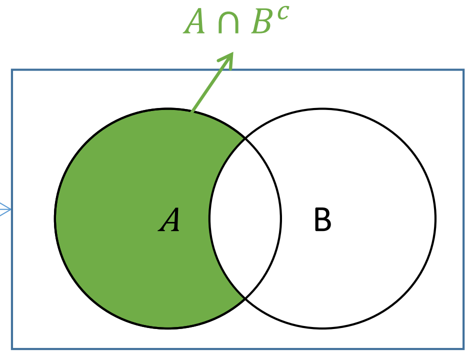
1. If A and B are independent, please show that $A$and B^c are also independent.



$$P(A \cap B^c) = P(A) - P(A \cap B)$$

A, B *are independent*
$\therefore P(A \cap B) = P(A)P(B)$

$$= P(A) - P(A)P(B)$$
$$= P(A)(1 - P(B))$$

$$= P(A)P(B^c)$$

$$\therefore A \text{ and } B^c \text{ are independent}$$

2. A class consisting of 4 graduate and 12 undergraduate students is randomly divided into four groups of 4.
What is the probability that each group includes exactly a graduate student?

方法一

$A_1 = \{\text{graduate students 1 and 2 are in different groups}\}$
$A_2 = \{\text{graduate students 1, 2 and 3 are in different groups}\}$
$A_3 = \{\text{graduate students 1, 2, 3 and 4 are in different groups}\}$

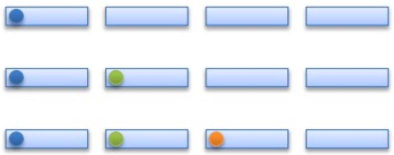
$$P(A_3) = P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2)$$

$$P(A_1) = \frac{12}{15}$$

$$P(A_2|A_1) = \frac{8}{14}$$

$$P(A_3|A_1 \cap A_2) = \frac{4}{13}$$

$$P(A_3) = \frac{12}{15} \times \frac{8}{14} \times \frac{4}{13}$$



方法二、三

– By partition

$$\frac{\dfrac{12!}{3! \times 3! \times 3! \times 3!} \times 4!}{\dfrac{16!}{4! \times 4! \times 4! \times 4!}}$$

– By combination

$$\frac{\binom{12}{3}\binom{9}{3}\binom{6}{3}\binom{3}{3} \times 4!}{\binom{16}{4}\binom{12}{4}\binom{8}{4}\binom{4}{4}}$$

Reference:2022-02-25 Conditional Probability,P14

3. Suppose events A and B are disjoint, please show that

$$P(A \cup B|C) = P(A|C) + P(B|C).$$

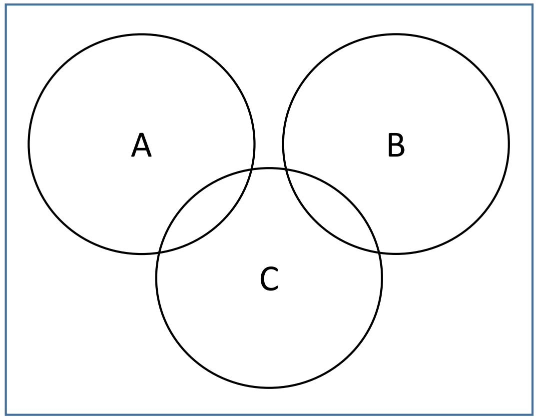
$$P(A \cup B|C) = \frac{P((A \cup B) \cap C)}{P(C)}$$

$$= \frac{P((A \cap C) \cup (B \cap C))}{P(C)}$$

$$= \frac{P(A \cap C) + P(B \cap C)}{P(C)}$$

$$= \frac{P(A \cap C)}{P(C)} + \frac{P(B \cap C)}{P(C)}$$

$$= P(A|C) + P(B|C)$$



A *and* B *are disjoint*

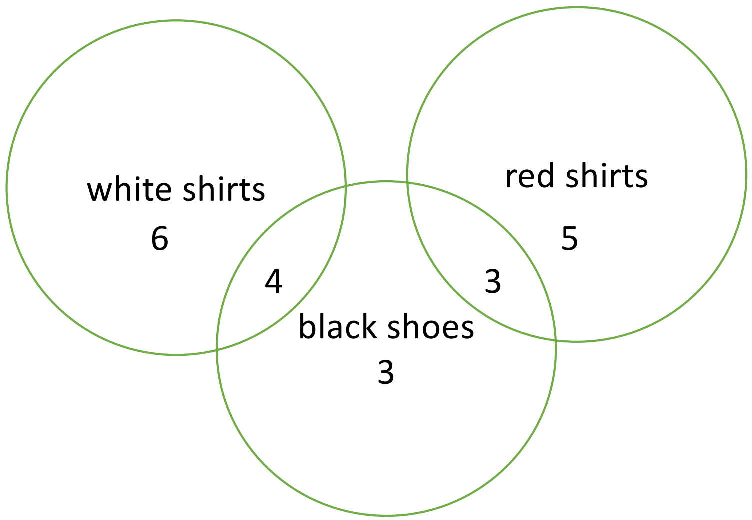
$\therefore (A \cap C)$ *and* $(B \cap C)$ *are disjoint*

$\therefore P((A \cap C) \cup (B \cap C)) = P(A \cap C) + P(B \cap C)$ **(additivity)**

2022-02-25 Conditional Probability P.5

4. In a party, there are 10 people with white shirts and 8 people with red shirts;
   4 people have black shoes and white shirts;
   3 people have black shoes and red shirts;
   the total number of people with white or red shirts or black shoes is 21.
   How many people have black shoes?

total=21



answer: 4 + 3 + 3 = 10

5. Suppose that I want to buy a new iphone SE. I can choose either a large or a small screen; a 64GB, 128GB, or 256GB storage capacity, and a black, white or red color. How many different options do I have?

$$(large, small)\times(64GB, 128GB, 256GB)\times(black, white, red)$$
$$= 2\times3\times3 = 18$$

6. A bag contains 30 red balls and 70 green balls. What is the probability of getting exactly $k (0 \leq k \leq 20)$ red balls in a sample of size 20 if the sampling is done with replacement (repetition allowed)?

$$P(A) = \frac{30^k \times 70^{(20-k)} \times \frac{20!}{k!\,(20-k)!}}{100^{20}}$$

$$or$$

$$P(A) = 0.3^k \times 0.7^{(20-k)} \times \frac{20!}{k!\,(20-k)!}$$

$$or$$

$$P(A) = 0.3^k \times 0.7^{(20-k)} \times C_k^{20}$$

7. A bag contains 30 red balls and 70 green balls. What is the probability of getting exactly $k(0 \leq k \leq 20)$ red balls in a sample of size 20 if the sampling is done without replacement (repetition not allowed)?

$$P(A) = \frac{C_k^{30} \times C_{20-k}^{70} \times 20!}{P_{20}^{100}}$$

$$or$$

$$p(A) = \frac{C_k^{30} \times C_{20-k}^{70}}{C_{20}^{100}}$$

8. If $P(A) = 0.4$, $P(B|A) = 0.35$, and $P(A \cup B) = 0.69$, please calculate $P(B) = ?$

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(B \cap A)}{0.4} = 0.35$$
$$P(B \cap A) = 0.4 \times 0.35 = 0.14$$

$$P(A \cup B)$$
$$= P(A) + P(B) - P(A \cap B)$$
$$= 0.4 + P(B) - 0.14 = 0.69$$
$$P(B) = 0.43$$

9.How many different words (letter sequences) can be obtained by rearranging the letters in the word DETARTRATED?

$$DETRATRATED$$

total: 11

$$A\times 2, D\times 2, E\times 2, R\times 2, T\times 3$$

$$\frac{11!}{2!\,2!\,2!\,2!\,3!}$$

10. Consider an experiment involving two successive rolls of a 6-sided die in which all 36 possible outcomes are equally likely and have probability $\frac{1}{36}$.
Are the events A={1st roll is a 1} and B={sum of the two rolls is a 6} independent? Please clearly state your comments.

$$P(A) = \frac{1}{6}$$

$$P(B) = \frac{5}{36}$$

$$P(A \cap B) = \frac{1}{36} \neq P(A) \times P(B)$$

$$A \text{ and } B \text{ are not independent}$$

2022-03-04 Independence P.7

11. A computer network connects two nodes $A$and Bthrough intermediate nodes C,D, E and F. For every pair of directly connected nodes, there is a given probability that the link is up. We assume that link failures are independent of each other. What is the probability that there is a path connecting $A$and Bin which all links are up?

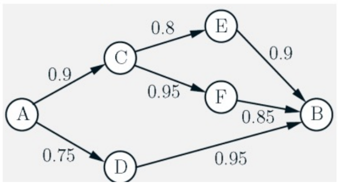
– The probability of the connections from C to B are all fail

$P(C \to E \to B) = 0.8 \times 0.9$

$P(C \to F \to B) = 0.95 \times 0.85$

$$P(\overline{C \to B}) = P(\overline{C \to E \to B} \cap \overline{C \to F \to B})$$
$$= (1 - (0.8 \times 0.9)) \times (1 - (0.95 \times 0.85))$$

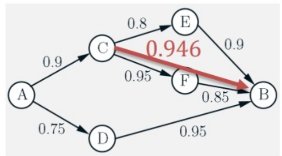
$\therefore P(C \to B) = 1 - P(\overline{C \to B}) = 0.946$



– The probabilities of the two successful paths

$P(A \to C \to B) = 0.9 \times 0.946 = 0.851$

$P(A \to D \to B) = 0.75 \times 0.95 = 0.712$



– The desired probability

$$\begin{aligned} P(A \to B) &= 1 - P(\overline{A \to B}) \\ &= 1 - P(\overline{A \to C \to B} \cap \overline{A \to D \to B}) \\ &= 1 - P(\overline{A \to C \to B}) \times P(\overline{A \to D \to B}) \\ &= 1 - (1 - 0.851) \times (1 - 0.712) \\ &= 0.957 \end{aligned}$$

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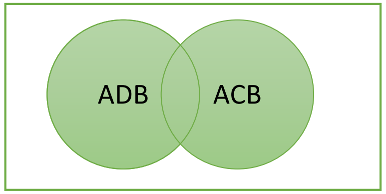
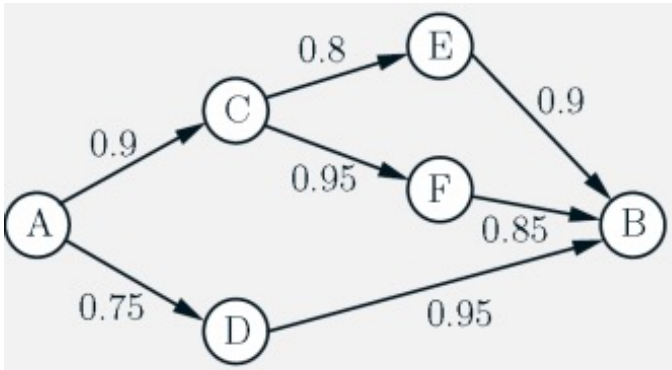
2022-03-04 Independence P.18



$$P(AB) = P(ADB \cup ACB) = P(ADB) + P(ACB) - P(ACB \cap ACB)$$
$$= 0.712 + 0.854 - 0.712 \times 0.854 = 0.957$$



$$P(ADB) = 0.75 \times 0.95 = 0.712$$

$$P(ACB) = P(AC \cap CB) = P(AC) \times P(CB)$$
$$= 0.9 \times 0.946 = 0.854$$

$$P(CB) = P(CFB \cup CEB) = P(CFB) + P(CEB) - P(CFB) \times P(CEB)$$
$$= 0.807 + 0.72 - 0.807 \times 0.72 = 0.946$$

$$P(CFB) = 0.95 \times 0.85 = 0.807$$

$$P(CEB) = 0.8 \times 0.9 = 0.72$$