Assignment 4 - Question 3

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Given: A is a $m \times n$ matrix, $P = A^T A$ and $Q = AA^T$.

Part (a)

For any $n \times 1$ column vector y,

$$y^t P y = y^t A^t A y = (Ay)^T A y = \|Ay\|^2 \geq 0$$

So, P is positive definite.

For any $m \times 1$ column vector z,

$$z^t Q z = z^t A A^t z = (A^t z)^T A^t z = \|A^t z\|^2 \geq 0$$

So, Q is positive definite.

Let λ be a eigenvalue of R, a positive definite matrix with eigenvector v, then $Pv = \lambda v$. Premultiply by v^T.

$$v^t R v = \lambda v^t v$$

$$\lambda = \frac{v^t R v}{\|v\|^2} \geq 0$$

because R is positive definite. Here both P and Q are positive definite. So, their eigenvalues are all positive.

Part (b)

Given

$$Pu = \lambda u$$

$$A^t A u = \lambda u$$

Premultiply by A

$$AA^t(Au) = \lambda(Au)$$

$$Q(Au) = \lambda(Au)$$

So, Au is a eigenvector of Q with eigenvalue λ

Given,

$$Qv = \mu v$$

$$AA^t v = \mu v$$

Premultiply by A^t

$$A^t(AA^t v) = \mu(A^T v)$$

$$P(A^T v) = \mu(A^t v)$$

So, $A^t u$ is a eigenvector of P with eigenvalue μ. Since P is $n \times n$, u is a $n \times 1$ matrix. and since Q is $m \times m$, v is a $m \times 1$ matrix.

Part (c)

Given,

$$Qv_i = \mu v_i$$

$$AA^t v_i = \mu v_i$$

$$Au_i = \frac{AA^t v_i}{\|A^t v_i\|} = \frac{\mu}{\|A^t v_i\|} v_i = \gamma_i v_i$$

where $\gamma_i = \frac{\mu}{\|A^t v_i\|} \geq 0$ because $\mu \geq 0$.

Part (d)

Given $V = [u_1|u_2|u_3|....|u_n]$. Then i, j th element of $V^T V = u_i^t u_j = 1$ if $i = j$ and $= 0$ if $i \neq j$. So, $V^T V = I$ i.e, $V^T = V^{-1}$ as V is a $n \times n$ matrix.

To show $A = U\Gamma V^t$, it is enough to show $AV = U\Gamma$ where Γ is a diagonal matrix containing γ_i's along diagonals.

We know from part (c), $Au_i = \gamma v_i$.

$$AV = A[u_1|u_2|u_3|\ldots|u_n] = [Au_1|Au_2|Au_3|....|Au_n]$$

$$= [\gamma_1 v_1|\gamma_2 v_2|\gamma_3 v_3|....|\gamma_n v_n] = [v_1|v_2|v_3|\ldots|v_n|]\Gamma = U\Gamma$$

. where Γ is a diagonal matrix containing γ_i's along diagonals.