CS2008302: Probability and Statistics

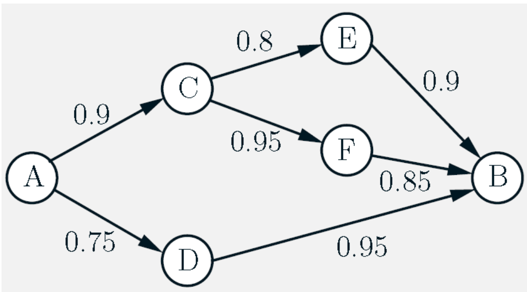
Quiz 4 @ 2022/05/27 10:20~12:10

Name:_______________

ID:_________________

1. (5%) A class consisting of 4 graduate and 12 undergraduate students is randomly divided into four groups of 4. What is the probability that each group includes exactly a graduate student?

2. (5%) A computer network connects two nodes A and B through intermediate nodes $C, D, E,$ and F. There is a given probability that the link is up for every pair of directly connected nodes. We assume that link failures are independent of each other. What is the probability that nodes A and B are connected?



3. (10%) Suppose X is a continuous random variable, its expectation $\mathbf{E}[X] = 0.5$, and its PDF is given as:

$$f_X(x) = \begin{cases} ax + bx^2, & \mathit{if}\ 0 < x < 1 \\ 0, & \mathit{otherwise} \end{cases},$$

where a and b are two constants.
   (a) Please find a and b
   (b) Please calculate the $\mathrm{var}(X) =?$

4. (10%) The definition of the law of total variance is $\mathrm{var}(X) = \mathbf{E}[\mathrm{var}(X|Y)] + \mathrm{var}(\mathbf{E}[X|Y])$.
   (a) Please show that the equation always holds.
   (b) Based on the law of total variance, if we have two independent random variables W and Z, please show that $\mathrm{var}(WZ) = (\mathbf{E}[W])^2\mathrm{var}(Z) + (\mathbf{E}[Z])^2\mathrm{var}(W) + \mathrm{var}(W)\mathrm{var}(Z)$.

5. (10%) X and Y are two continuous random variables, and their joint PDF is:

$$f_{X,Y}(x,y) = \begin{cases} 3x+1, & \text{if } 0 \le x, 0 \le y, x+y < 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find marginal PDF $f_Y(y)$

(b) Find $P\left(X \ge \frac{1}{3}\right)$

6. (15%) X and Y are two discrete random variables, and their joint PMF is given. We define a random variable Z as $Z = \mathbf{E}[X|Y]$.

	$Y=0$	$Y=1$	$Y=2$
$X=0$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{8}$
$X=1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{4}$

(a) Please find the PMF of Y

(b) Please find the PMF of Y given $X = 1$ (i.e., $P_{Y|X}(y|1)$)

(c) Please calculate the expectation of Z (i.e., $\mathbf{E}(Z)$)

7. (15%) X is a gaussian random variable, and its mean is 3 and the variance is 9. Givan a random variable $Y = 5 - X$, please use Φ to represent:

(a) $P(X > 2) =?$

(b) $P(-1 < Y < 3) =?$

(c) $P(X > 4|Y < 2) =?$

Hint: $P(X > 3) = \Phi(0)$

8. (5%) The pair of random variables (X,Y) takes the values $(1,0)$, $(0,-2)$, $(-3,0)$, and $(0,4)$, each with probability $\frac{1}{4}$, please calculate their covariance $\text{cov}(X,Y) =?$

9. (5%) X and Y are two random variables, and their variances are $\text{var}(X) = 4$ and $\text{var}(Y) = 9$, respectively. Suppose random variables $Z = 2X - Y$ and $W = X + Y$ are independent, please find $\text{cov}(X,Y) =?$

10. (5%) Let X, Y and Z are three independent Gaussian random variables, and their means and variances are all equal to 1. Please find $\mathbf{E}[XY|Y + Z = 1] =?$

11. (5%) If X and Y are two independent continuous random variables, and $\mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y]$. Please show that $\text{var}(X - Y) = \text{var}(X) + \text{var}(Y)$.

12. (5%) For a given unit-length stick, if you break it randomly into three pieces. What is the probability of making a triangle using the three pieces?

13. (5%) X and Y are two independent random variables, and their variances are $\text{var}(X) = 7$ and $\text{var}(Y) = 9$, respectively. Given random variables $Z = 7 + X + Y$ and $W = 1 + Y$, please find correlation coefficient $\rho(W, Z) =$?

14. (10%) The definition of the transform for a random variable X with a scalar parameter s is $M_X(s) = \mathbf{E}[e^{sX}]$.
    (a) Find the transform associated with an integer-valued (discrete) random variable Y that is uniformly distributed in the range $\{a, \cdots, b\}$.
    (b) Find the transform associated with a continuous random variable Z that is uniformly distributed in the range $[a, b]$.

    Hint: $\int e^{cx} dx = \frac{1}{c} e^{cx}$

15. (10%) Suppose the joint PDF of random variables X and Y is a constant c on the set S shown in the figure and is zero outside.
    (a) Please find the constant $c =$?
    (b) Please calculate $P(X \le 1, Y \le 1) =$?