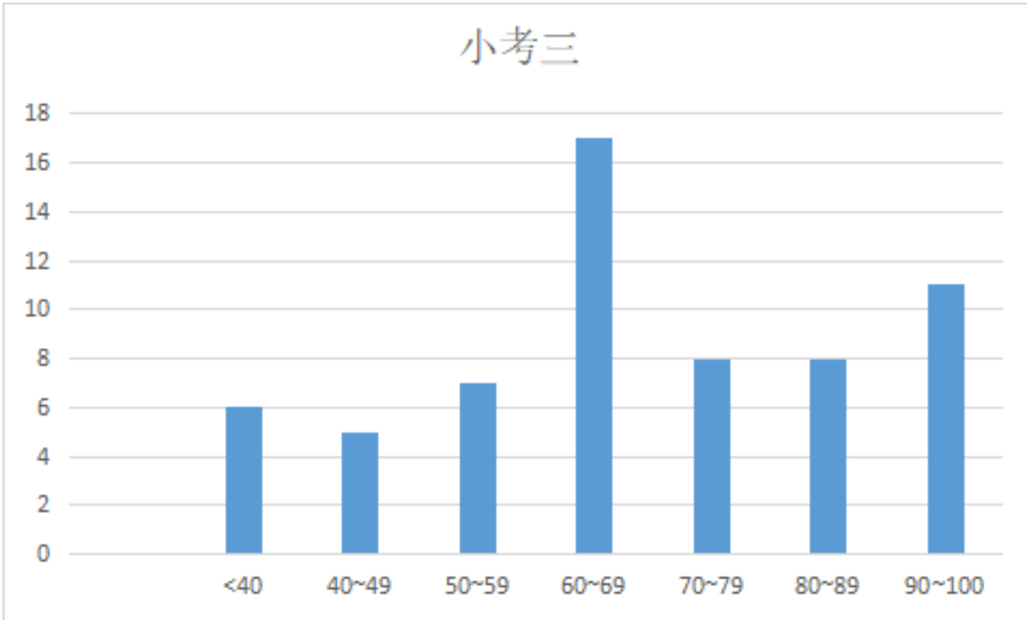

小考三
18
16
14
12
10
8
6
4
2
0
<40
40~49
50~59
60~69
70~79
80~89
90~100

1. (10%) Given a biased coin for which $P(Head) = t$. You toss the coin 20 times.

a) What is the probability that you obtain 8 heads and 12 tails?

b) What is the probability of obtaining more than 8 heads and more than 12 tails?

a) probability $= C_8^{20} t^8 (1-t)^{12}$

b) $\mathrm{P}(\text{head} > 8 \ \cap \ \text{tail} > 12)$

$= \mathrm{P}(12 > \text{head} > 8)$

$= C_9^{20} t^9 (1-t)^{11} + C_{10}^{20} t^{10} (1-t)^{10} + C_{11}^{20} t^{11} (1-t)^9$

2. (5%) A family has n children, $n \geq 2$. What is the probability that all children are girls, given that at least one of them is a girl?

$$\frac{P(all\ children\ are\ girls \cap at\ least\ one\ girl)}{P(at\ least\ one\ \ girl)}$$

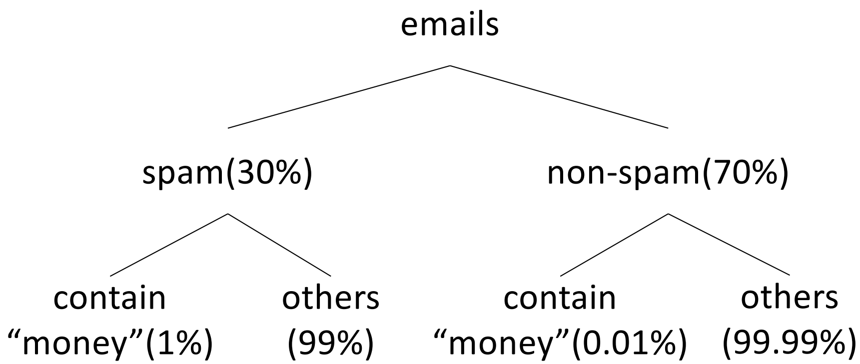
$$= \frac{P(all\ children\ are\ girls)}{1 - P(no\ girl)}$$

$$= \frac{(\frac{1}{2})^n}{1 - (\frac{1}{2})^n} = \frac{1}{2^n - 1}$$

3. (5%) Suppose we observe some words are more frequent in spam emails, and the information can be summarized as:

- 30% of emails are spam
- 1% of spam emails contain the term “money”
- 0.01% of non-spam emails have the word “money”

For a given email, which is checked and found to include the word “money,” what is the probability that the email is spam?



$$\frac{P(spam \cap \text{contain “money”})}{P(\text{contain “money”})} = \frac{0.3 \times 0.01}{0.3 \times 0.01 + 0.7 \times 0.0001}$$

4. (14%) X and Y are continuous random variables, and $Y = 2X$. Given the PDF of X:

$$f_X(x) = \begin{cases} \frac{1}{3}, 1 \leq x < 2 \\ ax^2, 0 \leq x < 1 \\ 0, otherwise \end{cases}$$

a) Please find the constant a

b) Please calculate $P(1 \leq Y \leq 3) =?$

a) $\int_{-\infty}^{\infty} f_X(x)dx = 1$

$$\Rightarrow \int_0^1 ax^2 dx + \int_1^2 \frac{1}{3} dx$$

$$= \frac{1}{3} ax^3 \Big|_0^1 + \frac{1}{3} x \Big|_1^2$$

$$= \frac{1}{3} a + \frac{1}{3} = 1$$

$$\therefore a = 2$$

b) $P(1 \leq Y \leq 3) = P(\frac{1}{2} \leq X \leq \frac{3}{2})$

$$= \int_{\frac{1}{2}}^{1} 2x^2 dx + \int_1^{\frac{3}{2}} \frac{1}{3} dx$$

$$= \frac{2}{3} x^3 \Big|_{\frac{1}{2}}^{1} + \frac{1}{3} x \Big|_1^{\frac{3}{2}} = \frac{3}{4}$$

5. (14%) X is a gaussian random variable whose expectation is 3 and variance is 9. Besides, we define a new random variable $Y = 5 - X$.

a) Please find $P(X > 1.5) =?$

b) Please find $P(-1 < Y < 5)$

Hint: $f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$

	.00	.01	.02	.03	.04	.05
0.0	.5000	.5040	.5080	.5120	.5160	.5199
0.1	.5398	.5438	.5478	.5517	.5557	.5596
0.2	.5793	.5832	.5871	.5910	.5948	.5987
0.3	.6179	.6217	.6255	.6293	.6331	.6368
0.4	.6554	.6591	.6628	.6664	.6700	.6736
0.5	.6915	.6950	.6985	.7019	.7054	.7088
0.6	.7257	.7291	.7324	.7357	.7389	.7422
0.7	.7580	.7611	.7642	.7673	.7704	.7734
0.8	.7881	.7910	.7939	.7967	.7995	.8023
0.9	.8159	.8186	.8212	.8238	.8264	.8289
1.0	.8413	.8438	.8461	.8485	.8508	.8531
1.1	.8643	.8665	.8686	.8708	.8729	.8749
1.2	.8849	.8869	.8888	.8907	.8925	.8944
1.3	.9032	.9049	.9066	.9082	.9099	.9115
1.4	.9192	.9207	.9222	.9236	.9251	.9265

a) $Z = \frac{X-3}{3}$ is a standard normal random variable

$P(X > 1.5) = P(Z > -0.5) = \Phi(0.5) = 0.6915$

b) $P(-1 < Y < 5) = P(0 < X < 6) = P(-1 < Z < 1)$

$= 2\times(\Phi(1) - \Phi(0)) = 2\times 0.3413 = 0.6826$

6. (21%) X and Y are two continuous random variables, and their joint PDF is:

$$f_{X,Y}(x,y) = \begin{cases} 3x+1, if\ 0 \le x, 0 \le y, x+y<1 \\ 0, \quad otherwise \end{cases}$$

a) Find marginal PDF $f_X(x)$

b) Are X and Y independent? Please explain your answer in detail.

c) Find $P(Y \ge \frac{1}{2})$

a) $f_X(x) = \int_0^{1-x} 3x + 1 dy$

$= (3x+1)|_0^{1-x}$

$= 3x(1-x) + 1 - x$

$= -3x^2 + 2x + 1$

$f_X(x)$

$= \begin{cases} -3x^2 + 2x + 1, if\ 0 \le x < 1 \\ 0, otherwise \end{cases}$

b) $f_Y(y) = \int_0^{1-y} 3x + 1 dx$

$= (\frac{3x^2}{2} + x)|_0^{1-y}$

$= \frac{3}{2}(1-y)^2 + 1 - y$

$= \frac{3}{2}y^2 - 4y + \frac{5}{2}$

$\because f_{X,Y}(x,y) = 3x + 1 \neq f_X(x) f_Y(y)$

$\therefore$ *They are not independent*

c) $P(Y \ge \frac{1}{2})$

$= \int_{\frac{1}{2}}^{1} \int_0^{1-y} f_{X,Y}(x,y) dx dy$

$= \int_{\frac{1}{2}}^{1} f_Y(y)\, dy$

$= \int_{\frac{1}{2}}^{1} \frac{3}{2}y^2 - 4y + \frac{5}{2} dy$

$= \frac{3}{16}$

7. (35%) X and Y are two discrete random variables, and their joint PMF is given. We define a random variable Z as $Z = E[X|Y]$.

a) Please find the PMF of X

b) Are X and Y independent? Please explain your answer in detail.

c) Please find the PMF of X given $Y = 0$ (i.e., $P_{X|Y}(x|0)$)

d) Please find the PMF of Z

e) Please calculate the expectation of ZZ (i.e., $E(Z)$)

	$Y = 0$	$Y = 1$	$Y = 2$
$X = 0$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{8}$
$X = 1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{4}$

a) $$p_X(x) = \begin{cases} \frac{11}{24}, if\ x = 0 \\ \frac{13}{24}, if\ x = 1 \\ 0, otherwise \end{cases}$$

b) $$p_{X,Y}(0,0) = \frac{1}{6} \neq p_X(0)p_Y(0) = \frac{11}{24} \times \frac{7}{24}$$

$$\therefore not\ independent$$

c) $$p_{X|Y}(x|0) = \frac{p_{X,Y}(x,0)}{p_Y(0)}$$

$$= \begin{cases} \frac{p_{X,Y}(0,0)}{p_Y(0)} = \frac{\frac{1}{6}}{\frac{1}{6} + \frac{1}{8}} = \frac{4}{7}, if\ x = 0 \\ \frac{p_{X,Y}(1,0)}{p_Y(0)} = \frac{\frac{1}{8}}{\frac{1}{6} + \frac{1}{8}} = \frac{3}{7}, if\ x = 1 \\ 0, otherwise \end{cases}$$

7. (35%) X and Y are two discrete random variables, and their joint PMF is given. We define a random variable Z as $Z = E[X|Y]$.

d) Please find the PMF of Z

e) Please calculate the expectation of Z (i.e., $E(Z)$)

$p_{X,Y}(x,y)$:

	$Y=0$	$Y=1$	$Y=2$
$X=0$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{8}$
$X=1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{4}$

d) $Z = E[X|Y] = \sum_x x p_{X|Y}(x|y)$

$$= \begin{cases} E[X|0] = 0 \times p_{X|Y}(0|0) + 1 \times p_{X|Y}(1|0) = \frac{3}{7} \\ E[X|1] = 0 \times p_{X|Y}(0|1) + 1 \times p_{X|Y}(1|1) = \frac{1}{2} \\ E[X|2] = 0 \times p_{X|Y}(0|2) + 1 \times p_{X|Y}(1|2) = \frac{2}{3} \end{cases}$$

$$p_Z(z) = \begin{cases} \frac{7}{24}, if\ z = \frac{3}{7} \\ \frac{1}{3}, if\ z = \frac{1}{2} \\ \frac{3}{8}, if\ z = \frac{2}{3} \\ 0, otherwise \end{cases}$$

e) $E(Z) = \sum_z z\, p_Z(z)$

$$= \frac{7}{24} \times \frac{3}{7} + \frac{1}{3} \times \frac{1}{2} + \frac{3}{8} \times \frac{2}{3} = \frac{13}{24}$$

8. (6%) X is a discrete random variable with the CDF:

$$F_X(x)\begin{cases} 0, if\ x < 0 \\ \frac{1}{6}, if\ 0 \le x < 1 \\ \frac{1}{2}, if\ 1 \le x < 2 \\ \frac{3}{4}, if\ 2 \le x < 3 \\ 1, if\ 3 \le x \end{cases}$$

Please find the PMF of X.

$$p_X(x) = \begin{cases} \frac{1}{6}, if\ x = 0 \\ \frac{1}{3}, if\ x = 1 \\ \frac{1}{4}, if\ x = 2 \\ \frac{1}{4}, if\ x = 3 \\ 0, otherwise \end{cases}$$