**Due:** February 11th, before class.

**Name: Solution**

**Collaborators:**

Document your work and note any collaborators. In your write-up, please neatly document any code you have written, alongside its output, and including any necessary commentary to answer the question. Please use an `.ipynb` notebook for any code, include commentary in markdown code blocks, and print the `.ipynb` notebook to PDF to attach to your submission. The LaTeXsource for this file may be helpful: https://www.overleaf.com/read/fyvskqcrsbdd#5b4577.

**Problems:**

1. Suppose that $\mathbf{A}$ is $n \times n$ and that $\|\mathbf{A}\| < 1$ in some induced matrix norm.

   (a) Show that $(\mathbf{I} - \mathbf{A})$ is nonsingular.

   We will prove this claim by contradiction. Suppose to the contrary that $\mathbf{I} - \mathbf{A}$ is singular, then there exists nontrivial vector $x$ such that $(\mathbf{I} - \mathbf{A})x = 0$. This implies that there exists $x$ such that $\mathbf{A}x = x$. We can take the norm of both sides to get $\|\mathbf{A}x\| = \|x\|$. However, according to norm inequalities

   $$\|\mathbf{A}x\| \leq \|\mathbf{A}\|\|x\| < \|x\|$$

   since $\|\mathbf{A}\| < 1$. Thus this is a contradiction: $\|\mathbf{A}x\|$ can't be equal to and smaller than $\|x\|$ at the same time. Thus $\mathbf{I} - \mathbf{A}$ is not singular.

   (b) Show that $\lim_{m\to\infty} \mathbf{A}^m = \mathbf{0}$. We say the sequence of matrices $\mathbf{B}_m$ has limit $\mathbf{L}$ if $\lim_{m\to\infty} \|\mathbf{B}_m - \mathbf{L}\| = 0$ for some induced matrix norm.

   By the matrix norm inequality we have

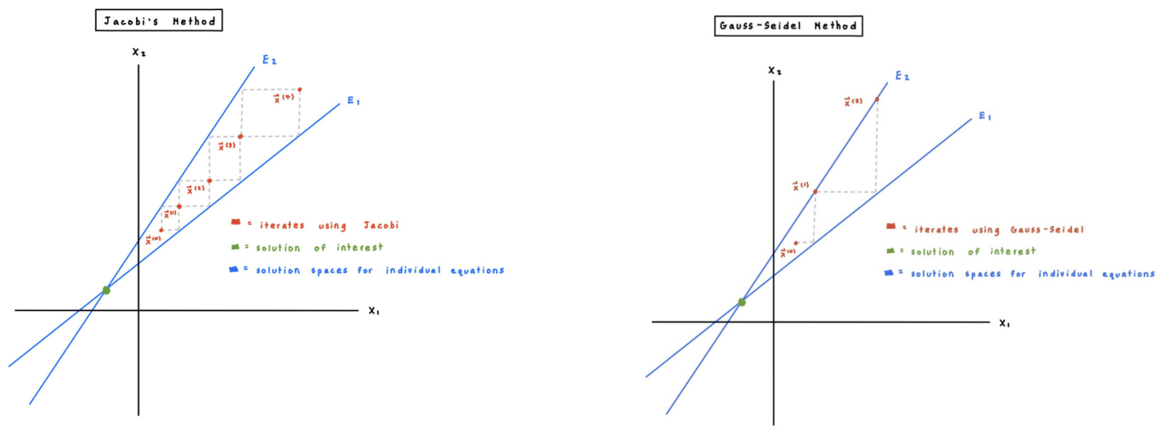
   $$\|\mathbf{A}^m\| \leq \|\mathbf{A}\|\|\mathbf{A}^{m-1}\| \leq \|\mathbf{A}\|^2\|\mathbf{A}^{m-2}\| \leq \ldots \leq \|\mathbf{A}\|^m.$$

   Thus
   $$\lim_{m\to\infty} \|\mathbf{A}^m - \mathbf{0}\| = \lim_{m\to\infty} \|\mathbf{A}^m\| \leq \lim_{m\to\infty} \|\mathbf{A}\|^m = 0$$

   since $\|\mathbf{A}\| < 1$. Additionally, since norms are nonnegative, this suggests that $\lim_{m\to\infty} \|\mathbf{A}^m - \mathbf{0}\| = 0$. Thus $\lim_{m\to\infty} \mathbf{A}^m = \mathbf{0}$.

In this case, we have $(\mathbf{I} - \mathbf{A})^{-1} = \sum_{k=0}^{\infty} \mathbf{A}^k$. (You don't need to show this – I just want you to know this. ☺)

2. Sketch an example linear system in two dimensions and an initial iterate $\mathbf{x}_0$ so that Jacobi's method and the Gauss-Seidel method will diverge on this system (the iterates will travel away from the solution). Note: a proof is not necessary here, just sketch a linear system and the behavior of the Jacobi and Gauss-Seidel methods like was done in Lecture 5.




(a) Jacobi (b) Gauss-Seidal

Figure 1: Plots (cred. Linda Nguyen)

3. Prove that for $n \times n$ nonsingular matrices $\mathbf{A}$ and $\mathbf{B}$, $\kappa(\mathbf{AB}) \leq \kappa(\mathbf{A})\kappa(\mathbf{B})$.

By the definition of condition number for matrices, we have

$$\kappa(\mathbf{AB}) = \|(\mathbf{AB})^{-1}\|\|\mathbf{AB}\| \quad \kappa(\mathbf{A}) = \|\mathbf{A}^{-1}\|\|\mathbf{A}\| \quad \kappa(\mathbf{B}) = \|\mathbf{B}^{-1}\|\|\mathbf{B}\|.$$

Thus we have

$$\begin{aligned}
\kappa(\mathbf{AB}) = \|(\mathbf{AB})^{-1}\|\|\mathbf{AB}\| &= \|\mathbf{B}^{-1}\mathbf{A}^{-1}\|\|\mathbf{AB}\| \quad \text{matrix inverse} \\
&\leq \|\mathbf{B}^{-1}\|\|\mathbf{A}^{-1}\|\|\mathbf{A}\|\|\mathbf{B}\| \quad \text{norm inequality} \\
&= \left(\|\mathbf{A}^{-1}\|\|\mathbf{A}\|\right)\left(\|\mathbf{B}^{-1}\|\|\mathbf{B}\|\right) \quad \text{regroup} \\
&= \kappa(\mathbf{A})\kappa(\mathbf{B}),
\end{aligned}$$

which completes the proof.

**Lab:** In a previous lab you learned to blur an image, represented as an $m \times n$ pixel intensity matrix $\mathbf{X}$, through multiplying by a matrix on each side. Let $\mathbf{V} = \mathbf{B}_m$ and $\mathbf{H} = \mathbf{B}_n^T$, where each $\mathbf{B}_p$ is a $p \times p$ local blurring matrix returned by your function `blurmatrix`. For some positive integer $k$,

$$\mathbf{Z} = \mathbf{V}^k \mathbf{X}\, \mathbf{H}^k \tag{1}$$

applies local blurring $k$ times in each direction to get the resulting $\mathbf{Z}$.
It's much more common, of course, to have a blurred image that you want to restore, or *deblur*. We can easily express the process using matrix inverses:

$$\mathbf{X} = \mathbf{V}^{-k}\mathbf{Z}\,\mathbf{H}^{-k}.$$

However, in computational practice we rarely use matrix inverses; instead, we solve linear systems. In Julia this is done by using \ (backslash) when multiplying by an inverse on the left (use $\mathbf{A}\backslash\mathbf{b}$ to solve $\mathbf{A}\mathbf{x} = \mathbf{b}$), and / (forward slash) when multiplying by an inverse on the right (use $\mathbf{b}/\mathbf{A}$ to solve $\mathbf{x}^\top\mathbf{A} = \mathbf{b}^\top$).
It's intuitively clear that blurring and deblurring are fundamentally different. While we can blur an image as much as we like, we expect that a severely blurred image cannot be accurately restored. In finite precision, this proves to be the case. This week we are learning how to characterize matrix multiplications like this that cannot easily be undone.

*Goals:* You will blur an image as before, then apply backslash and slash to deblur it. For a small amount of blur this will succeed, but not when you try to restore a more severely blurred image.

*Procedure:* Perform the following steps. Make sure your function `blurmatrix` is available.

(a) Using `blurmatrix`, let $\mathbf{V} = \mathbf{B}_{100}$. Make side-by-side plots of $\mathbf{V}$ and $\mathbf{V}^{-1}$ using `spy` (see Lecture 6). Note that while the nonzeros of $\mathbf{V}$ are all near the diagonal, the same is not true for $\mathbf{V}^{-1}$, so deblurring is not a simple local operation.

(b) Download and locally save an image. For best results, use a cartoon or line drawing of size 500 pixels or less on each side. Import the image and call the result $\mathbf{X}$, the original image.

(c) Make side-by-side plots of $(V^{-1})^6$ and $(V^6)^{-1}$. These should be equivalent, but your results will not be—our first sign of some trouble.

(d) Using $k = 1$, let $\mathbf{Z}$ be the blurred image as described in equation (1) . Using `subplot` and `heatmap`, make side by side pictures of the original and blurred images.

(e) Now let $\mathbf{Y}$ be the result of deblurring $\mathbf{Z}$. Use \ and /, as described in the introduction. Make side by side pictures of $\mathbf{X}$ and $\mathbf{Y}$. You should see apparently identical images.

(f) Repeat steps 4 and 5, but using $k = 6$. The restoration results should be *very* different!