M,[0 10],initconds);
101 %Solution of Differential Equations for Nonlinear Model
102 M=odeFunction(x_dot_full_nonlinear,x_full);
103 [t_nonlinear,sol_nonlinear] = ode45(M,[0 10],initconds);
104 %Plotting ouputs y1(t) and y2(t)
105 %y1(t)=phi=x1=x_full1(t)
106 plot(t_linear,sol_linear(:,1),'', 'Color',[0 0.4470 0.7410])
107 hold on;
108 %y2(t)=s bar=x3=x_full3(t)
109 plot(t_linear,sol_linear(:,3),'', 'Color',[0.9290 0.6940 0.1250])
110 hold on;
111 plot(t_nonlinear,sol_nonlinear(:,1),'—', 'Color',[0 0.4470 0.7410])
112 hold on;
113 %y2(t)=s bar=x3=x_full3(t)
114 plot(t_nonlinear,sol_nonlinear(:,3),'—', 'Color',[0.9290 0.6940 0.1250])
115 hold on;
116 % ylim([-15 15]);
117 grid on
118 title('Subplot 3: Initial Conditions ($\\frac{\\pi}{11}, \\frac{\\pi}{40}, 0.2, 0.1)', 'Interpreter', 'latex')
119 xlabel('Time')
120 lgd=legend('$y_1(t)=\phi_{L}', '$y_2(t)=\bar{s}', '$y_1(t)=\phi_{NL}', '$y_2(t)=\bar{s}_{NL}', 'Location'
121 title(lgd,'$\\textbf{L} : Linear Model - $\\textbf{NL} : Nonlinear Model', 'Interpreter', 'latex')
122
123 subplot(4,1,4);
124 %The cart moving to one side with some negative velocity and the bar leaning to the ...
125
126 %negative
127 %side not at rest
128 initconds=[-pi/11; pi/55; 0.5; -0.25];
129 %Always take the initial conditions of the observer to be [0,0,0,0]
130 initconds=[initconds;0;0;0];
131 %Solution of Differential Equations for Linear Model
132 M=odeFunction(x_dot_full,x_full);
133 [t_linear,sol_linear] = ode45(M,[0 10],initconds);
134 %Solution of Differential Equations for Nonlinear Model
135 M=odeFunction(x_dot_full_nonlinear,x_full);
136 [t_nonlinear,sol_nonlinear] = ode45(M,[0 10],initconds);
137 %Plotting ouputs y1(t) and y2(t)
138 %y1(t)=phi=x1=x_full1(t)
139 plot(t_linear,sol_linear(:,1),'', 'Color',[0 0.4470 0.7410])
140 hold on;
141 %y2(t)=s bar=x3=x_full3(t)
142 plot(t_linear,sol_linear(:,3),'', 'Color',[0.9290 0.6940 0.1250])

```

```

141 hold on;
142 plot(t_nonlinear,sol_nonlinear(:,1),'—','Color',[0 0.4470 0.7410])
143 hold on;
144 %y2(t)=s bar=x3=x_full3(t)
145 plot(t_nonlinear,sol_nonlinear(:,3),'—','Color',[0.9290 0.6940 0.1250])
146 hold on;
147 % ylim([-15 15]);
148 grid on
149 title('Subplot 4: Initial Conditions ($-\frac{\pi}{11}$, $\frac{\pi}{40}$, 0.2, -0.1)', 'Interpreter', 'latex')
150 xlabel('Time')
151 lgd=legend('$y_1(t)=\phi_L$', '$y_2(t)=\bar{s}_L$', '$y_1(t)=\phi_{NL}$', '$y_2(t)=\bar{s}_{NL}$', 'Location')
152 title(lgd, '$\textbf{L} : Linear Model - \textbf{NL} : Nonlinear Model', 'Interpreter', 'latex')
153
154 %%%%%%
155 %Problem 8 – Linearization about the Origin
156 A_augm=[0 1 0 0 0 0 0;
157 1 0 0 0 K(1) K(2) K(3) K(4);
158 0 0 0 1 0 0 0;
159 -1 0 0 0 -b*K(1) -b*K(2) -b*K(3) -b*K(4);
160 L(1,1) 0 L(1,2) 0 -L(1,1) 1 -L(1,2) 0;
161 L(2,1) 0 L(2,2) 0 1+K(1)-L(2,1) K(2) K(3)-L(2,2) K(4);
162 L(3,1) 0 L(3,2) 0 -L(3,1) 0 -L(3,2) 1;
163 L(4,1) 0 L(4,2) 0 -1-b*K(1)-L(4,1) -b*K(2) -b*K(3)-L(4,2) -b*K(4)];
164 eig(A_augm)
165 %%%%%%

```