Denoising Error Formulation

Clayton Sanford, Cyrus Cousins

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We introduce the following variables and functions:

- m different functions, each representing a denoising algorithm: $\phi_1, \ldots, \phi_m$, where $\phi_j : S_{[0,1]} \to S_{[0,1]}$, where $S_{[0,1]}$ is the set of finite sequences of real numbers in $[0, 1]$. Can be expressed as $\phi : S_{[0,1]} \to S_{[0,1]}^m$, where $\phi(x) = (\phi_1(x), \ldots, \phi_m(x))$ and $\phi(x)_\ell = (\phi_1(x)_\ell, \ldots, \phi_m(x)_\ell)$.
- N input/output pairs: $(x_1, y_1), \ldots, (x_N, y_N) \in S_{[0,1]}^2$, where the sequences in each pair are the same size. For each input pair (x_i, y_i), let $k_i = \texttt{length}(x_i) = \texttt{length}(y_i)$. Each sequence can then be indexed as $x_i = (x_{i,1}, \ldots, x_{i,k_i})$.
- The weight vector for the classifiers: $w \in [0, 1]^m$ such that $\sum_{j=1}^m w_j = 1$.

We wish to minimize the following quantity:

$$
\begin{aligned}
\sum_{i=1}^N \texttt{err}\left(y_i, w^T\phi(x_i)\right) &= \sum_{i=1}^N \frac{1}{k_i} \sum_{\ell=1}^{k_i} \left(y_{i,\ell} - w^T\phi(x_i)_\ell\right)^2 && \textsc{Error Function} \\
&= \sum_{i=1}^N \frac{1}{k_i} \sum_{\ell=1}^{k_i} \left(y_{i,\ell}^2 - 2y_{i,\ell}w^T\phi(x_i)_\ell + (w^T\phi(x_i)_\ell)^2\right) && \textsc{Square Expansion} \\
&= \sum_{i=1}^N \frac{1}{k_i} \sum_{\ell=1}^{k_i} \left(y_{i,\ell}^2 + L_{i,\ell}^T w + w^T\phi(x_i)_\ell\phi(x_i)_\ell^T w\right) && \begin{array}{r}\textsc{Substitution} \\ x^Ty = y^Tx \text{ for } x, y \in \mathbb{R}^n\end{array} \\
&= \sum_{i=1}^N \frac{1}{k_i} \sum_{\ell=1}^{k_i} \left(y_{i,\ell}^2 + L_{i,\ell}^T w + w^T Q_{i,\ell} w\right) && \textsc{Substitution} \\
&= w^TQw + L^Tw + \sum_{i=1}^N \frac{1}{k_i}\|y_i\|_2^2 && \begin{array}{r}\textsc{Substitution} \\ \textsc{L2-Norm Definition}\end{array}
\end{aligned}
$$

where:

$$\forall i, \ell,\ L_{i,\ell} := -2y_{i,\ell}\phi(x_i)_\ell,\ Q_{i,\ell} := \phi(x_i)_\ell\phi(x_i)_\ell^T$$

$$L := \sum_{i=1}^N \frac{1}{k_i} \sum_{\ell=1}^{k_i} L_{i,\ell},\ Q := \sum_{i=1}^N \frac{1}{k_i} \sum_{\ell=1}^{k_i} Q_{i,\ell}$$

We can then solve the following quadratic program:

$$\min_w w^T Q w + L^T w \quad \text{such that: } \sum_{j=1}^{m} w_j = 1$$
$$\forall j, \; w_j \geq 0$$