
Factoring large integers using Quadratic Sieve

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Abstract

This report will give you an insight to factorization of integers using quadratic sieve. It covers the basics of the factoring problem and mentions other commonly used algorithms, but the main focus is on the quadratic sieve which is the best algorithm for this problem up to 100 bit integers.

This report was written during the course DD2440 Advanced Algorithms at KTH 2013 and recieved the maximum amount of points in the scoring system KATTIS.

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I. INTRODUCTION

This report will cover the work of a integer factorization program written in the course DD2440 advanced algorithms. It was decided to focus primarily on quadratic sieve which should give a good understand of the problem in general.

I.1 Purpose

The goal of the program is to be able to factorize integers in an efficient way and also pass KATTIS test cases with as high score as possible, this is done by improving the program step-by-step and gradually increasing the performance and intelligence in the task at hand. In the report the methods used will be analyzed described. How they work independently and their correlation in our implementation.

I.2 Problem

An unknown set of 100 integers in varying size are used as input to the algorithm from KATTIS. The output is either the factors of the input if it is solved or the string fail otherwise. One of the problems is dealing with large integers within the timeframe but also finding every single non-trivial factor. The restrictions in kattis are 15 seconds and 64MB of memory.

I.3 Scope

There is no intention of reinventing any methods that already exists, the idea is to create a solver that can factorize unknown integers and in the end get a high score on KATTIS.

I.4 Statement of Collaboration

Some text.

II. PRELIMINARIES

Notation. By ' \log_b ' we denote the base b logarithm and the natural logarithm denotes by $\ln = \log_e$ with $e \approx 2.71828$. The largest integer $\leq x$ is denoted by ' $\lfloor x \rfloor$ '. The number of primes $\leq x$ is denoted by ' $\pi(x)$ ', and due the *Prime number theorem*[1] we know that $\pi(x) \approx x/\ln(x)$.

Smoothness. A positive integer is B -smooth if all its prime factors are lesser than a boundary B . Furthermore an integer, say x , is said to be *smooth with respect to S* , if x can be completely factored using integers by some set S alone.

Modular arithmetic. Throughout this paper ' $x \equiv y \pmod n$ ' means that $(x - y)$ is a multiple of n whereas x, y and $n \in \mathbb{N}_{\neq 0}$.

Similarly, ' $x \not\equiv y \pmod n$ ' mean that $(x - y)$ is not a multiple of n . Euclid's algorithm for finding the *greatest common divisor* of two non-negative integers, say x and z , is denoted by $\gcd(x, z)$.

III. BACKGROUND

As described in the scope the intention is not to find a new method of factoring integers but instead use existing methods and implement them in a intelligent way.

III.1 Trial Division

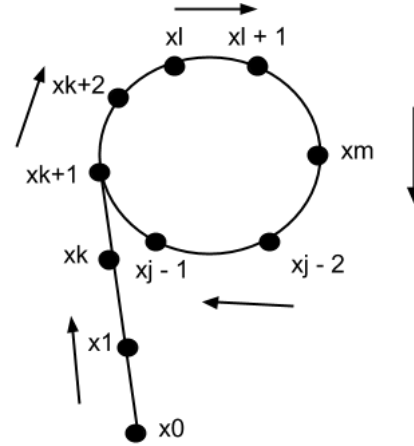
Trial division is an easy implemented algorithm for finding factors to N . Like its name suggests it uses tries to divide N by using a $x > 2$. If the remainder of $N/x = 0$ the factor is a non-trivial factor, if we cannot find a divisor x that gives the remainder 0 then N is a prime.

The restrictions for the algorithm are easy to find and improve, $N > 1$ and $x > 2$. This can be improved by saying that $N > x$ and since the factors of N cannot be larger than \sqrt{N} we can also say that $\sqrt{N} > x > 2$. Another improvement is to only do tests when x is a prime number. Although trial division is easy to implement and guaranteed to grant an answer it is not efficient for large a N .

III.2 Pollard's Rho

Pollard rho is a prime factorization algorithm which is based on Floyd's cycle-finding algorithm. The idea of pollard rho algorithm is to iterate a formula until it falls into a cycle. We want to find a x and y where the x makes twice as many iterations as the y using a function *modulo* N as a generator of a pseudo-random sequence. The $\text{GCD}|x - y|$ of N is taken each step, and the GCD reaches N we have not found an answer and the algorithm terminates with a failure.

We can write $N = p * q$, when x , which iterates twice as fast as y , catches up with y which will happen eventually, at this point factor p will be found. The time it will take cannot be proven matematically and can only be proven by heuristics. If the sequence behaves randomly it would take approximately p steps to find p , which is not very efficient. [2]



The figure above illustrates the Pollard's Rho cycle. The mapping of x_{i+1} is instead replaced with a function $x^2 + 1$ so that we get $x_i^2 + 1 \bmod N$ the factor p will be found after $O(\sqrt{p}) \in O(N^{1/4})$ steps. [2]

Pollard's rho algorithm can be improved further by implementing Brent's cycle finding method. [5]

III.3 Miller-Rabin primality test

Miller-Rabin primality test is a probabilistics algorithm which determines if the given input N is a prime. It is based on the properties of strong pseudoprimes, given an integer N , $N = 2^r * s + 1$ where s is odd you choose a random number a with the properties $1 \leq a \leq N - 1$. When $a^s \equiv 1 \bmod N$ or $a^{2^j s} \equiv -1 \bmod N$ where $0 \leq j \leq r - 1$, if the input number N is a prime it will pass the test with any random number a .

Since Miller-Rabin is a probabilistics method it is not completely true that N is a prime simply by passing the test, however the probability that the answer is true when N is a composite number is $1/4^N$ which grows quickly with N . It can be considered a very small trade-off to use

a probabilistics method because the algorithm executes at $O(k \log^3 N)$ where k is the number of different values of a tested. [6]

IV. QUADRATIC SIEVE

IV.1 Brief explanation

The quadratic sieve algorithm is today the algorithm of choice when factoring very large composite numbers with no small factors. The general idea behind quadratic factoring is based on Fermat's observation that a composite number n can be factored if one can find two integers $x, y \in \mathbb{Z}$, such that $x^2 \equiv y^2 \pmod{n}$ and $x \not\equiv \pm y \pmod{n}$. This would imply that,

$$n \mid x^2 - y^2 = (x - y) \cdot (x + y) \quad (1)$$

but n neither divides $(x - y)$ nor $(x + y)$. Furthermore it can be rewritten as $(x - y) \cdot (x + y) = k \cdot p \cdot q$ for some integer k , thus becoming two possible cases.

- either p divides $(x - y)$ and q divides $(x + y)$, or vice versa.
- or both p and q divides $(x - y)$ and neither of them divides $(x + y)$, or vice versa.

Hence, the greatest common divisor of $(x - y, n)$ and $(x + y, n)$ would in first case yield p or q and a non-trivial factor of n is found. In the second case we get n or 1 and trivial solution is found.

Carl Pomerance suggested a method to find such squares [3]. The first step in doing so is to define the polynomial

$$q(r) = (r + \lfloor \sqrt{n} \rfloor)^2 - n \approx \tilde{r}^2 - n \quad (2)$$

Now, consider we set of primes $P = \{p_1, p_2, \dots, p_k\}$ lesser than a bound B , i.e

$k < \pi(B)$. We then want to construct a subset of integers r_1, r_2, \dots, r_k such that $\forall i : q(r_i) = r_i^2 - n \pmod{n}$ is smooth in respect to P , more specifically $\forall i :$

$$q(r_i) = r_i^2 \equiv p_1^{e_{i1}} \cdot \dots \cdot p_k^{e_{i\pi(B)}} \pmod{n} \quad (3)$$

where e_{ij} is the exponent of p_j of factorization of $q(r_i)$. If the exponents for all primes sums to an even number we arrive at following relation:

$$\prod_{i=1}^n q(r_i) = \prod_{i=1}^n r_i^2 \equiv (p_1^{e_1} \cdot \dots \cdot p_k^{\pi(B)})^2 \pmod{n} \quad (4)$$

and the integers we x, y we sought to find are simply:

$$x = \prod_{i=1}^n q(r_i) \quad (5)$$

$$y = \prod_{i=1}^{\pi(B)} p_i^{e_i} \quad (6)$$

Lets consider a factor base P with the primes p_1, p_2, \dots, p_k that is $k \geq B$ and co-prime to n . We want to search for small r_i so that $q(r_i)$ is smooth in respect to P . If we find such r 's we say that $q(r_i) = (r + \lfloor \sqrt{n} \rfloor)^2 - n$ is B -smooth and we can factor completely over the factor base.

A prime factorization $p_1^{e_1} \cdot p_2^{e_2} \cdot \dots \cdot p_k^{e_{\pi(B)}}$ of a B -smooth number can then be expressed be entirely expressed by the factor base and exponent vector $(e_1, e_2, \dots, e_{\pi(B)})$.

Since a prime p only divides $q(r_i)$ if and only if it divides $q(r_i + kp)$ for any integer k , we can found these values efficiently using a sieve. For this reason its called the *sieving step* and because only primes $\left(\frac{n}{p_i}\right) = 1$ can divide $q(r_i)$, explains the definition of the factor base P . Naively one could simply randomly select r in a range of interest and verify that $q(r)$ is divisable

by all the primes in the factor base. Instead in quadratic sieve algorithm we can first solve the quadratic congruence $r^2 \equiv n \pmod{p}$ and then cleverly only divide by prime numbers corresponding to the interval r and $r + kp$ for some any integer k . Thus, finding these r 's becomes easier than just a simple trail division as we apriori know at that a B-smooth $q(r_i)$ is divisible by a subset of primes in P . This is the reason why trail division of B-smooth numbers preferable over is a plain naive trail division.

Then the product of subsequence of x i.e. $q(x_1) \cdot q(x_2) \cdot \dots \cdot q(x_k)$ produced a square *iff* the exponent vectors has only even entries. That is that it's the *null*-vector in mod 2 space. So with the collection of smooth numbers we want to form their exponent vectors, reduce them modulo 2. For that vector space (finite field of two elements) the sequence of vectors thus becomes linearly dependent and can be solved with Gaussian Elimination to find the non-empty subsequences. The exponent vectors is represented as the rows of a matrix. Hence, the column sums must be even.

IV.2 Deciding on factor base

To factorize during the sieving step we want all the $q(r)$'s to be smooth in respect to the factor base, i.e. $q(r)$ must be divisible by only primes in the factor base. If a prime p divides $q(r_i)$ earlier mentioned polynomial implies following

$$(r_i + \lfloor \sqrt{n} \rfloor)^2 \equiv n \pmod{p_i} \quad (7)$$

Hence, n is a quadratic residue modulo p_i and we only need to consider those primes. Thus the set of primes p_i for which

the Legendre symbol $\left(\frac{n}{p_i}\right)$ is 1 will form the factor base.

V. IMPLEMENTATION

For our implementation we used several papers as reference. One few that really come handy lecture notes by Carl Pomerance [4]

VI. RESULTS

VII. DISCUSSION

We chose to work with quadratic sieve since it's one of the most efficient integer factorization algorithms out there. It is the fastest algorithm for factorization up to approximately 110 bit integers, there was no doubt it would be a challenge, but since we had knowledge that Pollard's Rho combined with Brent's cycle finding granted approximately 75 points in KATTIS and that the code was not a great challenge to write we wanted to give ourselves a challenge and write a state of the art algorithm. We gave ourselves the time to investigate the more complex alternative and ended up succeeding but still have the basic knowledge about Pollard's Rho since that was our lifeline if our implementation would not have worked.

The result was as expected from a correctly implemented quadratic sieve method, however, on the first submit (that was working) we got a score of 85 points which was due to partly a lack of general optimization but the main reason was making sure the smoothness bounds worked correctly. There are a lot of mathematical parts to implementing this algorithm, although not very complex math in theory getting all the parts of the algorithm to work

well was a tough job. The understanding of the concept came as we went along and step-by-step we improved our code and realized the earlier errors we made and in the end realized that the concept is not hard to grasp if you divide the problem in to subproblems and follow pre-existing formulas to execute it.

VIII. CONCLUSION

In hindsight there is no regret of choosing the more complex solution to the problem, it did not only give us more understanding of how to solve the problem we were given in an intelligent way but we also managed to successfully implement the quadratic sieve algorithm to achieve maximum score in KATTIS. Of course the solution can be optimized further but considering the time constraints we are satisfied with our work.

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