Linear Algebra

- Vectors
  - linear dependency, basis, norm, projection
- Matrix
  - linear transformation, mat-vect, mat-mat,
    eigenvalues, eigenvectors, page rank

Use Case:
Loss function, Covariance matrix, SVM, PCA, SVD,
Image representation as tensors, convoluting & image
processing, etc

Vector

What is a vector?

CS View:

list of numbers, data, etc (list, array)

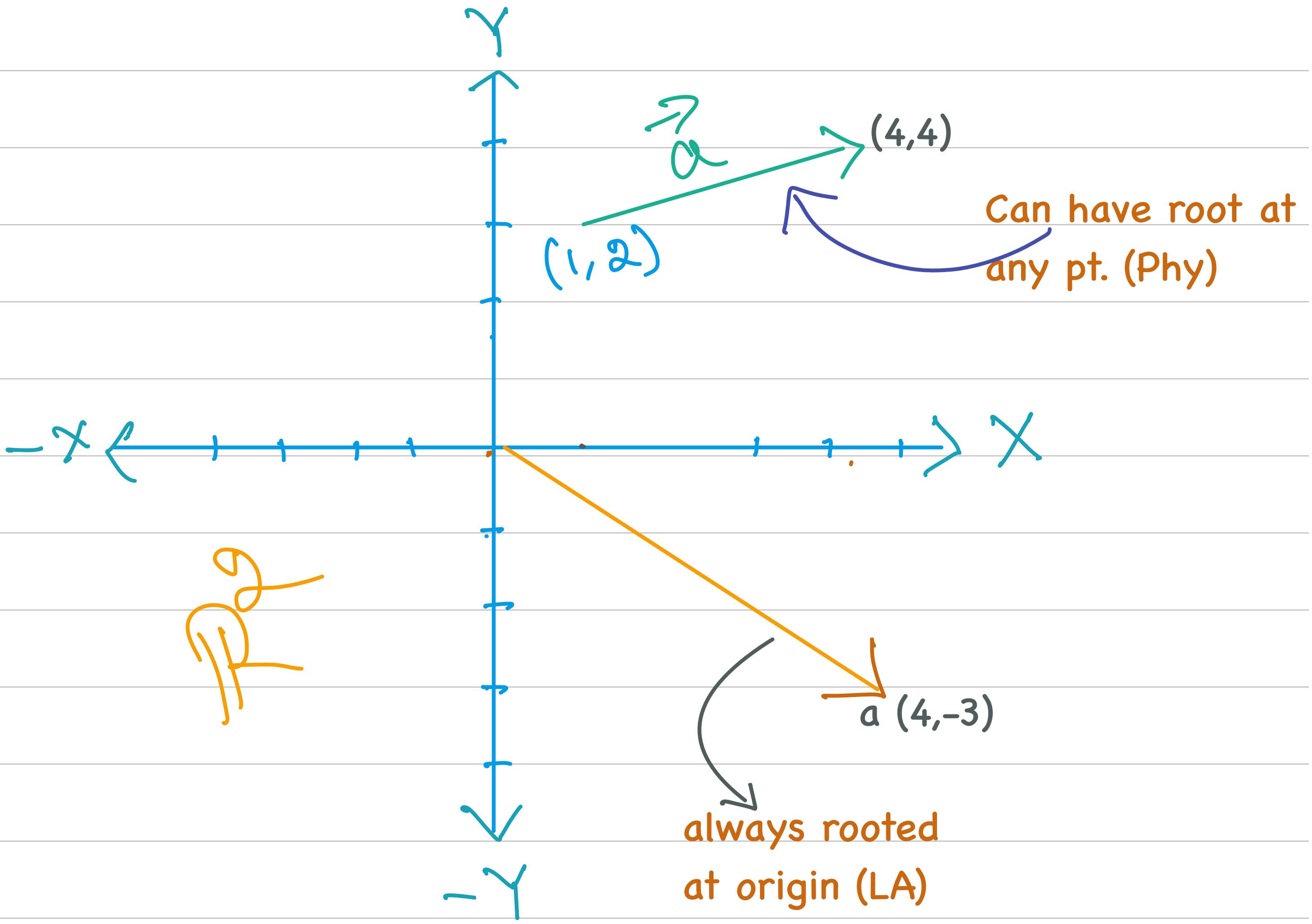
Math/Physics:

- A geometric object with magnitude and direction
- An element of a vector space
    - Vector space: a set V together with vector addition and scalar multiplication that satisfy associativity, commutativity, identity, inverse, distributivity

Data Science:

- A set of features of a data point

Consider a Coordinate system:



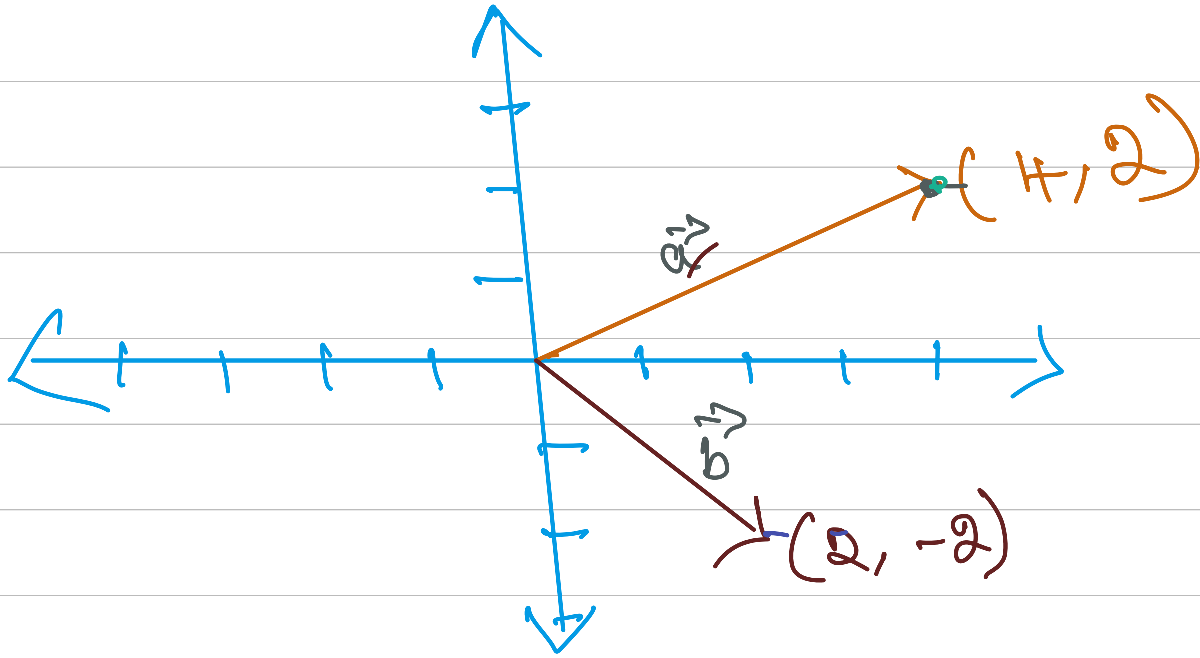
$$\vec{a} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$$

$$\vec{a} = 4\hat{i} - 3\hat{j}$$
$$= 4\begin{pmatrix} 1 \\ 0 \end{pmatrix} - 3\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

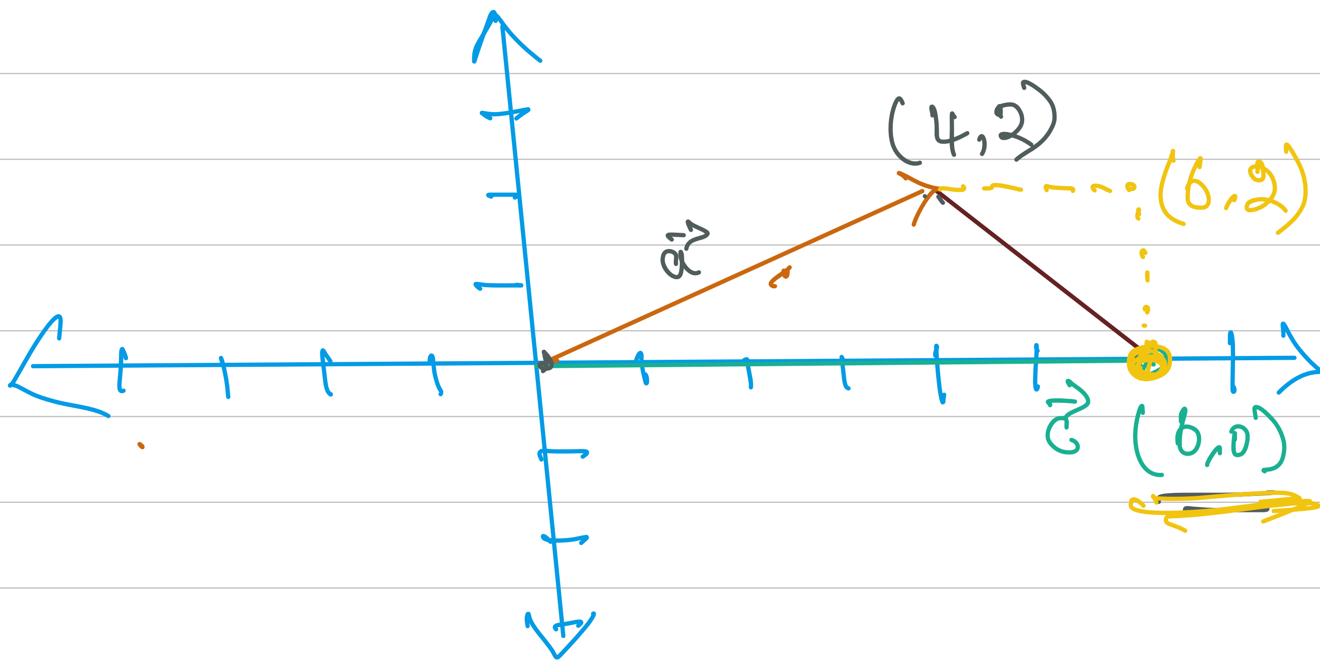
Let $\vec{c} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$, *then*

$$\vec{a} + \vec{c} = \begin{pmatrix} 4 \\ -3 \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ -1 \end{pmatrix}$$

$$2\vec{c} = 2\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$$



$\vec{a} + \vec{b} = (6, 0)$



Scalar

$\mathbb{R}$

- An element of a field.
- Field is a set on which addition, subtraction, multiplication and division are defined
- An object with only magnitude.
- A quantity that scales a vector

Vector Space

A vector space over a field F is a set V that is closed under finite vector addition and scalar multiplication.

Let V be a vector space, then the following conditions hold.

For all $x, y, z \in V \;\; \& \; a, b \in F$

1. Commutate: x + y = y + x
2. Associativity: (x + y) + z = x + (y + z)
3. Additive identity: 0 + x = x + 0 = x
4. Existence of additive inverse: x + (- x) = 0
5. Associativity of Scalar multiplication: a (b x) = (ab) x
6. Distributivity of scalar sums: (a + b) x = a x + b x
7. Distributivity of vector sums: a (x + y) = ax + ay
8. Scalar multiplicative identity: 1*x = x

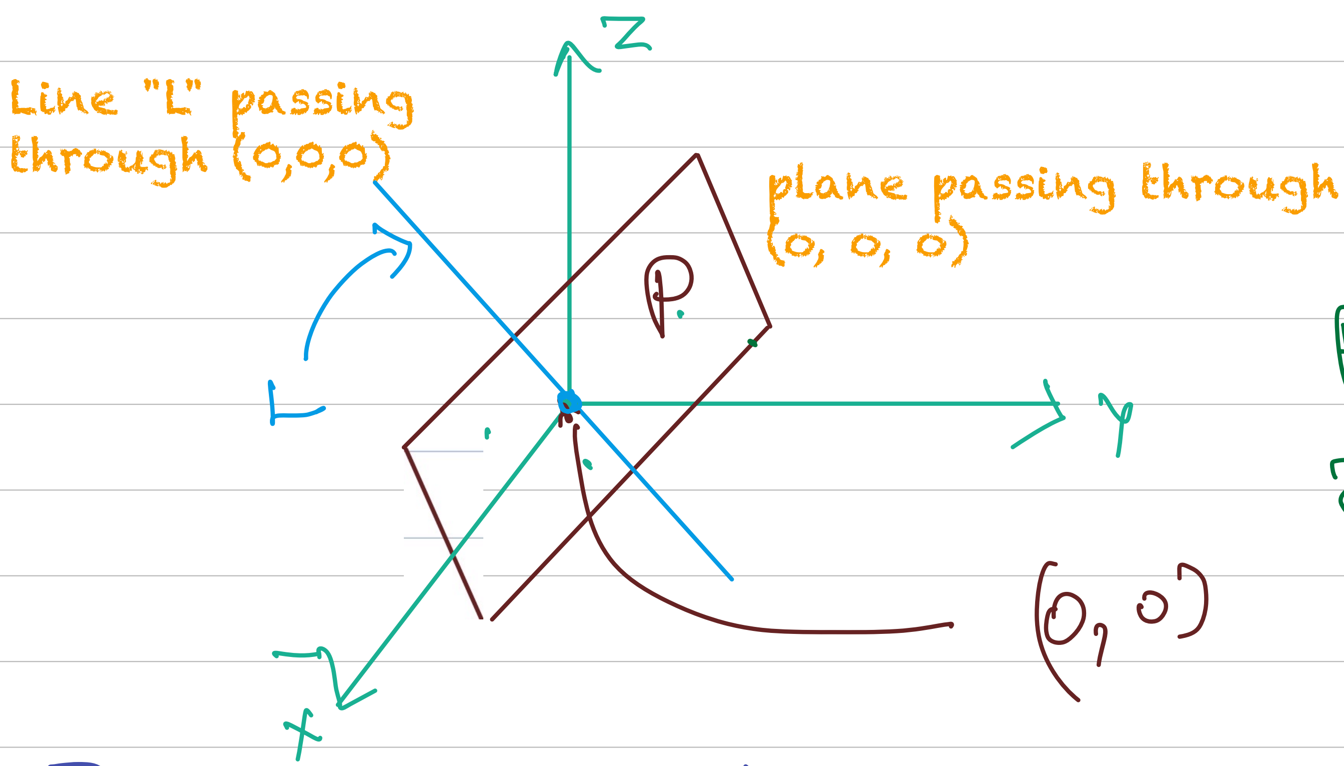
Linear subspace

Let V be a vector space over a field F, and W be a subset of V. Then, W is said to be a linear subspace of V if under of the same operations of V, W is also a vector space over F.

i.e., $\alpha w_1 + \beta w_2 \in W, \quad \forall \alpha, \beta \in F, \quad w_1, w_2 \in W$

Example

Let $V := \mathbb{R}^3$: $P \subseteq V$ & $L \subseteq V$



$P \subseteq \mathbb{R}^3$

$? \; L \subseteq P$

Is it a subspace?

Q: How about $P \cup L$? Q1: Y/N

" " $P \cap L$? Q2: Y/N

Kernel of a linear map

Let V and W be two vector spaces, and L be a linear map defined as -

$$L : V \to W$$

The kernel of the linear map (null space) is the linear subspace of the domain (V) of the map which is mapped to the zero vector, that is,

$$\ker(L) = \{v \in V | L(v) = 0\}.$$

Suppose A is a (m x n) matrix with coefficient in the field F. The kernel of the matrix is the set of solutions to the system Ax=0, where 0 is a zero-vector.

$$N(A) = Null(A) = \ker(A) = \{x \in F^n \mid Ax = 0\}$$

Span

Let S be a set of vectors. The span of S can be defined as the set of all finite linear combinations of elements of S.

Linearly Dependent vectors

Let $v_1, \; v_2, \; v_3, \ldots, v_n$

be a collection of vectors. The vectors are said to be linearly dependent, if there exist scalars $a_1, \; a_2, \; a_3, \ldots, a_n$ not all zero such that

$$a_1 v_1 + \ldots + a_n v_n = \overrightarrow{0}$$

Remark

Suppose a scalar, say "a_1' is nonzero, then

$$v_1 = -\frac{a_2}{a_1}\overrightarrow{v}_2 + \ldots + \frac{-a_n}{a_1}\overrightarrow{v}_n$$

$\Longrightarrow$ v_1 is a linear combination of $\{\overrightarrow{v}_2, \ldots, \overrightarrow{v}_n\}$

Linearly Independent

The set of vectors $v_1, \; v_2, \; v_3, \ldots, v_n$
are said to be linearly independent if Eq. (1) can be satisfied only with

$$a_i = 0, i = 1, \ldots, n.$$

$\Rightarrow$ No vector in the collection can be written as a linear combination of other vectors

Example:

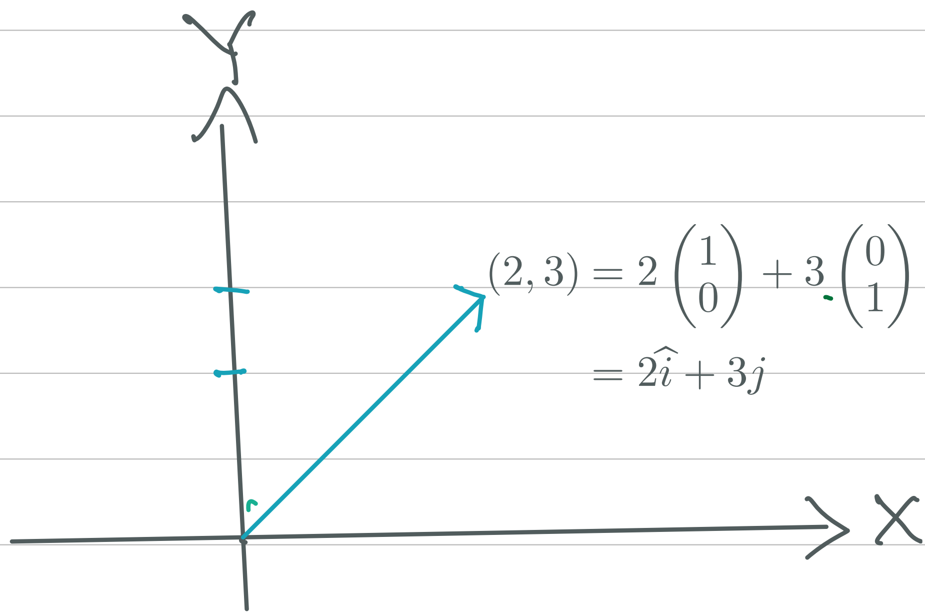
Basis

The basis of a vector space is the set of all linearly independent vectors that span the full space.

Q: Can we have more than one basis?

Basis of a vector space:

The basis of a vector space over a field is a linearly independent subset of the space that spans the space.



$$(2,3) = 2\begin{pmatrix}1\\0\end{pmatrix} + 3\begin{pmatrix}0\\1\end{pmatrix}$$
$$= 2\hat{i} + 3j$$

$\hat{i} = (1,0)$ - unit vector in x-direction

$\hat{j} = (0,1)$ - unit vector in y-direction

Any vector in xy-plane can be written as a linear combination of $\hat{i}$ & $\hat{j}$

$$\vec{d} = a_1\hat{i} + a_2\hat{j}, \quad a_1, a_2 \in \mathbb{R}$$

Q: other than (1, 0) & (0, 1), can we have any basis for R^2?

The set of all linear combinations of $\widehat{i}$ & $\widehat{j}$ is the span of $\widehat{i}$ & $\widehat{j}$

In general, the standard basis in $\mathbb{R}^d$ is given by

$$e_i = (0, 0, \ldots, 0, 1, 0 \ldots 0)$$

where $\quad i = 1, 2, \ldots, d$

ie $\{e_i\}_{i=1}^{d}$ is the standard basis of $\mathbb{R}^d$

Example (3D):

$$e_1 = (1, 0, 0)$$
$$e_2 = (0, 1, 0)$$
$$e_3 = (0, 0, 1)$$

Norm

Let $\overrightarrow{x} \in \mathbb{R}^n$, *that is,* $\overrightarrow{x} = (x_1, x_2, \ldots, x_n)^T$
then the $l_1 - norm$ *of* $\overrightarrow{x}$ *is defined by*

$$\|x\|_1 := \sum_{i=1}^{n} |x_i| \,,$$

where $|\cdot|$ *denotes the absolute*

Use case: LASSO regularization

Similarly, $\|\cdot\|_2$, *the* $l_2 - norm$ *of* x *is defined by*

$$\|x\|_2^2 = \sum_{i=1}^{n} |x_i|^2$$

$$= x^T x$$

$$\|x\|_2 = \sqrt{x^T x}$$

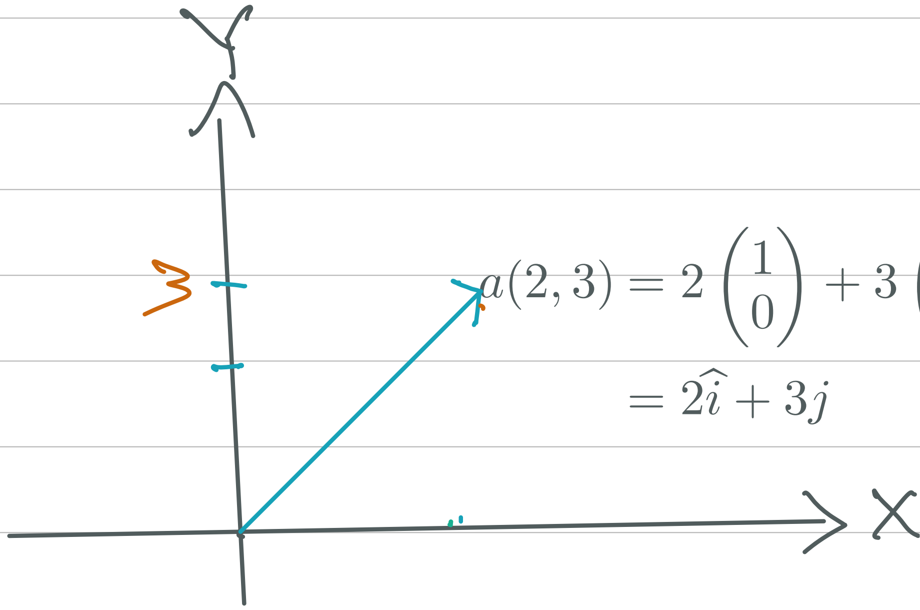
Use case: Ridge regularization

Similarly, $|\cdot|_\infty$, *the* $l_\infty - norm$ *of* x *is defined by*

$$|x|_\infty = \max_{i=1,\ldots n} \{|x_i|\}$$

Use case: Uniform convergence

What does the norm represent?



$$a(2,3) = 2\begin{pmatrix}1\\0\end{pmatrix} + 3\begin{pmatrix}0\\1\end{pmatrix}$$
$$= 2\hat{i} + 3j$$

$$\|a\|_1 = |2| + |3| = 5$$
$$\|a\|_2 = \sqrt{2^2 + 3^2} = \sqrt{13}$$
$$|a|_\infty = \max\{|2|, |3|\} = 3$$

All norms are equivalent in a finite dimensional space.

$$c_1 \|\cdot\|_1 \leq \|\cdot\|_2$$

Dot product (Inner Product)

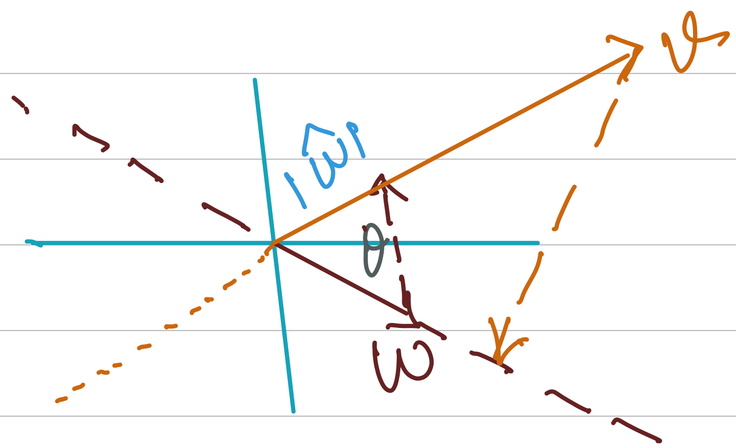
Let v and w be two vectors of dimension n.

$$v \cdot w = v^T w = \sum_{i=1}^{n} v_i w_i$$

$$= v_1 w_1 + v_2 w_2 + \ldots + v_n w_n$$

= (length of projected w on V) x (length of V)

$$v \cdot w = |v| \, |w| \cos\theta$$



Outer product

Let u and v be two vectors of dimension m and n , respectively.

$$u \otimes v = uv^T = \begin{pmatrix} u_1 v_1 & u_1 v_2 & \ldots u_1 v_n \\ u_2 v_1 & u_2 v_2 & \ldots u_2 v_n \\ \vdots & \vdots & \ddots \vdots \\ u_m v_1 & u_m v_2 & \ldots u_m v_n \end{pmatrix}$$

https://colab.research.google.com/drive/1XPWkRq3sWbQ7LIGL7MGdiLdyQ5oJbu9O?usp=sharing

Cosine Similarity

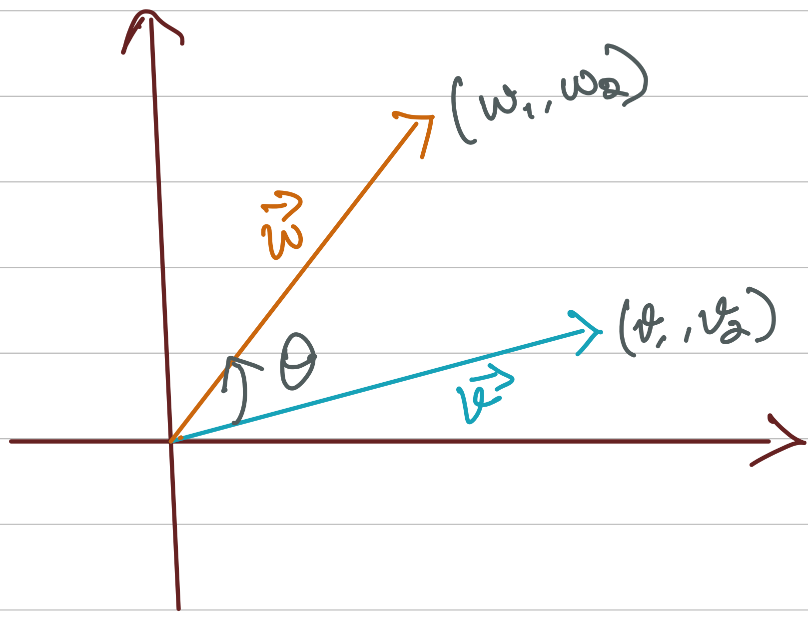
Recall Dot product

Let v and w be two vectors of dimension n.

$$v \cdot w = v^T w = \sum_{i=1}^{n} v_i w_i = |v||w| \cos\theta$$

The Cosine similarity between v and w is defined by

$$S_c(v, \omega) = \cos\theta = \frac{v \cdot \omega}{|v|\,|\omega|}$$



Application of Cosine Similarity to NLP

https://colab.research.google.com/drive/1XPWkRq3sWbQ7LIGL7MGdiLdyQ5oJbu9O?usp=sharing

```
from nltk.corpus import stopwords
from nltk.tokenize import word_tokenize
```

```
# X = input("Enter first string: ").lower(),
# Y = input("Enter second string: ").lower()
X ="I love India"
Y ="India is a beautiful country"
```

```
[ ] #tokenization
    X_list = word_tokenize(X)
    Y_list = word_tokenize(Y)
```

```
[ ] print (X_list)
    print (Y_list)
```

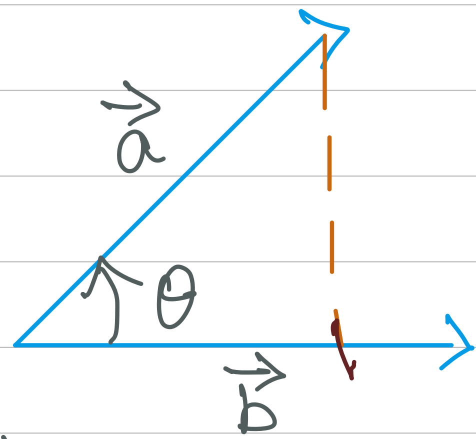
```
[ ] # sw contains the list of stopwords
    sw = stopwords.words('english')
    l1 =[];l2 =[]
```

```
[ ] # remove stop words from the string
    X_set = {w for w in X_list if not w in sw}
    Y_set = {w for w in Y_list if not w in sw}
```

Projection

Scalar Projection

Let a and b be two vectors



The scalar projection of "a" onto $\vec{}$"b" is given by

$$S = |\vec{a}| \cos\theta = \vec{a} \cdot \hat{b}$$

magnitude of a

unit vector of b

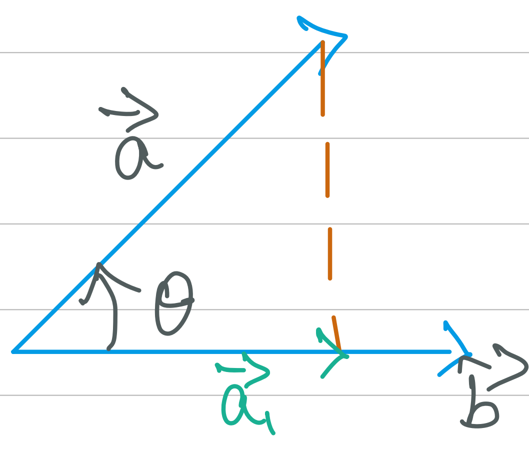
$$= \vec{a} \cdot \frac{\vec{b}}{|b|} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$$

$$a \cdot b = |a|\,|b| \cos\theta \Rightarrow \overset{S}{\overset{||}{|a| \cos\theta}} = \frac{\vec{a} \cdot \vec{b}}{|b|} = \vec{a} \cdot \hat{b}$$

The vector projection of "a" onto "b" is given by

$$\vec{a}_1 = s\hat{b} = s\frac{\vec{b}}{|b|}$$

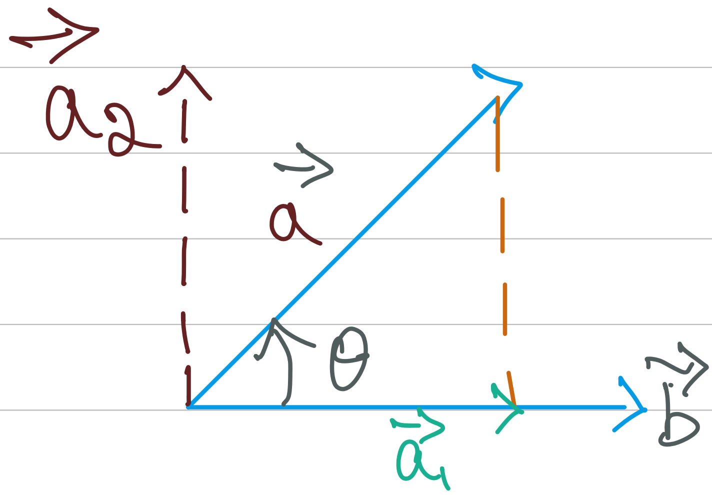
$$= \frac{\vec{a} \cdot \vec{b}}{|b|} \frac{\vec{b}}{1b1} = \left(\vec{a} \cdot \hat{b}\right) \hat{b}$$



Remark

The vector projection implies that $\vec{a}_1$ and $\vec{b}$ are parallel but may have different direction when $90° \leq \theta \leq 180°$.

Orthogonal Projection (vector rejection)



Let $\vec{a} = \vec{a_1} + \vec{a_2}$

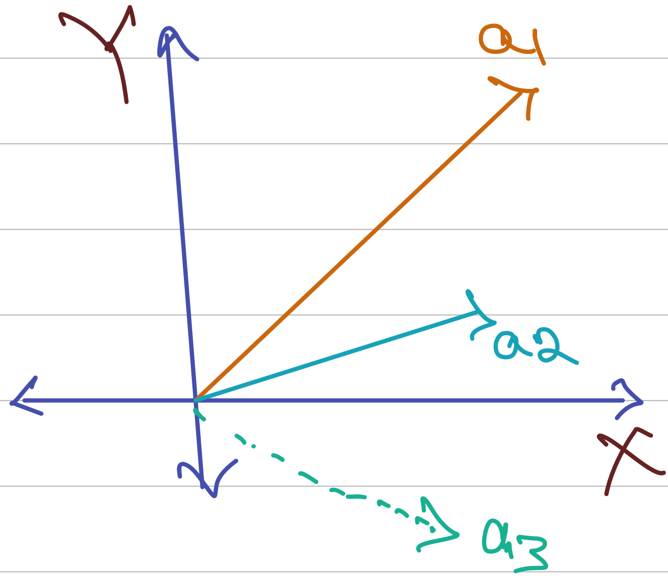
$\Rightarrow$ $a_2 = a - a_1$ is the orthogonal projection of $\vec{a}$ onto $\vec{b}$.

ie $a_2 = a - \left(\dfrac{a \cdot b}{b \cdot b}\right) b$

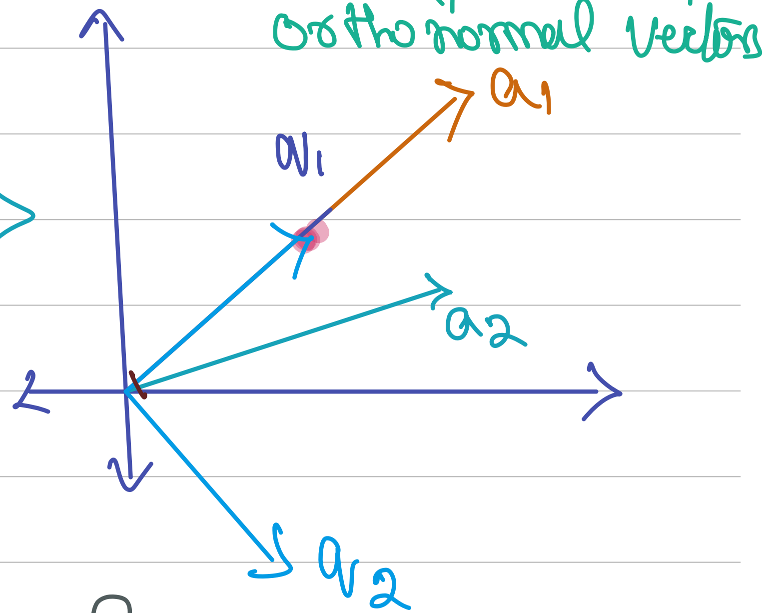
$\Rightarrow$ a_2 and b are orthogonal

Geometrical Interpretation

Given a_1 & a_2 Vectors find q_1 & q_2 orthonormal vectors!



$\Longrightarrow$



How to get q_1 & q_2?

$$q_1 = \frac{a_1}{\|a_1\|}$$

$$\tilde{q}_2 = a_2 - \frac{(a_2 \cdot q_1)}{q_1 \cdot q_1} q_1$$

$$= a_2 - (a_2 \cdot q_1) q_1$$

$$q_2 = \frac{\tilde{q}_2}{\|\tilde{q}_2\|}$$

$$a_1, a_2 \dots a_n\} \longrightarrow \{q_1, q_2 \dots q_n\}$$

1. Arnoldi
2. Gram-schmidt / m.GS

Recap Poll:

Q1: Any vector in R^n can be written as a linear combination of the basis of R^n (True/False)

Q2: Let u be a n-by-1 vector, then uu^T will be an n-by-n matrix with ‘n’ linearly independent rows. (True/False)

Q3: Standard basis is the only basis of a vector space. (True/False)

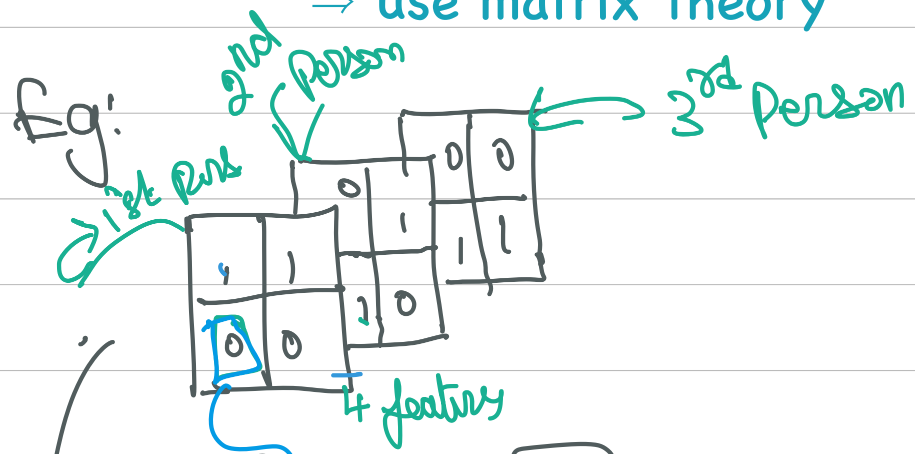
Q4: Vector projection of a vector “u” onto “v” increases the magnitude of “v” by the mag. Of u. (True/False)

Major challenge in Data science

- How to represent and perform operation on high-dimensional data?
- How to extract key features/ information from high-dimensional data "efficiently"?

$\Rightarrow$ use matrix theory

Eg:

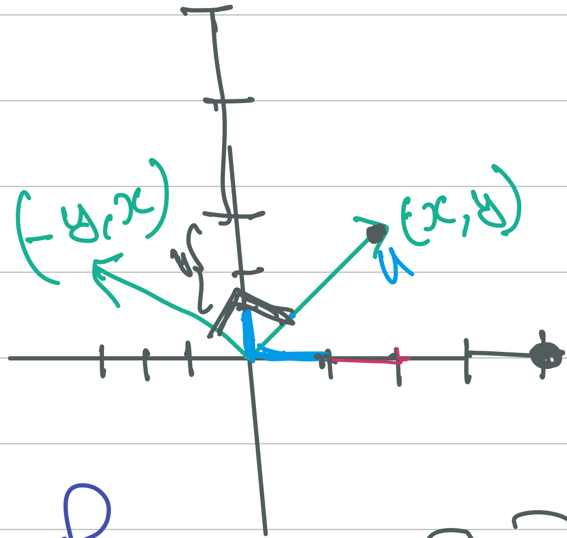


$$D = \begin{array}{ccc} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} & & \\ \text{1st person} & \text{2nd person} & \text{3rd person.} \end{array} \quad 4 \times 3$$

Matrix

Matrix is a linear transformation of (basis) vector.

Rotation



$$\underset{2\times2}{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}} \underset{2\times1}{\begin{bmatrix} x \\ y \end{bmatrix}} = \underset{2\times1}{\begin{bmatrix} -y \\ x \end{bmatrix}}$$

B

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 3y \end{bmatrix}$$

In general

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$\theta = 0 \Rightarrow ?$

$\theta = 90^0 \Rightarrow ?$

$\theta = 180^0 \Rightarrow ?$

Shear:

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ y \end{bmatrix}$$

How about

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

Definition

Let $m, n \in \mathbb{N}$. A real valued matrix A is a mn-tuple of elements $a_{ij} \in \mathbb{R}$, $1 \le i \le m$, $1 \le j \le n$, which is ordered as m rows & n columns.

ie $$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

$m = n \Rightarrow$ Square matrix.

Let A & $B \in \mathbb{R}^{n \times n}$

$$A + B = \begin{bmatrix} a_{11} \pm b_{11} & \cdots & a_{1n} \pm b_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} \pm b_{n1} & \cdots & a_{nn} \pm b_{nn} \end{bmatrix}$$

Matrix-vector multiplication

$y_1 = R_1 \cdot b$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{31} & a_{33} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = Y$$

$$\Rightarrow \quad Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} =$$

Column operation:

$$Y = b_1 \overset{C_1}{\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}} + b_2 \overset{C_2}{\begin{bmatrix} a_{21} \\ a_{22} \\ a_{23} \end{bmatrix}} + b_3 \overset{C_3}{\begin{bmatrix} a_{31} \\ a_{32} \\ a_{33} \end{bmatrix}}$$

$$= b_1 C_1 + b_2 C_2 + b_3 C_3$$

Matrix-Matrix multiplication

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{A} \begin{bmatrix} \underbrace{\begin{matrix} x_{11} \\ x_{21} \end{matrix}}_{C_1} & \underbrace{\begin{matrix} x_{12} \\ x_{22} \end{matrix}}_{C_2} \end{bmatrix}$$

$$= \begin{bmatrix} AC_1 & AC_2 \end{bmatrix}$$

where

$$AC_1 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} x_{11} + x_{21} \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$$

$$C = AB$$

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \qquad 1 \le i, j \le n$$

Note that

$$AB \neq BA$$

$$(AB)C = A(BC)$$

$$(A+B)C = AC + BC$$

$$A(B+C) = AB + AC$$

Identity matrix

The square matrix I is said to be an identity matrix of

$$I_n = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else.} \end{cases}$$

$$IA = AI = A$$

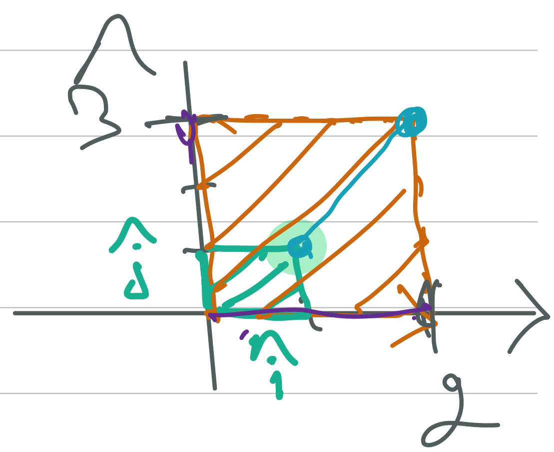
Inverse of a matrix

Let A be a square matrix of order n. Suppose a matrix B is said to be an inverse of A then

$$BA = I = AB.$$

Further, B is denoted as A^{-1}.

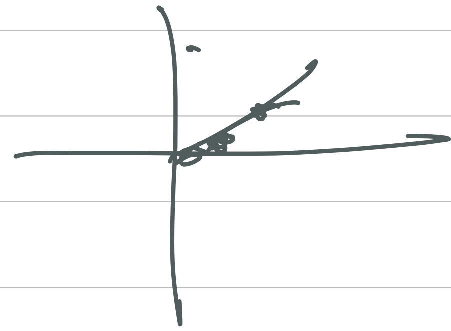
Determinant of a matrix:



$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$|A| = \det A = 6 \Rightarrow ?$$

Quantifies the change in the measure (area in 2d, volume in 3d) due to linear transformation.



How about $|A| = 0?$

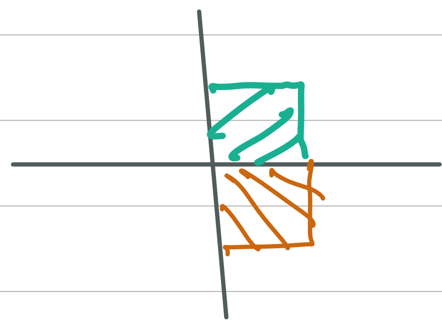
How about $|A| < 0?$

$\Rightarrow ?$

Same as above but the orientation got flipped

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$|A| = -1$



Suppose $$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$|A| = \det(A) = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & n \end{vmatrix}$$

$$= aei + bfg + cdn - ceg - bdi - afh$$

Suppose $|A| = 0 \Rightarrow$ singural matrix

& A^{-1} does not exists.

Transpose of a matrix

- writes Rows of A into Columns of A

$$AA^{-1} = I = A^{-1}A$$
$$(AB)^{-1} = B^{-1}A^{-1}$$
$$(A+B)^{-1} = A^{-1} + B^{-1}$$

$$(A^T)^T = A$$
$$(AB)^T = B^TA^T$$
$$(A+B)^T = A^T + B^T$$

Suppose $A = A^T$

$\Rightarrow$ A is symmetric

Multiplication of a matrix by a scalar.

- $\lambda \in \mathbb{R}, \& A \in \mathbb{R}^{n\times n}$

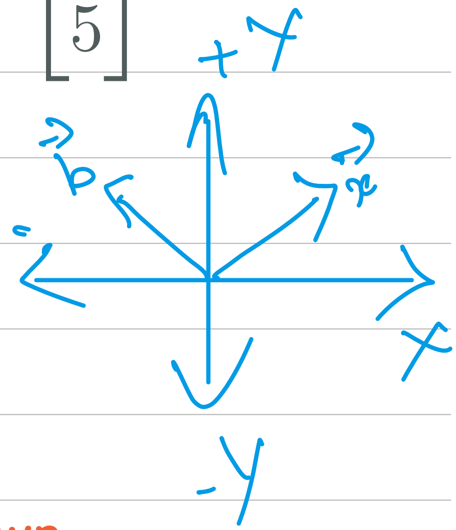
Then $\lambda A = K$ where

$$[k]_{ij} = \lambda a_{ij}, 1 \leq i, j \leq n$$

Suppose we have

$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 &= 5 \\ 6x_1 - 2x_2 - 7x_3 &= 2 \\ -5x_1 + 6x_2 + 8x_3 &= 5 \end{aligned}$$

$$\begin{bmatrix} 2 & 3 & 4 \\ 6 & -2 & -7 \\ -5 & 6 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 5 \end{bmatrix}$$



$$Ax = b$$

known

Unknown

known

We know the linear transformation (A), then question is where should we start (x) to land at an expected position (b)?

$$x = A^{-1}b$$

inverse transformation

Eigenvalues & Eigenvectors

Let A be a square matrix. Multiply the matrix A by a vector "X", where AX is parallel to X.

$$Ax = \lambda x, \lambda \in \mathbb{R}$$

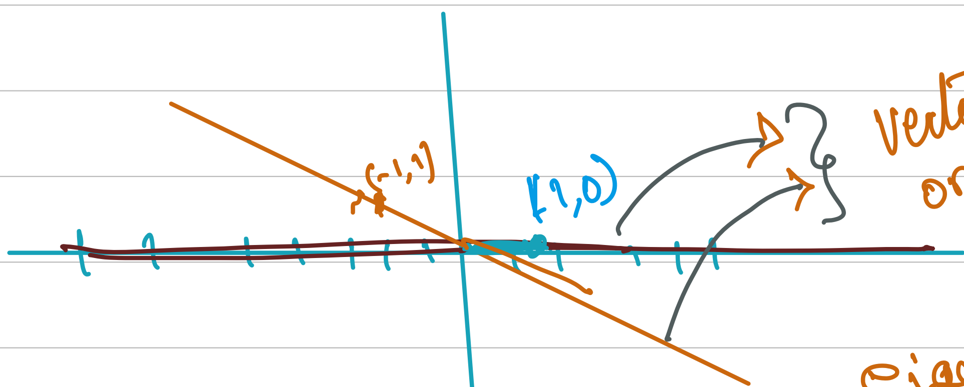
Such a vector x is an eigenvector, where lambda is an eigenvalue.

Let $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$

$$Ax = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} x_1 + \begin{pmatrix} 1 \\ 2 \end{pmatrix} x_2$$

Let $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow Ax = \begin{pmatrix} 3 \\ 0 \end{pmatrix} 1 + \begin{pmatrix} 1 \\ 2 \end{pmatrix} 0 = 3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 3x$

$x = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \Rightarrow Ax = \begin{pmatrix} -2 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 2x$



Vectors remain on its own span.

$\Downarrow$

eigen vectors

Recall:

$$Ax = \lambda x, \lambda \in \mathbb{R} \quad \text{--- } (1)$$

LHS : matrix - vector multiplication

RHS : scalar times a vector

looks awkward ?

$\Rightarrow$ Rewrite (1)

$$Ax = (\lambda \mathbb{I})x$$

$$\Rightarrow \quad Ax - \lambda \mathbb{I} x = 0$$

$$(A - \lambda \mathbb{I})x = 0 \quad \text{--- } (2)$$

- what does it mean?
- It will always be true when x=0
- how about for x != 0?

If there exists a non-trivial solution (x!=0) for (2), it implies that the matrix is not invertible or is singular, that is,

$$\det(A - \lambda I) = 0.$$

Remark:

A matrix need not have (real) eigenvectors always.

Eg: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (rotation matrix)

$$\begin{vmatrix} 0-\lambda & -1 \\ 1 & 0-\lambda \end{vmatrix} = \lambda^2 + 1 = 0$$

$$\Rightarrow \lambda = \pm i$$

$\Rightarrow$ no real eigenvectors exist.

A matrix has one eigenvalues with many eigenvectors

Eg: $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \Rightarrow \lambda = 1$

How about the eigenvalue & eigenvectors of a diagonal matrix?

- a is Eigen value & every vector in the xy-plane is an eigenvector
- all basis vectors are eigenvectors with diagonal entry being their eigenvalues

What is the significance of eigenvectors in Data matrix?

Let $Ax \rightarrow b$

Suppose x is an eigen vector

$$\Rightarrow \quad Ax = b = \lambda x$$

$$A^2 x = A(\lambda x) = \lambda A x = \lambda^2 x$$

$$\vdots$$

$$A^n x = \lambda^n x$$

⇒ The matrix "A" push the input vector "x" (initial guess of eigenvector) towards the true (dominant) eigenvector.

Applications:

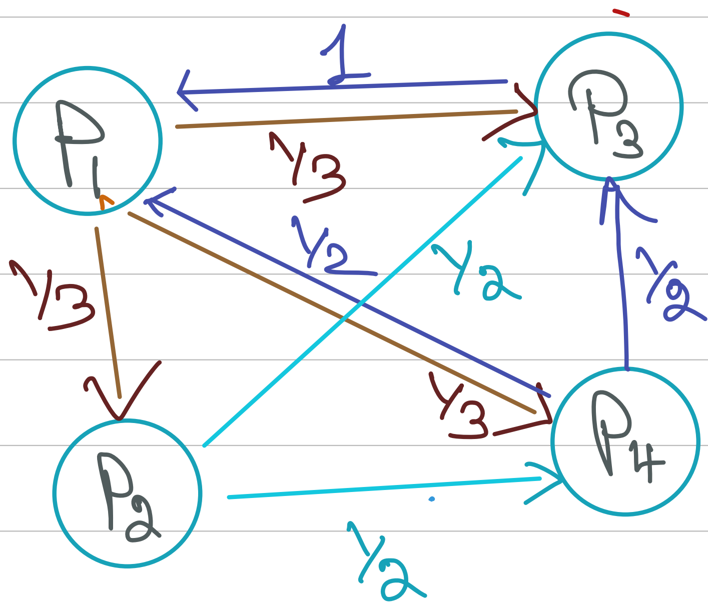
Pagerank, axis of rotation, etc.

Pagerank Algorithm

How to rank webpages?

A page that gets more referrals!

Eg.



==> the transition matrix

$$A = \begin{array}{c} \\ \end{array}\begin{array}{cc} \begin{array}{cccc} P_1 & P_2 & P_3 & P_4 \end{array} & \\ \begin{bmatrix} 0 & 0 & 1 & 1/2 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 1/2 & 0 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \end{bmatrix} & \begin{array}{c} P_1 \\ P_2 \\ P_3 \\ P_4 \end{array} \end{array}$$

We have 4-nodes, & have uniform rank initially, ie $v^0 (1/4, 1/4, 1/4, 1/4)^T$

$\left(\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}\right)$

New importance $\Rightarrow v^1 \leftarrow A v^0$

$$v^1 = Av^0 = \begin{bmatrix} 0 & 0 & 1 & 1/2 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 1/2 & 0 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \end{pmatrix} = \begin{pmatrix} 0.37 \\ 0.08 \\ 0.33 \\ 0.20 \end{pmatrix}$$

Check the new importance v_1^1 of P_1

$$v_1^1 = \underbrace{1 * 0.25}_{P_3} + \underbrace{\frac{1}{2} * 0.25}_{P_4} = 0.375$$

chance of going from

$\Rightarrow$ new importance of P_1

$$v^2 = A^2 v^0$$
$$\vdots$$
$$v^7 = A^7 v^0$$
$$v^8 = A^8 v^0$$

Stop: $\|v^7 - v^8\| < \varepsilon$

$$A^7 v^f \simeq A^8 v^f = \begin{pmatrix} 0.38 \\ 0.12 \\ 0.29 \\ 0.19 \end{pmatrix}$$

$$V^F = A^8 V_8^F = A^7 V_7^F \Rightarrow \boxed{A V^F = V^F}$$