EXAM #4

Strict partial order IFF R is transitive & irreflexive.
IFF R is transitive & asymmetric

Weak partial order IFF R is transitive, antisymmetric, and reflexive

Equivalence relation IFF R is reflexive, symmetric, and transitive

- Reflexive – $\forall x \in A \; xRx$ (all vertices have self loops)
- Irreflexive – $\neg[\exists x \in A, xRx]$ no self loops
- symmetry $[\forall x,y \in A, xRy \to yRx]$ if there is an edge from x to y, $\exists$ y to x
- asymmetry $[xRy \to \neg(yRx)]$ No self loops + at most one directed edge
- antisym $[\forall x \neq y \in A, xRy \to \neg(yRx)]$ Can have self loops, but otherwise, at most directed edge
- transitive $[xRy \;\&\; yRz \to xRz]$
- Linear xRy or yRx when $x \neq y$ Given 2 vertices, always an edge

Simple graph – nonempty vertices, some set of edges (can be empty), no self loops
edges are incident to its endpoints

Complete graph – edge b/w every pair of vertices

Isomorphism: $\langle u - v\rangle \in E(G)$ iff $\langle f(u) - f(v)\rangle \in E(H)$, $V(G) \to V(H)$
Ex. $f(a) ::= 2$ $f(b) ::= 3$ ……

Bipartite graph: graph whose vertices can be divided into two sets

– *MATCHING

Walk – can have repeat vertices
path – no repeat vertices
Cycle – distinct except beginning/end

HALL'S
– For there to be a matching, there must be no bottleneck.
bottleneck $|S| > |N(S)|$ → for some set S that contains elements of the domain, the neighbors is less than the subset itself.

deg constrained → matching – If every s maps to $\geq d$ r and every r maps to $\leq d$ s > no b

- Degree constrained – if vertex degrees on the left are at least as large as those on the right, implies matching *

tree – connected, acyclic graph
– leaf is deg 1

forest → acyclic graph

↳ degree constrained when $\deg(l) \geq \deg(r)$ for all l, r

MST: the gray thing
1. start w/ random coloring, draw the min weight gray edge.
2. Turn all connected groups the same color
3. repeat

Algorithms – select min weight edge

Kruskals → ① select min weight edge ② Continue selecting w/o cycles

Isomorphism preserved things
- vertices w/ specific degrees
- OR of 2 props
- # cycles

Prim's algorithm – Start w/ a single vertex, add edges to that tree (min weight)

Sums

$$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x} \qquad \sum_{i=1}^{n} i^2 = \frac{(2n+1)(n+1)n}{6}$$

$$\sum_{i=0}^{n} x^i = \frac{1-x^{n+1}}{1-x} \qquad \sum_{i=1}^{m} nx^n = \frac{(mx-m-1)x^{m+1}+x}{(1-x)^2}$$

$$H_n = \sum_{i=1}^{n} \frac{1}{i} \sim \ln(n) \qquad \sum_{i=1}^{m} x^n = \frac{n(n^m-1)}{n-1}$$

*empty relation = no arrows (SPO)

Asymptotic Notation

$f = o(g)$ $\quad \lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$

→ upper bound

$f = O(g)$ $\quad \limsup_{x\to\infty} \frac{|f(x)|}{g(x)} < \infty$ $\quad \exists c \in \mathbb{N} \, \exists n_0 \in \mathbb{N} \, \forall n \geq n_0 \; c \cdot g(n) \geq |f(n)|$

$f \sim g$ $\quad \lim_{x\to\infty} \frac{f}{g} = 1$

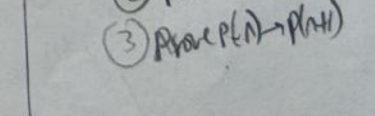
$f = \Theta(g) \to f = O(g)$ and $g = O(f)$

Tree props
- If the tree has 2+ vertices, it has 2+ leaves
- Adding an edge b/w nonadjacent nodes creates a cycle

If $S ::= \sum_{i=1}^{n} f(i)$

$I ::= \int_1^n f(x)\,dx$

$I + f(1) \leq S \leq I + f(n)$ (inc)



pf by inductions
1. Base case
2. $P(n) ::=$ ____
3. Prove $P(n) \to P(n+1)$

Ex. relation $a = b+1$
- Not reflexive
- $a=2, b=1$ $a=3, b=2$
  $a=3, b=1$ doesn't work — not tr
- no self loops $a=2, b=1$ $a=1, b=2$ does not work
  ↓
  asym

Ex. set of functions is ordered by $\leq$ by

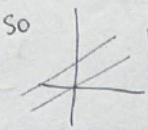
$[f \leq g] ::= \forall d \in D \; f(d) \leq g(d)$

$f(x) = ax + b$ in L

inf chain in $L \to f_1 \geq f_2 \geq f_3 \ldots$

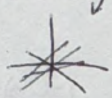
So $\forall_{d \in D}$ in $f_1(d) > \forall_{d \in D}$ in $f_2(d)$

So



b changes

inf antichain in $L \to$ no relationship



m changes

Ex Big O + relation

$f = O(g)$ & $\neg(g = O(f))$

Not reflexive b/c
$f = O(f)$ and $\neg(f = O(f))$ is F

Tr → $g = O(h)$ & $\neg(h = O(g)) \to f = O(h)$ & $\neg(h = O(f))$?

well if $f = O(g)$ & $g = O(h)$, then $f = O(h)$

$\neg(g = O(f))$ implies g is ∞ larger than f (same for h, g).

So h must ∞ f, and $\neg(h = O(f))$.

Asym → irr, and also $g = O(f)$ & $\neg(f = O(g))$ contradicts.

Ex - interest #5 on the pset

$100 \times 1.003^2 + 99 \times 1.003^1 + 98 \cdot 1.003^0 \ldots$

$S_n ::= \sum_{i=0}^{n-1} (100 - i)(1.003)^{n-i-1}$

Ex. $\sim$ w/ strongly connected blocks → WPO

① reflexive
- self loops b/c any vertex in a subgraph is reflexive

② antisym

If $A \to B$, then there cannot be $B \to A$, otherwise these vertices would be mutually connected and would be in the same component.

Ex. Bipartite matching pb
- Matching clubs to delegates → no student is member of ≥ 9 clubs
  → club has at least 13 members

Degree of each club → 13 Degree of every student → 9

degree constrained → no bottlenecks, matching g by Halls

Define $L(G), R(G)$
Look at subset $D \subseteq L(G)$

$\boxed{\text{Ex}}$ Round robin tourney (2m teams, 2m-1 days)

① Model as bipartite graph
$L(G)$ be $2m-1$ days
$R(G)$ be teams

For set D days $\subseteq L(G)$ let $N(D)$ be teams that won those days.

Bottleneck → Looking for some set where $|D| > |N(D)|$

But there must be a team that did not win on these days, which implies $|N(D)| \geq |D|$

By Halls, ∃ desired matching

Ex Holy Grail Pblm

$\frac{3}{4}$ days 2 gal ($\frac{1}{4}$, $\frac{1}{2}$ cache)

n gallons → $\frac{H_n}{2}$ days

① Build $n-1$ gallon cache − $\frac{1}{2n}$ days, dropping off $\frac{n-1}{n}$

$D_n = \frac{1}{2n} + D_{n-1}$
$= \frac{H_n}{2}$

Ex o/O

a) $f(n) = \frac{3n-7}{n+4}$ $g(n) = 4$

$g = O(f)$ b/c $\lim_{n \to \infty} \frac{4(n+4)}{3n-7} \to \frac{4}{3}$

$c \cdot f \geq |g(n)|$

$c \cdot \frac{3n-7}{n+4} \geq 4$
since $n \to \infty$, goes to 3

$c \geq 4/3$

so $c = 2$

$n = 15$

Ex limsup

$f(n) = 1 + \left(n \sin\left(\frac{n\pi}{2}\right)\right)^2$ $g(n) = 3n$

$\limsup_{n \to \infty} \frac{f}{g} = \frac{1 + n^2 \sin^2\left(\frac{n\pi}{2}\right)}{3n}$ → grows infinitely larger than $3n$

$\limsup_{n \to \infty} \frac{g}{f} = \frac{3n}{1 + n^2 \sin^2\left(\frac{n\pi}{2}\right)}$ → can be ∞ when denom is just 1

Exam 3

- A strict B iff NOT (A surj B)
- A strict B iff $|A|<|B|$
- countable ⇒ elements can be listed in order
  - countably infinite → IFF N bij C → $Z^+, Z, N\times N, Q^+, Z\times Z, Q$
  - countable → IFF N surj C
- Power Sets

Cantor → For any set A, A strict Pow(A)

$A_g ::= \{a \in A \mid a \notin g(a)\}$

* well defined = subset in pow(S)

A_g composed of a such that a is not in g(a)

if A_g is in range g, then $A_g = g(a_0)$

$a \in g(a_0)$ iff $a \in A_g$ iff $a \notin g(a)$

let $a = a_0$ → whoops

- Pow(N) is uncountable
- if U is uncountable and A surj U, then A is uncountable.
- If C is countable and C surj A, then A is countable.
- Diagonal argument

-Making sure elements on the diagonal are different such that

$r: A \to A$

$r(0)=1$

$r(1)=0$

$\vdots$

along diagonal, and all changed elements form a sequence not in original set/range!

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a|b : a divides b

- If a|b & a|c, → a|c
- a|b & a|c → a|sb+tc, ∀ s, t

Euclid's Algorithm gcd → Pulverizer

gcd(a,b) = gcd(b, rem(a,b))

a	b	rem	
125	45	35	= 125−2(45)
45	35	10	= 45−35 = 45−[125−2(45)]
35	10	5	= 35−3(10) = [125−2(45)] − 3[45−[125−2(45)]] = gcd(125, 45)
10	5	0	

↓

Linear combo of a and b $st + kb = \gcd(t,b)$

Multiplicative Inverse: $k^{-1} \cdot k \equiv 1 \pmod n$

→ k, n are rel prime

→ $sn + tk = 1$

→ Use pulverizer

if $a \equiv b \pmod n$

$m \equiv p \pmod n$

then $a \cdot m \equiv b \cdot p \pmod n$

$a + m \equiv b + p \pmod n$

- Fundamental Thm of Arithmetic: Every positive integer is a product of a unique weakly decreasing sequence of primes.

mod: $a \equiv b \pmod n$ iff $n \mid (a-b)$

iff $rem(a,n) = rem(b,n)$

gcd: $\gcd(au, av) = a\gcd(u,v)$ & gcd divides any int combo of m, n

$\gcd(au, v) = \gcd(u,v)$

for a rel. prime to v

$\gcd(u-v, v) = \gcd(u,v)$

For pfs containing gcd(a,b), consider gcd(a,b) = sa+tb!

Digraphs - G has nonempty set V(G) & E(G)
can be empty

Walk - any cnxn of edges, vertices

- unique vertices → path

DAG G → no cycles

→ topological sort →

vertex list	
a	vertex before any reachable vertices
b	
c	
d	

→ chain - set of vertices that are reachable

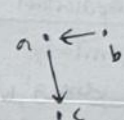
→ antichain → set of vertices not reachable

→ Greedy: tasks as soon as possible

→ depth k at step k

→ Critical path = longest chain

→ Adjacency matrix



	a	b	c
a	0	0	1
b	1	0	0
c	0	0	0

A^k shows # of k-length paths from one vertex to the other

Ex. divis by 9

$10 \equiv 1 \pmod 9$

$10^k \equiv 1^k \pmod 9 \rightarrow d_k \cdot 10^k \equiv d^k \pmod 9$

is rep'd as $d_k \cdot 10^k + d_{k-1} \cdot 10^{k-1} \ldots d_1 \cdot 10 + d_0$

$(d_k 10^k \text{———} - d_0) \equiv d_k + d_{k-1} \ldots d_0 \pmod 9$

Invariant → so like the invariant for the one problem was $\gcd(a,b) = e \gcd(x,y)$ and you had to prove it for $\gcd(a,b) = e \gcd(x,y) = e' \gcd(x', y')$ for all cases incl end case.

Ex. $\gcd(a,b) = e \gcd(x,y)$ for $(x,y,e) \rightarrow (\frac{x}{2}, \frac{y}{2}, 2e)$ if x,y [illegible]

$\gcd(\frac{x}{2}, \frac{y}{2}) = \gcd(x,y)/2$

$e g(x,y) = e \cdot 2 \cdot \frac{1}{2} \gcd(x', y')$ ✓

Ex. inverse 17 modulo 29 = $\boxed{12}$ pulverizer

↳ so $17x + 29y = \gcd(17, 29) = 1$

		rem
29	17	$12 = 29 - 17$
17	12	$5 = 17 - 12 = 17 - (29 - 17) = 2(17) - 29$
12	5	$2 = 12 - 2(5) = (29 - 17) - 2(2(17) - 29)$
5	2	$1 = 5 - 2(2) = (2(17) - 29) - 2((29 - 17) - 2(2(17) - 29))$

$= 12(17) - 7(29)$

$1 = 12(17) - 7(29)$

factor of ± 29 factor of ± 17

Ex. Big # remainder

remainder$(9876^{3456789} (9^{99})^{5555} - 6789^{3414259}, 14)$

*REPLACE BASES w/ REMAINDERS

rem$(9876, 14) = 6$

rem$(6789, 14) = 13$

remainder$(6^{3456789} (9^{99})^{5555} - 13^{3414259})$

*Look at patterns of exponentiation

rem$(6, 14) = 6$

rem$(6^2, 14) = 8$

rem$(6^3, 14) = 6$

⋮

So rem$(6^{odd}) = 6$

similarly

rem$(9^1, 14) = 9$

rem$(9^2, 14) = 11$

⋮

rem$(9^{99}, 14) = 1$

so rem$(1^{5555}, 14) = 1$

rem$(13^1, 14) = 13$

rem$(13^2, 14) = 1$

rem$(13 \cdot (13^2)^{3456787/2}, 14)$

↓ 1

So rem$(13, 14) = 13$

rem$(6 \cdot 1 - 13, 14) = $ rem$(-7, 14) = \boxed{7}$

Prove if $a \equiv b \pmod{14}$ then $a \equiv b \pmod{70}$

$a \equiv b \pmod 5$

$14 | a-b$ $5 | a-b$

14, 5 are relatively prime

so $14 \cdot 5 | a-b$

So $a \equiv b \pmod{70}$

DAG problem

- Longest chain in greedy strategy is # of levels
  - antichain is # in same level

If S is infinite, then so is pow(S), so if pow(S) was countable, then N bij pow(S). S must surj N bc it is the smallest inf set.

so S cannot surj pow(S).

Bijection w/ Z^+ and $Z^+ \times Z^+$:

~~(1,1)~~ ~~(1,2)~~ ~~(1,3)~~

~~(2,1)~~ ~~(2,2)~~ (2,3)

~~(3,1)~~ (3,2) (3,3)

⋱

ordering by sum and increasing first element creates bijection w/ Z^+

So Q^+, following, will be these (a,b) as $\frac{a}{b}$, deleting repeats.

Diagonal argument

Prove $\{1,2,3\}^\omega$ is uncountable.

well bc if N bij $\{1,2,3\}^\omega$, $\sigma: N \rightarrow \{1,2,3\}^\omega$

then lining up like

1	123123...
2	321 ----
3	1 --
4	2 ----
5	3 ----
⋮	

Taking all elements

$D: \langle \overline{S_{(0,0)}} \ldots, \overline{S_{(k,k)}} \ldots \ldots \rangle$

$\overline{S}: \begin{cases} 2 \text{ if } x=1 \\ 1 \text{ if } x=3 \\ 3 \text{ if } x=2 \end{cases}$

or define relation $r\{1,2,3\} \rightarrow \{1,2,3\}$

$r(1) ::= 2$

$r(2) ::= 3$ $diag[n] ::= r(\sigma(n)[n])$

$r(3) ::= 1$ so $diag[n] \neq \sigma(n)[n]$ for all [illegible]

$Pr[R \geq x] \leq \frac{Ex[R]}{x}$ → Markov's Bound

Random variable - total function whose domain is the sample space

→ Chebyshev's Thm

Indicator random var - maps every outcome to 0 or 1

$var = p(1-p) = E(x^2) - E^2(x)$ → variance

Note: $Ex[R^2] = \sum_{\omega \in S} \overset{\text{value}}{R^2(\omega)} Pr(\omega)$

$PDF_R(x) ::= \begin{cases} Pr[R=x] \\ 0 & \text{if } x \notin range(R) \end{cases}$

$Ex[R] ::= \underbrace{\sum_{\omega \in S} \underset{\text{Value}}{R(\omega)} \underset{\text{Probability of value}}{Pr[\omega]}}_{\text{values in set}}$ → Expectation

$Ex[R] = \sum_{x \in Range R} x \cdot Pr[R=x]$

Binomial Dist: $f_n(k) ::= \binom{n}{k} 2^{-n}$

$Ex[I_A] = 1 \cdot Pr[I_A=1] + 0 \cdot Pr[I_A=0]$ → Expectation of an indicator variable
$= Pr[I_A=1]$

$Ex[R|A] ::= \sum_{r \in range} r \cdot Pr[R=r|A]$ → Conditional expectation

$Ex[R] = \sum_i Ex[R|A_i] Pr[A_i]$ → Law of Total Expectation

$Ex[R_1+R_2] = Ex[R_1] + Ex[R_2]$ → Linearity of Expectation, for random variables R_1, R_2

$Ex[R_1 \cdot R_2] = Ex[R_1] \cdot Ex[R_2]$ → R_1, R_2 independent

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$Pr[E \cup F] = Pr[E] + Pr[F]$ for disjoint sets (no overlapping elements)

$Pr[X|Y] = \frac{Pr[X \cap Y]}{Pr[Y]}$ → Conditional probability
↳ given Y

$Pr[B|A] = \frac{Pr[A|B] \cdot Pr[B]}{Pr[A]}$ → Bayes' Rule