1. (5%) A family has 4 children. Each child has an equal probability of being a girl or a boy, independent of the other children. Please calculate the probability that there are exactly 3 girls.

$$probability = \frac{C_3^4}{2^4} = \frac{1}{4}$$

2. (10%) Suppose we draw 7 cards from a well-shuffled standard 52-card deck, then please find the probability that:

(a) The 7 cards include exactly 3 aces.

(b) The 7 cards have exactly 3 aces, 2 kings, or both.

$a)\ \frac{C_3^4\, C_4^{48}}{C_7^{52}}$ $\qquad$ $b)\ \frac{C_3^4\, C_4^{48} + C_2^4\, C_5^{48} - C_3^4\, C_2^4\, C_2^{44}}{C_7^{52}}$

3. (5%) There are n students in a probability class, and we assume that a year has 365 days. What is the probability that at least two of the students celebrate their birthdays on the same day?

$$= 1 - (n\ students\ \text{全都不同天生日})$$

$$= 1 - \frac{P_n^{365}}{365^n}$$

4. (10%) If X and Y are independent random variables, and a and b are two scalers, please show that $\mathrm{var}(aX + bY) = a^2\mathrm{var}(X) + b^2\mathrm{var}(Y)$.

$$Var(aX + bY) = E[(aX + bY)^2] - E[aX + bY]^2$$

$$= E[(aX)^2 + 2(aX)(bY) + (bY)^2] - (E[aX] + E[bY])^2$$

$$= E[(aX)^2] + E[2(aX)(bY)] + E[(bY)^2] - (E[aX]^2 + 2E[aX]E[bY] + E[bY]^2)$$

$\downarrow$ ∵ X and Y are independent

$$= E[(aX)^2] + 2E[aX]E[bY] + E[(bY)^2] - E[aX]^2 - 2E[aX]E[bY] - E[bY]^2$$

$$= Var(aX) + Var(bY)$$

$$= a^2Var(X) + b^2Var(Y)$$

5. (10%) X and Y are two discrete random variables and $Y = |X|$. The PMF of X is in the form of:

$$p_X(x) = \begin{cases} kx^2, & if\ x = -3,-2,-1,0,1,2,3 \\ 0, & \text{otherwise} \end{cases},$$

where k is a constant.

(a) Please determine the value of k.

(b) Please calculate the standard deviation σ_Y of Y.

a) $\sum_x p_X(x) = \sum_x kx^2 = 1$

$$\sum_x kx^2 = k(-3)^2 + k(-2)^2 + k(-1)^2 + k(0)^2 + k(1)^2 + k(2)^2 + k(3)^2$$
$$= 28k = 1$$

$$\therefore k = \frac{1}{28}$$

$$b)\ p_Y(y) = \begin{cases} 0,\ y = 0, \\ \frac{1}{14}, y = 1 \\ \frac{4}{14}, if\ y = 2 \\ \frac{9}{14}, if y = 3 \\ 0, otherwise \end{cases}$$

$$E[Y] = 1 \times \frac{1}{14} + 2 \times \frac{4}{14} + 3 \times \frac{9}{14} = \frac{18}{7}$$

$$\sigma_Y = \sqrt{Var(Y)} = \sqrt{E[Y^2] - E[Y]^2} = \sqrt{\frac{19}{49}}$$

$$\approx 0.62$$

6. (10%) Suppose X is a random variable, and its PMF is listed in the following table. Please compute the probability: (10%)

x	-3	-1	0	1	2	3	5	8
$p_X(x)$	0.10	0.20	0.15	0.20	0.10	0.15	0.05	0.05

(a) $P(X \geq 3|X > 0)$

(b) $P(X = -3|X \leq 0)$

a) $$P(X \geq 3|X > 0) = \frac{P(x\geq 3 \; and \; x>0)}{P(x>0)} = \frac{0.15+0.05+0.05}{0.2+0.1+0.15+0.05+0.05} \approx 0.45$$

b) $$P(X = -3|X \leq 0) = \frac{P(x=-3 \; and \; x\leq 0)}{P(x\leq 0)} = \frac{0.1}{0.1+0.2+0.15} \approx 0.22$$

7. (15%) Let X and Y are two random variables, and their joint PMF are given in the following table. Please compute the probability:

(a) Y is even

(b) XY is odd

(c) $X > 0$ and $Y \geq 0$

		Y			
		-2	1	2	6
X	-2	$\frac{1}{9}$	$\frac{1}{27}$	$\frac{1}{27}$	$\frac{1}{9}$
	1	$\frac{2}{9}$	0	$\frac{1}{9}$	$\frac{1}{9}$
	3	0	0	$\frac{1}{9}$	$\frac{4}{27}$

$a)\, probility = \frac{26}{27}$

		Y			
		-2	1	2	6
X	-2	$\frac{1}{9}$	$\frac{1}{27}$	$\frac{1}{27}$	$\frac{1}{9}$
	1	$\frac{2}{9}$	0	$\frac{1}{9}$	$\frac{1}{9}$
	3	0	0	$\frac{1}{9}$	$\frac{4}{27}$

$b)\, probility = 0$

		Y			
		-2	1	2	6
X	-2	$\frac{1}{9}$	$\frac{1}{27}$	$\frac{1}{27}$	$\frac{1}{9}$
	1	$\frac{2}{9}$	0	$\frac{1}{9}$	$\frac{1}{9}$
	3	0	0	$\frac{1}{9}$	$\frac{4}{27}$

$c)\, probility = \frac{13}{27}$

		Y			
		-2	1	2	6
X	-2	$\frac{1}{9}$	$\frac{1}{27}$	$\frac{1}{27}$	$\frac{1}{9}$
	1	$\frac{2}{9}$	0	$\frac{1}{9}$	$\frac{1}{9}$
	3	0	0	$\frac{1}{9}$	$\frac{4}{27}$

8. (20%) Consider a set of points in a plane, and we use random variables X and Y to denote their coordinates, which satisfy:

$$\{(x, y)|x, y \in \mathbb{Z}, x^2 + |y| \leq 2\}$$

Suppose we pick a point from these points at an entirely random. Please find:

(a) The marginal PMF of X.

(b) The conditional PMF of $p_{X|Y}(x|1)$.

(c) Are X and Y independent? Please clearly describe your answer.

(d) Please calculate $\mathbf{E}[XY^2] =?$

Hint: $\mathbb{Z}$ denotes the set of integers $\{\cdots, -2, -1, 0, 1, 2, \cdots\}$

$$points\ (x, y) \in \{(0,0), (0,1), (0,2), (0,-1), (0,-2), (1,0), (1,1), (1,-1), (-1,0), (-1,1), (-1,-1)\}$$

a) $$p_X(x) = \begin{cases} \frac{5}{11}, & if\ x = 0 \\ \frac{3}{11}, & if\ x = 1 \\ \frac{3}{11}, & if\ x = -1 \\ 0, & otherwise \end{cases}$$

b) $$points\ (x, 1) \in \{(0,1), (1,1), (-1,1)\}$$

$$p_{X|Y}(x|1) = \begin{cases} \frac{1}{3}, & if\ x = 0, 1, -1 \\ 0, & otherwise \end{cases}$$

8. (20%) Consider a set of points in a plane, and we use random variables X and Y to denote their coordinates, which satisfy:

$$\{(x, y) | x, y \in \mathbb{Z}, x^2 + |y| \leq 2\}$$

Suppose we pick a point from these points at an entirely random. Please find:

(a) The marginal PMF of X.

(b) The conditional PMF of $p_{X|Y}(x|1)$.

(c) Are X and Y independent? Please clearly describe your answer.

(d) Please calculate $\mathbf{E}[XY^2] = ?$

Hint: $\mathbb{Z}$ denotes the set of integers $\{\cdots, -2, -1, 0, 1, 2, \cdots\}$

$$points\ (x, y) \in \{(0,0), (0,1), (0,2), (0,-1), (0,-2), (1,0), (1,1), (1,-1), (-1,0), (-1,1), (-1,-1)\}$$

c) 方法一：

X and Y are independent if for all x and y satisfy $p_{X,Y}(x, y) = p_X(x) p_Y(y)$

$$p_{X,Y}(0,0) = \frac{1}{11}$$

$$p_X(0) p_Y(0) = \frac{5}{11} \times \frac{3}{11}$$

$$\because p_{X,Y}(0,0) \neq p_X(0) p_Y(0) \quad \therefore X\ and\ Y\ are\ not\ independent$$

方法二： show $p_{X|Y}(x|y) \neq p_X(x)$

8. (20%) Consider a set of points in a plane, and we use random variables X and Y to denote their coordinates, which satisfy:

$$\{(x, y) | x, y \in \mathbb{Z}, x^2 + |y| \leq 2\}$$

Suppose we pick a point from these points at an entirely random. Please find:

(a) The marginal PMF of X.

(b) The conditional PMF of $p_{X|Y}(x|1)$.

(c) Are X and Y independent? Please clearly describe your answer.

(d) Please calculate $\mathbf{E}[XY^2]$ =?

Hint: $\mathbb{Z}$ denotes the set of integers $\{\cdots, -2, -1, 0, 1, 2, \cdots\}$

$$points\ (x, y) \in \{(0,0), (0,1), (0,2), (0,-1), (0,-2), (1,0), (1,1), (1,-1), (-1,0), (-1,1), (-1,-1)\}$$

d) Let $Z = XY^2$

$$p_Z(z) = \begin{cases} \frac{7}{11}, if\ z = 0 \\ \frac{2}{11}, if\ z = 1 \\ \frac{2}{11}, if\ z = -1 \\ 0, otherwise \end{cases}$$

$$E[XY^2] = E[Z]$$
$$= 0 \times \frac{7}{11} + 1 \times \frac{2}{11} + -1 \times \frac{2}{11} = 0$$

9. (10%) Given a discrete random variable X and the PMF function:

$$p_X(x) = \begin{cases} a, & if\ x = 1 \\ 1 - a, & if\ x = 0 \\ 0, & \text{otherwise} \end{cases}$$

(a) Please calculate $\mathbf{E}[X] =?$

(b) Please calculate $\mathrm{var}(X) =?$

a) $E[X] = 1 \cdot a + 0 \cdot (1 - a) = a$

b) $Var(X) = E[X^2] - E[X]^2 = 1^2 \cdot a + 0^2 \cdot (1 - a) - a^2 = a - a^2$

10. (5%) Given a discrete random variable X and the PMF function:

$$p_X(x) = \begin{cases} a(1-a)^{x-1}, & if\ x = 1,2,\ldots \\ 0, & \text{otherwise} \end{cases}$$

Please calculate $\mathbf{E}[X] = ?$

Hint: $br^i + br^{i+1} + br^{i+2} + \cdots = \frac{br^i}{1-r}$

$$E[X] = \sum_{x=1}^{\infty} x \cdot a(1-a)^{x-1}$$

$$= 1 \cdot a(1-a)^0 + 2 \cdot a(1-a)^1 + 3 \cdot a(1-a)^2 + \cdots$$

$$= [a(1-a)^0 + a(1-a)^1 + a(1-a)^2 + \cdots] + [a(1-a)^1 + a(1-a)^2 + \cdots] + [a(1-a)^2 + \cdots] + \cdots$$

$$= [\frac{a(1-a)^0}{1-(1-a)}] + [\frac{a(1-a)^1}{1-(1-a)}] + [\frac{a(1-a)^2}{1-(1-a)}] + \cdots$$

$$= (1-a)^0 + (1-a)^1 + (1-a)^2 + \cdots$$

$$= \frac{1(1-a)^0}{1-(1-a)} = \frac{1}{a}$$

11. (5%) Given a discrete random variable X and the PMF function:

$$p_X(x) = \begin{cases} e^{-\lambda}\dfrac{\lambda^x}{x!}, & if\ x = 0,1,2,\ldots \\ 0, & \text{otherwise} \end{cases}$$

Please calculate $\mathbf{E}[X] =?$

$$\mathbf{E}[X] = \sum_{k=0}^{\infty} k e^{-\lambda}\frac{\lambda^k}{k!} = \sum_{k=1}^{\infty} k e^{-\lambda}\frac{\lambda^k}{k!} = \lambda \sum_{k=1}^{\infty} e^{-\lambda}\frac{\lambda^{k-1}}{(k-1)!} = \lambda \sum_{m=0}^{\infty} e^{-\lambda}\frac{\lambda^m}{m!} = \lambda$$