Factoring large integers using Quadratic Sieve

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Abstract

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I. Introduction

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II. Preliminaries

Notation. By '$\log_b$' we denote the base b logarithm and the natural logarithm denotes by ln=$\log_e$ with $e \approx 2.71828$. The largest integer $\leq x$ is denoted by '[x]'. The number of primes $\leq x$ is denoted by '$\pi(x)$', and due the *Prime number theorem*[1] we know that $\pi(x) \approx x/\ln(x)$.

Modular arithmetic. Throughout this paper '$x \equiv y \bmod n$' means that $(x-y)$ is a multiple of n whereas x, y and $n \in \mathbb{N}_{\neq 0}$. Similarly, '$x \not\equiv y \bmod n$' mean that $(x-y)$ is not a multiple of n. Euclid's algorithm for finding the *greatest common divisor* of two non-negative integers, say x and z, is denoted by $\gcd(x, z)$.

III. Quadratic Sieve

III.1 Brief explaination

The quadratic sieve algorithm is today the algorithm of choice when factoring very large composite numbers with no small factors. The general idea behind quadratic factoring is based on Fermat's observation that a composite number if one can find two find two integers $x, y \in Z$, such that $x^2 \equiv y^2 \pmod{n}$ and $x \not\equiv \pm y \pmod{n}$. This would imply that,

$$n \mid x^2 - y^2 = (x - y) \cdot (x + y) \qquad (1)$$

but n neither divides $(x - y)$ nor $(x + y)$. Furthermore it can be rewritten as $(x - y) \cdot (x + y) = k \cdot p \cdot q$ for some integer k, thus becoming two possible cases.

- either p divides $(x - y)$ and q divides $(x + y)$, or vice versa.
- or both p and q divides $(x - y)$ and neither of them divides $(x + y)$, or vice versa.

Hence, the greatest common divisor of $(x - y, n)$ and $(x + y, n)$ would with with a $1/2$ probability yield first case which is p or q and a non-trivial factor of n is found. In the second case we get n or 1 and trivial solution is found.

Carl Pomerance suggested a method to find these two distinct integers[2]. The first step in doing so is to is to define the polynomial

$$Q(x) = (x + \lfloor \sqrt{s} \rfloor)^2 - N \qquad (2)$$

References

[1] G.H. Hardy, E.M. Wright, D.R. Heath-Brown, and J. Silverman. *An Introduction to the Theory of Numbers*. Oxford mathematics. OUP Oxford, 2008.

[2] Carl Pomerance. The quadratic sieve factoring algorithm. In Thomas Beth, Norbert Cot, and Ingemar Ingemarsson, editors, *Advances in Cryptology*, volume 209 of *Lecture Notes in Computer Science*, pages 169–182. Springer Berlin Heidelberg, 1985.