Exam #1

• WOP - ① This pf is by WOP

② Assume some set that satisfies requirement. contradiction

③ Everything under some k, least element, must satisfy normal req.

④ Prove that either it can't be the smallest element or the true conditions of smaller numbers make it impossible for this element. ⑤ Conclude set is empty.

BASE CASES! to make sure it's in the domain

Pf by Contradiction

① Pf by Contradiction

② Suppose P is false

③ Deduce the contradiction

④ Therefore P must be true.

Pf by Cases

Pf by Induction
① Base
② $P(n) ::=$ ____
③ $P(n) \Rightarrow P(n+1)$

If and only if - prove both directions

Exam #2

- valid: predicate is true for all values in domain
- satisfiable: exists some combination that yields True

Conjunctive: $(A \vee B) \wedge (C \vee D)$ - Go down list of falses, reverse Nots

Disjunctive: $(A \wedge B) \vee (C \wedge D)$ - Go down list of trues

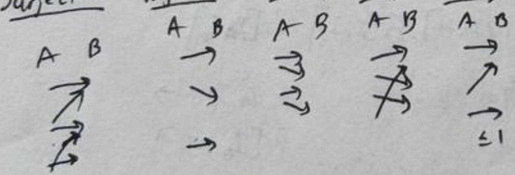
Ex	A	B	C	D	P
	T	F	F	T	T
	T	T	F	F	F

DNF: $(A \,\&\, \bar{B} \,\&\, \bar{C} \,\&\, D)$ or ...

CNF: $(\bar{A}$ or $\bar{B}$ or C or $D)$ AND ...

De morgan's Law: $\overline{R \wedge S} = \bar{R} \vee \bar{S}$

Surjective Injective Total Bij Func



- surj - surjective func
- inj - injective total relation

State Machines: States: {range}
start state
transitions

Preserved Invariant: if a property starts to be true, it stays true for all reachable states.

Stable Matching - Girls choose suitors, no rogue couples

Mating Ritual - reaches termination
- ends w/ everyone married - pf by cont.
- all are stable → use an example couple

Structural Induction

① Induction hypothesis

② Base Cases

③ Constructor Cases
- Assume by induction $P(r)$ & $P(s)$
- Prove $P(r), P(s) \Rightarrow P(c[r,s])$

Exam #3

- A strict B iff $(\neg A \text{ surj } B)$
- A strict B iff $|A| < |B|$
- Countable → elements can be listed in order
  - countably infinite IFF N bij C
  - countable IFF N surj C
- Power sets: Cantors - For any set A, A strict Pow(A)

$$A_g ::= \{a \in A \mid a \notin g(a)\}$$

→ Pow(N) is uncountable

→ If V is uncountable and A surj V, then A is uncountable

→ If C is countable and C surj A, then A is countable.

→ Diagonal argument:
Making sure elements on diagonal are dif ST $r: A \to A$
$r(0)=1$
$r(1)=0$
⋮

$a|b$: a divides b
- If $a|b$ & $a|c \to a|sb+tc \ \forall s,t$

Pulverizer

a	b	rem	
125	45	35	$= 125-2(45)$
45	35	10	$= 45-35 = 45-[125-2(45)]$
35	10	5	$= 35-3(10) = [125-2(45)]-3[45-[125-2(45)]]$
10	5	0	

↓ Linear combo of a, b

mod: $a \equiv b \pmod{n}$ iff $n|(a-b)$
iff $rem(a,n) = rem(b,n)$

gcd: $gcd(au, av) = a\,gcd(u,v)$
$gcd(au, v) = gcd(u,v)$ for a rel prime to v
$gcd(u-v, v) = gcd(u,v)$
☆ consider $gcd(a,b) = sa + tb$ ☆

Digraphs - G has a nonempty set V(G) & E(G)

→ Walk - any connection of edges, vertices

→ Path: unique vertices

DAG → no cycles

→ topological sort

vertex	vertex before other reachable vertices
a	
b	
c	
d	

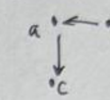
→ chain - set of vertices that are reachable

→ antichain - set of vertices not reachable

→ Greedy: tasks as soon as possible

→ Critical path: longest chain

→ Adjacency matrix



	a	b	c
a	0	0	1
b	1	0	0
c	0	0	0

A^k shows # of k-length paths from one vertex to the other

Exam #4

SPO - transitive & irreflexive
- transitive & asym

WPO - IFF R is transitive, antisym, reflexive

Eq relation - IFF R is reflexive, sym, transitive

- Reflexive - $\forall_{x \in A}\, xRx$ (self loops)
- Irreflexive: $\neg[\exists x \in A, xRx]$ no self loops
- Symmetry $[\forall_{x,y \in A}, xRy \to yRx]$
  - if there is an edge from x to y, ∃ y to x
- Asym $[xRy \to \neg(yRx)]$ No self loops, at most one directed edge
- Antisym $[\forall_{x \neq y \in A}, xRy \to \neg(yRx)]$
  - can have self loops, but only at most one DE
- Transitive $[xRy \,\&\, yRz \to xRz]$
- Linear xRy or yRx when $x \neq y$
  - given 2 vertices, always an edge

Simple graph - nonempty vertices, some set of edges, no self loops

Complete graph - edge b/w every pair of vertices

Isomorphism $\langle u-v \rangle \in E(G)$ iff $\langle f(u)-f(v) \rangle \in E(H)$, $V(G) \to V(H)$

Bipartite graph: graph whose vertices can be divided into 2 sets

Matching – there must be no bottlenecks

bottleneck $|S| > |N(S)|$

For some set S that contains elements of the domain, the |neighbors| is less than the subset itself.

deg constrained: If every S maps to $\geq d$ r and every r maps to $\leq d$ S, then no bottleneck

when $\deg(l) \geq \deg(r)$ for all l, r (l: left, r: right)

Tree – connected acyclic graph

leaf is deg 1

Forest – acyclic graph

MST: the gray thing

1. Start with random coloring
2. draw min weight gray edge
3. Turn all connected groups same color
4. repeat

Algorithms → select min weight edge

Kruskals → select w/o cycles

Prims → start w/ a vertex, add edges like that

Sums $\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$ $\quad \sum_{i=1}^{n} i^2 = \frac{(2n+1)(n+1)n}{6}$

$\sum_{i=0}^{n} x^i = \frac{1-x^{n+1}}{1-x}$ $\quad \sum_{i=1}^{m} nx^n = \frac{(mx-m-1)x^{m+1}+x}{(1-x)^2}$

$\sum_{i=1}^{m} x^n = \frac{n(n^m-1)}{n-1}$

Asymptotic Notation

$f = o(g)$ $\quad \lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$

$f = O(g)$ $\quad \limsup_{x\to\infty} \frac{|f(x)|}{g(x)} < \infty$ → upper bound

$f \sim g$ $\quad \lim_{x\to\infty} \frac{f}{g} = 1$

$f = \Theta(g) \to f = O(g)$ & $g = O(f)$

Post Exam #4

- 4 Step Method
  - define outcomes
  - define events of interest
  - define outcome probabilities
  - define event probabilities
- Balls in a Box (splitting up a group)

  $\times \times | \times \times | \times$ $\quad \binom{6}{2}$ dif ways (6 → spots for sticks, 2 → sticks)
- Pigeonhole Principle – If there are more pigeons than holes they occupy, then at least two pigeons must be in the same hole.
- Inclusion/Exclusion – If M, E, P are disjoint, then $|M \cup E \cup P| = |M| + |E| + |P|$

  If not, then $|M \cup E| = |M| + |E| - |M \cap E|$

  For 3 its like $|M \cup E \cup K| = |M| + |E| + |K| - |M \cap E| - |M \cap K| - |K \cap E| + |M \cap K \cap E|$
- $Pr[X|Y]$ – Given Y
- $Pr[X|Y] = \frac{Pr[X \wedge Y]}{Pr[Y]}$
- $Pr[B|A] = \frac{Pr[A|B] \cdot Pr[B]}{Pr[A]}$ Bayes Thm $\quad$ Bayes Factor: $\frac{Pr[A|B]}{Pr[A|\bar{B}]} = \text{Bayes}(A,B)$
- Law of Total Probability: $Pr[A] = Pr[A|E] \cdot Pr[E] + Pr[A|\bar{E}] \cdot Pr[\bar{E}]$
- Independence of events $Pr[A|B] = Pr[A]$
- Mutually independent: $Pr[A_1 \cap A_2 \cap \ldots] = Pr[A_1] Pr[A_2] \ldots$
- K way independence: Check to see that each combo is mutually independent within each k size subset
- Random variable – total function whose domain is in sample space
  - Indicator maps every outcome to 0 or 1
- Binomial Dist: $f_n(k) ::= \binom{n}{k} 2^{-n}$ (like n coin tosses w/ k heads)

- $Ex[R] ::= \sum R(\omega) Pr[\omega] \to \sum x \cdot Pr[R=x]$

  $Ex[I_A] = 1 \cdot Pr[I_A = 1] + 0 \cdot [Pr[I_A = 0]] = Pr[I_A = 1]$. $Ex[I_A I_B] = Pr[I_A I_B]$ if I_A, I_B ind → $Pr[I_A] \cdot Pr[I_B]$
- $Ex[R|A] ::= \sum r \cdot Pr[R = r|A]$ $\quad Var[R] = E(x^2) - (E(x))^2 = p(1-p)$
- $Ex[R] = \sum_i Ex[R|A_i] \cdot Pr[A_i]$ $\quad E(x^2) = \sum p \cdot I(p)^2$
- $Ex[R_1 + R_2] = Ex[R_1] + Ex[R_2]$
- $Ex[R_1 \cdot R_2] = Ex[R_1] \cdot Ex[R_2]$ → independent

$Pr[R \geq x] \leq \frac{Ex[R]}{x}$ → Markov's Bound

– if lower bound is provided – let $T ::= R - \text{lowerbound}$ ST $Ex[T] = Ex[R] - \text{lower}$

Plug in for R for $Pr[R \geq x]$, redefine x

Chebyshev's Thm

$Pr[|R| \geq x] \leq \frac{Ex[|R|^z]}{x^z}$

Useful: $Pr[|R - Ex[R]| \geq x] \leq \frac{Var[R]}{x^2}$

$\left[\sum_{i=0}^{n} x_i\right]^2 \to \sum_{i=0}^{n} x_i^2 + 2 \sum_{0 \leq i < j \leq n} x_i x_j$

Markov/Cheb – Cards

– Given $Ex[W] = 108$, find Markov Bound on at least 216 hands.

$Pr[W \geq 216] \leq \frac{108}{216} \leq \frac{1}{2}$

– Variance on hands won

$Var = p(1-p) \cdot \text{rounds\#}$

$= 240 \cdot \frac{1}{6}(1 - \frac{1}{6}) \ldots = \frac{1046}{15}$

– Chebyshev upper bound w/ at least 216.

$Pr[W \geq 216] = Pr[W - 108 \geq 108] \leq \frac{Var[W]}{108^2}$

Markov Lower Bound – Cows

$Ex = 85$ – Above 70 – prob above 90?

$Pr[T \geq 90] = Pr[T - 70 \geq 20] \leq \frac{Ex[T-70]}{20}$

$Pr[T - 70 \geq 20] \leq \frac{15}{20}$

Exp w/ Indicators – Hats

Given $S_n = \sum_{n=0}^{n} X_i$

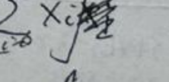
$Ex[S_n] = \sum Ex[X_i] = \sum Pr[X_i] = n \cdot \frac{1}{n} = 1$

Expression for $Ex[X_i \cdot X_j]$:

Note: $Pr[X_2 | X_1] = \frac{1}{n-1}$

$Pr[X_i \cdot X_j] = Pr[X_i \cap X_j] = Pr[X_i | X_j] \cdot Pr[X_j] = \frac{1}{n} \cdot \frac{1}{n-1}$

Stationary Dist – Arrows Into!



$d(w) = \frac{9}{10} d(z)$

$d(z) = d(w) + 0.1 d(z)$

$d(w) + d(z) = 1$

Solve

Let $\Pr[B]=1/1000$

$\Pr[Y|B]=0.99$

$\Pr[\bar{Y}|\bar{B}]=0.97$

– So $\Pr[\bar{B}]=1-\frac{1}{1000}$.

$\Pr[Y|\bar{B}]=1-\Pr[\bar{Y}|\bar{B}]=0.03$

– What is $\Pr[Y]$?

Total Prob: $\Pr[Y]=\Pr[Y|B]\Pr[B]+\Pr[Y|\bar{B}]\Pr[\bar{B}] \approx 3\%$

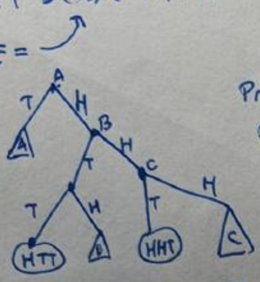
– What is $\Pr[B|Y]$?

$$\Pr[B|Y]=\frac{\Pr[Y|B]\cdot\Pr[B]}{\Pr[Y]}=\frac{\Pr[Y|B]\cdot\Pr[B]}{}$$

$\approx \frac{1}{32}$

Coin HTT before HHT?

E =



$\Pr[E]=\Pr[E|A]$

$\Pr[E|A]=\Pr[E|A]\Pr[T]+\Pr[E|B]\cdot\Pr[H]$

$\Pr[E|A]=\Pr[E|B]$

$\Pr[E|B]=\Pr[E|B_{TT}]\cdot\Pr[TT]+\Pr[E|B_{TH}]\cdot\Pr[TH]+\Pr[E|B_H]\cdot\Pr[H]$

$=1\cdot\frac{1}{4}+\Pr[E|B]\cdot\frac{1}{4}+\Pr[E|C]\cdot\frac{1}{2}$

$\downarrow$ 0

so $\Pr[E|B]=1/3$

WOP: $4a^3+2b^3=c^3$ – Prove no solutions.

Let S be the set of positive integers of a ST b, c satisfy eqn.

Assume S is not empty – by WOP – lowest element a_0

$4a_0^3+2b_0^3=c_0^3$

c_0 is even.

⋮

a_0, b_0, c_0 are all even. Thus there exists another solution that is smaller, which contradicts assumption, so S must be empty.

Structural Induction – MVAOs

Base: A single prop variable, constants T, F

Constructor: If $G, H \in$ MVAO, then (G AND H) and (G or H) are MVAOs

Def: False decreasing

Prove every MVAO is false decreasing $\rightarrow P(G)$

Base: Making any var false will increase falseness

T/F – stays the same

Constructor: ~~[illegible]~~ Assume P(G) and P(H) Prove $\rightarrow$ P(G AND H)

P(G AND H) – if some assignment makes G false (or H), then it will continue to make (G AND H) false.

---

inf binary string $b=b_0b_1b_2\ldots b_n\ldots$ OK when

$$b_i=\begin{cases}0 & \text{if } i\in\{0,1,4,9,\ldots n^2\ldots\}\\ 0 \text{ or } 1 & \text{otherwise}\end{cases}$$

a) Prove set of ok strings is uncountable.

Take ~~[illegible]~~

$g(b)::=b_0b_1b_4\ldots b_{n^2}\ldots$

$\downarrow$

A bijection to $\{0,1\}^\omega$, which is uncountable.

b) Prove that a set w/ an uncountable subset must be uncountable.

If $A\subseteq B$, then B surj A.

Since A is uncountable, $\mathbb{N}$ does not surj A.

So $\mathbb{N}$ does not surj B, and B is uncountable.

GCD: Prove gcd(m, n) is the minimum pos. value of any integer linear combo of m, n.

Pf: gcd(m, n) is a lin combo of m, n, and is positive. Also gcd(m, n) divides all lin combos, so it must be the smallest.

Remainder Arith: ~~[illegible]~~

$\text{remainder}(9876^{3456784}(9^{99})^{5595}-6789^{341072}, 14)$

① Replace bases w/ remainders

$\text{rem}(9876,14)=6\ldots$

$\text{remainder}(6^{345\ldots}(9^{99})^{5595}-13^{342254}, 14)$

② Look at patterns of exponentiation

$\text{rem}(6,14)=6$

$\text{rem}(6^2,14)=8$

⋮

replace terms.

$\text{remainder}(6\cdot 1-13, 14)=7$

If a simple graph has e edges, v vertices, k components then it has $\geq e-v+k$ cycles.

$P(e)::=\forall v,c,k\in\mathbb{N}$

$c\geq e-v+k$

Base: $e=0$

$v=k\rightarrow c\geq 0$

Inductive step: Let G be a graph w/ e+1 edges. Take away an edge. c, v

Case 1 – Edge is part of a cycle.

$c-1\geq e-v+k$

So adding 1 edge holds.

Case 2 – Edge is not part of a cycle., so $k\uparrow 1$

$c\geq e-v+(k+1)$

$c\geq(e+1)-v+k$

P(e+1) holds in both cases, completing the inductive step.

limsup

$f(n)=1+\left(n\sin\left(\frac{n\pi}{2}\right)\right)^2 \quad g(n)=3n$

$\limsup_{n\to\infty}\frac{f}{g}=\frac{1+(n^2\sin^2(\frac{n\pi}{2}))}{3n}$ → grows inf w/ n^2.

$\frac{g}{f}=\frac{3n}{1+\sin^2}$ ↝ whole thing → ∞

↝ can go to 0

Prove something is a preserved invariant

Transition ::= $(r,s,a)\to(\frac{r}{2}, s\cdot 2, a^2)$

Assume $P(r,s,a)$ and $(r,s,a)\to(r',s',a')$

Get (r',s',a') in terms of (r,s,a)

show $P(r',s',a')$ holds.

Use the WOP to prove that $T_n=\frac{3^{n+1}+(-1)^n}{4}$ given $T_n=2T_{n-1}+3T_{n-2}$

$P(n)::=\left[T_n=\frac{3^{n+1}+(-1)^n}{4}\right]$

let $C::=\{n\geq 0 \mid \neg P(n)\}$ let m be the smallest elem. by WOP

First note $T_0=1$, $T_1=2=\frac{3^{1+1}+(-1)^1}{4}$ ✓

so $m>1$. Since $m-1$, $m-2$ are smaller, then $P(m-1)$, $P(m-2)$

$T_m=2T_{m-1}+3T_{m-2}$ ⋯ math

So $P(m)$ is true, contradicting def of m.

C must be empty, which proves that $P(n)$ is true for all $n\geq 0$.

↳ can be solved w/ strong induction

Strong Induction: 12¢ postage w/ 3¢/7¢

$S(n)::=n+12$¢ using only 3¢, 7¢

Base $S(0)=12$¢ ✓

$S(1)=13$¢ ✓

$S(2)=14$¢ ✓

Inductive Step: Assume $S(0)\to S(n)$ ~~Prove~~ → $S(n+1)$

for $n\geq 2$

$S(n-2)$ must be true. So $n+10$¢ can be made.

Add 3¢ to achieve $S(n+1)$. So by strong induction, $S(n)$ holds for $n\in\mathbb{Z}^+$.

Proving equivalences of sets (chains of IFFs)

$A=(A-B)\cup(A\cap B)$

let $x\in(A-B)\cup(A\cap B)$ IFF

$(x\in(A-B))$ or $x\in(A\cap B)$

$(x\in A$ and $\neg x\in B)$ or $(x\in A$ AND $x\in B)$

* R AND (P or Q) IFF (R AND P) or (R AND Q)

$x\in A$ AND $(x\in\bar{B}$ or $x\in B)$

1 ↝ Truth by validity

so $x\in A$

EXAM #4

Strict partial order IFF R is transitive & irreflexive.
IFF R is transitive & asymmetric

Weak partial order IFF R is transitive, antisymmetric, and reflexive

Equivalence relation IFF R is reflexive, symmetric, and transitive

- Reflexive – $\forall x \in A \; xRx$ (all vertices have self loops)
- Irreflexive – $\neg[\exists x \in A, xRx]$ no self loops
- symmetry $[\forall x,y \in A, xRy \to yRx]$ if there is an edge from x to y, $\exists$ y to x
- asymmetry $[xRy \to \neg(yRx)]$ No self loops + at most one directed edge
- antisym $[\forall x \neq y \in A, xRy \to \neg(yRx)]$ Can have self loops, but otherwise, at most directed edge
- transitive $[xRy \;\&\; yRz \to xRz]$
- Linear xRy or yRx when $x \neq y$ Given 2 vertices, always an edge

Sums

$$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x} \qquad \sum_{i=1}^{n} i^2 = \frac{(2n+1)(n+1)n}{6}$$

$$\sum_{i=0}^{n} x^i = \frac{1-x^{n+1}}{1-x} \qquad \sum_{i=1}^{m} nx^n = \frac{(mx-m-1)x^{m+1}+x}{(1-x)^2}$$

$$H_n = \sum_{i=1}^{n} \frac{1}{i} \sim \ln(n) \qquad \sum_{i=1}^{m} x^n = \frac{n(n^m-1)}{n-1}$$

* empty relation = no arrows (SPO)

Asymptotic Notation

$f = o(g)$ $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$

upper bound

$f = O(g)$ $\limsup_{x\to\infty} \frac{|f(x)|}{g(x)} < \infty$ $\exists c \in \mathbb{N} \, \exists n_0 \in \mathbb{N} \, \forall n \geq n_0 \; c \cdot g(n) \geq |f(n)|$

$f \sim g$ $\lim_{x\to\infty} \frac{f}{g} = 1$

$f = \Theta(g) \to f = O(g)$ and $g = O(f)$

Simple graph – nonempty vertices, some set of edges (can be empty), no self loops
edges are incident to its endpoints

Complete graph – edge b/w every pair of vertices

Isomorphism: $\langle u - v \rangle \in E(G)$ iff $\langle f(u) - f(v) \rangle \in E(H)$, $V(G) \to V(H)$
Ex. f(a) ::= 2 f(b) ::= 3 ……

Bipartite graph: graph whose vertices can be divided into two sets
– * MATCHING

Walk – can have repeat vertices
path – no repeat vertices
Cycle – distinct except beginning/end

– For there to be a matching, there must be no bottleneck. HALL'S
bottleneck $|S| > |N(S)|$ → for some set S that contains elements of the domain, the neighbors is less than the subset itself.

Tree props
- If the tree has 2+ vertices, it has 2+ leaves
- Adding an edge b/w nonadjacent nodes creates a cycle

deg constrained → matching – If every s maps to $\geq d$ r and every r maps to $\leq d$ s > no b

• Degree constrained – if vertex degrees on the left are at least as large as those on the right, implies matching *

↳ degree constrained when $\deg(l) \geq \deg(r)$ for all l, r

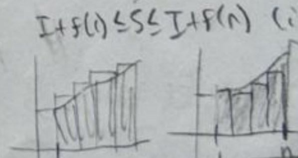
tree – connected, acyclic graph
– leaf is deg 1

forest → acyclic graph

If $S ::= \sum_{i=1}^{n} f(i)$

$I ::= \int_1^n f(x)dx$

$I + f(1) \leq S \leq I + f(n)$ (inc)



since $f(n) \geq f(1)$

MST: the gray thing
1. Start w/ random coloring, draw the min weight gray edge.
2. Turn all connected groups the same color
3. repeat

Isomorphism preserved things
- vertices w/ specific degrees
- OR of 2 props
- # cycles

pf by inductions
1. Base case
2. P(n) ::= ______
3. Prove $P(n) \to P(n+1)$

Algorithms – select min weight edge

Kruskals → ① select min weight edge ② Continue selecting w/o cycles

Prim's algorithm – Start w/ a single vertex, add edges to that tree (min weight)

Ex. relation $a=b+1$
- Not reflexive
- $a=2, b=1$ $a=3, b=2$
  $a=3, b=1$ doesn't work — not tr
- no self loops $a=2, b=1$ $a=1, b=2$ does not work

↓

asym

Ex Big O relation

$f=O(g)$ & $\neg(g=O(f))$

Not reflexive bc
$f=O(f)$ and $\neg(f=O(f))$ is F

Tr → $g=O(h)$ & $\neg(h=O(g))$ → $f=O(h)$ & $\neg(h=O(f))$?

well if $f=O(g)$ & $g=O(h)$, then $f=O(h)$

$\neg(g=O(f))$ implies g is ∞ larger than f (same for h, g).

So h must ∞ f, and $\neg(h=O(f))$.

Asym → irr, and also $g=O(f)$ & $\neg(f=O(g))$ contradicts.

Ex. set of functions is ordered by $\leq$ by

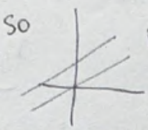
$[f \leq g] ::= \forall d \in D\ f(d) \leq g(d)$

$f(x)=ax+b$ in L

inf chain in L → $f_1 \geq f_2 \geq f_3 \ldots$

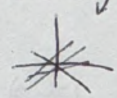
So $\forall_{d \in D}$ in $f_1(d) > \forall_{d \in D}$ in $f_2(d)$

So



b changes

inf antichain in L → no relationship



m changes

Ex - interest #5 on the pset

$100 \times 1.003^2 + 99 \times 1.003^1 + 98 \cdot 1.003^0 \ldots$

$S_n ::= \sum_{i=0}^{n-1} (100-i)(1.003)^{n-i-1}$

Ex. ~~AAA~~ ~ w/ strongly connected blocks → WPO

① reflexive
- self loops bc any vertex in a subgraph is reflexive

② antisym

If $A \to B$, then there cannot be $B \to A$, otherwise these vertices would be mutually connected and would be in the same component.

Ex. Bipartite matching pb
- Matching clubs to delegates → no student a member of ≥ 9 clubs
  → club has at least 13 members

Degree of each club → 13 Degree of every student → 9

degree constrained → no bottlenecks, matching g by Halls

& Define $L(G), R(G)$
& Look at subset $D \subseteq L(G)$

$\boxed{\text{Ex}}$ Round robin tourney ($2m$ teams, $2m-1$ days)

① Model as bipartite graph
$L(G)$ be $2m-1$ days
$R(G)$ be teams

For set D days $\subseteq L(G)$ let $N(D)$ be teams that won those days.

Bottleneck → Looking for some set where $|D| > |N(D)|$
But there must be a team that did not win on these days, which implies $|N(D)| \geq |D|$
By Halls, $\exists$ desired matching

Ex Holy Grail Pblm

$\frac{3}{4}$ days 2 gal ($\frac{1}{4}$, $\frac{1}{2}$ cache)

① Build $n-1$ gallon cache — $1/2n$ days, dropping off $\frac{n-1}{n}$

$D_n = \frac{1}{2n} + D_{n-1}$
$= \frac{H_n}{2}$

n gallons → $\frac{H_n}{2}$ days

Ex o/O

a) $f(n) = \frac{3n-7}{n+4}$ $g(n)=4$

$g=O(f)$ bc $\lim_{n\to\infty} \frac{4(n+4)}{3n-7} \to \frac{4}{3}$

$c \cdot f \geq |g(n)|$

$c \cdot \frac{3n-7}{n+4} \geq 4$

since $n \to \infty$, goes to 3

$c \geq 4/3$

so $c=2$

$n=15$

Ex limsup

$f(n) = 1 + \left(n \sin\left(\frac{n\pi}{2}\right)\right)^2$ $g(n)=3n$

$\limsup_{n\to\infty} \frac{f}{g} = \frac{1+n^2\sin^2\left(\frac{n\pi}{2}\right)}{3n}$ → grows infinitely larger than $3n$

$\limsup_{n\to\infty} \frac{g}{f} = \frac{3n}{1+n^2\sin^2\left(\frac{n\pi}{2}\right)}$ → can be ∞ when denom is just 1

Exam 3

- A strict B iff NOT (A surj B)
- A strict B iff $|A| < |B|$
- Countable → elements can be listed in order
  - countably infinite → IFF N bij C → $\mathbb{Z}^+, \mathbb{Z}, N \times N, \mathbb{Q}^+, \mathbb{Z} \times \mathbb{Z}, \mathbb{Q}$
  - countable → IFF N surj C
- Power Sets

Cantor → For any set A, A strict Pow(A)

$A_g ::= \{a \in A \mid a \notin g(a)\}$

*well defined = subset in pow(s)

A_g composed of a such that a is not in g(a)

If A_g is in range g, then $A_g = g(a_0)$

$a \in g(a_0)$ iff $a \in A_g$ iff $a \notin g(a)$

let $a = a_0$ → whoops

- Pow(N) is uncountable
- if U is uncountable and A surj U, then A is uncountable.
- If C is countable and C surj A, then A is countable.
- Diagonal argument

-Making sure elements on the diagonal are different such that

$r: A \to A$

$r(0) = 1$

$r(1) = 0$

$\vdots$

along diagonal, and all changed elements form a sequence not in original set/range!

Digraphs - G has nonempty set V(G) & E(G) can be empty

Walk - any cnxn of edges, vertices

-unique vertices → path

DAG G → no cycles

→ topological sort → | vertex list: a, b, c, d | vertex before any reachable vertices

→ chain - set of vertices that are reachable

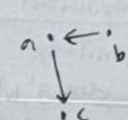
→ antichain → set of vertices not reachable

→ Greedy: tasks as soon as possible

→ depth k at step k

→ critical path = longest chain

→ Adjacency matrix



	a	b	c
a	0	0	1
b	1	0	0
c	0	0	0

A^k shows # of k-length paths from one vertex to the other

---

a | b : a divides b

- If a|b & a|c, → a|c
- a|b & a|c → a|sb+tc, ∀ s, t

Euclid's Algorithm gcd → Pulverizer

gcd(a,b) = gcd(b, rem(a,b))

a	b	rem	
125	45	35	= 125 − 2(45)
45	35	10	= 45 − 35 = 45 − [125 − 2(45)]
35	10	5	= 35 − 3(10) = [125 − 2(45)] − 3[45 − [125 − 2(45)]] = gcd(125, 45)
10	5	0	

Linear combo of a and b st + kb = gcd(t, b)

Multiplicative Inverse: $k^{-1} \cdot k \equiv 1 \pmod n$

→ k, n are rel prime

→ sn + tk = 1

→ Use pulverizer

if $a \equiv b \pmod n$

$m \equiv p \pmod n$

then $a \cdot m \equiv b \cdot p \pmod n$

$a + m \equiv b + p \pmod n$

- Fundamental Thm of Arithmetic: Every positive integer is a product of a unique weakly decreasing sequence of primes.

mod: $a \equiv b \pmod n$ iff $n \mid (a-b)$

iff rem(a,n) = rem(b,n)

gcd: gcd(au, av) = a gcd(u, v) & gcd divides any int combo of m, n

gcd(au, v) = gcd(u, v)

for a rel. prime to v

gcd(u−v, v) = gcd(u, v)

For pfs containing gcd(a,b), consider gcd(a,b) = sa + tb!

Ex. divis by 9

$10 \equiv 1 \pmod 9$

$10^k \equiv 1^k \pmod 9 \rightarrow d_k \cdot 10^k \equiv d^k \pmod 9$

is rep'd as $d_k \cdot 10^k + d_{k-1} \cdot 10^{k-1} \ldots d_1 \cdot 10 + d_0$

$(d_k 10^k \text{——} - d_0) \equiv d_k + d_{k-1} \cdots d_0 \pmod 9$

Ex. inverse (17 modulo 29) = $\boxed{12}$ pulverizer

↳ so $17x + 29y = \gcd(17, 29) = 1$

		rem
29	17	$12 = 29 - 17$
17	12	$5 = 17 - 12 = 17 - (29-17) = 2(17) - 29$
12	5	$2 = 12 - 2(5) = (29-17) - 2(2(17) - 29)$
5	2	$1 = 5 - 2(2) = (2(17) - 29) - 2((29-17) - 2(2(17)-29))$

$= 12(17) - 7(29)$

$1 = 12(17) - 7(29)$

factor of ± 29 factor of ± 17

Ex. Big # remainder

remainder$(9876^{3456789} (9^{99})^{5555} - 6789^{3414259}, 14)$

*REPLACE BASES w/ REMAINDERS

rem$(9876, 14) = 6$

rem$(6789, 14) = 13$

remainder$(6^{3456789} (9^{99})^{5555} - 13^{3414259})$

*Look at patterns of exponentiation

rem$(6, 14) = 6$

rem$(6^2, 14) = 8$

rem$(6^3, 14) = 6$

⋮

So rem$(6^{odd}) = 6$

similarly

rem$(9^1, 14) = 9$

rem$(9^2, 14) = 11$

⋮

rem$(9^{99}, 14) = 1$

so rem$(1^{5555}, 14) = 1$

rem$(13^1, 14) = 13$

rem$(13^2, 14) = 1$

rem$(13 \cdot (13^2)^{3456787\ldots/2}, 14)$

↓

1

So rem$(13, 14) = 13$

rem$(6 \cdot 1 - 13, 14) = $ rem$(-7, 14) = \boxed{7}$

Invariant → so like the invariant for the one problem was

$\gcd(a,b) = e \gcd(x,y)$ and you had to

prove it for $\gcd(a,b) = e\gcd(x,y) = e' \gcd(x', y')$

for all cases incl end case.

Ex. $\gcd(a,b) = e\gcd(x,y)$ for $(x,y,e) \to (\frac{x}{2}, \frac{y}{2}, 2e)$ if x, y even

$\gcd(\frac{x}{2}, \frac{y}{2}) = \gcd(x,y)/2$

$eg(x,y) = e \cdot 2 \cdot \frac{1}{2} \gcd(x', y')$ ✓

Prove if $a \equiv b \pmod{14}$ then $a \equiv b \pmod{70}$

$a \equiv b \pmod 5$

$14 | a-b$ $5 | a-b$

14, 5 are relatively prime

so $14 \cdot 5 | a-b$

so $a \equiv b \pmod{70}$

DAG problem

- Longest chain in greedy strategy is # of levels
  - antichain is # in same level

If S is infinite, then so is pow(S), so if pow(S) was countable, then N bij pow(S). S must surj N bc it is the smallest inf. set.

so S cannot surj pow(S).

Bijection w/ Z^+ and $Z^+ \times Z^+$:

(1,1) (1,2) (1,3)

(2,1) (2,2) (2,3)

(3,1) (3,2) (3,3)

⋱

ordering by sum and increasing first element creates bijection w/ Z^+

So Q^+, following, will be these (a,b) as $\frac{a}{b}$, deleting repeats.

Diagonal argument

Prove $\{1,2,3\}^\omega$ is uncountable.

well bc if N bij $\{1,2,3\}^\omega$, $\sigma: N \to \{1,2,3\}^\omega$

then lining up like

1	123123…
2	321 …
3	1 …
4	2 …
5	3 …
⋮	

Taking all elements

$D: \langle \bar{s}_{(0,0)} \ldots, \bar{s}_{(k,k)} \ldots \rangle$

$\bar{s}: \begin{cases} 2 & \text{if } x=1 \\ 1 & \text{if } x=3 \\ 3 & \text{if } x=2 \end{cases}$

or define relation $r\{1,2,3\} \to \{1,2,3\}$

$r(1) ::= 2$

$r(2) ::= 3$ $\quad$ diag$[n] ::= r(\sigma(n)[n])$

$r(3) ::= 1$ $\quad$ so diag$[n] \neq \sigma(n)[n]$ for all …

$Pr[R \geq x] \leq \frac{Ex[R]}{x}$ → Markov's Bound

→ Chebyshev's Thm

Random variable - total function whose domain is the sample space

Indicator random var - maps every outcome to 0 or 1

$var = p(1-p) = E(x^2) - E^2(x)$ → variance

Note: $Ex[R^2] = \sum_{\omega \in S} \overset{\nearrow \text{value}^2}{R^2(\omega)} Pr(\omega)$

$PDF_R(x) ::= \begin{cases} Pr[R=x] \\ 0 & \text{if } x \notin range(R) \end{cases}$

$Ex[R] ::= \underbrace{\sum_{\omega \in S} \underset{\downarrow \atop \text{value}}{R(\omega)} \; \underset{\downarrow \atop \text{Probability of value}}{Pr[\omega]}}_{\text{values in set}}$ → Expectation

$Ex[R] = \sum_{x \in Range R} x \cdot Pr[R=x]$

Bhanial Dist: $f_n(k) ::= \binom{n}{k} 2^{-n}$

$Ex[I_A] = 1 \cdot Pr[I_A = 1] + 0 \cdot Pr[I_A = 0]$ → Expectation of an indicator variable

$= Pr[I_A = 1]$

$Ex[R|A] ::= \sum_{r \in range} r \cdot Pr[R=r|A]$ → Conditional expectation

$Ex[B] = \sum_i Ex[R|A_i] Pr[A_i]$ → Law of Total Expectation

$Ex[R_1 + R_2] = Ex[R_1] + Ex[R_2]$ → Linearity of Expectation, for random variables R_1, R_2

$Ex[R_1 \cdot R_2] = Ex[R_1] \cdot Ex[R_2]$ → R_1, R_2 independent

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$Pr[E \cup F] = Pr[E] + Pr[F]$ for disjoint sets (no overlapping elements)

$Pr[X|Y] = \frac{Pr[X \cap Y]}{Pr[Y]}$ → Conditional probability

↳ given Y

$Pr[B|A] = \frac{Pr[A|B] \cdot Pr[B]}{Pr[A]}$ → Bayes' Rule