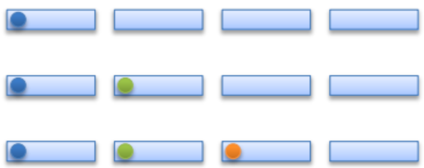
1. A class consisting of 4 graduate and 12 undergraduate students is randomly divided into four groups of 4. What is the probability that each group includes exactly a graduate student?

$A_1 = \{\text{graduate students 1 and 2 are in different groups}\}$
$A_2 = \{\text{graduate students 1, 2 and 3 are in different groups}\}$
$A_3 = \{\text{graduate students 1, 2, 3 and 4 are in different groups}\}$

$$P(A_3) = P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2)$$



$$P(A_1) = \frac{12}{15}$$

$$P(A_2|A_1) = \frac{8}{14}$$

$$P(A_3|A_1 \cap A_2) = \frac{4}{13}$$

$$P(A_3) = \frac{12}{15} \times \frac{8}{14} \times \frac{4}{13}$$

– By partition

$$\frac{\frac{12!}{3! \times 3! \times 3! \times 3!} \times 4!}{\frac{16!}{4! \times 4! \times 4! \times 4!}}$$

– By combination

$$\frac{\binom{12}{3}\binom{9}{3}\binom{6}{3}\binom{3}{3} \times 4!}{\binom{16}{4}\binom{12}{4}\binom{8}{4}\binom{4}{4}}$$

Reference: 2022-02-25 Conditional Probability,P14

2. A computer network connects two nodes A and B through intermediate nodes C, D, E and F. For every pair of directly connected nodes, there is a given probability that the link is up. We assume that link failures are independent of each other. What is the probability that there is a path connecting A and B in which all links are up?

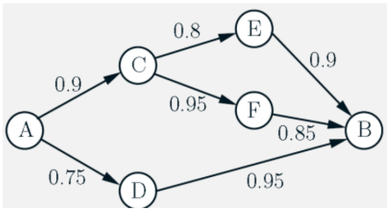
- The probability of the connections from C to B are all fail

$$P(C \to E \to B) = 0.8 \times 0.9$$

$$P(C \to F \to B) = 0.95 \times 0.85$$

$$P\left(\overline{C \to B}\right) = P\left(\overline{C \to E \to B} \cap \overline{C \to F \to B}\right)$$
$$= (1 - (0.8 \times 0.9)) \times (1 - (0.95 \times 0.85))$$

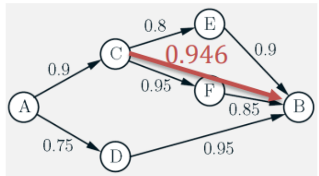
$$\therefore P(C \to B) = 1 - P\left(\overline{C \to B}\right) = 0.946$$



- The probabilities of the two successful paths

$$P(A \to C \to B) = 0.9 \times 0.946 = 0.851$$

$$P(A \to D \to B) = 0.75 \times 0.95 = 0.712$$



- The desired probability

$$P(A \to B) = 1 - P\left(\overline{A \to B}\right)$$
$$= 1 - P\left(\overline{A \to C \to B} \cap \overline{A \to D \to B}\right)$$
$$= 1 - P\left(\overline{A \to C \to B}\right) \times P\left(\overline{A \to D \to B}\right)$$
$$= 1 - (1 - 0.851) \times (1 - 0.712)$$
$$= 0.957$$

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Reference: 2022-03-04 Independence, P.18

3. Suppose X is a continuous random variable, its expectation $E[X] = 0.5$, and its PDF is given as:

$$f_X(x) = \begin{cases} ax + bx^2, if\ 0 < x < 1 \\ 0, otherwise \end{cases}$$

where a and b are two constants.

a) Please find a and b.

b) Please calculate the $var(X)$ =?

Ans:

a) $\because PDF\ property: \int_{-\infty}^{\infty} f_X(x)\,dx = 1$

$$\therefore \int_0^1 ax + bx^2\,dx = 1$$

$$\Rightarrow 3a + 2b = 6$$

$$\because E[X] = \int_0^1 x \cdot f_X(x)\,dx = 0.5$$

$$\therefore \int_0^1 x(ax + bx^2)\,dx = 0.5$$

$$\Rightarrow 4a + 3b = 6$$

solve equations: $\begin{cases} 3a + 2b = 6 \\ 4a + 3b = 6 \end{cases}$

$\Rightarrow a = 6, b = -6$

b) $E[X^2] = \int_0^1 x^2(6x - 6x^2)\,dx = 0.3$

$$\begin{aligned} Var(X) &= E[X^2] - E[X]^2 \\ &= 0.3 - 0.5^2 \\ &= 0.05 \end{aligned}$$

4. (10%) The definition of the law of total variance is $\mathrm{var}(X) = \mathbf{E}[\mathrm{var}(X|Y)] + \mathrm{var}(\mathbf{E}[X|Y])$.

(a) Please show that the equation always holds.

(b) Based on the law of total variance, if we have two independent random variables W and Z, please show that $\mathrm{var}(WZ) = (\mathbf{E}[W])^2\mathrm{var}(Z) + (\mathbf{E}[Z])^2\mathrm{var}(W) + \mathrm{var}(W)\mathrm{var}(Z)$.

a. 目標: $\mathrm{var}(X) = E[\mathrm{var}(X|Y)] + \mathrm{var}(E[X|Y])$

$\mathrm{var}(X) = E[X^2] - E[X]^2$ ($E[X]$ to $E[E[X|Y]]$)

$= \underbrace{E[E[X^2|Y]]}_{\downarrow} - E[E[X|Y]]^2$

$E[\mathrm{var}(X|Y) + E[X|Y]^2] \leftarrow (E[X^2|Y] = \mathrm{var}(X|Y) + E[X|Y]^2)$

$= E[\mathrm{var}(X|Y)] + E[E[X|Y]^2] - E[E[X|Y]]^2$

$= E[\mathrm{var}(X|Y)] + \mathrm{var}(E[X|Y])$

4. (10%) The definition of the law of total variance is $\mathrm{var}(X) = \mathbf{E}[\mathrm{var}(X|Y)] + \mathrm{var}(\mathbf{E}[X|Y])$.

(a) Please show that the equation always holds.

(b) Based on the law of total variance, if we have two independent random variables W and Z, please show that $\mathrm{var}(WZ) = (\mathbf{E}[W])^2\mathrm{var}(Z) + (\mathbf{E}[Z])^2\mathrm{var}(W) + \mathrm{var}(W)\mathrm{var}(Z)$.

b. let $A = WZ$

$$var(A) = var(E[A|W]) + E[var(A|W)]$$

$$\because E[A|W] = E[WZ|W] = WE[Z]$$
$$\therefore var(E[A|W]) = var(WE[Z]) = (E[Z])^2 var(W)$$

$$\because var(A|W) = var(WZ|W) = W^2 var(Z|W) = W^2 var(Z)$$
$$\therefore E[var(A|W)] = E\left[W^2 var(Z)\right] = var(Z)E\left[W^2\right]$$
$$= var(Z)(var(W) + (E[W])^2)$$
$$= var(Z)var(W) + var(Z)(E[W])^2$$

$$var(WZ) = var(A) = var(E[A|W]) + E[var(A|W)]$$
$$= (E[Z])^2 var(W) + var(Z)var(W) + var(Z)(E[W])^2$$

5. (10%) X and Y are two continuous random variables, and their joint PDF is:

$$f_{X,Y}(x,y) = \begin{cases} 3x+1, & \text{if } 0 \le x, 0 \le y, x+y<1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find marginal PDF $f_Y(y)$

(b) Find $P\left(X \ge \frac{1}{3}\right)$

a.

$$f_y(Y) = \int_0^{1-y} 3x+1 \, dx$$

$$= \left[\frac{3}{2}x^2 + x \right]_0^{1-y}$$

$$= \frac{3}{2}(1-y)^2 + (1-y)$$

$$= \frac{3}{2}(1-2y+y^2) + (1-y)$$

$$= \frac{3}{2}y^2 - 4y + \frac{5}{2}$$

b.

$$P(x \ge \tfrac{1}{3}) = \int_{\frac{1}{3}}^{1} \int_0^{1-x} 3x+1 \, dy dx$$

$$= \int_{\frac{1}{3}}^{1} (3x+1)(1-x) \, dx$$

$$= \int_{\frac{1}{3}}^{1} (-3x^2+2x+1) \, dx$$

$$= -x^3+x^2+x \Big|_{\frac{1}{3}}^{1}$$

$$= 1 - \left(-\frac{1}{27} + \frac{1}{9} + \frac{1}{3}\right) = \frac{16}{27} \text{ ※}$$

6. (15%) X and Y are two discrete random variables, and their joint PMF is given. We define a random variable Z as $Z = \mathbf{E}[X|Y]$.

(a) Please find the PMF of Y

(b) Please find the PMF of Y given $X = 1$ (i.e., $P_{Y|X}(y|1)$)

(c) Please calculate the expectation of Z (i.e., $\mathbf{E}(Z)$)

	$Y=0$	$Y=1$	$Y=2$
$X=0$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{8}$
$X=1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{4}$

6.

a. $P_Y(y) = \begin{cases} \frac{1}{6}+\frac{1}{8}=\frac{7}{24}, & y=0 \\ \frac{1}{6}+\frac{1}{6}=\frac{1}{3}, & y=1 \\ \frac{1}{8}+\frac{1}{4}=\frac{3}{8}, & y=2 \\ 0, & \text{otherwise} \end{cases}$

b. $P(X=1) = \frac{1}{8}+\frac{1}{6}+\frac{1}{4} = \frac{13}{24}$

$P_{Y|X}(Y|1) \begin{cases} \frac{1}{8} \div \frac{13}{24} = \frac{3}{13}, & y=0 \\ \frac{1}{6} \div \frac{13}{24} = \frac{4}{13}, & y=1 \\ \frac{1}{4} \div \frac{13}{24} = \frac{6}{13}, & y=2 \\ 0, & \text{otherwise} \end{cases}$

c. $Z = E[X|Y] = E[E[X|Y]] = E[X]$

$P_{X=1} = \frac{13}{24}$

$P_{X=0} = \frac{11}{24}$

$E[X] = 0 \cdot \frac{11}{24} + 1 \cdot \frac{13}{24}$

$= \frac{13}{24}$

7. (15%) X is a gaussian random variable, and its mean is 3 and the variance is 9. Givan a random variable $Y = 5 - X$, please use Φ to represent:

(a) $P(X > 2) =?$

(b) $P(-1 < Y < 3) =?$

(c) $P(X > 4|Y < 2) =?$

Hint: $P(X > 3) = \Phi(0)$

a) $P(X > 2) = 1 - \Phi\left(\frac{2-3}{3}\right) = 1 - \Phi\left(\frac{-1}{3}\right) = \Phi\left(\frac{1}{3}\right)$

b) $\because Y = 5 - X$

$\therefore E[Y] = 2, var(Y) = 9$

$$P(-1 < Y < 3) = \Phi\left(\frac{3-2}{3}\right) - \Phi\left(\frac{(-1)-2}{3}\right) = \Phi\left(\frac{1}{3}\right) - \Phi(-1)$$

c) $P(X > 4|Y < 2) = P(X > 4|X > 3) = \frac{P(X>4\,,X>3)}{P(X>3)} = \frac{P(X>4)}{P(X>3)}$

$$= \frac{1 - \Phi\left(\frac{4-3}{3}\right)}{1 - \Phi\left(\frac{3-3}{3}\right)} = \frac{1 - \Phi\left(\frac{1}{3}\right)}{1 - 0.5} = 2\left(1 - \Phi\left(\frac{1}{3}\right)\right)$$

8. (5%) The pair of random variables (X, Y) takes the values $(1,0)$, $(0,-2)$, $(-3,0)$, and $(0,4)$, each with probability $\frac{1}{4}$, please calculate their covariance $\operatorname{cov}(X, Y) = ?$

$$\operatorname{cov}(X,Y) = E[XY] - E[X]E[Y]$$

$$= \sum_{x,y} xy \cdot P_{XY}(x,y) - \left(\sum_{x} x \cdot P_X(x)\right)\left(\sum_{y} y \cdot P_Y(y)\right)$$

$$= \left(1\times 0\times \tfrac{1}{4} + 0\times -2\times \tfrac{1}{4} + -3\times 0\times \tfrac{1}{4} + 0\times 4\times \tfrac{1}{4}\right)$$

$$- \left(1\times \tfrac{1}{4} + 0\times \tfrac{1}{4} + -3\times \tfrac{1}{4} + 0\times \tfrac{1}{4}\right)\left(0\times \tfrac{1}{4} + -2\times \tfrac{1}{4} + 0\times \tfrac{1}{4} + 4\times \tfrac{1}{4}\right)$$

$$= 0 - \left(-\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) = \tfrac{1}{4}$$ #

9. (5%) X and Y are two random variables, and their variances are $\text{var}(X) = 4$ and $\text{var}(Y) = 9$, respectively. Suppose random variables $Z = 2X - Y$ and $W = X + Y$ are independent, please find $\text{cov}(X, Y) =?$

$$E[ZW] = E[Z]E[W]$$

$$Cov(Z, W) = Cov(2X - Y, X + Y) = Cov(2X, X) + Cov(2X, Y) - Cov(Y, X) - Cov(Y, Y) = 0$$

$$2var(x) + Cov(X, Y) - var(Y) = 0$$

$$8 + Cov(X, Y) - 9 = 0$$

$$Cov(X, Y) = 1$$

10. (5%) Let X, Y and Z are three independent Gaussian random variables, and their means and variances are all equal to 1. Please find $\mathbf{E}[XY|Y+Z=1]=?$

$$\because \mathrm{E}[Y|Y+Z=1]=\mathrm{E}[Z|Y+Z=1]$$

$$\therefore \mathrm{E}[Y|Y+Z=1]+\mathrm{E}[Z|Y+Z=1]=\mathrm{E}[Y+Z|Y+Z=1]=1$$

$$\mathrm{E}[Y|Y+Z=1]=\mathrm{E}[Z|Y+Z=1]=\frac{1}{2}$$

$$\mathrm{E}[XY|Y+Z=1]=\mathrm{E}[X]\mathrm{E}[Y|Y+Z=1]=1\times\frac{1}{2}=\frac{1}{2}$$

11. (5%) If X and Y are two independent continuous random variables, and $\mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y]$. Please show that $\mathrm{var}(X - Y) = \mathrm{var}(X) + \mathrm{var}(Y)$.

$$\mathrm{var}(X-Y) = E[(X-Y)^2] - (E[X-Y])^2$$

$$= E[X^2 - 2XY + Y^2] - (E[X] - E[Y])^2$$

$$= \left(E[X^2] \cancel{-2E[X]E[Y]} + E[Y^2]\right) - \left(E[X]^2 \cancel{-2E[X]E[Y]} + E[Y]^2\right)$$

$$= \left(E[X^2] - E[X]^2\right) + \left(E[Y^2] - E[Y]^2\right)$$

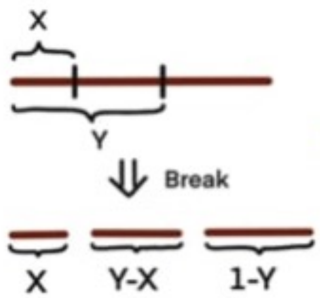
$$= \mathrm{Var}(X) + \mathrm{Var}(Y)$$

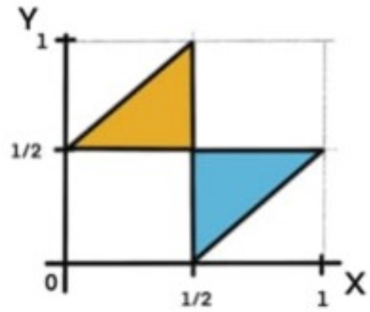
12. (5%) For a given unit-length stick, if you break it randomly into three pieces. What is the probability of making a triangle using the three pieces?

Assuming Y > X: (yellow region)

$$X + (Y - X) > (1 - Y)$$
$$(Y - X) + (1 - Y) > X$$
$$X + (1 - Y) > (Y - X)$$

$$\Rightarrow X > \frac{1}{2},\ Y < \frac{1}{2},\ X < Y + \frac{1}{2}$$





Probability: $\frac{1}{8} + \frac{1}{8} = \frac{1}{4}$

Assuming X > Y: (blue region)

等式與 (Y > X) 相反。

13. (5%) X and Y are two independent random variables, and their variances are $\text{var}(X) = 7$ and $\text{var}(Y) = 9$, respectively. Given random variables $Z = 7 + X + Y$ and $W = 1 + Y$, please find correlation coefficient $\rho(W, Z) =?$

$$\rho(W,Z) = \frac{\text{cov}(W,Z)}{\sqrt{\text{Var}(W)\,\text{var}(Z)}}$$

$$= \frac{\text{cov}(7+X+Y,\ 1+Y)}{\sqrt{\text{var}(7+X+Y)\,\text{var}(1+Y)}}$$

$$= \frac{\text{cov}(X+Y,\ Y)}{\sqrt{\text{Var}(X+Y)\,\text{var}(Y)}}$$

$$= \frac{\text{cov}(X+Y) + \text{cov}(Y,Y)}{\sqrt{\text{Var}(X+Y)\,\text{Var}(Y)}}$$

$\because$ X and Y are independent

$$\therefore \rho(W,Z) = \frac{0 + \text{cov}(Y,Y)}{\sqrt{[\text{Var}(X)+\text{Var}(Y)]\,\text{Var}(Y)}}$$

$$= \frac{0 + \text{Var}(Y)}{\sqrt{[\text{Var}(X)+\text{Var}(Y)]\,\text{Var}(Y)}}$$

$$= \frac{0+9}{\sqrt{(7+9)\times 9}}$$

$$= \frac{3}{4}\ \#$$

14. (10%) The definition of the transform for a random variable X with a scalar parameter s is $M_X(s) = \mathbf{E}[e^{sX}]$.

(a) Find the transform associated with an integer-valued (discrete) random variable Y that is uniformly distributed in the range $\{a, \cdots, b\}$.

(b) Find the transform associated with a continuous random variable Z that is uniformly distributed in the range $[a, b]$.

Hint: $\int e^{cx}dx = \frac{1}{c}e^{cx}$

Ans:

a) $$P_Y(y) = \begin{cases} \frac{1}{b-a+1}, if\ y\ in\ the\ set{:}\ \{a, a+1, \cdots, b-1,\ b\} \\ 0, otherwise \end{cases}$$

$$M_Y(s) = E[e^{sY}] = \sum_y e^{sy} \cdot P_Y(y) = e^{sa} \cdot \frac{1}{b-a+1} + e^{s(a+1)} \cdot \frac{1}{b-a+1} + \cdots + e^{sb} \cdot \frac{1}{b-a+1}$$

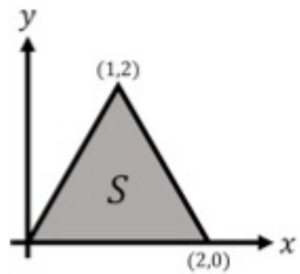
b) $$f_Z(z) = \begin{cases} \frac{1}{b-a}, if\ a \leq y \leq b \\ 0, otherwise \end{cases}$$

$$M_Z(s) = E[e^{sZ}] = \int_{-\infty}^{\infty} e^{sZ} \cdot f_Z(z)dz = \int_a^b e^{sZ} \cdot \frac{1}{b-a}dz = \frac{1}{b-a}\int_a^b e^{sZ}dz = \frac{1}{b-a} \cdot (\frac{1}{s}e^{sZ}|_a^b) = \frac{e^{sb} - e^{sa}}{s(b-a)}$$

15. (10%) Suppose the joint PDF of random variables X and Y is a constant c on the set S shown in the figure and is zero outside.

(a) Please find the constant c =?

(b) Please calculate $P(X \leq 1, Y \leq 1)$ =?

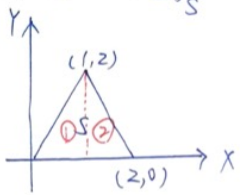


$f_{X,Y}(x,y) = \begin{cases} c, & \text{if } (x,y) \text{ in } S \\ 0, & \text{otherwise} \end{cases}$

normalization property : $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$

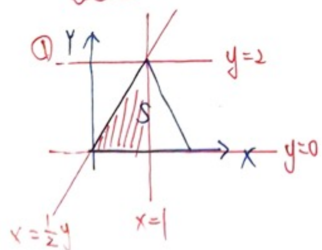
由於在S以外的地方機率為0，可以化簡成 $\Rightarrow \iint_S c\,dx\,dy = 1$

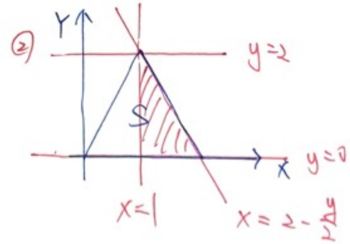
將S拆成兩部分來積分 $\Rightarrow$



寫法一：

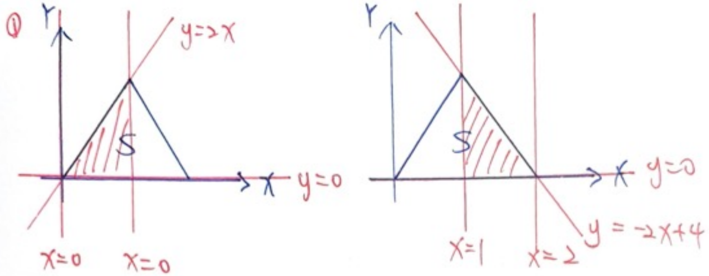
$$\underbrace{\int_0^2\int_{\frac{1}{2}y}^{1} c\,dx\,dy}_{①} + \underbrace{\int_0^2\int_{1}^{2-\frac{y}{2}} c\,dx\,dy}_{②} = 1 \Rightarrow c = \frac{1}{2} \#$$





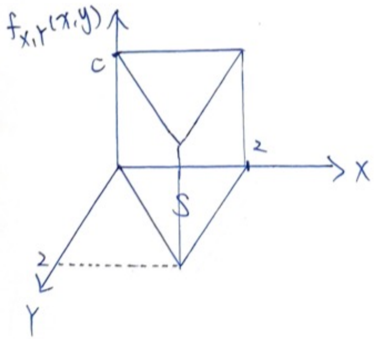
寫法二：

$$\underbrace{\int_0^1\int_0^{2x} c\,dy\,dx}_{} + \underbrace{\int_1^2\int_0^{-2x+4} c\,dy\,dx}_{} = 1 \quad\Rightarrow\quad c=\frac{1}{2}\ \#$$



寫法三：

不會積分也沒關係，因為剛才的積分就是在求下圖的三角柱體積



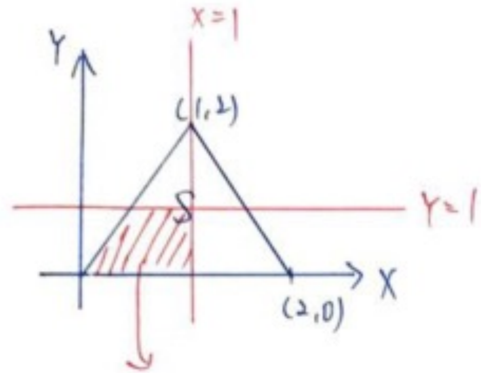
$\therefore$ 體積 $= \frac{2\times 2}{2}\times C$

$=$ 剛才的積分 $= 1$

$\therefore\ C=\frac{1}{2}$ #

15.

(b)



$X\le 1$, $Y\le 1$ 這塊只有原本三角形的 $\frac{3}{8}$

$\therefore P(X\le 1, Y\le 1)$ 爲原三角柱 $\frac{3}{8}$ 的體積 $=\frac{3}{8}$

A: $\frac{3}{8}$