Linear Algebra

Final Exam (June 13, 2022)

1. (5 %) Let A be a nonsingular matrix. Show that

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

2. Let A be a 4×4 matrix. If

$$adj\ A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 4 & 3 & 2 \\ 0 & -2 & -1 & 2 \end{bmatrix}$$

   a. (6 %) Calculate the value of det(adj A). What should the value of det(A) be? [*Hint*: $\det(\text{adj } A) = (\det(A))^{n-1}$ for $n \times n$ matrix with $n > 1$]
   b. (4 %) Find A.

3. (5 %) Let A be an $n \times n$ matrix. Show that if $B = S^{-1}AS$ for some nonsingular matrix S, then $\det(B) = \det(A)$.

4. (5 %) Determine the null space of matrix

$$A = \begin{bmatrix} 2 & 3 & 6 \\ 1 & 2 & -1 \\ -2 & -3 & -6 \end{bmatrix}$$

5. Determine whether the following are spanning sets for $\mathbb{R}^2$:

   a. (5 %) $\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \end{bmatrix}\right\}$

   b. (5 %) $\left\{\begin{bmatrix} 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$

6. Determine whether the following vectors are linearly independent in $\mathbb{R}^{2\times2}$:

   a. (5 %) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

   b. (5 %) $\begin{bmatrix} 4 & 2 \\ 6 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}$

7. (5 %) Find the transition matrix representing the change of coordinates on P_3 from the ordered basis $[1, x, x^2]$ to the ordered basis $[1 + 2x + x^2, 1 + 2x, 1]$.

8. Given a matrix

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 4 \\ 4 & 7 & 8 \end{bmatrix}$$

a. (5 %) Find a basis for the row space.

b. (5 %) Find a basis for the column space.

c. (5 %) Find a basis for the null space.

9. (5 %) Let $L: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear operator. If

$$L\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \quad \text{and} \quad L\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

Find the value of $L\left(\begin{bmatrix} 8 \\ 7 \end{bmatrix}\right)$.

10. (10 %) Let

$$\boldsymbol{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{b}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{b}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

and let L be the linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^3$ defined by

$$L(\boldsymbol{x}) = x_1\boldsymbol{b}_1 + x_2\boldsymbol{b}_2 + (x_1 + x_2)\boldsymbol{b}_3$$

Find the matrix A representing L with respect to the ordered bases $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ and $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3\}$.

11. (10 %) Prove that if A is similar to B and B is similar to C, then A is similar to C.

12. Let

$$\boldsymbol{u}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \boldsymbol{u}_2 = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

and

$$\boldsymbol{v}_1 = \begin{bmatrix} 7 \\ 5 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

and let L be a linear operator on $\mathbb{R}^2$ whose matrix representation with respect to the ordered basis $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ is

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$$

a. (5 %) Determine the transition matrix from the basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ to the basis $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$.

b. (5 %) Find the matrix representation of L with respect to $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$.