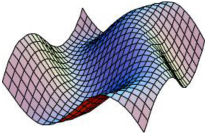
THE UNIVERSITY OF MELBOURNE

SCHOOL OF MATHEMATICS AND STATISTICS

MAST10006 Calculus 2

Lecture Notes



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usually written in FORMULAS, not in written text

Standard Abbreviations

1. such that or given that: $|$ usually in sets: $\{x \mid x^2 = 1\}$
2. for all: $\forall$ "for all x larger than zero" $\forall x > 0$
3. there exists: $\exists$ "there exists an x larger than zero" $\exists x > 0$
4. equivalent to: $\equiv$ $A \equiv B$
5. that is: *i.e* $x^2 = 1$ and $x > 0$; i.e. $x = 1$
6. approximate: $\approx$ $\sqrt{2} \approx 1.4$
7. much smaller than: $\ll$ $10^{-8} \ll 10^8$

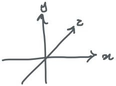
Standard Notation for Sets of Numbers

⚠ 0 not included

1. natural numbers: $\mathbb{N} = \{1, 2, 3, \ldots\}$ if you want 0 to be included: $\mathbb{N}_0 = \{0, 1, 2, 3 \ldots\}$
2. integers: $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$
3. rational numbers: $\mathbb{Q} = \{\frac{m}{n} \mid m, n \in \mathbb{Z}, n \neq 0\}$
4. real numbers: $\mathbb{R}$ (rational numbers plus irrational numbers)

   $\sqrt{2}, \pi, e$
5. complex numbers: $\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R},\ i^2 = -1\}$
6. $\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$ (xy plane)

7. $\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$ (3 dimensional space)

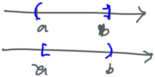


Standard Notation for Intervals

1. element of: $\in$
   so $a \in X$ means "a is an element of the set X"

2. open interval: (a, b)
   so $x \in (0, 1)$ means "$0 < x < 1$"
   

3. closed interval: $[a, b]$
   so $x \in [0, 1]$ means "$0 \leq x \leq 1$"
   

4. partial open and closed interval: $(a, b]$ or $[a, b)$
   so $x \in [0, 1)$ means "$0 \leq x < 1$"
   

⚠ WILL APPEAR REGULARLY

5. not including: $\setminus$
   so $x \in \mathbb{R} \setminus \{0\}$ means "x is any real number excluding 0".
   Alternatively, we could write $(-\infty, 0) \cup (0, \infty)$ where $\cup$ means the "union of the two intervals".

   $\mathbb{R} \setminus [a, b] = (-\infty, a) \cup (b, \infty)$

More Standard Notation

1. natural logarithm: $\log x$
   base 10 logarithm: $\log_{10} x$, $ld\ x$
   base 2 logarithm: $\log_2 x$, $lb\ x$
   Alternative notations for natural logarithms used in textbooks: $\log_e x, \ln x$

2. inverse trigonometric functions: $\arcsin x$, $\arctan x$ etc

   Alternative notations used in textbooks: $\sin^{-1} x$, $\tan^{-1} x$ etc

3. implies: $\Rightarrow$
   so $p \Rightarrow q$ means "p implies q"

4. if and only if (iff): $\Leftrightarrow$ (means both $\Leftarrow$ and $\Rightarrow$)
   so $p \Leftrightarrow q$ means "p implies q" AND "q implies p"

5. approaches: $\rightarrow$
   so $f(x) \rightarrow 1$ as $x \rightarrow 0$ means "$f(x)$ approaches 1 as x approaches 0"

Greek Alphabet

α	alpha
β	beta
γ	gamma
δ	delta
ϵ or ε	epsilon
ζ	zeta
η	eta
θ	theta ϑ
ι	iota
κ	kappa
λ	lambda
μ	mu
ν	nu
ξ	xi
o	omicron
π	pi
ρ	rho
σ	sigma
τ	tau
υ	upsilon
ϕ	phi φ
χ	chi
ψ	psi
ω	omega

EXAMPLE 1: Consider $f_a(x) = x - a$

WRITTEN TEXT: "For any real number a, there is a real number x_0 so that $f_a(x_0) = 0$"

FORMULA: $\forall a \in \mathbb{R} \ \exists x_0 \in \mathbb{R}$ so that $f_a(x_0) = 0$

⚠ do not mix text & formula

EXAMPLE 2:

WRITTEN TEXT: "The expression A is equivalent to B iff and only if B is equivallent to A"

FORMULA: $A \equiv B \iff B \equiv A$

FORMULA is usually shorter

write FORMULAS in exams & worksheets

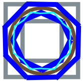
Section 1: Limits, Continuity, Sequences, Series

Why do we study limits and continuity in calculus?

- SOLAR SYSTEM
  - meta-stable
- FINANCE
  - stock graphs are continuous & non-differential

- AREA UNDER CURVE

- APPROXIMATING CIRCUMFERENCE OF CIRCLE



Definition of limit of a function

Let f be a real-valued function.

The limit of $f(x)$ as x approaches a is L, written

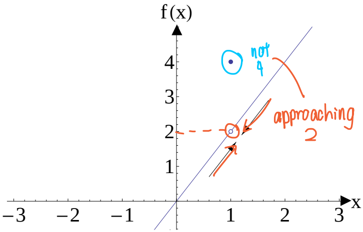
$$\lim_{x \to a} f(x) = L$$

if $f(x)$ gets arbitrarily close to L whenever x is close enough to a but $\boxed{x \neq a}$.

Note:

If it exists, the limit L must be a unique finite real number.

Example 1.1: If $f(x) = \begin{cases} 2x & x \neq 1 \\ 4 & x = 1 \end{cases}$, evaluate $\lim_{x \to 1} f(x)$.



Solution:

f(x) gets arbitrarily close to 2 whenever x is close enough to 1 but $x \neq 1$

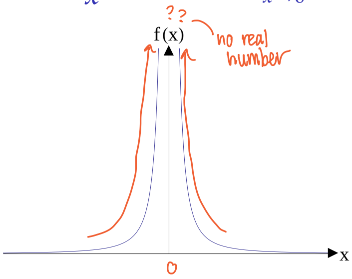
Therefore,

$$\lim_{x \to 1} f(x) = 2$$

Note:

The limit of f as x approaches a does not depend on $f(a)$. The limit can exist even if f is undefined at $x = a$.

Example 1.2: If $f(x) = \dfrac{1}{x^2}$, evaluate $\lim_{x\to 0} f(x)$.



Solution:

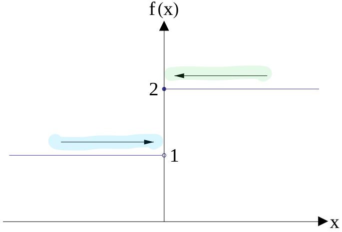
$f(x)$ is unbounded as x approaches 0

Hence, $f(x)$ cannot be arbitrarily close to any real number as x approaches 0

$\Rightarrow$ This shows that $\lim_{x\to 0} f(x)$ does not exist

⚠ VERY IMPORTANT

Example 1.3: If $f(x) = \begin{cases} 1 & x < 0 \\ 2 & x \geq 0 \end{cases}$, evaluate $\lim_{x \to 0} f(x)$.



Solution:

- $f(x)$ approaches 1 when x approaches 0 from the left
- $f(x)$ approaches 2 when x approaches 0 from the right

Hence, $\lim_{x \to 0} f(x)$ DOES NOT EXIST

We can describe this behaviour in terms of one-sided limits.
We write

$$\lim_{x \to 0^-} f(x) = 1 \qquad \text{(left hand limit)}$$

$$\lim_{x \to 0^+} f(x) = 1 \qquad \text{(right hand limit)}$$

Theorem:

$\lim\limits_{x\to a} f(x) = L$ $\boxed{\text{if and only if}}$ $\underline{\lim\limits_{x\to a^-} f(x) = L}$ and $\underline{\lim\limits_{x\to a^+} f(x) = L}$.

Thus the limit exists if and only if the left and right hand limits exist and are equal.

Limit Laws ⚠ LEARN BY ♡

Let f and g be real-valued functions and let $c \in \mathbb{R}$ be a constant. If $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ exist, then

1. $\lim\limits_{x\to a} [f(x) + g(x)] = \lim\limits_{x\to a} f(x) + \lim\limits_{x\to a} g(x).$

2. $\lim\limits_{x\to a} [cf(x)] = c \lim\limits_{x\to a} f(x).$

3. $\lim\limits_{x\to a} [f(x)g(x)] = \lim\limits_{x\to a} f(x) \cdot \lim\limits_{x\to a} g(x).$

4. $\lim\limits_{x\to a} \left[\dfrac{f(x)}{g(x)}\right] = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$ provided $\lim\limits_{x\to a} g(x) \neq 0.$

5. $\lim\limits_{x\to a} c = c.$

6. $\lim\limits_{x\to a} x = a.$

⚠ WRITE DOWN WHICH LIMIT LAWS USED

Example 1.4: Use the limit laws to evaluate $\lim_{x\to 2} \dfrac{x^3+2x^2-1}{5-3x}$.

Solution: $\lim_{x\to 2} \dfrac{x^3+2x^2-1}{5-3x} = \dfrac{\lim_{x\to 2}(x^3+2x^2-1)}{\lim_{x\to 2}(5-3x)}$ (Limit Law 4)

$$= \frac{\lim_{x\to 2} x^3 + 2\lim_{x\to 2} x^2 + \lim_{x\to 2}(-1)}{\lim_{x\to 2} 5 - 3\lim_{x\to 2} x} \quad \text{(Limit laws 1,2)}$$

$$= \frac{\left(\lim_{x\to 2} x\right)^3 + 2\left(\lim_{x\to 2} x\right)^2 + \lim_{x\to 2}(-1)}{\lim_{x\to 2} 5 - 3\lim_{x\to 2} x} \quad \text{(Limit Law 3)}$$

$$= \frac{2^3+2\cdot 2^2-1}{5-3\cdot 2} \quad \text{(Limit Laws 5,6)}$$

$$= \frac{8+8-1}{-1}$$

$$= -15$$

EXAMPLE: Consider: $\lim_{x\to 1} \frac{x^2-1}{x^2+1}$

$$\lim_{x\to 1} \frac{x^2-1}{x^2+1} = \frac{\lim_{x\to 1}(x^2-1)}{\lim_{x\to 1}(x^2+1)} \quad \text{(Limit Law 4)}$$

$$= \frac{\lim_{x\to 1} x^2 + \lim_{x\to 1}(-1)}{\lim_{x\to 1} x^2 + \lim_{x\to 1} 1} \quad \text{(Limit Law 1)}$$

$$= \frac{(\lim_{x\to 1} x)^2 + \lim_{x\to 1}(-1)}{(\lim_{x\to 1} x)^2 + \lim_{x\to 1} 1} \quad \text{(Limit Law 3)}$$

$$= \frac{1^2-1}{1^2+1} \quad \text{(Limit Laws 5, 6)}$$

$$= 0$$

inverse of

What do you think will be the limit $\lim_{x\to 1} \frac{x^2+1}{x^2-1}$? Does the limit exist?

$$\lim_{x\to 1} \frac{x^2+1}{x^2-1} = \frac{1^2+1}{1^2-1}$$

$$= \frac{2}{0}$$ <does not exist>

Limits as x Approaches Infinity

The **limit of $f(x)$ as x approaches positive infinity is L**,

$$\lim_{x\to\infty} f(x) = L$$

if $f(x)$ gets arbitrarily close to L whenever x is sufficiently large and positive.

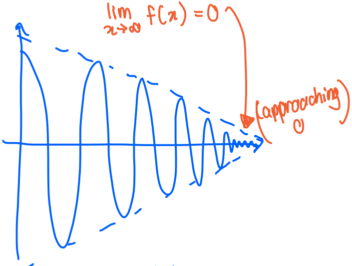
The **limit of $f(x)$ as x approaches negative infinity is M**,

$$\lim_{x\to-\infty} f(x) = M$$

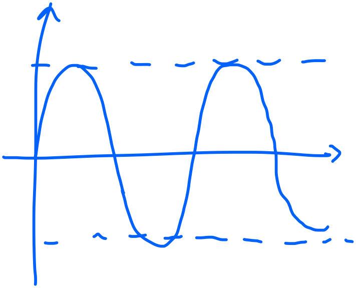
if $f(x)$ gets arbitrarily close to M whenever x is sufficiently large and negative.

Note:

1. L and M must be finite.
2. Limit laws (1)-(5) apply. ⚠ NOT (6)

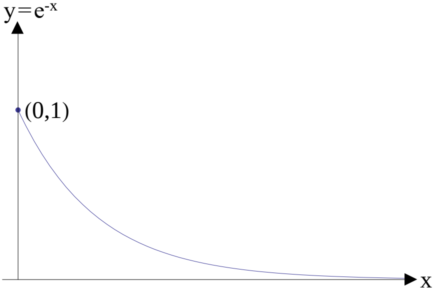


Damped oscillations



$\lim_{x \to \infty} f(x)$ does not exist

Example 1.5: Evaluate $\lim_{x\to\infty} e^{-x}$.



Solution:

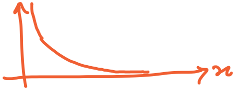
e^{-x} gets arbitrarily close to 0 as x gets larger.

Therefore $\lim_{x\to\infty} e^{-x} = 0$ //

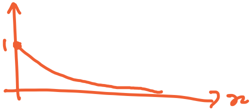
Some Standard Limits

We can use the following standard limits without further proof:

(1) $\lim_{x \to \infty} \frac{1}{x^p} = 0 \quad (p > 0)$



(2) $\lim_{x \to \infty} r^x = 0 \quad (0 \leq r < 1)$



We will see more standard limits later.

Terminology

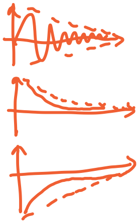
The following are all equivalent ways of expressing $\lim_{x\to a} f(x) = L$:

same

- The limit of $f(x)$ as x approaches a is L
- $f(x)$ converges to L as x approaches a
- $f(x) \to L$ as $x \to a$

These all mean the same thing.

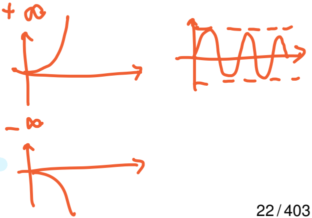
CONVERGE:



If the limit $\lim_{x\to a} f(x)$ exists, we say that
$f(x)$ converges as x approaches a.

DIVERGE:

goes to infinity: oscillates



If the limit $\lim_{x\to a} f(x)$ does not exist, we say that
$f(x)$ diverges as x approaches a.

Similarly for limits as $x \to \pm\infty$.

Note:

Diverges simply means does not converge.

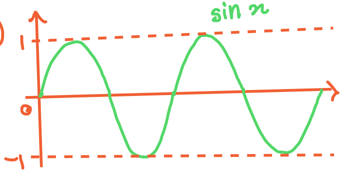
More about divergence

How do we show that a limit $\lim_{x\to a} f(x)$ diverges?

Example 1.6: Explain why the following limits diverge.

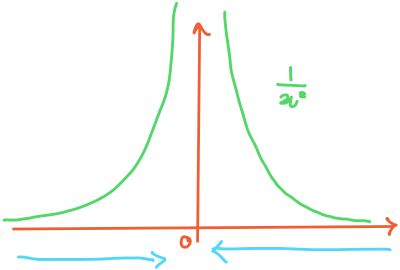
1. $\lim\limits_{x\to\infty} \sin x$
2. $\lim\limits_{x\to 0} \dfrac{1}{x^2}$

Solution:

1 

sin(x) oscillates between -1 and 1 so it does not approach a single real number as $x \to \infty$. Therefore, $\lim\limits_{x\to\infty} \sin(x)$ diverges/(does not exist).

②



$\frac{1}{x^2}$ is unbounded when x is close to 0. Therefore, $\lim_{x \to 0} \frac{1}{x^2}$ diverges/ (does not exist).

More about divergence

How do we show that a limit $\lim_{x \to a} f(x)$ diverges?

Example 1.7: Explain why the following limits diverge.

1. $\lim_{x \to \infty} \sin x$
2. $\lim_{x \to 0} \frac{1}{x^2}$

Solution:

?? repeated page?

Notation and ∞

∞ is not a number, and should not appear as a number in your writing.

In other textbooks, you will often see the notation $\lim_{x \to 0} \frac{1}{x^2} = \infty$ to mean that $\frac{1}{x^2}$ diverges to infinity.

You should not use this notation in Calculus 2 and will lose notation marks for this! $\ominus$marks!

It is notation that is often misunderstood.

We say a function $\dfrac{f(x)}{g(x)}$ has indeterminate form $\dfrac{\infty}{\infty}$ as $x \to a$ if $f(x) \to \infty$ and $g(x) \to \infty$.

Example 1.8: Evaluate $\displaystyle\lim_{x\to\infty} \frac{3x^2 - 2x + 3}{x^2 + 4x + 4}$.

Solution: (As $x \to \infty$ it has the form $\frac{\infty}{\infty}$)

$$\lim_{x\to\infty} \frac{3x^2-2x+3}{x^2+4x+4} = \lim_{x\to\infty} \frac{x^2\left(3-\frac{2}{x}+\frac{3}{x^2}\right)}{x^2\left(1+\frac{4}{x}+\frac{4}{x^2}\right)}$$

$$= \lim_{x\to\infty} \frac{3-\frac{2}{x}+\frac{3}{x^2}}{1+\frac{4}{x}+\frac{4}{x^2}}$$

$$= \frac{\lim_{x\to\infty}\left(3-\frac{2}{x}+\frac{3}{x^2}\right)}{\lim_{x\to\infty}\left(1+\frac{4}{x}+\frac{4}{x^2}\right)} \quad \text{(Law 4)}$$

$$= \frac{3}{1} \quad \text{since } \lim_{x\to\infty}\frac{1}{x} = 0;\ \lim_{x\to\infty}\frac{1}{x^2} = 0$$

$$= 3$$

NOTE: It is not always $\frac{\infty}{\infty}$ = finite number

$\therefore$ ∞ is not a number

We say a function $f(x) - g(x)$ has indeterminate form $\infty - \infty$ as $x \to a$ if $f(x) \to \infty$ and $g(x) \to \infty$.

if you change this, $\infty - \infty \neq$ finite number

Example 1.9: Evaluate $\lim_{x\to\infty} \left(\underbrace{\sqrt{x^2+1} - x}_{\text{unbounded}} \right)$.

Solution: (As $x \to \infty$ it has the form $\infty - \infty$)

$$\lim_{x\to\infty}\left(\sqrt{x^2+1} - x\right) = \lim_{x\to\infty}\left[\left(\sqrt{x^2-1} - x\right)\frac{\left(\sqrt{x-1} + x\right)}{\left(\sqrt{x-1} + x\right)}\right]$$

$$= \lim_{x\to\infty} \frac{x^2+1 + x\sqrt{x^2+1} - x\sqrt{x^2+1} - x^2}{\sqrt{x^2+1} + x}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{x^2+1} + x}$$

sandwich theorem

$$= 0$$

NOTE: It is not always $\infty - \infty$ = finite number

$\therefore$ ∞ is not a number

We will finish this example later. <Page 32>

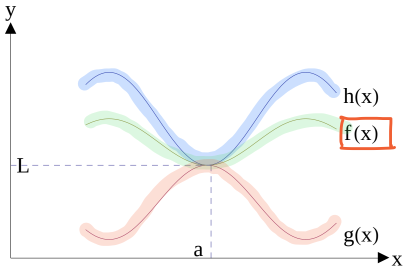
Sandwich Theorem:

If $g(x) \leq f(x) \leq h(x)$ when x is near a but $x \neq a$, and

$$\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$$

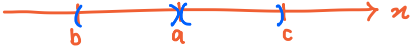
then

$$\lim_{x \to a} f(x) = L.$$



∴ sandwiched in between h(x) & g(x)

Note:



1. "x is near a but $x \neq a$" means that x lies in $(b, a) \cup (a, c)$ for some $b < a < c$.

2. The validity of Sandwich Theorem is based on the fact that
$$g(x) \leq f(x) \leq h(x)$$
$$\Rightarrow |f(x) - L| \leq |g(x) - L| + |h(x) - L| \text{ for all } x.$$
Can you prove this inequality or even the stronger conclusion that $|f(x) - L| \leq \max\{|g(x) - L|, |h(x) - L|\}$?

3. Sandwich Theorem also works for limits as $x \to \infty$.
For example, if $g(x) \leq f(x) \leq h(x)$ when $x \in (c, \infty)$ for some real number c, and $\lim_{x\to\infty} g(x) = \lim_{x\to\infty} h(x) = L$, then
$$\lim_{x\to\infty} f(x) = L.$$
Similarly, the theorem holds for $x \to -\infty$. Can you write down the state for $x \to -\infty$?.

Example 1.10: Evaluate $\lim_{x\to 0}\left[x^2 \sin\left(\frac{1}{x}\right)\right]$.

Solution: Since $-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$, $x \neq 0$

Then $\underbrace{-x^2}_{g(x)} \leq \underbrace{x^2 \sin\left(\frac{1}{x}\right)}_{f(x)} \leq \underbrace{x^2}_{h(x)}$, $x \neq 0$

Now, $\lim_{x\to 0}(-x^2) = 0$ and $\lim_{x\to 0}(x^2) = 0$

AKA: $\lim_{x\to 0} g(x) = 0$ $\lim_{x\to 0} h(x) = 0$

SANDWICH THEOREM

if $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$

in this case: $\lim_{x\to 0} g(x) = \lim_{x\to 0} h(x) = 0$

then $\lim_{x\to a} f(x) = L$

in this case: $\lim_{x\to 0} f(x) = 0$

Therefore $\lim_{x\to 0}\left[x^2 \sin\left(\frac{1}{x}\right)\right] = 0$, by the Sandwich Theorem //

Example 1.11: Evaluate $\lim_{x\to 0}\left[x\sin\left(\frac{1}{x}\right)\right]$.

Solution: Since $-1 \le \sin\left(\frac{1}{x}\right) \le 1$, $x \ne 0$

Then $\underbrace{-|x|}_{g(x)} \le \underbrace{x\sin\left(\frac{1}{x}\right)}_{f(x)} \le \underbrace{|x|}_{h(x)}$, $x \ne 0$

Now $\lim_{x\to 0}(-|x|)=0$ and $\lim_{x\to 0}|x|=0$

Therefore, $\lim_{x\to 0}\left[x\sin\left(\frac{1}{x}\right)\right]=0$, by the Sandwich Theorem.

Example 1.9 (continued)

$$\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right) = \lim_{x\to\infty}\frac{1}{\sqrt{x^2+1}+x}$$

$$\underbrace{\sqrt{x^2+1}}_{\text{always } \geq 0} + x \underbrace{\geq}_{\text{because}} x \; ; \; x > 0$$

$$\underbrace{0}_{g(x)} \leq \underbrace{\frac{1}{\sqrt{x^2+1}+x}}_{f(x)} \leq \underbrace{\frac{1}{x}}_{h(x)}$$

Now, $\lim_{x\to\infty} 0 = 0$ and $\lim_{x\to\infty}\frac{1}{x} = 0$

So, $\lim_{x\to\infty}\left[\frac{1}{\sqrt{x^2+1}+x}\right] = 0$, by the Sandwich Theorem

Hence, $\lim_{x\to\infty}\frac{1}{\sqrt{x^2+1}+x} = 0$

Continuity

Definition of continuity

Let f be a real-valued function.

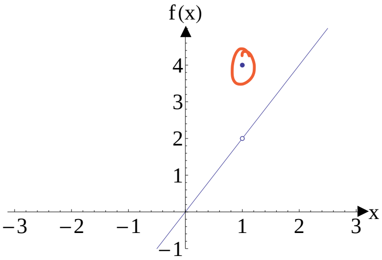
The function f is continuous at $x = a$ if

$$\lim_{x \to a} f(x) = f(a).$$

Example 1.12: Let

$$f(x) = \begin{cases} 2x & x \neq 1 \\ 4 & x = 1. \end{cases}$$

Is f continuous at $x = 1$?



Solution: since $\lim_{x \to 1} f(x) = 2 \neq 4 = f(1)$

Therefore, f is not continuous at $x = 1$

Example 1.13: Let $f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & x \neq 2 \\ 4 & x = 2. \end{cases}$

Is f continuous at $x = 2$?

Solution: Since $\displaystyle\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}}$

$$= \lim_{x\to 2} (x+2)$$

$$= 4 = f(2)$$

Therefore, f is continuous at $x=2$

Let f and g be real-valued functions and let $c \in \mathbb{R}$ be a constant.

Continuity Theorem 1:

VERY IMPORTANT

If the functions f and g are continuous at $x = a$, then the following functions are continuous at $x = a$:

1. $f + g$,
2. cf,
3. fg,
4. $\frac{f}{g}$ if $g(a) \neq 0$.

corresponds with Limit Laws 1–4

may not exist

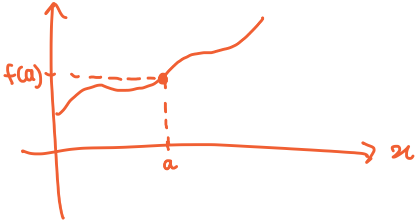
Note:

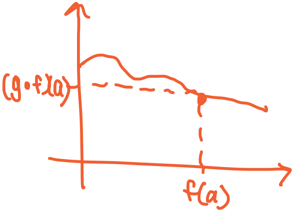
The theorem follows from limit laws.

Recall that $(g \circ f)(x) = g(f(x))$.

Continuity Theorem 2:

If f is continuous at $x = a$ and g is continuous at $x = f(a)$, then $g \circ f$ is continuous at $x = a$.





Continuity Theorem 3: !

The following function types are continuous at every point in their domains:

only when $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \dots$

- 1 polynomials
- 2 trigonometric functions: $\sin x$, $\cos x$, $\tan x$, $\sec x$, $\operatorname{cosec} x$, $\cot x$, $\arcsin x$, $\arccos x$, $\arctan x$

  $x \in [-1, 1]$

  only when $x \neq 0, \pm\pi, \pm 2\pi, \dots$
- 3 exponential functions: a^x for $a > 0$
- 4 logarithm functions: $\log_a x$ for $a > 0$, $x > 0$
- 5 nth root functions: $\sqrt[n]{x}$ for $n \in \{2, 3, 4, \dots\}$, $x \geq 0$
- 6 hyperbolic functions: $\sinh x$, $\cosh x$, $\tanh x$, $\operatorname{sech} x$, $\operatorname{cosech} x$, $\coth x$, $\operatorname{arcsinh} x$, $\operatorname{arccosh} x$, $\operatorname{arctanh} x$

  $x \neq 0$

  $x \geq 1$

Example 1.14: Let $f(x) = \dfrac{\sin\left(x^2+1\right)}{\log x}$.

For which values of x is f continuous?

Solution:

- x^2+1 is continuous for all $x \in \mathbb{R}$ as it is a polynomial 1
- $\sin(x)$ is continuous for all $x \in \mathbb{R}$ as it is a trigonometric function 2

$\Rightarrow$ Theorem 2: $\sin(x^2+1)$ is continuous for all $x \in \mathbb{R}$, as it is a composition of continuous functions

- $\log x$ is continuous for $x > 0$ as logarithmic functions are continuous

$\Rightarrow$ Theorem 1.4: $\dfrac{\sin(x^2+1)}{\log(x)}$ is continuous for $\underbrace{x > 0}_{\text{smallest domain}}$, as it is a quotient of continuous functions, except when $\log(x) = 0 \Rightarrow x = 1$

$$\boxed{\text{So } \frac{\sin(x^2+1)}{\log(x)} \text{ is continuous for } x \in (0, \infty) \setminus \{1\}}$$

Note:

This set can also be written as

- $x \in (0,1) \cup (1, \infty)$
- $x > 0$ with $x \neq 1$

Example 1.15: $f(x) = \begin{cases} x^3 - cx + 8, & x \le 1 \\ x^2 + 2cx + 2, & x > 1. \end{cases}$

For which values of c is f continuous? Justify your answer.

Solution:

- Since both branches of f are polynomials, the function f is continuous for $x<1$ and $x>1$ regardless of c.
- For continuity at $x=1$, we need

$$\lim_{x\to 1} f(x) = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$$

$$= 1 - c + 8$$

$$= 9 - c$$

- For left hand limit

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x^3 - cx + 8)$$

$$= 1^3 - c\cdot 1 + 8$$

$$= 9 - c$$

$$= f(1) \checkmark$$

- For right hand limit

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x^2 + 2cx + 2)$$

$$= 1^2 + 2c\cdot 1 + 2$$

$$= 3 + 2c$$

- $\lim_{x\to 1} f(x) = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$

$$9 - c = 3 + 2c$$

$$3c = 6$$

$$c = 2$$

When $c=2$, f is continuous at $x=1$ as $\lim_{x\to 1} f(x) = f(1)$

$\Rightarrow$ When $c=2$, f is continuous for all $x \in \mathbb{R}$ //

Theorem:

If f is continuous at b and $\lim_{x \to a} g(x) = b$ then

$$\lim_{x \to a} f[g(x)] = f\left[\lim_{x \to a} g(x)\right] = f(b).$$

Note:

This theorem also holds for limits as $x \to \infty$, as long as $b \in \mathbb{R}$ is finite.

Example 1.16: Evaluate $\lim_{x \to \infty} \sin(e^{-x})$.

Solution: $\lim_{x\to\infty} \sin(e^{-x}) = \sin\left[\lim_{x\to\infty} e^{-x}\right]$

sin(x) is continuous for all $x \in \mathbb{R}$

$\lim_{x\to\infty} e^{-x} = 0$

$= \sin(0)$

$= 0$

Differentiability

Definition of derivative

Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function. The derivative of f at $x = a$ is defined by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

The function f is differentiable at $x = a$ if this limit exists.

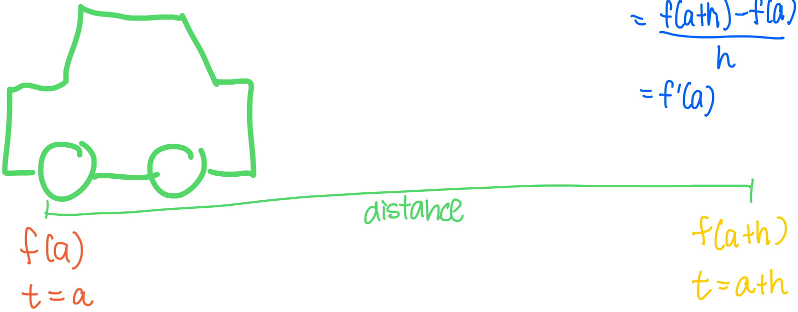
Geometrically, f is differentiable at $x = a$ if the graph $y = f(x)$ has a *tangent line* at $x = a$ given by

$$\boxed{y - f(a) = f'(a)(x - a)}$$

which gives a good approximation to the graph near $x = a$.

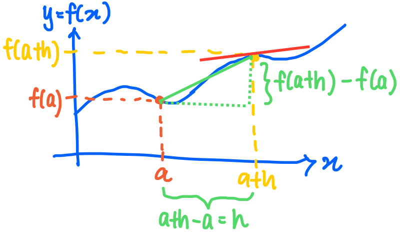
Instantaneous speed $= \lim_{h \to a} f'(a)$

$$\text{speed} = \frac{\text{distance}}{\text{time}} = \frac{f(a+h) - f(a)}{a+h - a} = \frac{f(a+h)-f(a)}{h} = f'(a)$$



Where does this definition come from?

Before seeing what is filled in from the lectures, try to come up with a picture that explains where the definition comes from.



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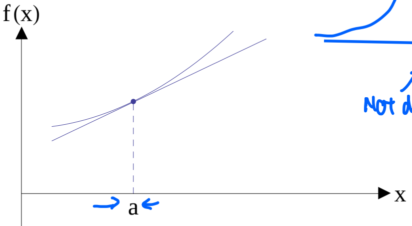
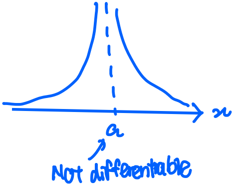
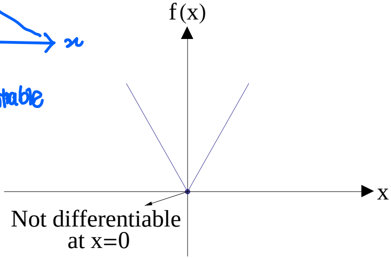


Differentiable at x=a





Not differentiable at x=0

If f is differentiable at $x = a$, the linear approximation of f near $x = a$ is given by

$$f(x) \approx f(a) + f'(a)(x - a)$$

Theorem:

If f is differentiable at $x = a$, then f is continuous at $x = a$.

L'Hôpital's Rule

Let f and g be differentiable functions near $x = a$, and $g'(x) \neq 0$ at all points x near a with $x \neq a$. If

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

has the indeterminate form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$ then

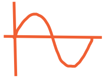
$$\boxed{\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}} \quad g'(x) \neq 0$$

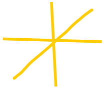
if the limit involving the derivatives exists.

Note:

L'Hôpital's Rule also holds when x approaches infinity.

Example 1.17: Evaluate $\lim_{x \to 0} \frac{\sin x}{x}$.





Solution:

$$\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{(\sin(x))'}{(x)'}$$ (L'Hôpital's Rule $(\frac{0}{0})$)

$$= \lim_{x \to 0} \frac{\cos(x)}{1}$$

$$= \lim_{x \to 0} \cos(x)$$

$\cos(x)$ is continuous for all x

$$= \cos\left(\lim_{x \to 0} x\right)$$

$$= \cos(0)$$

$$= 1$$

Example 1.18: Evaluate $\lim_{x\to\infty} \frac{3x^2 - 2x + 3}{x^2 + 4x + 4}$. $= \lim_{x\to\infty} \frac{\cancel{x^2}(3 - \frac{2}{x} + \frac{3}{x^2})}{\cancel{x^2}(1 + \frac{4}{x} + \frac{4}{x^2})} = 3$

ANOTHER WAY: divide by highest denominator

Solution:

$$\lim_{x\to\infty} \frac{3x^2-2x+3}{x^2+4x+4}$$

L'Hôpital's Rule $(\frac{\infty}{\infty})$ ①

$$= \lim_{x\to\infty} \frac{(3x^2-2x+3)'}{(x^2+4x+4)'}$$

$$= \lim_{x\to\infty} \frac{6x-2}{2x+4}$$

L'Hôpital's Rule $(\frac{\infty}{\infty})$ ②

$$= \lim_{x\to\infty} \frac{(6x-2)'}{(2x+4)'}$$

$$= \lim_{x\to\infty} \frac{6}{2}$$

Do L'Hôpital's until limit can be found

$$= \lim_{x\to\infty} 3$$

$$= 3$$

Example 1.19: Evaluate $\lim_{x\to\infty}\left(x^{-\frac{1}{3}}\log x\right)$. $(0\cdot\infty)$

Solution:

$$\lim_{x\to\infty}\left(x^{-\frac{1}{3}}\log x\right) = \lim_{x\to\infty}\frac{\log x}{x^{\frac{1}{3}}}$$

$\downarrow$ L'Hôpital's Rule $\left(\frac{\infty}{\infty}\right)$ $\downarrow$

$$= \lim_{x\to\infty}\frac{(\log x)'}{(x^{\frac{1}{3}})'}$$

$$= \lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{3}x^{-2/3}}$$

$$= \lim_{x\to\infty}\frac{1}{x}\times\frac{3}{x^{-2/3}}$$

$$= \lim_{x\to\infty}\frac{3}{x^{1/3}}$$

$$= 0$$

$$\lim_{x\to\infty}\frac{1}{x^r} = 0$$

Aside - What is a limit really?*

Recall our definition of limit:

$\lim_{x \to a} f(x) = L$ if $f(x)$ gets arbitrarily close to L whenever x is close enough to a but $\boxed{x \neq a}$.

What do '*arbitrarily close*' and '*close enough*' mean?

More formally,

for any arbitrary positive real number ε, there is a positive real number δ such that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad \boxed{0 < |x - a| < \delta.}$$

This formal definition of limit is covered in MAST20026 Real Analysis, but as a taster we do two examples.

** Slides 53-55 are not examinable in MAST10006.*

Aside - What is a limit really?*

Example 1.20: Using the definition, prove that

$$\lim_{x\to 1} 2x = 2$$

Solution:

For an arbitrary positive real number ε,

$$|f(x) - 2| = 2|x - 1| < \varepsilon \text{ if and only if } |x - 1| < \frac{1}{2}\varepsilon = \delta.$$

This shows that $|f(x) - 2|$ can be arbitrarily small whenever $|x - 1|$ is small enough but not equal to 0.

In other words, $f(x)$ can be arbitrarily close to 2 whenever x is close enough to 1 but $x \neq 1$.

Therefore,

$$\lim_{x\to 1} f(x) = 2.$$

Aside - What is a limit really?*

Example 1.21: Sketch a proof of the limit law

$$\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

Solution:

Suppose $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$.

For an arbitrary positive real number ε, to make

$$|f(x)+g(x)-(L+M)| < \varepsilon$$

$|f(x)-L+g(x)-M| \leq |f(x)-L| + |g(x)-M|$

we only need to make $|f(x)-L| < \frac{\varepsilon}{2}$ and $|g(x)-M| < \frac{\varepsilon}{2}$.

These will be satisfied whenever x is close enough to a but $x \neq a$ since $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$.

Hence $f(x)+g(x)$ can be arbitrarily close to $L+M$ whenever x is close enough to a but $x \neq a$, which means that

$$\lim_{x\to a}[f(x)+g(x)] = L+M = \lim_{x\to a} f(x) + \lim_{x\to a} g(x).$$

Sequences

Definition of sequence

A sequence is a function $f : \mathbb{N} \to \mathbb{R}$/$\mathbb{C}$/etc.
It can be thought of as an ordered list of real numbers

$$a_1, a_2, a_3, a_4, \ldots, a_n \ldots$$

if you change places, it is a different sequence - because different order

Thus, $f(n) = a_n$.
The sequence is denoted by $\{a_n\}$, where a_n is the n^{th} term.

Example

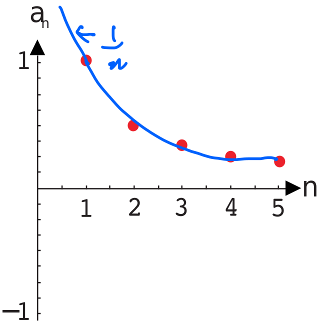
$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots \qquad a_n = \frac{1}{n} = f(n)$$

Example

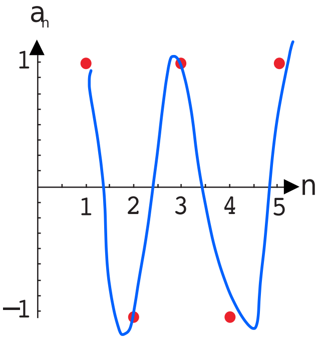
$$1, -1, 1, -1, 1, -1, \ldots \qquad a_n = (-1)^{n-1}$$

The graph of a sequence $\{a_n\}$ can be plotted on a set of axes with n on the x-axis and a_n on the y-axis.

Example: $a_n = \frac{1}{n}$



Example: $a_n = (-1)^{n-1}$



LIMIT DOES NOT EXIST

oscillating

Limits of Sequences

Definition of limit of sequence

A sequence $\{a_n\}$ has the limit L if a_n can be made arbitrarily close to L by making n sufficiently large.

We write

$$\boxed{\lim_{n\to\infty} a_n = L}$$

or $a_n \to L$ as $n \to \infty$.

If the limit exists we say that the sequence converges. Otherwise, we say that the sequence diverges.

Note:

If it exists, L must be a unique finite real number.

Example 1.22: Determine whether the following sequences converge or diverge: (a) $\left\{\frac{1}{n}\right\}$ (b) $\left\{(-1)^{n-1}\right\}$ (c) $\{n\}$

$\boxed{\text{3 CASES}}$

Solution:

(a) $\left\{\frac{1}{n}\right\}$ converges to 0

(b) $\left\{(-1)^{n-1}\right\}$ oscillates between +1 and -1, so it diverges, but it is bounded

(c) $\{n\}$ diverges to ∞ as it is unbounded

The only difference between $\lim_{n\to\infty} a_n = L$ and $\lim_{x\to\infty} f(x) = L$ is that n is a natural number whereas x is a real number.

Theorem:

Let $f : \mathbb{R} \to \mathbb{R}$ be a real function and $\{a_n\}$ be a sequence of real numbers such that $a_n = f(n)$.

$$\text{If} \quad \lim_{x\to\infty} f(x) = L \quad \text{then} \quad \lim_{n\to\infty} a_n = L.$$

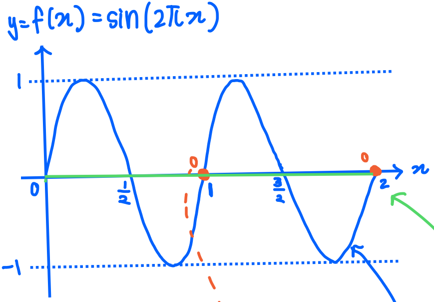
This means that we can use the techniques for evaluating limits of functions to evaluate limits of sequences.

Note:

$$\boxed{\lim_{n\to\infty} a_n = L \quad \not\Rightarrow \quad \lim_{x\to\infty} f(x) = L.}$$

eg. $a_n = \sin(2\pi n)$, $f(x) = \sin(2\pi x)$.

$= 0$ $\qquad$ = does not exist – oscillating



$a_n = \sin(2\pi n) = 0$

$a_n = f(n)$; $f(n) = \sin(2\pi n)$

$a_n = g(n)$; $g(x) = 0$

both fns satisfy the sequence (& many more fns)

$\therefore \lim_{n\to\infty} a_n = L \not\Rightarrow \lim_{x\to\infty} f(x) = L$

⚠ Theorem (Limit Laws):

Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers and $c \in \mathbb{R}$ a constant.

If $\lim\limits_{n\to\infty} a_n$ and $\lim\limits_{n\to\infty} b_n$ exist, then

1. $\lim\limits_{n\to\infty} [a_n + b_n] = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n.$

2. $\lim\limits_{n\to\infty} [ca_n] = c \lim\limits_{n\to\infty} a_n.$

3. $\lim\limits_{n\to\infty} [a_n b_n] = \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n.$

4. $\lim\limits_{n\to\infty} \left[\dfrac{a_n}{b_n}\right] = \dfrac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$ provided $\lim\limits_{n\to\infty} b_n \neq 0$; $b_n \neq 0$ for sufficiently large n

5. $\lim\limits_{n\to\infty} c = c.$

Sandwich Theorem:

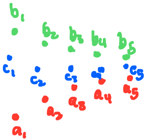
Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences of real numbers.

If $a_n \leq c_n \leq b_n$ for all $n > N$ for some N, and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$$

then

$$\lim_{n\to\infty} c_n = L.$$



The Factorial Function

The factorial function $n!$ $(n = 0, 1, 2, \ldots)$ is defined by

$$n! = n(n-1)!\ , \quad 0! = 1$$

or

$$n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$$

Therefore

$$\begin{gathered} 1! = 1 \\ 2! = 2 \times 1 = 2 \\ 3! = 3 \times 2 \times 1 = 6 \\ 4! = 4 \times 3 \times 2 \times 1 = 24 \end{gathered}$$

Binomial f^ns have factorials too

Example

$$(2n+2)! = (2n+2) \times (2n+1) \times (2n) \times (2n-1) \times \ldots \times 3 \times 2 \times 1$$

or

$$(2n+2)! = (2n+2) \times (2n+1) \times (2n)!$$

Standard Limits

conditions are important

$(0 \leq r < 1)$

$$(1)\ \lim_{n\to\infty} \frac{1}{n^p} = 0 \quad (p > 0)$$

$$(2)\ \lim_{n\to\infty} r^n = 0 \quad (|r| < 1)$$

$e^{\frac{1}{n}\log(n)}$

$$(3)\ \lim_{n\to\infty} a^{\frac{1}{n}} = 1 \quad (a > 0)$$

$$(4)\ \lim_{n\to\infty} n^{\frac{1}{n}} = 1$$

$$(5)\ \lim_{n\to\infty} \frac{a^n}{n!} = 0 \quad (a \in \mathbb{R})$$

$$(6)\ \lim_{n\to\infty} \frac{\log n}{n^p} = 0 \quad (p > 0)$$

$$(7)\ \lim_{n\to\infty} \left(1 + \frac{a}{n}\right)^n = \mathrm{e}^a \quad (a \in \mathbb{R})$$

$e^{x \cdot \log(1+\frac{a}{n})}$

$$(8)\ \lim_{n\to\infty} \frac{n^p}{a^n} = 0 \quad (p \in \mathbb{R}, a > 1)$$

Note:

Standard limits (1), (3), (4), (6), (7), (8) also hold for limits of real-valued functions as $x \to \infty$. Standard limit (2) also holds for $x \to \infty$ when $0 \leq r < 1$.

Example 1.23: Evaluate $\lim_{n\to\infty}\left[\left(\frac{n-2}{n}\right)^n + \frac{4n^2}{3^n}\right]$.

Solution: $\lim_{n\to\infty}\left[\left(\frac{n-2}{n}\right)^n + \frac{4n^2}{3n}\right] = \lim_{n\to\infty}\left(1-\frac{2}{n}\right)^n + \lim_{n\to\infty}\frac{4n^2}{3^n}$ (Limit Law 1)

$= \lim_{n\to\infty}\left(1-\frac{2}{n}\right)^n + 4\lim_{n\to\infty}\frac{n^2}{3^n}$ (Limit Law 2)

$= e^{-2} + 4(0)$

$= \frac{1}{e^2}$

(Standard Limit
7: $\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n = e^a \quad (a \in \mathbb{R})$
8: $\lim_{n\to\infty}\frac{n^p}{a^n} = 0 \quad (p \in \mathbb{R}, a > 1)$)

Example 1.24: Find the limit of the sequence

$$a_n = \frac{3^n + 2}{4^n + 2^n}, \quad n \geq 1.$$

Solution:

$$\lim_{n\to\infty} \frac{3^n+2}{4^n+2^n} = \lim_{n\to\infty} \frac{\frac{3^n}{4^n} + \frac{2}{4^n}}{1 + \frac{2^n}{4^n}}$$ (divide by 4^n)

$$= \frac{\lim_{n\to\infty} \frac{3^n}{4^n} + 2 \lim_{n\to\infty} \frac{1}{4^n}}{\lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{2^n}{4^n}}$$ (Limit Laws 1, 2, 4)

$$= \frac{\lim_{n\to\infty} \left(\frac{3}{4}\right)^n + 2 \lim_{n\to\infty} \left(\frac{1}{4}\right)^n}{1 + \lim_{n\to\infty} \left(\frac{1}{2}\right)^n}$$ (Limit Law 5)

$$= \frac{0 + 2(0)}{1 + (0)}$$ (Standard limit 2: $\lim_{n\to\infty} r^n = 0$, $|r| < 1$)

$$= 0$$ //

Note:

The order hierarchy can be used to help identify the largest term in an expression:

$$(\log n)^q \ll n^p \ll a^n \ll (n!)^q \ll e^{\ln^2}$$

logarithmic growth

algebraic growth

exponential growth

factorial growth

$; q, p > 0, a > 1$

Example 1.25: Prove Standard Limit 6:

$$\lim_{n\to\infty} \frac{\log n}{n^p} = 0 \quad (p > 0)$$

$x \to x \in \mathbb{R}_+$, $a_n = f(n)$; $f(x) = \frac{\log(x)}{x^p}$; $x > 0$

Solution:

$$\lim_{n\to\infty} \frac{\log(n)}{n^p} = \lim_{x\to\infty} \frac{\log(x)}{x^p}$$

L'Hôpital's Rule $\left(\frac{\infty}{\infty}\right)$ if $p > 0$

$$= \lim_{x\to\infty} \frac{(\log(x))'}{(x^p)'}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{p x^{p-1}}$$

Standard Limit 1: $\lim_{x\to\infty} \frac{1}{x^p} = 0$; $p > 0$

$$= \frac{0}{p(0)} = 0$$

Note:

We must change from a discrete variable n to a real variable x before applying L'Hôpital's rule.

Example 1.26: Evaluate $\lim_{n\to\infty}\left[\log(3n^2+2)-\log(n^2)\right]$.

Solution:

$$\lim_{n\to\infty}\left[\log(3n^2+2)-\log(n^2)\right] = \lim_{n\to\infty}\log\left(\frac{3n^2+2}{n^2}\right) \quad \text{(Log Law)}$$

$$= \lim_{n\to\infty}\log\left(3+\frac{2}{n^2}\right) \quad \left(n \to x \in \mathbb{R};\ a_n = f(x);\ f(x) = \log\left(3+\frac{2}{x^2}\right)\right)$$

$$= \lim_{x\to\infty}\log\left(3+\frac{2}{x^2}\right)$$

$$= \log\left(\lim_{x\to\infty}\left(3+\frac{2}{x^2}\right)\right) \quad \text{(Continuity Theorem 2)}$$

$$= \log\left(\lim_{x\to\infty}3 + 2\lim_{x\to\infty}\frac{1}{x^2}\right) \quad \text{(Limit Law 1,2)}$$

$$= \log\left(3+2(0)\right) \quad \left(\text{Limit Law 5, Standard Limit 1: } \lim_{x\to\infty}\frac{1}{x^p}=0,\ p>0\right)$$

$$= \log(3)$$

Note:

We must change from a discrete variable n to a real variable x before applying the continuity theorem.

Example 1.27: Evaluate $\lim\limits_{n\to\infty} \dfrac{1+\sin^2\left(\frac{n\pi}{3}\right)}{\sqrt{n}}$.

Solution:

Using Sandwich Theorem:

Since $0 \leq \sin^2\left(\frac{n\pi}{3}\right) \leq 1$ add 1

$1 \leq 1+\sin^2\left(\frac{n\pi}{3}\right) \leq 2$ divide by $\sqrt{n}$

$$\underbrace{\frac{1}{\sqrt{n}}}_{\{a_n\}} \leq \underbrace{\frac{1+\sin^2\left(\frac{n\pi}{3}\right)}{\sqrt{n}}}_{\{c_n\}} \leq \underbrace{\frac{2}{\sqrt{n}}}_{\{b_n\}}$$

$\lim\limits_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ (Standard Limit 1)

$\lim\limits_{n\to\infty} \frac{2}{\sqrt{n}} = 2\lim\limits_{n\to\infty} \frac{1}{\sqrt{n}}$ (Limit Law 2)

$= 2(0)$ (Standard Limit 1)

$= 0$

$$\Rightarrow \lim_{n\to\infty} \frac{1+\sin^2\left(\frac{n\pi}{3}\right)}{\sqrt{n}} = 0$$, by Sandwich theorem.

Adding Infinitely Many Numbers

Starting with any sequence $\{a_n\}$, adding the a_n's together in order gives a sequence $\{s_n\}$:

$$\begin{aligned} s_1 &= a_1, \\ s_2 &= a_1 + a_2, \\ s_3 &= a_1 + a_2 + a_3, \\ \vdots \quad & \quad \vdots \qquad \vdots \qquad \vdots \end{aligned}$$

The sequence of partial sums $\{s_n\}$ may or may not converge. If it does converge, we call

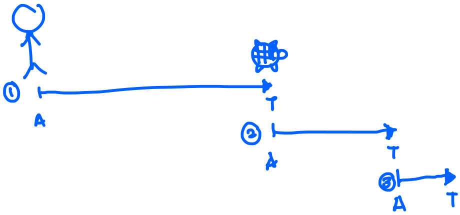
$$S = \lim_{n\to\infty} s_n = \lim_{n\to\infty} (a_1 + a_2 + \ldots + a_n)$$

the sum of the a_n's.

ZENO's PARADOX

Archilles and the tortoise

When Archilles (A) will reach the position where the tortoise (T) started, the tortoise will have moved further ahead

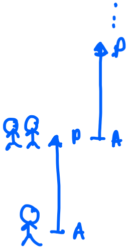


Can Archilles take over the tortoise?

⇒ Yes, because the sequence converges

AGE

When A reaches A's parents' age, parents have already gotten older



Can the age of A overtake A's parents'?

⇒ No, because the sequence diverges

Example 1.28: Find the sum S of $a_n = \left(\frac{1}{2}\right)^n, n \geq 1$.

Solution:

Since $a_1 = \frac{1}{2}$, $a_2 = \frac{1}{4}$, $a_3 = \frac{1}{8}$, ...

then

$$s_1 = a_1 = \frac{1}{2}$$

$$s_1 = a_1 + a_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$s_2 = a_1 + a_2 + a_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

$$\vdots$$

$$S_n = \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^n} = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n}$$

Standard Law (2) and Limit Laws (1) and (5)

$$\Rightarrow S = \lim_{n\to\infty} \sum_{j=1}^{n} a_j = 1 - \lim_{n\to\infty} \frac{1}{2^n} = 1$$

Series

A series with terms a_n is denoted by the sum

$$\sum_{n=1}^{\infty} a_n = \underbrace{\lim_{N\to\infty} \sum_{n=1}^{N} a_n}_{s_N}$$

L= finite&unique number

If $\lim_{n\to\infty} s_n$ exists, we say that the series converges. Otherwise we say that the series diverges.

unbounded

oscillates

Example

The sequence $\{n\} = 1, 2, 3, 4, \ldots$

The series $\sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + \ldots$

The sequence and series both diverge to infinity, so the series diverges.

Application: Decimals

The decimal representation of a number is actually a series.

Example

The sequence $\left\{\frac{1}{10^n}\right\} = 0.1, 0.01, 0.001, \ldots$.

The series $\sum_{n=1}^{\infty} \frac{1}{10^n} = 0.1 + 0.01 + 0.001 + \ldots = 0.11111111\ldots$

The sequence converges to 0 while the series converges to $\frac{1}{9}$.

In General

For a number $x \in (0, 1)$ with decimal digits $d_1, d_2, d_3, d_4, \ldots$

$$x = 0.d_1 d_2 d_3 d_4 \ldots = \sum_{n=1}^{\infty} \frac{d_n}{10^n}$$

also works for binary numbers

Show: $0.\overline{9} = 1$

$$0.\bar{9} = 0.9999\ldots = 9 \cdot 0.1 + 9 \cdot 0.01 + 9 \cdot 0.001 + \ldots$$

$$= 9 \cdot \sum_{n=1}^{\infty} (0.1)^n = 9 \cdot \sum_{n=1}^{\infty} \frac{1}{10^n} = \cancel{9} \cdot \frac{1}{\cancel{9}} = 1.$$

Properties of Series

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series, and $c \in \mathbb{R}\backslash\{0\}$ a constant.

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge then

1. $\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$ converges. (From Limit Law 1: $\lim_{N\to\infty}\left(\sum_{N=1}^{N} a_n + \sum_{N=1}^{N} b_n\right)$)

2. $\sum_{n=1}^{\infty}(ca_n) = c\sum_{n=1}^{\infty} a_n$ converges. (From Limit Law 2)

If $\sum_{n=1}^{\infty} a_n$ diverges then $\sum_{n=1}^{\infty}(ca_n)$ diverges.

Note:

These follow from the properties of the limits of sequences.

Geometric Series

$$\sum_{n=0}^{\infty} a(1)^n = a \sum_{n=0}^{\infty} 1 \text{ diverges}$$

$$\sum_{n=0}^{\infty} a(-1)^n = \lim_{N\to\infty} a \sum_{n=0}^{N} (-1)^n = \lim_{N\to\infty} \frac{a}{2}\left[1+(-1)^N\right]$$

diverges (oscillates btwn a & 1)

A geometric series has the form

$$\sum_{n=0}^{\infty} ar^n = \sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \ldots$$

where $a \in \mathbb{R}\backslash\{0\}$ and $r \in \mathbb{R}$.

The series converges if $|r| < 1$ and diverges if $|r| \geq 1$.

If $|r| < 1$, we have

$$\boxed{\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}.}$$

Note: (geometric sum)

This follows from the fact that $\sum_{k=0}^{n} ar^k = \frac{a(1-r^{n+1})}{1-r}$ for $r \neq 1$.

Example 1.29: What does the series

$$\sum_{n=0}^{\infty} 1 \cdot \left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \dots$$

converge to?

GEOMETRIC SERIES:

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

Solution:

Geometric Series with $a=1$ and $r=\frac{1}{2}$, so

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2$$

$\Rightarrow$ The series converges to 2 //

Harmonic p Series

A harmonic p series has the form

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^p}.}$$

The series converges if $p > 1$ and diverges if $p \leq 1$.

Example

$\boxed{\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges}}$ BUT $\boxed{\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges.}}$

Divergence Test

If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ diverges. (Necessary Condition)

Note:

If $\lim_{n\to\infty} a_n = 0$ then

but

1. $\sum_{n=1}^{\infty} a_n$ may converge or diverge.
2. The Divergence Test is not applicable, so we need to use another test to determine if $\sum_{n=1}^{\infty} a_n$ converges or diverges.

Example 1.30: Does the series $\sum_{n=1}^{\infty} \underbrace{\frac{n+1}{n}}_{a_n}$ converge?

Solution:

Since $\lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)$

$= \lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{1}{n}$ (Limit Law 1)

$= 1 + 0$ (Standard Limit 1 & Limit Law 5)

$= 1 \neq 0$

$\Rightarrow \sum_{n=1}^{\infty} \frac{n+1}{n}$ diverges by the Divergence Test

Comparison Test

(consequence of Sandwich Theorem)

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be positive term series.

$\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$

1. If $a_n \leq b_n$ for all n and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

2. If $a_n \geq b_n$ for all n and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

$\sum_{n=1}^{\infty} a_n \geq \sum_{n=1}^{\infty} b_n$

To apply the comparison test we compare a given series to a harmonic p series or geometric series.

Example 1.31: Does $\sum_{n=1}^{\infty} \frac{3+\frac{5}{n}}{2n^2+n+2}$ converge or diverge?

Solution:

For large n, $\frac{3+\frac{5}{n}}{2n^2+n+2} = \frac{3+\frac{5}{n}}{n^2(2+\frac{1}{n}+\frac{2}{n^2})} \approx \frac{3}{2n^2}$

Now we have $\underbrace{\frac{3+\frac{5}{n}}{2n^2+n+2}}_{a_n} \leq \frac{3+5}{2n^2} = \frac{8}{4n^2} = \underbrace{\frac{4}{n^2}}_{b_n}$ for $n \geq 1$

$\sum_{n=1}^{\infty} \frac{4}{n^2}$ converges at it is a multiple of a harmonic p series ($p=2$). Hence it converges.

$\Rightarrow \sum_{n=1}^{\infty} \frac{3+\frac{5}{n}}{2n^2+n+2}$ converges by the Comparison Test (1)

Example 1.32: Does $\sum_{n=1}^{\infty} \frac{n^2+4}{n^3+5}$ converge or diverge?

Solution:

For large n, $\frac{n^2+4}{n^3+5} = \frac{n^2\left(1+\frac{4}{n^2}\right)}{n^3\left(1+\frac{5}{n^3}\right)} \approx \frac{n^2}{n^3} = \frac{1}{n}$

So we expect divergence because $\frac{1}{n}$ is harmonic p series ($p=1$)

Now $\underbrace{\frac{n^2+4}{n^3+5}}_{a_n} \geq \frac{n^2}{n^3+5} \overset{n \geq 1}{\geq} \frac{n^2}{n^3+5n^3} = \frac{1}{6}\frac{n^2}{n^3} = \underbrace{\frac{1}{6n}}_{b_n}$

Since $\sum_{n=1}^{\infty} \frac{1}{6n}$ is a multiple of a harmonic p series ($p=1$), it diverges.

$\Rightarrow \sum_{n=1}^{\infty} \frac{n^2+4}{n^3+5}$ diverges by Comparison Test (2)

Ratio Test

$$\sum_{n=1}^{\infty} \frac{a_n \quad a_{n-1} \quad a_{n-2} \cdots}{a_{n-1} \quad a_{n-2} \quad a_{n-3}}$$

< I have no idea what he writes here after this ;-; >

Let $\sum_{n=1}^{\infty} a_n$ be a positive term series and

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

1. If $L < 1$, $\sum_{n=1}^{\infty} a_n$ converges.
2. If $L > 1$, $\sum_{n=1}^{\infty} a_n$ diverges.
3. If $L = 1$, the ratio test is inconclusive.

The ratio test is useful if a_n contains an exponential or factorial function of n.

Example 1.33: Does $\sum_{n=1}^{\infty} \underbrace{\frac{10^n}{n!}}_{a_n}$ converge or diverge?

Solution:

$a_n = \frac{10^n}{n!}$

Using the Ratio Test:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{10^{n+1}}{(n+1)!} \cdot \frac{n!}{10^n}$$

$$= \lim_{n\to\infty} \left(\frac{10^n}{n!} + 1 \right) \div \frac{10^n}{n!}$$

$$= \lim_{n\to\infty} \frac{10^{n+1}}{(n+1)!} \times \frac{n!}{10^n}$$

$$= \lim_{n\to\infty} \frac{10 \cdot \cancel{10^n} \cdot \cancel{n!}}{(n+1) \cdot \cancel{n!} \cdot \cancel{10^n}}$$

$$= \lim_{n\to\infty} \frac{10}{(n+1)}$$

$$= 10 \lim_{n\to\infty} \frac{1}{(n+1)}$$ (Limit Law 2)

$$= 10 \cdot 0$$ (Standard Limit 1)

$$= 0 = L < 1$$

$\Rightarrow$ Since $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$, $\sum_{n=1}^{\infty} \frac{10^n}{n!}$ converges by the Ratio Test

Example 1.34: Does $\displaystyle\sum_{n=1}^{\infty} \underbrace{\frac{(2n)!}{n!\ n!}}_{a_n}$ converge or diverge?

Solution:

Using Ratio Test:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left[\left(\frac{(2n)!}{n!\,n!}+1\right) \div \frac{(2n)!}{n!\,n!}\right]$$

$$= \lim_{n\to\infty}\left[\frac{(2n+2)!}{(n+1)!\,(n+1)!} \times \frac{n!\,n!}{(2n)!}\right]$$

$$= \lim_{n\to\infty} \frac{2\cancel{(n+1)}\,(2n+1)\,\cancel{(2n)!}\,\cancel{n!\,n!}}{\cancel{(n+1)}\,\cancel{n!}\,(n+1)\,\cancel{n!}\,\cancel{(2n)!}}$$

$$= \lim_{n\to\infty} \frac{2(2n+1)}{(n+1)}$$

$$= \lim_{n\to\infty} 2\cdot\frac{\left(2+\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)}$$

$$= 2\,\frac{\lim_{n\to\infty} 2 + \lim_{n\to\infty}\frac{1}{n}}{\lim_{n\to\infty} 1 + \lim_{n\to\infty}\frac{1}{n}} \qquad \text{(Limit Laws 1, 2, 4)}$$

$$= 2 \cdot \frac{2+0}{1+0}$$ (Limit Law 5 & standard Limit 1)

$$= 4 = L > 1$$

$\Rightarrow \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$, $\sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$ diverges, by the Ratio Test.

~~ROOT TEST~~

X NOT IN EXAM
<but still useful>

Let $\sum_{n=1}^{\infty} a_n$ be a positive term series and

$$L = \lim_{n\to\infty} \sup_{m\geq n} (a_m)^{\frac{1}{m}}$$

supreme rule

(1) If $L<1$, $\sum_{n\geq 1}^{\infty} a_n$ converges

(2) If $L>1$, $\sum_{n\geq 1}^{\infty} a_n$ diverges

(3) If $L=1$, $\sum_{n\geq 1}^{\infty} a_n$, the series is inconclusive

EXAMPLE

Does $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(\log(n+1))^n}$ converge or diverge?

$$a_n = \frac{1}{[\log(n+1)]^n}$$

$\Rightarrow a_n^{\frac{1}{n}} = \dfrac{1}{\log(n+1)}$ is strictly monotonically decreasing (because $\log$ is monotonically increasing)

$a_m^{\frac{1}{m}} \leq a_n^{\frac{1}{n}} \ \forall m \geq n$

$$\Rightarrow \sup_{m \geq n} (a_m)^{\frac{1}{m}} = \sup_{m \geq n} \frac{1}{\log(m+1)} = \frac{1}{\log(n+1)}$$

$$\Rightarrow L = \lim_{n \to \infty} \sup_{m \geq n} (a_m)^{\frac{1}{m}} = \lim_{n \to \infty} \frac{1}{\log(n+1)} = 0 < 1$$

$\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{1}{[\log(n+1)]^n}$ converges by the root test.

Section 2: Hyperbolic Functions

Any function on real line can be decomposed into a composition of odd & even functions

$$g(x) = \underbrace{\frac{g(x)+g(-x)}{2}}_{\text{even}} + \underbrace{\frac{g(x)-g(x)}{2}}_{\text{odd}}$$

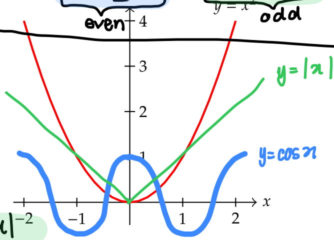
Even Functions

A function f is an even function if

$$f(-x) = f(x)$$

Example

$f(x) = \cos x$ and $f(x) = x^2$ and $f(x) = |x|$



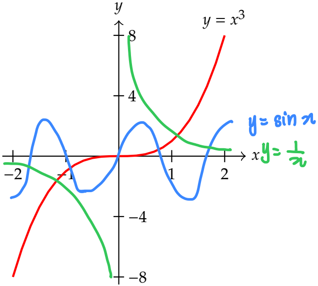
Odd Functions

A function f is an odd function if

$$f(-x) = -f(x)$$

Example

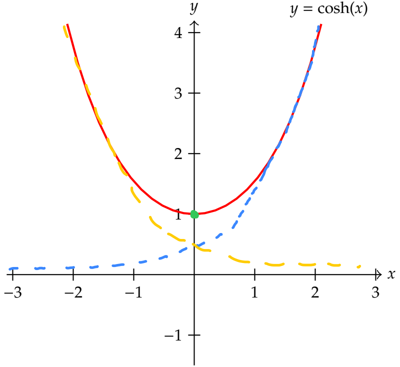
$f(x) = \sin x$ and $f(x) = x^3$ and $f(x) = \frac{1}{x}$



$g(x) = e^x$

We define the hyperbolic cosine function:

$$\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right), \quad x \in \mathbb{R}$$

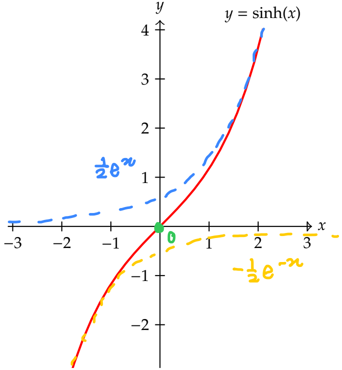


Properties

- $\cosh(0) = \frac{1}{2}(e^0 + e^{-0}) = \frac{1}{2}(1+1) = 1$
- $\cosh(-x) = \frac{1}{2}(e^{-x} + e^{-(-x)}) = \frac{1}{2}(e^x + e^{-x}) = \cosh(x)$ ~ EVEN FUNCTION
- $\lim_{x\to\infty} \frac{\cosh(x)}{\frac{1}{2}e^x} = \lim_{x\to\infty} \frac{\frac{1}{2}(e^x+e^{-x})}{\frac{1}{2}e^x} = \lim_{x\to\infty} (1+e^{-2x}) = 1$
- $\lim_{x\to-\infty} \frac{\cosh(x)}{\frac{1}{2}e^{-x}} = \lim_{x\to-\infty} \frac{\frac{1}{2}(e^x+e^{-x})}{\frac{1}{2}e^{-x}} = \lim_{x\to-\infty} (e^{2x}+1) = 1$

We define the hyperbolic sine function:

$$\sinh x = \frac{1}{2}\left(e^{x} - e^{-x}\right), \quad x \in \mathbb{R}$$

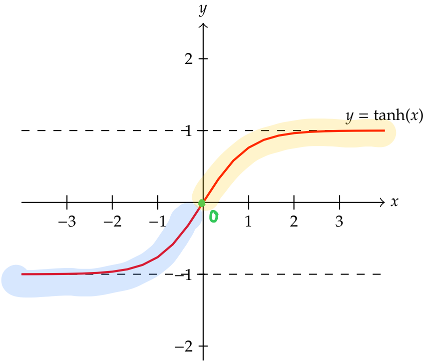


Properties

- $\sinh(0) = \frac{1}{2}(e^{0} - e^{-0}) = \frac{1}{2}(1-1) = 0$
- $\sinh(-x) = \frac{1}{2}(e^{-x} - e^{-(-x)}) = \sinh(x)$ ~ ODD FUNCTION
- $\lim_{x\to\infty} \frac{\sinh(x)}{\frac{1}{2}e^{x}} = \lim_{x\to\infty}(1-e^{-2x}) = 1$
- $\lim_{x\to-\infty} \frac{\sinh(x)}{-\frac{1}{2}e^{-x}} = \lim_{x\to-\infty}(-e^{2x}+1) = 1$

We define the hyperbolic tangent function:

$$\tanh x = \frac{\sinh x}{\cosh x} \quad \in (-1, 1)$$
$$= \frac{\frac{1}{2}(e^x - e^{-x})}{\frac{1}{2}(e^x + e^{-x})}$$
$$= \frac{e^x - e^{-x}}{e^x + e^{-x}}, \quad x \in \mathbb{R}.$$



Properties

- $\tanh(0) = \frac{\sinh(0)}{\cosh(0)} = \frac{0}{1} = 0$
- $\tanh(-x) = \frac{\sinh(-x)}{\cosh(-x)} = \frac{-\sinh(x)}{\cosh(x)} = \tanh(x)$ ~ ODD FUNCTION
- $\lim_{x\to\infty} \tanh(x) = \lim_{x\to\infty} \frac{1-e^{-2x}}{1+e^{-2x}} = \frac{1}{1} = 1$
- $\lim_{x\to-\infty} \tanh(x) = \lim_{x\to-\infty} \frac{e^{2x}-1}{e^{2x}+1} = -1$

Why call them hyperbolic functions?

Let $x = \cosh t$ and $y = \sinh t$ then

$$x^2 - y^2 = \cosh^2(t) - \sinh^2(t)$$

$$= \left[\frac{1}{2}\left(e^t + e^{-t}\right)\right]^2 - \left[\frac{1}{2}\left(e^t - e^{-t}\right)\right]^2$$

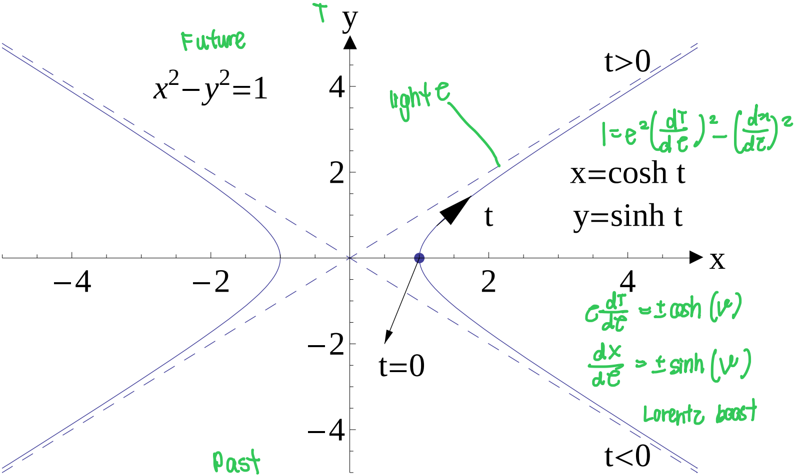
$$= \frac{1}{4}\left[\cancel{e^{2t}} + 2 + \cancel{e^{-2t}} - \left(\cancel{e^{2t}} - 2 + \cancel{e^{-2t}}\right)\right]$$

$$= \frac{1}{4}\left[2 + 2\right]$$

$$= 1$$

$$\Rightarrow x^2 - y^2 = 1$$

So $(x, y) = (\cosh t, \sinh t)$ denotes a point on the hyperbola $x^2 - y^2 = 1$. Since $x \geq 1$, the right hand branch of the hyperbola can be parametrised by $x = \cosh t, y = \sinh t, t \in \mathbb{R}$.



CHAOS THEORY

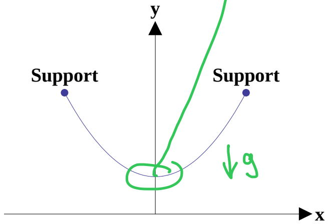
- small deviation in a system can lead to large changes

Application: Catenary

A flexible, heavy cable of uniform mass per length ρ and tension T at its lowest point has shape

$$y = \frac{T}{\rho g} \cosh\left(\frac{\rho g x}{T}\right)$$

where g is the acceleration due to gravity.



Example 2.1: If $\cosh x = \frac{13}{12}$ and $x < 0$ find $\sinh x$ and $\tanh x$.

Solution:

Now $\cosh^2(x) - \sinh^2(x) = 1$

$$\sinh^2(x) = \cosh^2(x) - 1$$

$$= \left(\frac{13}{12}\right)^2 - 1 = \frac{169}{144} = \frac{25}{144}$$

$\Rightarrow \sinh(x) = \pm\sqrt{\frac{25}{144}} = \pm\frac{5}{12}$

$\xrightarrow{x<0} \sinh(x) = -\frac{5}{12}$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{-\frac{5}{\cancel{12}}}{\frac{13}{\cancel{12}}} = -\frac{5}{13} \quad //$$

Addition Formulae ⚠

⚠ signs are different for trigonometric

$$
\begin{aligned}
\sinh(x+y) &= \sinh x \cosh y + \cosh x \sinh y \\
\cosh(x+y) &= \cosh x \cosh y + \sinh x \sinh y \\
\sinh(x-y) &= \sinh x \cosh y - \cosh x \sinh y \\
\cosh(x-y) &= \cosh x \cosh y - \sinh x \sinh y
\end{aligned}
$$

Example 2.2: Prove the $\sinh(x+y)$ addition formula.

Solution:

$$\sinh(x)\cosh(y)+\cosh(x)\sinh(x)$$

$$=\frac{1}{2}(e^{x}-e^{-x})\frac{1}{2}(e^{y}+e^{-y})+\frac{1}{2}(e^{x}+e^{-x})\frac{1}{2}(e^{y}-e^{-y})$$

$$=\frac{1}{4}\left(e^{x+y}+\cancel{e^{x-y}}\cancel{-e^{-x+y}}-e^{-x-y}+e^{x+y}\cancel{-e^{x-y}}\cancel{+e^{-x+y}}-e^{-x-y}\right)$$

$$=\frac{1}{4}\cdot 2\left(e^{x+y}-e^{-x-y}\right)$$

$$=\frac{1}{2}\left(e^{x+y}-e^{-x-y}\right)$$

$$=\sinh(x+y)_{//}$$ <proven>

Double Angle Formulae

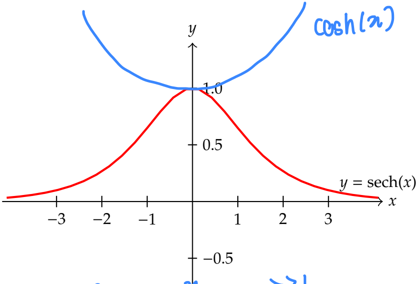
$$\begin{aligned}
\sinh(2x) &= 2\sinh x \cosh x \\
\cosh(2x) &= \cosh^2 x + \sinh^2 x \\
\cosh(2x) &= 2\cosh^2 x - 1 \\
\cosh(2x) &= 2\sinh^2 x + 1
\end{aligned}$$

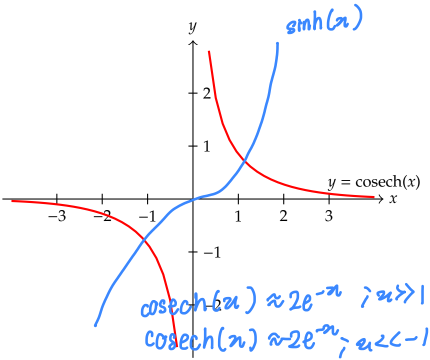
These can be proved using the addition formulae.

Reciprocal Hyperbolic Functions

We define the three reciprocal hyperbolic functions:

$$\operatorname{sech} x = \frac{1}{\cosh x}, \quad x \in \mathbb{R} \qquad \operatorname{cosech} x = \frac{1}{\sinh x}, x \in \mathbb{R} \setminus \{0\}$$





$\operatorname{sech}(x) \approx 2e^{-x}$; $x >> 1$

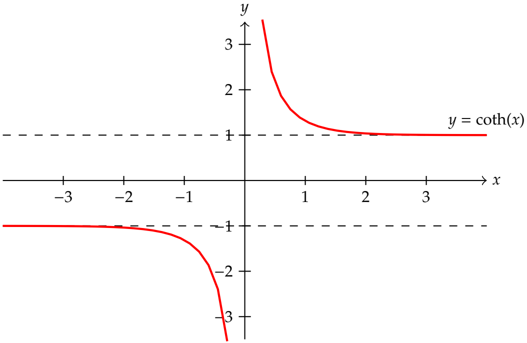
$\operatorname{sech}(x) \approx 2e^{x}$; $x << -1$

$\operatorname{cosech}(x) \approx 2e^{-x}$; $x >> 1$

$\operatorname{cosech}(x) \approx -2e^{-x}$; $x << -1$

Reciprocal Hyperbolic Functions

$$\coth x = \frac{1}{\tanh x} = \frac{\cosh x}{\sinh x} \,,$$
$$x \in \mathbb{R} \setminus \{0\}$$



Basic Identities

$$\begin{aligned} \cosh^2 x - \sinh^2 x &= 1 \\ \coth^2 x - 1 &= \operatorname{cosech}^2 x \\ 1 - \tanh^2 x &= \operatorname{sech}^2 x \end{aligned}$$

Derivatives of Hyperbolic Functions

$\frac{d}{dx} e^{\alpha x} = \alpha e^{\alpha x}; \alpha \in \mathbb{R}$

$\frac{d}{dx} \frac{1}{2}(e^{x} + e^{-x}) = \frac{1}{2}(e^{x} - e^{-x})$

trigonometric identy not the same as hyperbolic

$\boxed{\frac{d}{dx} \cos(x) = -\sin(x)}$

$$\frac{d}{dx}(\cosh x) = \sinh x, \quad x \in \mathbb{R}$$

$\frac{d}{dx} \frac{1}{2}(e^{x} - e^{-x}) = \frac{1}{2}(e^{x} - (-e^{-x}))$

$$\frac{d}{dx}(\sinh x) = \cosh x, \quad x \in \mathbb{R}$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x, \quad x \in \mathbb{R}$$

$$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x, \quad x \in \mathbb{R}$$

$$\frac{d}{dx}(\operatorname{cosech} x) = -\operatorname{cosech} x \coth x, \quad x \in \mathbb{R} \setminus \{0\}$$

$$\frac{d}{dx}(\coth x) = -\operatorname{cosech}^2 x, \quad x \in \mathbb{R} \setminus \{0\}$$

Example 2.3: Prove that $\dfrac{d(\cosh x)}{dx} = \sinh x$.

Solution:

$$\frac{d}{dx}\cosh x = \frac{d}{dx}\left[\frac{1}{2}\left(e^{x}+e^{-x}\right)\right]$$

$$= \frac{1}{2}\left(e^{x}-e^{-x}\right)$$

$$= \sinh(x)$$

Example 2.4: Let $y = \sqrt{\sinh(6x)}$, $x > 0$. Find $\dfrac{dy}{dx}$.

Solution:

$$\frac{dy}{dx} = \frac{d}{dx}\left(\left(\sinh(6x)\right)^{\frac{1}{2}}\right)$$

$$= \frac{1}{2}\left(\sinh(6x)\right)^{-\frac{1}{2}} \cdot \frac{d}{dx}\left(\sinh(6x)\right) \qquad \text{(CHAIN RULE)}$$

$$= \frac{1}{\cancel{2}} \frac{1}{\sqrt{\sinh(6x)}} \cdot \cancel{6}^{\,3} \cosh(6x)$$

$$= \frac{3\cosh(6x)}{\sqrt{\sinh(6x)}}$$

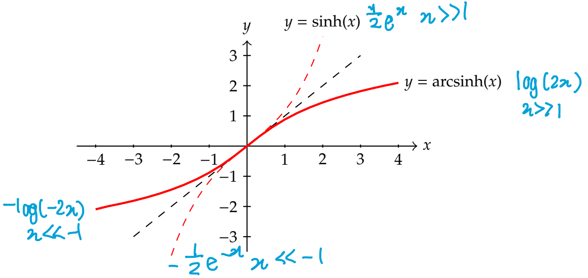
Inverses of Hyperbolic Functions

We define three inverse hyperbolic functions.

1. Inverse hyperbolic sine function: $\operatorname{arcsinh} x$

Since $\sinh x$ is a 1-1 function

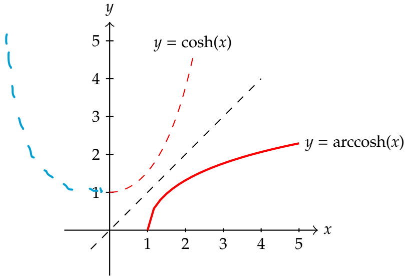
$$\begin{aligned}
\text{domain } \operatorname{arcsinh} x &= \text{range } \sinh x = \mathbb{R}. \\
\text{range } \operatorname{arcsinh} x &= \text{domain } \sinh x = \mathbb{R}. \\
\operatorname{arcsinh}(\sinh x) &= x, \quad x \in \mathbb{R}. \\
\sinh(\operatorname{arcsinh} x) &= x, \quad x \in \mathbb{R}.
\end{aligned}$$



2. Inverse hyperbolic cosine function: $\operatorname{arccosh} x$

Restrict domain of $\cosh x$ to be $[0, \infty)$ to give a 1-1 function. Then

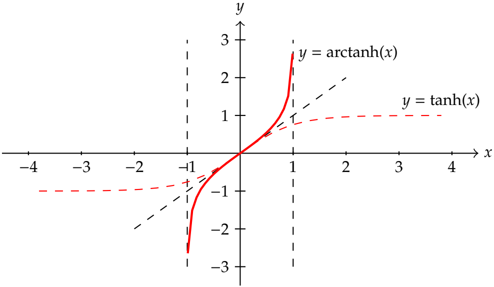
$$\begin{aligned}
\text{domain } \operatorname{arccosh} x &= \text{range } \cosh x = [1, \infty). \\
\text{range } \operatorname{arccosh} x &= \text{restricted domain } \cosh x = [0, \infty). \\
\cosh(\operatorname{arccosh} x) &= x, \quad x \geq 1. \\
\operatorname{arccosh}(\cosh x) &= x, \quad x \geq 0.
\end{aligned}$$



3. Inverse hyperbolic tangent function: $\operatorname{arctanh} x$

Since $\tanh x$ is a 1-1 function

$$\begin{aligned}
\text{domain } \operatorname{arctanh} x &= \text{range } \tanh x = (-1, 1). \\
\text{range } \operatorname{arctanh} x &= \text{domain } \tanh x = \mathbb{R}. \\
\tanh(\operatorname{arctanh} x) &= x, \quad -1 < x < 1. \\
\operatorname{arctanh}(\tanh x) &= x, \quad x \in \mathbb{R}.
\end{aligned}$$



The inverse hyperbolic functions can be expressed in terms of natural logarithms.

$$\boxed{\begin{aligned}
\operatorname{arcsinh} x &= \log\left(x + \sqrt{x^2+1}\right), && x \in \mathbb{R} \\
\operatorname{arccosh} x &= \log\left(x + \sqrt{x^2-1}\right), && x \geq 1 \\
\operatorname{arctanh} x &= \frac{1}{2}\log\left(\frac{1+x}{1-x}\right), && -1 < x < 1
\end{aligned}}$$

We can also define inverse reciprocal hyperbolic functions:

- $\operatorname{arcsech} x \qquad (0 < x \leq 1)$
- $\operatorname{arccosech} x \qquad (x \neq 0)$
- $\operatorname{arccoth} x \qquad (x < -1$ or $x > 1)$

Example 2.5: Proof of $\operatorname{arcsinh} x$ relation.

Solution: $y = \operatorname{arcsinh}(x) \Rightarrow x = \sinh(y)$

$\Rightarrow x = \frac{1}{2}(e^y - e^{-y}) \quad 1 \cdot 2e^y$

$e^{2y} - 2x e^y - 1 = 0$ (quadratic eqn e^y)

$\Rightarrow e^y = x \pm \sqrt{x^2 + 1}$

Now, we know that $e^y > 0$ (because it is an exponential fn)

$x^2 + 1 > x^2 \Rightarrow \sqrt{x^2+1} > x$

$\Rightarrow e^y = x + \sqrt{x^2+1}$

$\Rightarrow y = \log(x + \sqrt{x^2+1})$ //

Example 2.6: Simplify $\cosh(\operatorname{arcsinh} x)$ for $x \in \mathbb{R}$.

Solution: Let $y = \operatorname{arcsinh}(x) \Rightarrow x = \sinh(y)$

We know that $\cosh^2(y) - \sinh^2(y) = 1$

$\Rightarrow \cosh^2(y) = 1 + \sinh^2(x)$

$\Rightarrow \cosh^2(\operatorname{arcsinh}(x)) = 1 + \sinh^2(\operatorname{arcsinh}(x))$

$= 1 + x^2$

$\Rightarrow \cosh(\operatorname{arcsinh}(x)) = \pm\sqrt{1+x^2}$

$\cosh(y) > 0$ for all $y \in \mathbb{R}$

$\Rightarrow \cosh(\operatorname{arcsinh}(x)) = \sqrt{1+x^2}$ //

Derivatives

$$\frac{d}{dx}(\operatorname{arcsinh} x) = \frac{1}{\sqrt{x^2+1}} \quad (x \in \mathbb{R})$$

$$\frac{d}{dx}(\operatorname{arccosh} x) = \frac{1}{\sqrt{x^2-1}} \quad (x > 1)$$

$$\frac{d}{dx}(\operatorname{arctanh} x) = \frac{1}{1-x^2} \quad (-1 < x < 1)$$

Each formula is derived using implicit differentiation or by differentiating the logarithm definition of each function.

Example 2.7: Prove that $\frac{d}{dx}(\operatorname{arcsinh} x) = \frac{1}{\sqrt{x^2+1}}$.

Solution: Let $y = \operatorname{arcsinh}(x) \Rightarrow x = \sinh(y)$

Differentiate both sides with respect to x

$$\frac{d}{dx}(x) = \frac{d}{dx}(\sinh(y))$$

Chain rule

$$\Rightarrow \quad 1 = \cosh(y) \cdot \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cosh(y)} = \frac{1}{\cosh(\operatorname{arcsinh}(x))}$$

From Example 2.6:

$$\frac{1}{\cosh(\operatorname{arcsinh}(x))} = \frac{1}{\sqrt{1+x^2}} \quad //$$

Example 2.8: Find $\frac{d}{dx}(\operatorname{arctanh}(2x)\cosh(3x))$.

Solution: $\frac{d}{dx}(\operatorname{arctanh}(2x)\cosh(3x))$

Apply Product Rule: $\frac{d}{dx}(uv) = v\,du + u\,dv$

$= \cosh(3x)\frac{d}{dx}(\operatorname{arctanh}(2x)) + \operatorname{arctanh}(2x)\frac{d}{dx}(\cosh(3x))$

From Derivatives (Slide 119) ~ $\frac{d}{dx}(\operatorname{arctanh} x) = \frac{1}{1-x^2}$ $(-1<x<1)$

$= \cosh(3x)\frac{2}{1-(2x)^2} + \operatorname{arctanh}(2x) \cdot 3\sinh(3x)$

$= \frac{2\cosh(3x)}{1-4x^2} + 3\sinh(3x) \cdot \operatorname{arctanh}(2x)$

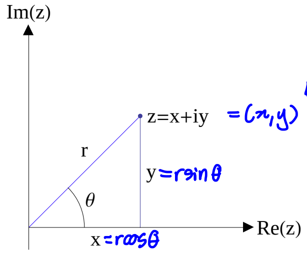
Section 3: Complex Numbers

The Cartesian form of a complex number $z \in \mathbb{C}$ is

$$\boxed{z \;=\; x + iy} \quad \text{where} \quad x, y \in \mathbb{R}$$

and

- $x = \mathrm{Re(z)}$ is the real part of z, $x \in \mathbb{R}$
- $y = \mathrm{Im(z)}$ is the imaginary part of z, $y \in \mathbb{R}$
- $i^2 = -1$. imaginary



Cartesian representation

The complex number can be written as

$$\boxed{z \;=\; r(\cos\theta + i\sin\theta)}$$

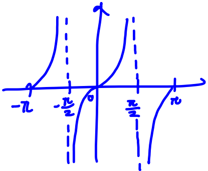
where

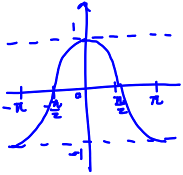
- $r = |z| = \sqrt{x^2 + y^2}$
- $\tan\theta = \dfrac{y}{x}$, $\cos\theta = \dfrac{x}{\sqrt{x^2+y^2}}$, $\sin\theta = \dfrac{y}{\sqrt{x^2+y^2}}$

Note:

The angle θ is not unique – only defined up to multiples of 2π. We choose θ such that $-\pi < \theta \leq \pi$ and call this angle the principal argument of z.

⚠ in exam, always compute principal value





The Complex Exponential

We define the complex exponential using Euler's formula

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

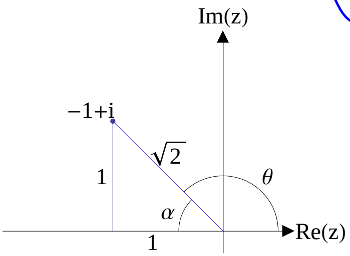
for $\theta \in \mathbb{R}$.

We can then write the $\boxed{\text{polar form}}$ of a complex number as

$$\boxed{z = re^{i\theta}} \quad , \; r > 0$$

Example 3.1: Write $z = -1 + i$ in polar form.

$z = re^{i\theta}$



Solution:
- $x = -1 \qquad y = 1$
- $r = \sqrt{x^2 + y^2} = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$
- $\cos\theta = \frac{x}{r} = \frac{-1}{\sqrt{2}}$
- $\sin\theta = \frac{y}{r} = \frac{1}{\sqrt{2}}$
- $\theta = \frac{3\pi}{4}$

$\Rightarrow z = \sqrt{2}\, e^{\frac{3\pi}{4} i}$

Properties of the Complex Exponential

1. $\boxed{e^{i0} = 1}$

Proof:

$e^{i0} = \cos 0 + i \sin 0 = 1.$

2. $\boxed{e^{i\theta} e^{i\phi} = e^{i(\theta+\phi)}}$

Proof:

$$
\begin{aligned}
e^{i\theta} e^{i\phi} &= (\cos\theta + i\sin\theta)\big(\cos\phi + i\sin\phi\big) \\
&= \cos\theta\cos\phi + i\cos\theta\sin\phi + i\sin\theta\cos\phi - \sin\theta\sin\phi \\
&= \big(\cos\theta\cos\phi - \sin\theta\sin\phi\big) + i\big(\cos\theta\sin\phi + \sin\theta\cos\phi\big) \\
&= \cos\big(\theta+\phi\big) + i\sin(\theta+\phi) \\
&= e^{i(\theta+\phi)}.
\end{aligned}
$$

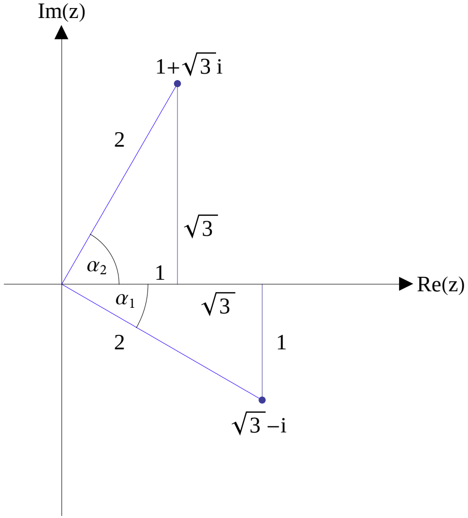
Products and Division in Polar Form

If $z = r_1 e^{i\theta}$ and $w = r_2 e^{i\phi}$ then

$$\boxed{\begin{aligned} zw &= r_1 r_2 e^{i(\theta+\phi)} \\ \frac{z}{w} &= \frac{r_1}{r_2} e^{i(\theta-\phi)} \end{aligned}}$$

Example 3.2: Using the complex exponential, simplify

$$\underbrace{(\sqrt{3}-i)}_{z_1}\underbrace{(1+\sqrt{3}i)}_{z_2} \quad \text{and} \quad \frac{\sqrt{3}-i}{1+\sqrt{3}i}. = \frac{z_1}{z_2}$$



Solution:

- For $z_1 = \underbrace{\sqrt{3}}_{x_1} \underbrace{-1}_{y_1}$; $r_1 = \sqrt{x_1^2 + y_1^2} = \sqrt{3+1} = \sqrt{4} = 2$

$$\cos\theta_1 = \frac{x_1}{r_1} = \frac{\sqrt{3}}{2}, \quad \sin\theta_1 = \frac{y_1}{r_1} = -\frac{1}{2} \Rightarrow \theta_1 = -\frac{\pi}{6}$$

$$\Rightarrow z_1 = 2e^{-i\frac{\pi}{6}}$$

- For $z_2 = \underbrace{1}_{x_2} + \underbrace{\sqrt{3}}_{y_2} i$; $r_2 = \sqrt{x_2^2 + y_2^2} = \sqrt{1+3} = \sqrt{4} = 2$

$$\cos\theta_2 = \frac{x_2}{r_2} = \frac{1}{2}, \quad \sin\theta_2 = \frac{y_2}{r_2} = \frac{\sqrt{3}}{2} \Rightarrow \theta_2 = \frac{\pi}{3}$$

$$\Rightarrow z_2 = 2e^{i\frac{\pi}{3}}$$

(a) $(\sqrt{3}-i)(1+\sqrt{3}\,i) = z_1 z_2$

$$= 2e^{-i\frac{\pi}{6}}\, 2e^{i\frac{\pi}{3}}$$

$$= 4e^{i\pi\left(-\frac{1}{6}+\frac{1}{3}\right)}$$

$$= 4e^{i\frac{\pi}{6}} \quad \text{(Polar Form)}$$

$$= 4\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)$$

$$= 4\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)$$

$$= 2\sqrt{3} + 2i \quad \text{(Cartesian Form)}$$

b) $$\frac{\sqrt{3}-i}{1+\sqrt{3}\,i} = \frac{z_1}{z_2}$$

$$= \frac{2e^{-i\frac{\pi}{6}}}{2e^{i\frac{\pi}{3}}}$$

$$= e^{-i\pi\left(\frac{1}{6}+\frac{1}{3}\right)}$$

$$= e^{-i\pi\frac{3}{6}}$$

$$= e^{-i\frac{\pi}{2}} \quad \text{(Polar Form)}$$

$$= \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)$$

$$= -i \quad \text{(Cartesian Form)}$$

De Moivre's Theorem:

If $z = re^{i\theta}$ and n is a positive integer then

$$\boxed{z^n = \left(re^{i\theta}\right)^n = r^n e^{in\theta}}$$

Example 3.3: Evaluate $(\underbrace{1+\sqrt{3}i}_{z_2})^{15}$.

Solution: $z_2 = 2e^{i\frac{\pi}{3}}$

$$(1+\sqrt{3}i)^{15} = z_2^{15}$$
$$= \left(2e^{i\frac{\pi}{3}}\right)^{15}$$
$$= 2^{15} e^{i\frac{15\pi}{3}} \qquad \text{(De Moivere's Theorem)}$$
$$= 2^{15}\left(\cos(5\pi) + i\sin(5\pi)\right)$$
$$= -2^{15}$$

Exponential Form of $\sin\theta$ and $\cos\theta$

$$\text{Now } e^{i\theta} = \cos\theta + i\sin\theta \qquad (1)$$

$$\Rightarrow e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$$

$$\Rightarrow e^{-i\theta} = \cos\theta - i\sin\theta \qquad (2)$$

Equation (1) + (2) gives

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta$$

$$\Rightarrow \boxed{\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)}$$

Equation (1) – (2) gives

$$e^{i\theta} - e^{-i\theta} = 2i\sin\theta$$

$$\Rightarrow \boxed{\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)}$$

Note:

These formulae give a connection between the hyperbolic and trigonometric functions.

$$\begin{aligned} \cosh(i\theta) &= \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) = \cos\theta \\ \sinh(i\theta) &= \frac{1}{2}\left(e^{i\theta} - e^{-i\theta}\right) = i\sin\theta \end{aligned}$$

Example 3.4: Express $\sin^5\theta$ in terms of the functions $\sin(n\theta)$ for integers n.

Solution:

Differentiation via the Complex Exponential

If $z = x + yi$ where $x, y \in \mathbb{R}$ then we define

$$e^z = e^{x+iy} = e^x\, e^{iy} = e^x(\cos y + i \sin y).$$

Derivatives of functions from $\mathbb{R}$ to $\mathbb{C}$ are defined similarly as those from $\mathbb{R}$ to $\mathbb{R}$.

Differentiation to functions from $\mathbb{R}$ to $\mathbb{C}$ is also linear and follows the product law.

Show that $\dfrac{d}{dt}\left(e^{kt}\right) = ke^{kt}$ when $k = a + bi \in \mathbb{C}$.

$$\frac{d}{dt}\left[e^{(a+bi)t}\right] = \frac{d}{dt}\left[e^{at}e^{ibt}\right]$$

$$
\begin{aligned}
&= \frac{d}{dt}\left[\underbrace{e^{at}}_{u}\,\overbrace{\underbrace{(\cos(bt) + i\sin(bt))}_{v}}^{e^{ibt}}\right] \\
&= \underbrace{ae^{at}}_{u}\,\underbrace{[\cos(bt) + i\sin(bt)]}_{dv} + \underbrace{e^{at}}_{v}\,\underbrace{[-b\sin(bt) + bi\cos(bt)]}_{du} \qquad \text{(Product Rule)} \\
&= ae^{at}\,[\cos(bt) + i\sin(bt)] + e^{at}\left[bi^2\sin(bt) + bi\cos(bt)\right] \\
&= ae^{at}\,[\cos(bt) + i\sin(bt)] + bie^{at}\,[\cos(bt) + i\sin(bt)] \\
&= (a + bi)e^{at}\,\underbrace{[\cos(bt) + i\sin(bt)]} \\
&= (a + bi)e^{at}\overbrace{e^{ibt}} \\
&= (a + bi)e^{(a+ib)t}.
\end{aligned}
$$

Example 3.5: Find $\dfrac{d^{56}}{dt^{56}}\left(e^{-t}\cos t\right)$.

Complex Exponential

$$\underbrace{e^{i\theta} = \cos\theta}_{Re} + \underbrace{i\sin\theta}_{Im}$$

$$\cos\theta = Re\left(e^{i\theta}\right)$$

$$\therefore \cos t = Re\left(e^{it}\right)$$

Solution:

$$e^{-t}\cos(t) = e^{-t}\underbrace{Re\left(e^{it}\right)}_{\frac{e^{it}+e^{-it}}{2}}$$

$$= Re\left(e^{-t}e^{it}\right)$$

$$= Re\left(e^{(-1+i)t}\right)$$

$$\Rightarrow \frac{d^{56}}{dt^{56}}\left(e^{-t}\cos t\right) = \frac{d^{56}}{dt^{56}} Re\left[e^{(-1+i)t}\right]$$

$$= Re\left[\quad (-1+i)^{56} e^{(-1+i)t}\right]$$

$$(-1+i)^{56} = \left(\sqrt{2}\; e^{\frac{3\pi i}{4}}\right)^{56}$$

$$= \left(\sqrt{2}\right)^{56} e^{56\cdot\frac{3\pi i}{4}} \quad \text{(De Moivre)}$$

$$= 2^{28} e^{42\pi i}$$

$$= 2^{28}\left(\underbrace{\cos(42\pi)}_{1} + i\underbrace{\sin(42\pi)}_{0}\right)$$

$$= 2^{28}$$

$$\Rightarrow \frac{d^{56}}{dt^{56}}\left(e^{-t}\cos t\right) = \mathrm{Re}\left[2^{28} e^{(-1+i)t}\right]$$

$$= 2^{28}\,\mathrm{Re}\left[e^{(-1+i)t}\right]$$

$$= 2^{28} e^{-t}\cos(t)$$

Note:

Example 3.5 also gives the answer to $\dfrac{d^{56}}{dt^{56}}\left(e^{-t}\sin t\right)$.

Since $e^{-t}\sin t = \operatorname{Im}\left(e^{(-1+i)t}\right)$

$$\Rightarrow \frac{d^{56}}{dt^{56}}\left(e^{-t}\sin t\right) = \operatorname{Im}\left[\frac{d^{56}}{dt^{56}}\left(e^{(-1+i)t}\right)\right]$$

$$= \operatorname{Im}\left[2^{28} e^{(-1+i)t}\right]$$

$$= 2^{28} e^{-t}\sin t$$

Integration via the Complex Exponential

$x \in \mathbb{R}$

Since $\frac{d}{dx}\left(e^{kx}\right) = k\,e^{kx}$ if $k = a + bi\ (a, b \in \mathbb{R})$, then

constant (can be complex)

$$\int k\,e^{kx}\,dx \quad = \quad e^{kx} + C$$

$k \neq 0$

$$\Rightarrow \int e^{kx}\,dx \quad = \quad \frac{1}{k}e^{kx} + D$$

$D = \frac{C}{k} \in \mathbb{C}$

Example 3.6: Evaluate $\int e^{3x}\sin(2x)\,dx$.

Solution: $e^{3x}\sin(2x) = e^{3x}\underbrace{\mathrm{Im}\left[e^{2ix}\right]}_{\frac{e^{2ix}-e^{-2ix}}{2i}}$

$$= \mathrm{Im}\left[e^{3x}e^{2ix}\right]$$

$$= \mathrm{Im}\left[e^{(3+2i)x}\right]$$

$$\int e^{3x}\sin(2x)\,dx = \int \mathrm{Im}\left[e^{(3+2i)x}\right]dx$$

$$= \mathrm{Im}\left[\int e^{(3+2i)x}\,dx\right.$$

$$= \mathrm{Im}\left[\frac{1}{3+2i}e^{(3+2i)x} + c_R + ic_I\right]$$

$c_R, c_I \in \mathbb{R}$ constant

$$= \operatorname{Im}\left[\frac{3-2i}{(3+2i)(3-2i)} e^{3x} e^{2ix} + C_R + iC_I\right]$$

$$= \operatorname{Im}\left[\frac{3-2i}{13} e^{3x} (\cos(2x) + i\sin(2x)) + C_R + iC_I\right]$$

$$= \operatorname{Im}\left[\frac{e^{3x}}{13}(3\cos(2x) + 3i\sin(2x) - 2i\cos(2x) + 2\sin(2x)) + C_R + iC_I\right]$$

Only take complex numbers

$$\rightarrow \int e^{3x}\sin(2x)\,dx = \frac{e^{3x}}{13}(3\sin(2x) - 2\cos(2x)) + C_I$$

Note:

Example 3.6 also gives the answer to $\int e^{3x}\cos(2x)\,dx$.

Since $\int e^{3x}\sin(2x)\,dx = \operatorname{Re}\left[e^{(3+2i)x}\right]$

$$\int e^{3x}\cos(2x)\,dx = \operatorname{Re}\left[\int e^{(3+2i)x}\,dx\right]$$

$$= \operatorname{Re}\left[\frac{e^{3x}}{13}\left(3\cos(2x) + 3i\sin(2x) - 2i\cos(2x) + 2\sin(2x)\right) + C_R + iC_i\right]$$

$$= \frac{e^{3x}}{13}\left[3\cos(2x) + 2\sin(2x)\right] + C_R$$

$x \in \mathbb{R}$

Derivative Substitutions

To evaluate

$$\int f[g(x)]g'(x)dx$$

put $u = g(x) \Rightarrow \frac{du}{dx} = g'(x)$.

Then

$$\begin{aligned}\int f[g(x)]g'(x)dx &= \int f(u)\frac{du}{dx}\,dx \\ &= \int f(u)\,du\end{aligned}$$

Example 4.1: Evaluate $\int (6x^2 + 10)\sinh(x^3 + 5x - 2)dx$.

Solution: Let $u = x^3 + 5x - 2$

$$\frac{du}{dx} = 3x^2 + 5$$

$$\int (6x^2+10)\sinh(x^3+5x-2)\,dx$$

$$= \int 2\frac{du}{dx}\sinh(u)\,dx$$

$$= \int 2\sinh(u)\,du$$

$$= 2\cosh(u) + c$$

$$= 2\cosh(x^3+5x-2)\,dx$$

Example 4.2: Evaluate $\int \frac{\operatorname{sech}^2(3x)}{10 + 2\tanh(3x)} dx$.

Solution: Let $u = 10 + 2\tanh(3x)$

$$\frac{du}{dx} = 6\operatorname{sech}^2(3x)$$

$$\int \frac{\operatorname{sech}^2(3x)}{10 + 2\tanh(3x)} dx = \int \frac{1}{u} \cdot \frac{1}{6} \frac{du}{dx} dx$$

$$= \frac{1}{6} \int \frac{1}{u} du$$

$$= \frac{1}{6} \log|u| + c$$

$$= \frac{1}{6} \log|10 + 2\tanh(3x)| + c$$

Trigonometric and Hyperbolic Substitutions

We can use trigonometric and hyperbolic substitutions to integrate expressions containing

$$\sqrt{a^2 - x^2}, \quad \sqrt{a^2 + x^2}, \quad \sqrt{x^2 - a^2},$$

where a is a positive real number.

Method:

Put $x = g(\theta)$. Then

$$\int f(x)\,dx = \int f[g(\theta)]g'(\theta)\,d\theta$$

Integrand	Substitution
$\sqrt{a^2 - x^2}$, $\dfrac{1}{\sqrt{a^2 - x^2}}$, $\left(a^2 - x^2\right)^{\frac{3}{2}}$ etc.	$x = a\sin\theta$ or $x = a\cos\theta$
$\sqrt{a^2 + x^2}$, $\dfrac{1}{\sqrt{a^2 + x^2}}$, $\left(a^2 + x^2\right)^{-\frac{3}{2}}$ etc.	$x = a\sinh\theta$
$\sqrt{x^2 - a^2}$, $\dfrac{1}{\sqrt{x^2 - a^2}}$, $\left(x^2 - a^2\right)^{\frac{5}{2}}$ etc.	$x = a\cosh\theta$
$\dfrac{1}{a^2 + x^2}$, $\dfrac{1}{(a^2 + x^2)^2}$ etc.	$x = a\tan\theta$

Example 4.3: Evaluate $\displaystyle\int \frac{1}{\sqrt{x^2-25}}\,dx$ using a substitution.

Solution: Let $x = 5\cosh\theta \Rightarrow \theta = \operatorname{arccosh}\left(\frac{x}{5}\right)$

$$\frac{dx}{d\theta} = 5\sinh\theta$$

Substitution is valid for $\theta \geq 0$

$\frac{x}{5} \geq 1$

$$\frac{1}{\sqrt{x^2-25}} = \frac{1}{\sqrt{25\cosh^2\theta - 25}}$$

$$= \frac{1}{5\sqrt{\cosh^2\theta - 1}}$$

$$= \frac{1}{5\sqrt{\sinh^2\theta}}$$

$$= \frac{1}{5|\sinh\theta|}$$

$$= \frac{1}{5\sinh\theta}, \text{ since } \theta \geqslant 0$$

Therefore:

$$\int \frac{1}{\sqrt{x^2-25}}\,dx = \int \frac{1}{5\sinh\theta}\,dx$$

$$= \int \frac{1}{5\sinh\theta}\,\frac{dx}{d\theta}\,d\theta$$

$$= \int \frac{1}{5\sinh\theta} \times 5\sinh\theta \, d\theta$$

$$= \int 1 \, d\theta$$

$$= \theta + C$$

$$= \operatorname{arccosh}\left(\frac{x}{5}\right) + C$$

Must discuss / write down domain

for which the trig / hyp. function

is valid

Note:

Note the identity

$$\boxed{\sqrt{x^2} = |x|} \qquad (x \in \mathbb{R})$$

Example 4.4: Evaluate $\int \sqrt{9-4x^2}\,dx$ if $|x| \le \frac{3}{2}$.

Solution: $\int \sqrt{9-4x^2}\,dx = \int 2\sqrt{\frac{9}{4}-x^2}\,dx$

Let $x = \frac{3}{2}\sin\theta$

$\frac{dx}{d\theta} = \frac{3}{2}\cos\theta$

$\theta = \arcsin\left(\frac{2x}{3}\right)$

Substitution valid for:

$-\frac{3}{2} \le x \le \frac{3}{2}, \quad \theta \in \mathbb{R}$

$$2\sqrt{\frac{9}{4}-x^2} = 2\sqrt{\frac{9}{4}-\frac{9}{4}\sin^2\theta}$$

$$= 2\left(\frac{3}{2}\right)\sqrt{1-\sin^2\theta}$$

$$= 3\sqrt{\cos^2\theta}$$

$$= 3|\cos\theta|$$

$$= 3\cos\theta, \text{ since } \theta \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}]$$

$$\int \sqrt{9-4x^2}\,dx = \int 3\cos\theta\,dx$$

$$= \int 3\cos\theta \cdot \frac{3}{2}\cos\theta \; d\theta$$

$$= \frac{9}{2}\int \cos^2\theta \; d\theta$$

$$= \frac{9}{2}\int \frac{\cos(2\theta)+1}{2}\,d\theta$$

$$= \frac{9}{4}\left[\frac{\sin 2(\theta)}{2} + \theta\right] + C$$

$$= \frac{9}{4}\left(\sin\theta\cos\theta + \theta\right) + C$$

$$= \frac{9}{4}\left(\sin\theta\sqrt{1-\sin^2\theta} + \theta\right) + C, \quad \cos\theta \geq 0$$

$$= \frac{9}{4}\left(\frac{x}{2}\sqrt{1-\frac{x^2}{4}} + \arcsin\left(\frac{x}{2}\right)\right) + C$$

Example 4.5: Evaluate $\int \left(x^2+1\right)^{\frac{3}{2}}\,dx$.

Solution:

Let $x = \sinh\theta$

- $\theta = \operatorname{arcsinh}\theta$, which is valid for all $x \in \mathbb{R}$ and for all $\theta \in \mathbb{R}$
- $\frac{dx}{d\theta} = \cosh\theta$
- $x^2+1 = \sinh^2\theta + 1 = \cosh^2\theta$

$$\Rightarrow (x^2+1)^{3/2} = \left(\sqrt{\cosh^2\theta}\right)^3$$

$$= |\cosh\theta|^3$$

$$= \cosh^3\theta, \text{ since } \cosh\theta \geq 1$$

$$\int (x^2+1)^{3/2} = \int \cosh^3\theta \cdot \cosh\theta \, d\theta$$

$$= \int \cosh^4\theta \, d\theta$$

<continued on slide 160>

Powers of Hyperbolic Functions

Consider the integral:

$$\int \sinh^m x \cosh^n x \, dx$$

where m, n are integers (≥ 0).

• If m or n is odd, use a "derivative substitution" after rewriting one of the odd power terms using identities if necessary.

• If m and n are even, use double angle formulae.

Example 4.6: Evaluate $\int \cosh^4\theta \, d\theta$.

Solution: $\int \cosh^4\theta \, d\theta = \int (\cosh^2\theta)^2 \, d\theta$

$$= \int \left(\frac{\cosh(2\theta)+1}{2}\right)^2 d\theta$$

$$= \frac{1}{4}\int \big(\underbrace{\cosh^2(2\theta)}_{\frac{1}{2}\cosh(4\theta+1)} + 2\cosh(2\theta) + 1\big)\, d\theta$$

$$= \frac{1}{4}\int \left(\frac{1}{2}\cosh(4\theta) + 2\cosh(2\theta) + \frac{3}{2}\right) d\theta$$

$$= \frac{1}{4}\left[\frac{1}{8}\sinh(4\theta) + \sinh(2\theta) + \frac{3}{2}\theta + C\right]$$

with $C \in \mathbb{R}$, a constant

$$= \frac{1}{32}\sin(4\theta) + \frac{1}{4}\sin(2\theta) + \frac{3}{8}\theta + \frac{C}{4}$$

$$\forall\ \theta \in \mathbb{R}$$

Finish Example 4.5:

$$\int (x^2+1)^{3/2}\,dx = \int \cosh^4\theta\, d\theta$$

$$= \frac{1}{32}\sinh(4\theta) + \frac{1}{4}\sinh(2\theta) + \frac{3}{8}\theta + \frac{C}{4}$$

$\forall\, \theta = \operatorname{arcsinh}(x) \in \mathbb{R}$

$$= \frac{1}{32}\big(2\sinh(2\theta)\cosh(2\theta)\big) + \frac{1}{4}\big(2\sinh(\theta)\cosh(\theta)\big) + \frac{3}{8}\theta + \frac{C}{4}$$

$$= \frac{1}{16}(2\sinh\theta\cosh\theta)(2\cosh^2\theta - 1) + \frac{1}{2}\sinh\theta\cosh\theta + \frac{3}{8}\theta + \frac{C}{4}$$

$$= \frac{1}{8}x\sqrt{x^2+1}\,(2x^2+1) + \frac{1}{2}x\sqrt{x^2+1} + \frac{3}{8}\operatorname{arcsinh}(x) + \frac{C}{4}$$

$, \forall\, x \in \mathbb{R}$

Example 4.7: Evaluate $\displaystyle\int \sinh^5 x \cosh^6 x \, dx$.

Solution: $\displaystyle\int \sinh^5(x)\cosh^6(x)\,dx = \int \sinh(x)\left(\sinh^2(x)\right)^2 \cosh^6(x)\,dx$

$$= \int \sinh(x)\left(\cosh^2(x)-1\right)^2 \cosh^6(x)\,dx$$

$u = \cosh(x) \Rightarrow \dfrac{du}{dx} = \sinh(x)$ only valid for $x \geqslant 0$ and $u \geqslant 1$

$$= \int (u^2-1)^2 u^6 \, du$$

$$= \int (u^{10} - 2u^8 + u^6)\, du$$

$$= \frac{1}{11}u^{11} - \frac{2}{9}u^9 + \frac{1}{7}u^7 + C, \text{ with } C \in \mathbb{R} \text{ a constant}$$

$$= \frac{1}{11}\cosh^{11}(x) - \frac{2}{9}\cosh^5(x) + \frac{1}{7}\cosh^7(x) + C$$

$$\forall\, x \geqslant 0$$

Example 4.8: Evaluate $I = \int \sinh^5 x \cosh^7 x \, dx$.

Solution:

METHOD 1 :

$I = \int \sinh(x) \left(\sinh^2(x)\right)^2 \cosh^7(x) \, dx$

$u = \cosh(x) \Rightarrow \frac{du}{dx} = \sinh x \quad \forall x \geq 0 \Rightarrow u \geq 1$

$= \int \sinh(x) \left(\cosh^2(x) - 1\right)^2 \cosh^7(x) \, dx$

$= \int (u^2 - 1)^2 u^7 \, du$

etc.

METHOD 2 : (better) because

$$I = \int \sinh^5(x)\,(\cosh^2(x))^3 \cosh(x)\,dx$$

$$= \int \sinh^5(x)\,(1+\sinh^2(x))^3 \cosh(x)\,dx$$

$u = \sinh(x) \Rightarrow \frac{du}{dx} = \cosh(x)$ $\boxed{\text{valid } \forall x \in \mathbb{R} \Rightarrow u \in \mathbb{R}}$

$$= \int u^5 (1+u^2)^3\,du$$

Integration by Parts

The product rule for differentiation is

$$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}$$

Integrate

$$\int \frac{d}{dx}(uv)\,dx = \int \left(\frac{du}{dx}v + u\frac{dv}{dx}\right)dx$$

$$\Rightarrow uv = \int \frac{du}{dx}v\,dx + \int u\frac{dv}{dx}\,dx$$

$$\Rightarrow \quad \boxed{\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx}$$

Example 4.9: Evaluate $\int x^2 \log x \, dx \qquad (x > 0)$.

Solution: Let $u = \log x$, $\frac{dv}{dx} = x^2$

$\Rightarrow \frac{du}{dx} = \frac{1}{x}$, $v = \frac{1}{3}x^3$, <don't need constant in v, add C in answer>

$$\int \frac{dv}{dx} \; u \; dx = u \; v - \int v \frac{du}{dx} \, dx$$

$$\int x^2 \log(x) \, dx = \log(x) \cdot \frac{1}{3}x^3 - \int \frac{1}{3} x^{\cancel{3}\,2} \cdot \frac{1}{\cancel{x}} \, dx$$

$$= \frac{1}{3} x^3 \log(x) - \frac{1}{3} \int x^2 \, dx$$

$$= \frac{1}{3} x^3 \log(x) - \frac{1}{9} x^3 + C$$, with $C \in \mathbb{R}$ a ~~constant~~ and $x > 0$

Example 4.10: Evaluate $\int xe^{5x}\,dx$.

Solution: Let $u = x \quad ; \quad \frac{dv}{dx} = e^{5x}$

$$\Rightarrow \frac{du}{dx} = 1 \quad ; \quad v = \frac{1}{5}e^{5x}$$

$$\Rightarrow \int x e^{5x}\,dx = x \cdot \frac{1}{5}e^{5x} - \int \frac{1}{5}e^{5x} \cdot 1 \, dx$$

$$= \frac{1}{5}xe^{5x} - \frac{1}{5}\int e^{5x}\,dx$$

$$= \frac{1}{5}xe^{5x} - \frac{1}{5}\left(\frac{1}{5}e^{5x} + c\right)$$

$$= \frac{1}{5}xe^{5x} - \frac{1}{25}e^{5x} + C$$

with $c \in \mathbb{R}$ a constant and $x \in \mathbb{R}$

Example 4.11: Evaluate $\displaystyle\int \log x \, dx \qquad (x > 0).$

Solution: $\displaystyle\int \log x \, dx = \int 1 \cdot \log x \, dx$

Let $u = \log(x)$; $\dfrac{dv}{dx} = 1$

$\Rightarrow \dfrac{du}{dx} = \dfrac{1}{x}$; $v = x$

$$\Rightarrow \int 1 \cdot \log x \, dx = \log(x) \cdot x - \int \cancel{x} \cdot \frac{1}{\cancel{x}} \, dx$$

$= x \log(x) - x + C$, with $C \in \mathbb{R}$ a constant and $x > 0$

Note:

This technique can also be used to integrate inverse trigonometric functions and inverse hyperbolic functions.

Example 4.12: Evaluate $\int e^{3x}\sin(2x)\,dx$.

Solution: Let $u = e^{3x}$; $\frac{dv}{dx} = \sin(2x)$

$\Rightarrow \frac{du}{dx} = 3e^{3x}$; $v = -\frac{1}{2}\cos(2x)$

$$\Rightarrow \int e^{3x}\sin(2x)\,dx = e^{3x}\cdot -\frac{1}{2}\cos(2x) - \int -\frac{1}{2}\cos(2x)\cdot 3e^{3x}\,dx$$

$$= -\frac{1}{2}e^{3x}\cos(2x) + \frac{3}{2}\int e^{3x}\cos(2x)\,dx + C$$

Integrate $\int e^{3x}\cos(2x)\,dx$

Let $u = e^{3x}$; $\frac{dv}{dx} = \cos(2x)$

$\Rightarrow \frac{du}{dx} = 3e^{3x}$; $v = \frac{1}{2}\sin(2x)$

$$\Rightarrow \int e^{3x}\cos(2x)\,dx = e^{3x}\cdot\frac{1}{2}\sin(2x) - \int \frac{1}{2}\sin(2x)\cdot 3e^{3x}\,dx$$

$$= \frac{1}{2}e^{3x}\sin(2x) - \frac{3}{2}\int \sin(2x)e^{3x}\,dx + d$$

$$\int e^{3x}\sin(2x)\,dx - \frac{3}{2}\int e^{3x}\cos(2x)\,dx = -\frac{1}{2}e^{3x}\cos(2x) + C \quad (1)$$

$$\frac{3}{2}\int e^{3x}\sin(2x)\,dx + \int e^{3x}\cos(2x)\,dx = \frac{1}{2}e^{2x}\sin(2x) + d \quad (2)$$

$(1) \times \frac{2}{3}$: $$\frac{2}{3}\int e^{3x}\sin(2x)\,dx - \int e^{3x}\cos(2x)\,dx = -\frac{1}{3}e^{3x}\cos(2x)$$

$(2) + \left((1) \times \frac{2}{3}\right)$:

$$\underbrace{\left(\frac{2}{3} + \frac{3}{2}\right)}\int e^{3x}\sin(2x)\,dx = \frac{1}{2}e^{2x}\sin(2x) - \frac{1}{3}e^{3x}\cos(2x) + f$$, with $f \in \mathbb{R}$ a constant

$$\frac{4}{6} + \frac{9}{6} = \frac{13}{6}$$

$$\int e^{3x}\sin(2x)\,dx = \frac{3}{13}e^{2x}\sin(2x) - \frac{2}{13}e^{3x}\cos(2x) + f$$, with $f \in \mathbb{R}$ a constant and $x \in \mathbb{R}$

Partial Fractions

Let $f(x)$ and $g(x)$ be polynomials, then

$$\frac{f(x)}{g(x)} \quad \begin{array}{l} \longrightarrow \text{ degree } n \\ \longrightarrow \text{ degree } d \end{array}$$

can be written as the sum of partial fractions.

Case 1: $n < d$

1. Factorise g over the real numbers.
2. Write down partial fraction expansion.
3. Find unknown coefficients

$$A, A_1, A_2, \ldots, A_r, B, B_1, B_2, \ldots, B_r.$$

Denominator Factor	Partial Fraction Expansion
$(x-a)$	$\dfrac{A}{x-a}$
$(x-a)^r$	$\dfrac{A_1}{x-a}+\dfrac{A_2}{(x-a)^2}+\cdots+\dfrac{A_r}{(x-a)^r}$
(x^2+bx+c)	$\dfrac{Ax+B}{x^2+bx+c}$
$(x^2+bx+c)^r$	$\frac{A_1x+B_1}{x^2+bx+c}+\frac{A_2x+B_2}{(x^2+bx+c)^2}+\cdots+\frac{A_rx+B_r}{(x^2+bx+c)^r}$

Example 4.13: Evaluate $\int \frac{4}{x^2(x+2)}\,dx \quad (x \neq 0, -2)$.

Solution:

$$\frac{4}{x^2(x+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2}$$

$$= \frac{A(x)(x+2) + B(x+2) + C(x^2)}{x^2(x+2)}$$

$$= \frac{\cancel{Ax^2} + 2Ax + Bx + 2B + \cancel{Cx^2}}{x^2(x+2)}$$

$$= \frac{(A+C)x^2 + (2A+B)x + 2B}{x^2(x+2)}$$, should hold for all $x \neq 0, -2$

Comparing the coefficients:

x^2: $A + C = 0$ (1)

x^1: $2A + B = 0$ (2)

x^0: $2B = 4$ (3)

③ $\Rightarrow \quad B = 2$

$B = 2$ in ② $\Rightarrow \quad 2A + 2 = 0 \quad \Rightarrow \quad A = -1$

$A = -1$ in ① $\Rightarrow \quad -1 + C = 0 \quad \Rightarrow \quad C = 1$

The coefficients are : $A = -1$, $B = 2$, $C = 1$

$$\Rightarrow \frac{4}{x^2(x+2)} = -\frac{1}{x} + \frac{2}{x^2} + \frac{1}{x+2}$$

$$\int \frac{4}{x^2(x+2)}\,dx = \int -\frac{1}{x} + \frac{2}{x^2} + \frac{1}{x+2}\,dx$$

$$= -\int \frac{1}{x}\,dx + 2\int \frac{1}{x^2}\,dx + \int \frac{1}{x+2}\,dx$$

$$= -\log|x| - \frac{2}{x} + \log|x+2| + c$$

with $c \in \mathbb{R}$ a constant and $x \neq 0, -2$

Example 4.14: Evaluate $\displaystyle\int \frac{4x}{(x^2+4)(x-2)}\,dx \quad (x \neq 2).$

Solution:

$$\frac{4x}{(x^2+4)(x-2)} = \frac{Ax+B}{x^2+4} + \frac{C}{x-2}$$

$$= \frac{(Ax+B)(x-2) + C(x^2+4)}{(x^2+4)(x-2)}$$

$$= \frac{\cancel{Ax^2} \cancel{-2Ax} + \cancel{Bx} \cancel{-2B} + \cancel{Cx^2} + \cancel{4C}}{(x^2+4)(x-2)}$$

$$= \frac{(A+C)x^2 + (B-2A)x + 4C-2B}{(x^2+4)(x-2)},$$ holds for all $x \neq 2$

comparing the coefficients:

x^2: $A+C=0$ ① $\Rightarrow A=-C$

x^1: $B-2A=4$ ②

x^0: $4C-2B=0$ ③ $\Rightarrow B=2C$

$B = 2C$ in ② $\Rightarrow$ $2C - 2A = 4$ ④

$A = -C$ in ④ $\Rightarrow$ $2C + 2C = 4$ $\Rightarrow$ $C = 1$

$C = 1$ in ① $\Rightarrow$ $A = -1$

$C = 1$ in ③ $\Rightarrow$ $B = 2$

The coefficients are : $A = -1$, $B = 2$, $C = 1$

$$\int \frac{4x}{(x^2+4)(x-2)}\,dx = \int \frac{-x+2}{(x^2+4)} + \frac{1}{x-2}\,dx$$

$$= \int \frac{-x+2}{(x^2+4)}\,dx + \int \frac{1}{x-2}\,dx$$

$$= \underbrace{\int \frac{2}{(x^2+4)}\,dx}_{\frac{d}{dx}\arctan\left(\frac{x}{2}\right)} - \underbrace{\int \frac{x}{(x^2+4)}\,dx}_{\frac{d}{dx}\left(\frac{1}{2}\log(x^2+4)\right)} + \underbrace{\int \frac{1}{x-2}\,dx}_{\frac{d}{dx}\log(x-2)}$$

$$= \arctan\left(\frac{x}{2}\right) - \frac{1}{2}\log\left(x^2+4\right) + \log(x-2) + C$$

with $C \in \mathbb{R}$ a constant and $x \neq 2$

Case 2: $n \geq d$

Use long division, then apply case 1.

Example 4.15: Find

$$\int \frac{5x^4 + 13x^3 + 6x^2 + 4}{x^3 + 2x^2}\, dx \quad (x \neq 0, -2).$$

Solution:

Long Division:

$$\begin{array}{r|l} & 5x + 3 \\ \hline x^3 + 2x^2 & 5x^4 + 13x^3 + 6x^2 + 4 \\ & -(5x^4 + 10x^3) \\ \hline & 3x^3 + 6x^2 + 4 \\ & -(3x^3 + 6x^2) \\ \hline & 4 \end{array}$$

$$\Rightarrow \frac{5x^4+13x^3+6x^2+4}{x^3+2x^2} = 5x+3 + \frac{4}{x^2(x+2)}$$

$$\Rightarrow \int \frac{5x^4+13x^3+6x^2+4}{x^3+2x^2}\,dx = \int \Big[5x+3+\underbrace{\frac{4}{x^2(x+2)}}_{-\frac{1}{x}+\frac{2}{x^2}+\frac{1}{x+2}}\Big]\,dx$$

<from Example 4.13>

$$= \frac{5}{2}x^2+3x-\log|x|-\frac{2}{x}+\log|x+2|+C$$

with $C \in \mathbb{R}$ a constant and $x \neq 0,-2$

- $\int f(g(x))\, g'(x)\, dx \overset{y=g(x)}{=} \int f(y)\, dy$
  - $g(x)$ does not need to be one-to-one
  - but you need to check whether it is differentiable in the domain of integration
- $\int f(y)\, dy \overset{y=g(x)}{=} \int f(g(x))\, g'(x)\, dx$
  - check whether $g(x)$ is one-to-one
  - AND differentiable within the domain of integration
    - Reason: we need to plug $g^{-1}(y)$ into the resulting expression.

Section 5: First Order Differential Equations

Ordinary Differential Equations

An ordinary differential equation (ODE) is an equation of the form

$$f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}\right) = 0$$

1 variable

1 fn

The order of an ODE is the order of the highest derivative occurring in the ODE.

Example 5.1: What order is $3\frac{d^4y}{dx^4} = \left(\frac{dy}{dx}\right)^2 + 2x^2y$?

=> Fourth Order

A solution of an ODE is a function $y(x)$ that satisfies the ODE for all x in some interval.

Example 5.2: Verify by substitution that $y(x) = x^2 + \dfrac{2}{x}$ is a solution of $\dfrac{dy}{dx} + \dfrac{y}{x} = 3x$ for all $x \in \mathbb{R} \setminus \{0\}$.

Solution:

$$y(x) = x^2 + \frac{2}{x} \quad \Rightarrow \frac{dy}{dx} = 2x - \frac{2}{x^2} \qquad (x \neq 0)$$

$$\Rightarrow \frac{dy}{dx} + \frac{y}{x} = 2x - \frac{2}{x^2} + \frac{1}{x}\left(x^2 + \frac{2}{x}\right)$$

$$= 2x - \cancel{\frac{2}{x^2}} + x + \cancel{\frac{2}{x^2}}$$

$$= 3x$$

$\Rightarrow y(x) = x^2 + \frac{2}{x}$ is a solution of the ODE for all $x \in \mathbb{R} \setminus \{0\}$

First Order ODEs

The general form of a first order ODE is $\dfrac{dy}{dx} = f(x, y)$.

Example 5.3: Solve $\dfrac{dy}{dx} = x^3$.

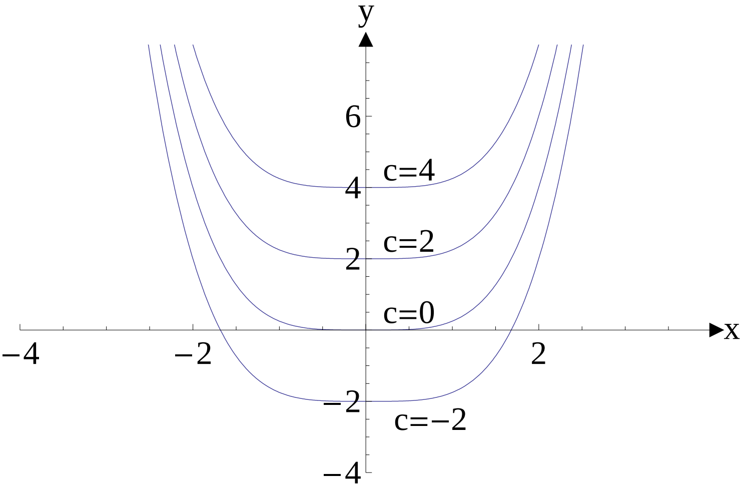
Solution: $y = \int x^3 \, dx$

$\Rightarrow y(x) = \frac{1}{4} x^4 + C$, with $C \in \mathbb{R}$ a constant and $x \in \mathbb{R}$

A solution like this which contains an arbitrary constant $c \in \mathbb{R}$ is called a general solution.

The general solution represents a family of solutions, where each value of c corresponds to a different solution of the ODE.

Sketch of the family of solutions of $\dfrac{dy}{dx} = x^3$



Initial value problem for a first order ODE

Solve $\dfrac{dy}{dx} = f(x,y)$ subject to the condition $\boxed{y(x_0) = y_0}$.

Example 5.4: Solve $\dfrac{dy}{dx} = x^3$ given $y(0) = 2$.

Solution: General solution: $y(x) = \frac{1}{4}x^4 + C$

solution has to satisfy $y(0) = 2$

$$\frac{1}{4}(0)^4 + C = 2$$

$$C = 2$$

$\Rightarrow$ Therefore solution is $y(x) = \frac{1}{4}x^4 + 2$ for all $x \in \mathbb{R}$

Separable ODEs

A separable first order ODE has the form:

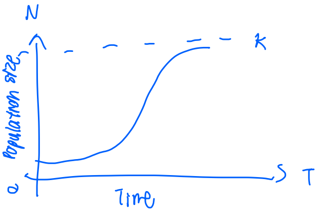
$$\frac{dy}{dx} = \overbrace{\mathcal{M}(x)\mathcal{N}(y)}^{f(x,y)}, \quad (\mathcal{M}(x) \neq 0, \;\; \mathcal{N}(y) \neq 0)$$

To solve, use *separation of variables*

$$\begin{aligned}
&& \frac{dy}{dx} &= \mathcal{M}(x)\mathcal{N}(y) \\
\Rightarrow && \frac{1}{\mathcal{N}(y)}\frac{dy}{dx} &= \mathcal{M}(x) \qquad (\mathcal{N}(y) \neq 0) \\
\Rightarrow && \int \frac{1}{\mathcal{N}(y)}\frac{dy}{dx}\,dx &= \int \mathcal{M}(x)dx \\
\Rightarrow && \int \frac{1}{\mathcal{N}(y)}dy &= \int \mathcal{M}(x)\,dx
\end{aligned}$$

LOGISTIC EQUATION

$$\frac{dN}{dt} = \underbrace{r_{max}}_{M(T)} \cdot \underbrace{N \cdot \frac{(k-N)}{k}}_{\mathcal{N}(N)} \quad ; N \in (0, k)$$



Let $M(T) = r_{max}$

$\mathcal{N}(N) = N \cdot \frac{(k-N)}{k}$

$$\int \underbrace{\frac{k}{N(k-N)}}_{\text{partial fractions}} \frac{dN}{dT} \, dT = \int r_{max} \, dT$$

$$\int \left[\frac{1}{N} + \frac{1}{k-N}\right] dN = r_{max} T + d \; ; \text{ with } d \in \mathbb{R} \text{ a constant}$$

$$\log(N) + \log(k-N) + c = r_{max} T + d \; ; \text{ with } c \in \mathbb{R} \text{ a constant}$$

$$f = d - c \in \mathbb{R} \text{ a constant}$$

$$\log\left(\frac{N}{k-N}\right) = r_{max} T + f$$

$$\frac{N}{k-N} = e^{r_{max} T} + f$$

$$N = ke^{r_{max} T + f} - Ne^{r_{max} T + f}$$

$$N(1 + e^{r_{max} T + f}) = ke^{r_{max} T} + f$$

$$N(T) = \frac{ke^{r_{max} T} + f}{1 + e^{r_{max} T} + f} \quad ; \; k > 0, \; r_{max} > 0, \; f \in \mathbb{R} \; ; \; T \in \mathbb{R}$$

$$N(T) \in (0, k)$$

Satisfies

Example 5.5: Solve $\dfrac{dy}{dx} = \dfrac{y}{1+x} \qquad (x \neq -1).$

Solution: $\dfrac{dy}{dx} = \left(\dfrac{1}{1+x}\right) \cdot (y)$ (Separable ODE)

Using separation of variables

Let $M(x) = \dfrac{1}{1+x}$

$N(y) = y$

$\Rightarrow \dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{1}{1+x}$ (for $y \neq 0$)

$\Rightarrow \displaystyle\int \frac{1}{y}\frac{dy}{dx}\,dx = \int \frac{1}{1+x}\,dx$

$\Rightarrow \displaystyle\int \frac{1}{y}\,dy = \int \frac{1}{1+x}\,dx$

$\Rightarrow \log|y| = \log|1+x| + c$, with $c \in \mathbb{R}$ a constant.

$\Rightarrow e^{\log|y|} = e^{\log|1+x| + c}$

$$\Rightarrow \quad |y| = e^c \cdot e^{\log|1+x|}$$

$$\Rightarrow \quad |y| = e^c \cdot |1+x|$$

$$\Rightarrow \quad y = \pm e^c \cdot (1+x)$$

$A = \pm e^c \in \mathbb{R} \setminus \{0\}$ constant

$$\Rightarrow y(x) = A(1+x)$$, general solution for $y \neq 0$ and all $x \neq -1$

For $y(x) = 0$

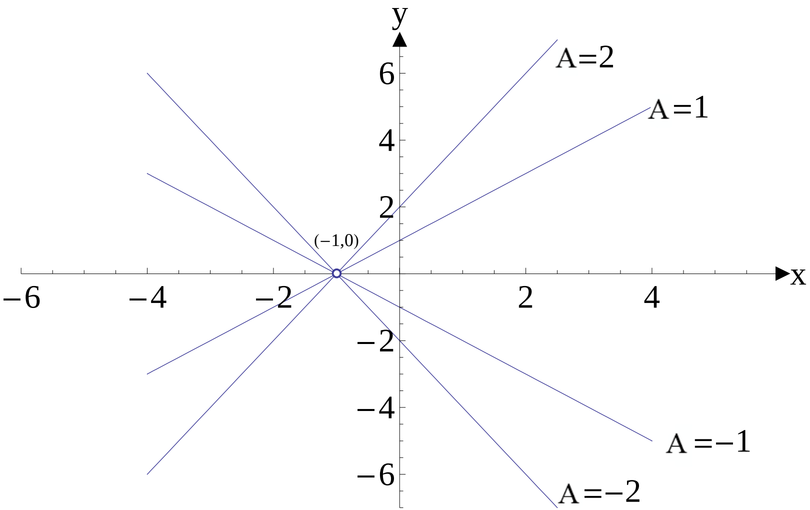
$$0 = \frac{dy}{dx} = \frac{0}{1+x} = 0$$

Thus, $y(x) = 0$ is also a solution.

Therefore, we can combine this with our general solution for $y(x) \neq 0$:

$y(x) = A(1+x)$ with a constant $A \in \mathbb{R}$ and for all $x \neq -1$

Family of solutions



Example 5.6: Solve

$$\frac{dy}{dx} = \frac{1}{2y\sqrt{1-x^2}} \qquad (-1 < x < 1,\ y \neq 0)$$

if $y(0) = 3$.

Solution:

Using separation of variables

$$\frac{dy}{dx} = \underbrace{\frac{1}{\sqrt{1-x^2}}}_{M(x)} \cdot \underbrace{\frac{1}{2y}}_{N(y)} \qquad \text{(separable ODE)}$$

$$\Rightarrow 2y\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \int 2y \frac{dy}{dx}\, dx = \int \frac{1}{\sqrt{1-x^2}}\, dx$$

$$\Rightarrow \int 2y\, dy = \int \frac{1}{\sqrt{1-x^2}}\, dx$$

$$\Rightarrow y^2 = \arcsin(x) + c, \text{ with } c \text{ a constant}$$

$y \neq 0 \Rightarrow y^2 > 0 \Rightarrow \arcsin(x) + c > 0$

$c > -\arcsin(x)$; $-\arcsin(x) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$1 > x > -\sin(c)$

$\Rightarrow$ $y^2 = \arcsin(x) + c$ is the general solution with a constant $c > -\frac{\pi}{2}$ for all $1 > x > \begin{cases} -\sin(c), & c \in [-\frac{\pi}{2}, \frac{\pi}{2}] \\ -1, & c \geq \frac{\pi}{2} \end{cases}$

$\Rightarrow$ General Sol$^{\underline{n}}$ $y(x) = \pm\sqrt{\arcsin(x) + c}$

Plug in $x = 0$ because $y(0) = 3$:

$$(y(0))^2 = 3^2 = 9$$

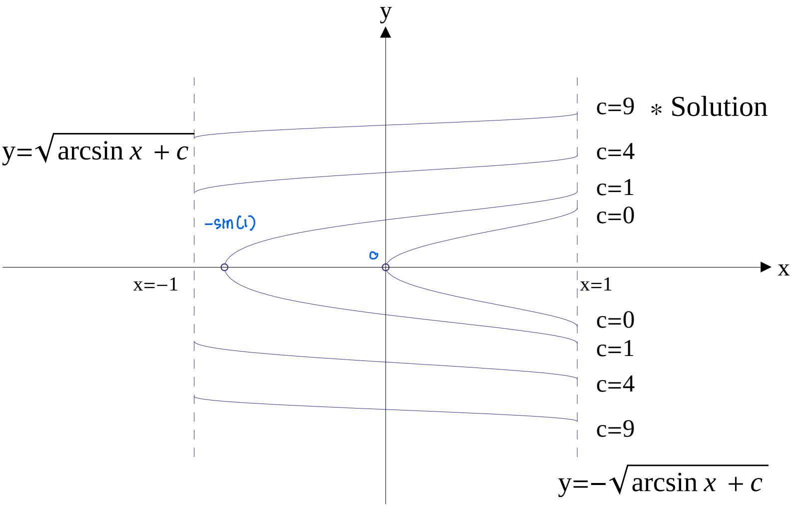
$$\arcsin(0) + c = 9$$

$$\Rightarrow c = 9$$

As $y(0) = 3 > 0 \Rightarrow$ take "+" sign

$\Rightarrow y(x) = \sqrt{\arcsin(x) + 9}$, for all $x \in (-1, 1)$

Family of solutions



Linear First Order ODEs

Example 5.7: Solve $x\dfrac{dy}{dx} + y = e^x$. $\Rightarrow \dfrac{dy}{dx} + \dfrac{y}{x} = \dfrac{e^x}{x} \quad (x \neq 0)$

Solution: $x\dfrac{dy}{dx} + y = e^x$

Reverse Product Rule

$\Rightarrow \dfrac{d}{dx}(xy) = e^x$

$\Rightarrow \displaystyle\int \frac{d}{dx}(xy)\,dx = \int e^x\,dx$

$\Rightarrow xy = e^x + c$, with $c \in \mathbb{R}$ a constant

$\Rightarrow y(x) = \dfrac{e^x}{x} + \dfrac{c}{x}$ is the general solution for all $x \neq 0$

(For $x = 0$ it is $0 \cdot \dfrac{dy}{dx} + y(0) = e^0$)

A linear first order ODE has the form: (standard form)

$$\frac{dy}{dx} + \mathcal{P}(x)y = \mathcal{Q}(x)$$

inhomogenity

To solve:

Multiply ODE by $\mathcal{I}(x)$

$$\mathcal{I}(x)\frac{dy}{dx} + \mathcal{P}(x)\mathcal{I}(x)y = \mathcal{Q}(x)\mathcal{I}(x)$$

If the left side can be written as the derivative of $y(x)\mathcal{I}(x)$, then

$$\frac{d}{dx}\left[y(x)\mathcal{I}(x)\right] = \mathcal{Q}(x)\mathcal{I}(x)$$

which can be solved by integrating with respect to x.

Aim:

Find an integrating factor $\mathcal{I}$ so the left side will be the derivative of $y\mathcal{I}$. Then

$$\frac{d}{dx}(y\mathcal{I}) \equiv \mathcal{I}\frac{dy}{dx} + \mathcal{P}\mathcal{I}y$$

$$\Rightarrow \quad \cancel{\frac{dy}{dx}\mathcal{I}} + y\frac{d\mathcal{I}}{dx} = \cancel{\mathcal{I}\frac{dy}{dx}} + \mathcal{P}\mathcal{I}y$$

$$\Rightarrow \quad y\frac{d\mathcal{I}}{dx} = \mathcal{P}\mathcal{I}y$$

To solve for all y

$$\Rightarrow \quad \boxed{\frac{d\mathcal{I}}{dx} = \overset{p(x)}{\overset{\parallel}{\mathcal{P}}}\mathcal{I}} \qquad \text{(separable)}$$

homegeneous first order linear ODE

$$
\begin{aligned}
\Rightarrow \frac{1}{\mathcal{I}}\frac{d\mathcal{I}}{dx} &= \mathcal{P} && \text{for } I \neq 0 \\
\Rightarrow \int \frac{1}{\mathcal{I}} d\mathcal{I} &= \int \mathcal{P} dx \\
\Rightarrow \log|\mathcal{I}| &= \int \mathcal{P}\, dx + c && \text{, with } c \in \mathbb{R} \text{ a constant} \\
\Rightarrow |\mathcal{I}| &= e^{\int \mathcal{P} dx + c} \\
&= e^{\int \mathcal{P} dx} \cdot e^{c} \\
\Rightarrow \mathcal{I} &= \underbrace{\pm e^{c}}_{\text{constant}} \cdot e^{\int \mathcal{P} dx}
\end{aligned}
$$

So one integrating factor is $(A = 1)$

$$\boxed{\mathcal{I}(x) = e^{\int \mathcal{P} dx}}$$

Note:

Since we only need one integrating factor $\mathcal{I}$, we can neglect the '$+c$' and modulus signs when calculating $\mathcal{I}$.

General Solution of $\frac{dy}{dx} + P(x)y = Q(x)$

$$\Rightarrow \frac{d}{dx}\left[y \; e^{\int P dx}\right] = Q(x)\, I(x)$$

$$\Rightarrow y \, e^{\int P dx} = \int Q(x)\, e^{\int P dx} \, dx + C$$

$$\Rightarrow y(x) = e^{-\int P dx} \cdot \left[\int Q(x)\, e^{\int P dx} \, dx + c\right]$$

Example 5.8: Find the general solution of

$$\frac{dy}{dx} + \frac{y}{x} = \sin x \qquad (x \neq 0).$$

Solution: The ODE is linear with $p(x) = \frac{1}{x}$ and $Q(x) = \sin x$

Use the Integrating Factor Method:

- Integrating factor:

$$I(x) = Ae^{\int p(x)\,dx}, \quad A \in \mathbb{R} \setminus \{0\} \text{ a constant}$$

$$= Ae^{\int \frac{1}{x}\,dx}$$

$$= Ae^{\log|x| + c}, \quad \text{with } c \in \mathbb{R} \text{ a constant}$$

$$= Ae^{c}|x|$$

Choose for $x > 0$; $Ae^{c} = 1$ and for $x < 0$; $Ae^{c} = -1$ $\Big\} \Rightarrow I(x) = x$

- Multiply ODE by I:

$$x \frac{dy}{dx} + y = x \sin(x)$$

- Apply the Product Rule in reverse:

$$\frac{d}{dx}(yx) = x \sin(x)$$

- Integrate

$$yx = \int x \sin(x)\, dx + C \quad , \text{with } C \in \mathbb{R} \text{ a constant}$$

- Integrate by parts with $u(x) = x$ and $\frac{dv}{dx}(x) = \sin(x)$

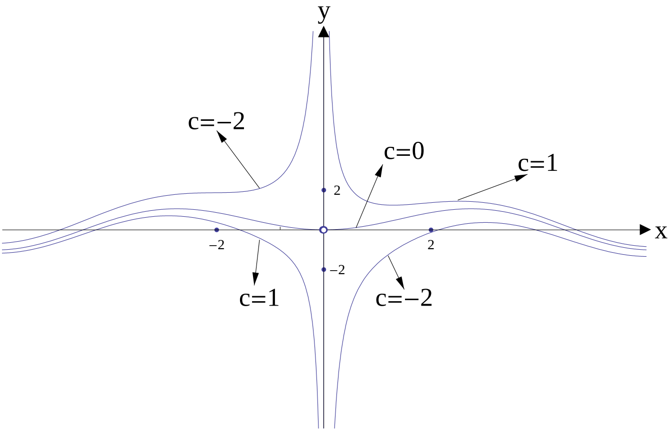
$$\Rightarrow \frac{du}{dx} = 1 \quad \text{and} \quad v(x) = -\cos(x)$$

$$yx = -x \cdot \cos(x) - \int (-\cos(x) \cdot 1\, dx + C$$

$$= -x\cos(x) + \sin(x) + C$$

$\Rightarrow$ General solution: $y(x) = -\cos(x) + \frac{\sin(x)}{x} + \frac{C}{x}$, for $C \in \mathbb{R}$ and for any $x \neq 0$

Family of solutions



$$y(x) = -\cos x + \frac{1}{x}\sin x + \frac{c}{x}$$

Example 5.9: Solve $\frac{1}{2}\frac{dy}{dx} - xy = x$ if $y(0) = -3$.

Solution:

- Write in standard form: $\frac{dy}{dx} - 2xy = 2x$
- The ODE is linear with $p(x) = -2x$ and $Q(x) = 2x$

$\Rightarrow$ Use Integrating factor Method:

$$I(x) = e^{\int p\, dx}$$
$$= e^{\int (-2x)\, dx}$$
$$= e^{-x^2} + C$$

- Multiply ODE in standard form by I:

$$e^{-x^2}\frac{dy}{dx} - 2xe^{-x^2}y = 2xe^{-x^2}$$

- Apply product rule in reverse:

$$\frac{d}{dx}\left(y e^{-x^2}\right) = 2x e^{-x^2}$$

- Integrate:

$$y e^{-x^2} = \int 2x e^{-x^2}\, dx + C,$$

with $C \in \mathbb{R}$ a constant

- Using Derivative Substitution

$u = -x^2 \qquad \frac{du}{dx} = -2x$

$$\Rightarrow y e^{-x^2} = \int e^{-x}\, du + C$$

$$= -e^{-u} + C$$

$$= -e^{-x^2} + C$$

$\Rightarrow$ General solution: $y(x) = -1 + c e^{x^2}$, with $c \in \mathbb{R}$ a constant and for all $x \in \mathbb{R}$

- Apply Initial Condition $y(0) = -3$

$$\rightarrow -3 = y(0) = -1 + c e^{0^2} =$$

$$= 1 + c$$

$$c = -2$$

$$\Rightarrow y(x) = -1 - 2e^{x^2} \text{ for all } x \in \mathbb{R}$$

Note:

The ODE is also separable:

$$\frac{1}{2} \frac{dy}{dx} = x + xy$$

$$= \underbrace{x}_{M(x)} \underbrace{(1+y)}_{N(y)}$$

Other First Order ODEs

Sometimes it is possible to make a substitution to reduce a general first order ODE to a separable or linear ODE.

- A homogeneous type ODE has the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \iff \frac{d\lambda y}{d\lambda x} = f\left(\frac{\lambda y}{\lambda x}\right) \quad \text{for any constant } \lambda \neq 0$$

Substituting $\boxed{u = \frac{y}{x}}$ reduces the ODE to a separable ODE.

- Bernoulli's equation has the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

Substituting $\boxed{u = y^{1-n}}$ reduces the ODE to a linear ODE.

Example 5.10: Solve the homogeneous type differential equation

$$\frac{dy}{dx} = \overbrace{\frac{y}{x} + \cos^2\left(\frac{y}{x}\right)}^{f\left(\frac{y}{x}\right)} \qquad \left(-\frac{\pi}{2} < \frac{y}{x} < \frac{\pi}{2}\right)$$

by substituting $u = \dfrac{y}{x}$.

Solution:

Let $u = \frac{y}{x}$. Express y and $\frac{dy}{dx}$ in terms of u and x.

$\Rightarrow y = ux \quad \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(xu) = u + x\frac{du}{dx}$ (Product Rule)

• Express the ODE in terms of u and x

$$\frac{dy}{dx} = \frac{y}{x} + \cos^2\left(\frac{y}{x}\right)$$

$$= u + x\frac{du}{dx} \quad (1)$$

$$= u + \cos^2(u) \quad (2)$$

$(1) = (2)$

$$\cancel{u} + x\frac{du}{dx} = \cancel{u} + \cos^2(u)$$

$$\frac{du}{dx} = \frac{1}{x}\cos^2(u) \qquad \text{(separable, use separation of variables)}$$

- Solve for $u(x)$

$$\Rightarrow \frac{1}{\cos^2(u)}\frac{du}{dx} = \frac{1}{x} \qquad ; \; u \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\Rightarrow \int \sec^2(u)\,du = \int \frac{1}{x}\,dx$$

$$\Rightarrow \tan(u) = \log|x| + c, \text{ with } c \in \mathbb{R} \text{ a constant}$$

- $u = \frac{y}{x}$ and write solution as $y(x)$

$$\Rightarrow \tan\left(\frac{y}{x}\right) = \log|x| + c$$

$$\Rightarrow \frac{y}{x} = \arctan\left(\log|x| + c\right)$$

General solution: $y(x) = x\arctan\left(\log|x| + c\right)$, with $c \in \mathbb{R}$ a constant and for all $x \neq 0$

Example 5.11: Solve the Bernoulli equation

$$\frac{dy}{dx} + y = e^{3x}y^4 \qquad (y \neq 0)$$

by substituting $u = y^{-3}$.

Solution:

- Let $u = y^{-3}$
- Express y and $\frac{dy}{dx}$ in terms of u and x

$\Rightarrow y = u^{-\frac{1}{3}}$

$\Rightarrow \frac{dy}{dx} = -\frac{1}{3} u^{-\frac{4}{3}} \frac{du}{dx}$ (Chain Rule)

- Express the ODE in terms of u and x

$$\frac{dy}{dx} + y = e^{3x} y^4$$

$$-\frac{1}{3}u^{-\frac{4}{3}}\frac{du}{dx}+u^{-\frac{1}{3}}=e^{3x}\,u^{-\frac{4}{3}}$$

$$\Rightarrow \frac{du}{dx}-3u=3e^{3x}$$ (Linear ~ use integrating Factor)

• Using Integrating Factor

$P(x)=-3$ and $Q(x)=-3e^{3x}$

• Integrating Factor: $I(x)=e^{\int P\,dx}$

$$=e^{\int -3\,dx}$$

$$=e^{-3x}$$

• Multiply Linear ODE by I:

$$\Rightarrow e^{-3x}\frac{du}{dx}-3e^{-3x}u=-3$$

- Apply Product Rule in Reverse:

$$\frac{d}{dx}\left(u\, e^{-3x}\right) = -3$$

- Integrate

$$u e^{-3x} = \int -3\, dx$$

$$= -3x + c \quad \text{, with } c \in \mathbb{R} \text{ a constant}$$

$$\Rightarrow \quad u(x) = e^{3x}(-3x + c)$$

- Put $u = y^{-3}$ and write solution as $y(x)$:

$$\Rightarrow y^{-3} = e^{3x}(-3x + c)$$

General Solution: $\Rightarrow y(x) = (-3x + c)^{-\frac{1}{3}} e^{-x}$, with $c \in \mathbb{R}$ constant and $x \neq \frac{c}{3}$

Population Models

Malthus (Doomsday) model / linear response model

Rate of growth is proportional to the population p at time t.

$$\begin{aligned} \boxed{\frac{dp}{dt} \propto p} \\ \Rightarrow \frac{dp}{dt} = \underbrace{k}_{\text{rate}}p \qquad \text{(separable/linear)} \end{aligned}$$

where k is a constant of proportionality representing net births per unit population per unit time.

If the initial population is $p(0) = p_0$, then the solution is

$$p(t) = p_0 e^{kt}$$

You should check that you can derive this on your own!

PROOF

$$\frac{dp}{dt} = kp \quad ; \quad p(0) = p_0$$

For $p \neq 0$: $\dfrac{1}{p} \dfrac{dp}{dt} = k$

$$\int_0^t \frac{1}{p} \frac{dp}{dt}\, dt = \int_0^t k \, dt$$

$$\int_0^{p(t)} \frac{1}{p}\, dt = kt$$

$$\log(p(t)) - \log(p(0)) = kt$$

$$\log(p(t)) = kt + \log(p(0))$$

$$p(t) = p(0)\, e^{kt}$$

$$= p_0\, e^{kt} \quad \text{for all } t \in \mathbb{R}$$

Note:

The Doomsday model is unrealistic since if

- $k > 0$ – unbounded exponential growth
- $k < 0$ – population dies out
- $k = 0$ – population stays constant

Equilibrium Solutions

Definition

An equilibrium is a constant solution of an ODE.

Note:

For the ODE $\frac{dx}{dt} = f(x,t)$, this means

- $x(t) = C$ where C is a constant
- $\frac{dx}{dt} = 0$ $\iff$ $f(x,t) = 0 \quad \forall t$

Terminology

The plural form of equilibrium is equilibria.

Example 5.12: Find all equilibrium solutions of the ODE

$$\frac{dx}{dt} = 3x - 2.$$

Solution:

Let $x(t)$ be a constant. Then $\frac{dx}{dt} = 0$

$\Rightarrow \frac{dx}{dt} = 3x - 2 = 0$

$\Rightarrow x(t) = \frac{2}{3} \quad \forall t \in \mathbb{R}$

The only equilibrium is $x(t) = \frac{2}{3}$

Phase plots

A phase plot is a plot of $\frac{dx}{dt}$ as a function of x.

A phase plot will give

- the equilibria
- the behaviour of solutions close to the equilibria

Note:

Phase plots are only useful for ODEs of the form

$$\frac{dx}{dt} = f(x)$$

i.e., when the right-hand side has no explicit dependence on t.

ODEs of this form are called autonomous.

Example 5.13: For the ODE

$$\frac{dx}{dt} = 3x - 2,$$

(a) Draw a phase plot

(b) Sketch the family of solutions of the ODE, including any equilibria.

(c) Describe the long-term behaviour of solutions with initial conditions

time

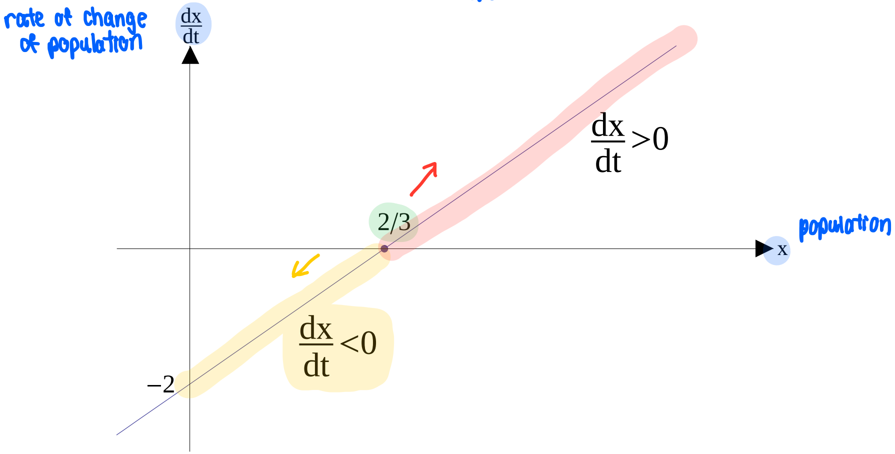
i. $x(0) = \frac{1}{2}$

ii. $x(0) = 1$

Solution:

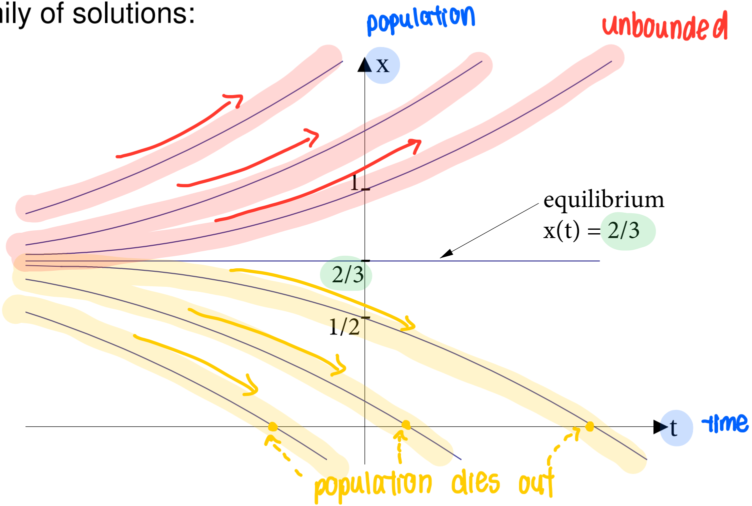
(a). The phase plot is

$\frac{dx}{dt} = 3x - 2$



(b).

Family of solutions:



(c) From the phase plot and family of solutions, we can see that

- $x(0) = \frac{1}{2} \rightarrow x(t)$ diverges to $-\infty$ (for population dies out)
- $x(0) = 1 \rightarrow x(t)$ diverges to $+\infty$ (unbounded growth)

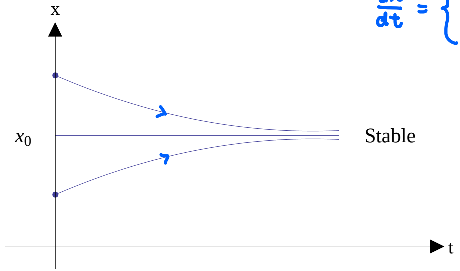
Note:

We do not need to solve the ODE analytically to determine the qualitative behaviour of the solutions.

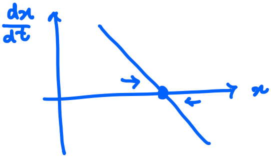
Stability of equilibria

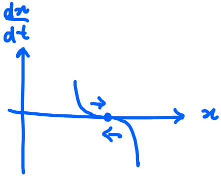
An equilibrium is stable if solutions that start nearby move closer to the equilibrium as t increases.

$$\frac{dx}{dt} = \begin{cases} >0 \ ; & x < x_0 \\ <0 \ ; & x > x_0 \end{cases}$$



On a phase plot:

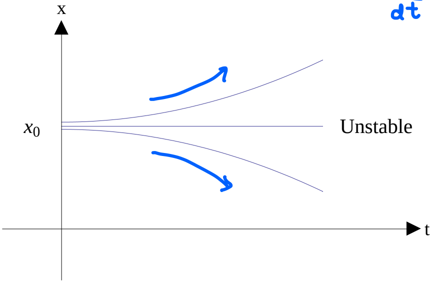




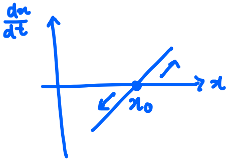
Stability of equilibria

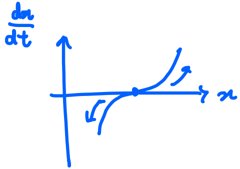
An equilibrium is unstable if solutions that start nearby move further away as t increases.

$$\frac{dx}{dt} = \begin{cases} <0, & x < x_0 \\ >0, & x > x_0 \end{cases}$$



On a phase plot:

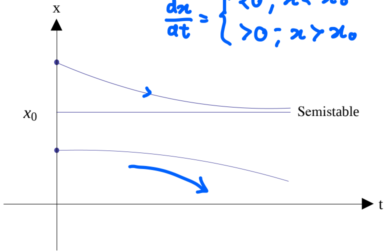




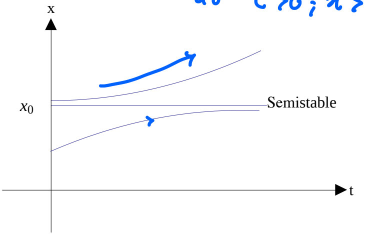
Stability of equilibria

An equilibrium is semistable if on one side of the equilibrium, solutions that start nearby move closer as t increases, whereas on the other side, solutions move further away as t increases.

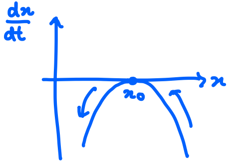
$$\frac{dx}{dt} = \begin{cases} <0; & x < x_0 \\ >0; & x > x_0 \end{cases}$$

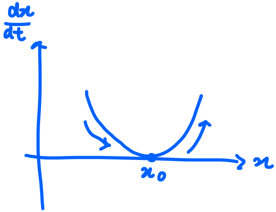


$$\frac{dx}{dt} = \begin{cases} >0; & x < x_0 \\ >0; & x > x_0 \end{cases}$$



On a phase plot:





★ In Assignment/Exam!

Example 5.13 (continued):

(d) Determine the stability of the equilibrium.

Solution:

Solutions that start nearby to $x = \frac{2}{3}$ move away from $x_0 = \frac{2}{3}$

$\Rightarrow$ Therefore, $x_0 = \frac{2}{3}$ is an unstable equilibrium

Doomsday model with harvesting.

Remove some of the population at a constant rate.

$$\frac{dp}{dt} = kp - h, \quad h > 0.$$

k, h are constant in time

'death' rate

Example 5.14: A pharmaceutical company grows engineered yeast to produce a drug. The yeast is continuously harvested to collect the drug.

The population p (in millions of yeast cells) at time t days is described by

rate of death per unit time

$$\frac{dp}{dt} = 3p - 2 \qquad (p \geq 0, t \geq 0)$$

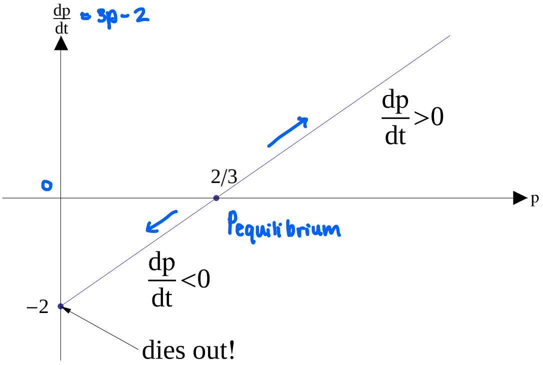
birth rate per unit population & per time

For what initial population sizes $p(0)$ will the yeast population eventually die out?

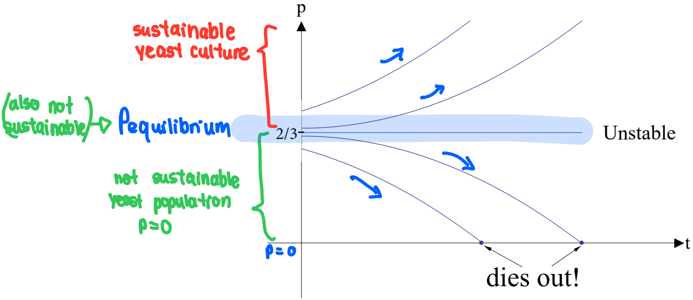
Solution:

From Example 5.13 we know

- the equilibrium is
- the phase plot is



- the family of solutions is



The population will die out if $0 < p(0) < \frac{2}{3}$

Logistic model.

Include "competition" term in Malthus' model since overcrowding, disease, lack of food and natural resources will cause more deaths.

net growth rate per population
and per time

$$\frac{dp}{dt} = \underbrace{kp}_{\text{net birth rate}} - \underbrace{\frac{k}{a}p^2}_{\text{competition term}} = kp\left(1 - \frac{p}{a}\right)$$

where $a > 0$ is the carrying capacity.

Example 5.15: Find the equilibrium solutions and their stability for the ODE

$$\frac{dp}{dt} = p\left(1 - \frac{p}{4}\right) \qquad (k = 1, a = 4)$$

Solution:

- Equilibrium solutions

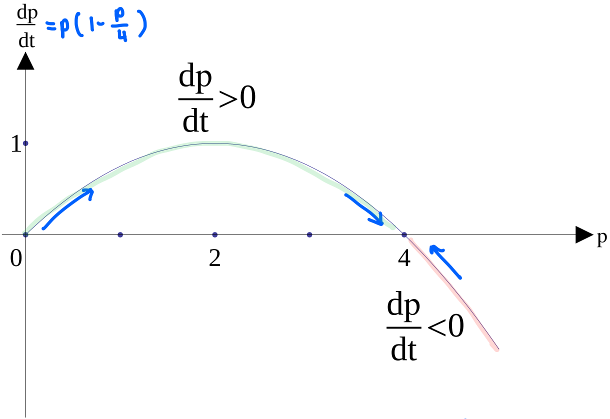
Let $p(t)$ be constant. Then

$$\frac{dp}{dt} = p\left(1 - \frac{p}{4}\right) = 0$$

$\Rightarrow p = 0$ and $p = 4$

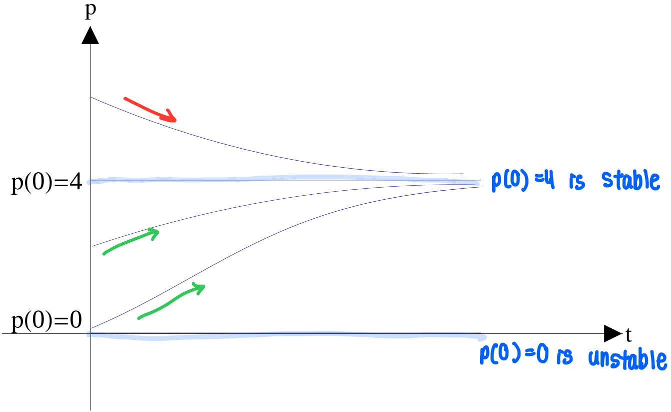
The equilibria are at $p(t) = 0$ and $p(t) = 4$

- Phase plot



- If $0 < p < 4$; $\frac{dp}{dt} > 0 \Rightarrow p$ increases with time (population is growing)
- If $p > 4$; $\frac{dp}{dt} < 0 \Rightarrow p$ decreases with time (population is shrinking)

- Family of solutions:



Note:

The logistic model accurately describes

- population in a limited space (e.g. bacteria culture).
- population of USA from 1790-1950.

Logistic model with harvesting.

Remove some of the population at constant rate:

$$\frac{dp}{dt} = kp\left(1 - \frac{p}{a}\right) - h, \quad h > 0, \; a > 0$$

Example 5.16: For the logistic model with harvesting

$$\frac{dp}{dt} \quad = \quad p\left(1 - \frac{p}{4}\right) - \frac{3}{4} \qquad \left(a = 4, \; k = 1, \; h = \frac{3}{4}\right)$$

time

determine the long-term consequences for the population predicted by the model.

Solution:

- Find equilibria:

Let $p(t)$ be constant $\Rightarrow \frac{dp}{dt} = 0$

$$\Rightarrow p\left(1 - \frac{p}{4}\right) - \frac{3}{4} = 0$$

$$-\frac{p^2}{4} + p - \frac{3}{4} = 0$$

$$-\frac{1}{4}\left(p^2 - 4p + 3\right) = 0$$

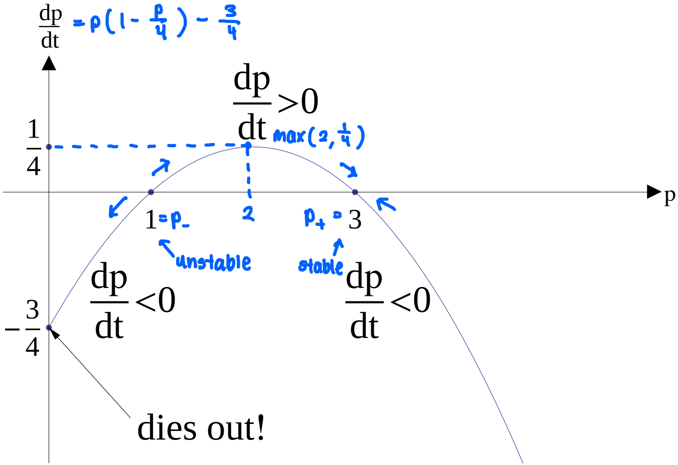
$$p_{+/-} = 2 \pm \sqrt{4-3}$$

$$p_+ = 2 + \sqrt{1} = 3$$

$$p_- = 2 - \sqrt{1} = 1$$

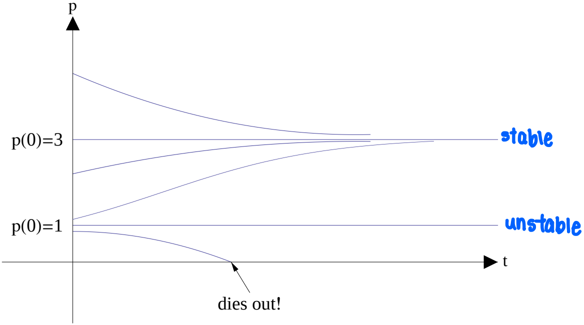
Equilibria are at $p(t) = 3$ and $p(t) = 1$

- Phase plot:


$\frac{dp}{dt} = p(1-\frac{p}{4}) - \frac{3}{4}$
$\frac{dp}{dt} > 0$
max$(2, \frac{1}{4})$
$\frac{1}{4}$
p
$1 = p_-$
2
$p_+ = 3$
unstable
stable
$\frac{dp}{dt} < 0$
$\frac{dp}{dt} < 0$
$-\frac{3}{4}$
dies out!

- Stability and family of solutions:

- If $0<p<1$ or $p>3$, then $\frac{dp}{dt}<0$
  $\Rightarrow p$ decreases in time (population shrinks)
- If $1<p<3$, then $\frac{dp}{dt}>0$
  $\Rightarrow p$ increases in time (population is growing)



- Interpretation:

(1) If $0 < p < 1$ then the population is shrinking and eventually will die out

(2) If $p(0) = 1$ then the population will remain constant

(3) If $1 < p(0) < 3$ then the population is growing and approaches $p = 3$ as $t \to \infty$

(4) If $p(0) = 3$ then the population stays constant

'best case'

(5) If $p(0) > 3$ then the population decreases and approaches $p = 3$ as $t \to \infty$

Note:

Case (2) is unrealistic in practice, as even a small deviation from $p(0) = 1$ will result in the solution following case (1) or (3).

Example 5.16 (continued): Find the time taken until the population dies out if $p(0) = \frac{1}{2}$.

Solve analytically for $p(t)$ and set $p(t) = 0$.

$$\frac{dp}{dt} = -\frac{1}{4}(p-3)(p-1)$$ (separable - use separation of variables)

$$\Rightarrow \int \frac{-4}{(p-3)(p-1)} \frac{dp}{dt}\, dt = \int 1\, dt$$

$$\Rightarrow \int \left(\frac{-2}{p-3} + \frac{2}{p-1} \right) dp = \int 1\, dt$$

(Derivative substitutions & partial fractions)

$$\Rightarrow -2\log|p-3| + 2\log|p-1| = t + C$$

$$\Rightarrow 2\log\left|\frac{p-1}{p-3}\right| = t + C$$

Plug in $t=0$ with $p(0) = \frac{1}{2}$

$$2\log\left|\frac{\frac{1}{2}-1}{\frac{1}{2}-3}\right| = 0 + C \Rightarrow C = 2\log\left(\frac{1}{5}\right)$$

$$\Rightarrow 2\log\left|\frac{p-1}{p-3}\right| = t + 2\log\left(\frac{1}{5}\right)$$

$$t = 2\log\left|\frac{p-1}{p-3}\right| - 2\log\left(\frac{1}{5}\right)$$

Population dies out when $p=0$

$$t_0 = 2\log\left|\frac{0-1}{0-3}\right| + 2\log(5)$$

$$= 2\log\left(\frac{1}{3}\right) + 2\log(5)$$

$$= 2\log\left(\frac{5}{3}\right)$$

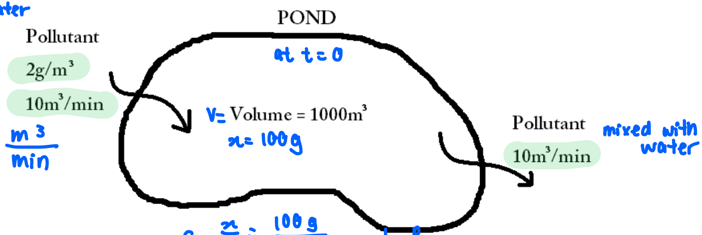
The population dies out at $t_0 = 2\log\left(\frac{5}{3}\right)$

Mixing Problems

Example 5.17: Effluent (pollutant concentration $2g/m^3$) flows into a pond (volume $1000m^3$, initially $100g$ pollutant) at a rate of $10m^3/min$. The pollutant mixes quickly and uniformly with pond water and flows out of pond at a rate of $10m^3/min$.

Find the concentration of pollutant in the pond at any time, and interpret the long-term behaviour of the system.

mixed with water



$2\frac{g}{m^3} \cdot 10\frac{m^3}{min}$

$= 20\frac{g}{min}$

$C = \frac{x}{V} = \frac{100g}{1000m^3} = \frac{1}{10}\frac{g}{m^3}$

Derive the ODE:

Let x be the amount (grams) of pollutant in pond at time t minutes. Then $C = \dfrac{x}{V}$ is the concentration of pollutant in pond $(grams/m^3)$, where V is the volume of the pond (m^3) at time t.

$$\boxed{\frac{dx}{dt} = \text{rate pollutant flows in - rate pollutant flows out}}$$

Initial condition: $x(t=0) = 100g$; $c(t=0) = \frac{1}{10}\frac{g}{m^3}$

rate in $= \left(\begin{array}{c}\text{concentration}\\ \text{in}\end{array}\right) \cdot \left(\text{flow rate in}\right)$

$= 2\,\frac{g}{m^3} \cdot 10\,\frac{m^3}{min}$

$= 20\,\frac{g}{min}$

• Rate out = $\left(\begin{matrix}\text{concentration}\\ \text{out}\end{matrix}\right) \cdot \left(\begin{matrix}\text{Hour rate}\\ \text{out}\end{matrix}\right)$

$$= \frac{x(t)}{v(t)} \cdot 10 \frac{m^3}{min}$$

$$\text{Volume} = v(t) = v(0) + \left(\begin{matrix}\text{Flow rate}\\ \text{in}\end{matrix}\right)t - \left(\begin{matrix}\text{Flow rate}\\ \text{out}\end{matrix}\right)t$$

$$= 1000\,m^3 + \cancel{\left(10 \frac{m^3}{min}\right)t} - \cancel{\left(10 \frac{m^3}{min}\right)t}$$

$$= 1000\,m^3$$

$$\Rightarrow \text{Rate out} = \frac{x\,g}{1000\,m^3} \cdot 10 \frac{m^3}{min}$$

$$= \frac{x}{100} \frac{g}{min}$$

$$\Rightarrow \frac{dx}{dt} = 20 \frac{g}{min} - \frac{x}{100\,min}$$

Solve analytically:

$\left(\frac{g}{min}\right)$

$$\frac{dx}{dt} + \frac{x}{100} = 20 \qquad \text{(Linear/Separable)}$$

General solution is

$$x(t) = 2000 + ce^{\frac{-t}{100}}$$

(working omitted).

$100 = 2000 + ce^{\frac{-0}{100}} = 2000 + c$

Initial condition is $x(0) = 100 \quad \Rightarrow \quad c = -1900$

$$\Rightarrow x(t) = 2000 - 1900e^{\frac{-t}{100}} \qquad (g)$$

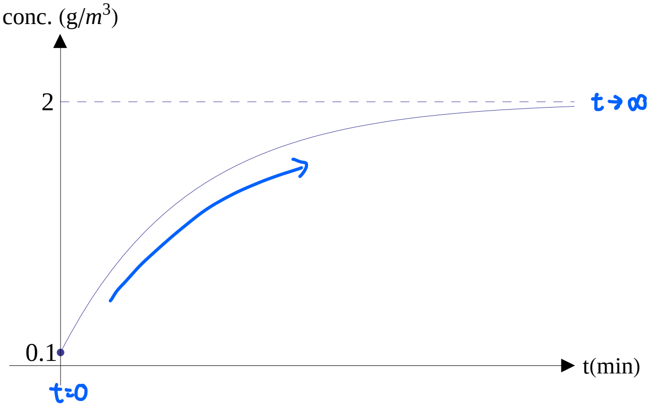
Find concentration:

Concentration : $C(t) = \frac{\text{amount}}{\text{volume}} = \frac{x(t)}{v(t)} = \frac{x(t)}{1000}$

$\Rightarrow C(t) = 2 - \frac{19}{10} e^{-t/100} \quad \left(\frac{g}{m^3}\right)$

Interpret:

As $t \to \infty$, the concentration increases and converges to a constant concentration of $2 \frac{g}{m^3}$



Definitions

1. Transient terms: terms decaying to 0 as $t \to \infty$.

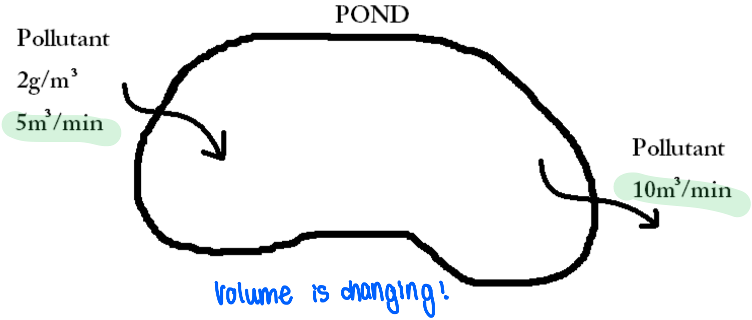
2. Steady state terms: terms NOT decaying to 0 as $t \to \infty$.

The solution for the concentration can be classified as follows.

Transient term: $-\frac{19}{10} e^{-t/100}$

Steady state term: 2

Example 5.18: Derive an ODE describing the amount x (kg) of pollutant in the lake at time t (minutes), if the input flow rate is decreased to $5m^3/min$.



Let V be the volume in pond (m^3) at time t minutes.

$$v(t) = v(0) + \begin{pmatrix}\text{flow rate}\\ \text{in}\end{pmatrix} t - \begin{pmatrix}\text{flow rate}\\ \text{out}\end{pmatrix} t$$

$$= 1000\, m^3 + \left(5 \frac{m^3}{min}\right) t - \left(10 \frac{m^3}{min}\right) t$$

$$\Rightarrow v(t) = 1000 - 5t \quad (m^3), \text{ for } 0 \leq t \leq 200 \quad (min)$$

• $\dfrac{dx}{dt}$ = rate in − rate out

$$= 2 \frac{g}{m^3} \cdot 5 \frac{m^3}{min} - \frac{x(t)}{v(t)} \cdot 10 \frac{m^3}{min}$$

$$\Rightarrow \frac{dx}{dt} = 10 - \frac{10x}{1000-5t} = 10 - \frac{2x}{200-t} \quad \left(\frac{g}{min}\right), \text{ for } 0 \leq t \leq 200$$

(Linear — Use Integrating Factor Method)

- Integrating Factor:

$$I = \exp\left[\int \frac{2}{200-t}\,dt\right] = \exp\left[-2\log|200-t|\right]$$

$$= \frac{1}{(200-t)^2}$$

- Multiply with ODE:

$$\frac{1}{(200-t)^2}\frac{dx}{dt} + \frac{2x}{(200-t)^3} = \frac{10}{(200-t)^2}$$

- Reverse product Rule:

$$\frac{d}{dt}\left[\frac{x}{(200-t)^2}\right] = \frac{10}{(200-t)^2}$$

- Integrate ODE:

$$\frac{x}{(200-t)^2} = \frac{10}{200-t} + c \quad , \quad c \in \mathbb{R} \text{ a constant}$$

$$\Rightarrow x(t) = 10(200-t) + c(200-t)^2 \; , \quad t \in [0, 200]$$

Initial condition $x(t=0) = 100 \quad (g)$

$$100 = x(0) = 10(200-0) + c\,(200-0)^2$$

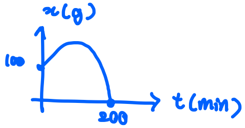
$$= 2000 + 40\,000\, C$$

$$C = -\frac{1900}{40\,000} = -\frac{19}{400}$$

$$\Rightarrow x(t) = 10(200-t) - \frac{19}{400}(200-t)^2 \; ; \; 0 \leqslant t \leqslant 200$$

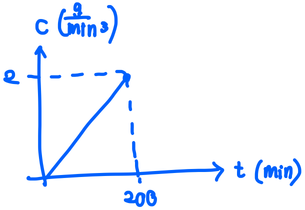
The concentration is :

$$c(t) = \frac{x(t)}{v(t)} = \frac{10(20-t) - \frac{19}{400}(200-t)^2}{1000-5t}$$



$$= 2 - \frac{19}{2000}(200-t)$$

$$= \frac{19}{2000}t + \frac{1}{10}$$



Section 6: Second Order Differential Equations

A second order ODE has the form

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0$$

what is the form of a

The general form of a linear second order ODE is

$$\frac{d^2y}{dx^2} + \mathcal{P}(x)\frac{dy}{dx} + \mathcal{Q}(x)y = \mathcal{R}(x)$$

How do you know if a linear second order ODE is ?

- If $\mathcal{R}(x) = 0$, the ODE is homogeneous (H).

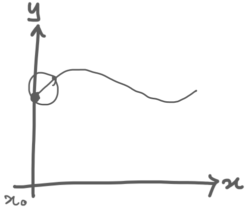
- If $\mathcal{R}(x) \neq 0$, the ODE is inhomogeneous (IH).

Note:

A homogeneous linear ODE is different to a homogeneous type first order ODE.

The general solution of a second order ODE typically has two arbitrary constants.

(1) Initial value problem for a second order ODE

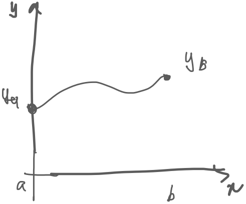


Solve

$$\frac{d^2y}{dx^2} + \mathcal{P}(x)\frac{dy}{dx} + \mathcal{Q}(x)y = \mathcal{R}(x)$$

subject to the conditions $y(x_0) = y_0$ and $y'(x_0) = y_1$.

(2) Boundary value problem for a second order ODE



Solve

$$\frac{d^2y}{dx^2} + \mathcal{P}(x)\frac{dy}{dx} + \mathcal{Q}(x)y = \mathcal{R}(x)$$

subject to the conditions $y(a) = y_0$ and $y(b) = y_1$.

Homogeneous 2^{nd} Order Linear ODEs

Homogeneous 1st order Linear ODE:

$$y' + p(x)\,y = 0 \Rightarrow y = \underbrace{ce^{-\int p\,dx}}_{y_1(x)}$$

Theorem:

The general solution of

$$y'' + \mathcal{P}(x)y' + \mathcal{Q}(x)y = 0$$

is given by what fn?

is the function y given by

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

where

- y_1, y_2 are two linearly independent solutions of the homogeneous ODE,
- $c_1, c_2 \in \mathbb{R}$ are arbitrary constants.

Definition:

when are 2 f^{n}s, y_1 and y_2, linearly independent?

Two functions y_1 and y_2 are linearly independent if

for all $x \in \mathbb{R}$

$$c_1 y_1(x) + c_2 y_2(x) = 0 \quad \Rightarrow \quad c_1 = c_2 = 0$$

or equivalently, if neither function is a non-zero constant multiple of the other function.

Example 6.1:

(a) Are $y_1(x) = x^2,\ y_2(x) = 2x^2$ linearly independent?

$c_1 x^2 + c_2(2x^2) = (c_1 + 2c_2)x^2 = 0 \quad \forall x \in \mathbb{R}$

Choose all $x \neq 0 \Rightarrow c_1 + 2c_2 = 0$ can be satisfied by $c_1 = -2$, $c_2 = 1$

$\Rightarrow y_1$ and y_2 are linearly dependent (not linearly independent)

(b) Are $y_1(x) = e^{2x},\ y_2(x) = xe^{2x}$ linearly independent?

$c_1 y_1(x) + c_2 y_2(x) = c_1 e^{2x} + c_2 x e^{2x} = (c_1 + c_2 x) e^{2x} = 0 \quad \forall x \in \mathbb{R}$

Choose $\left.\begin{array}{l} x = 1 \Rightarrow c_1 + c_2 = 0 \\ x = -1 \Rightarrow c_1 - c_2 = 0 \end{array}\right\} \Rightarrow c_1 = c_2 = 0 \Rightarrow$ y_1 and y_2 are linearly independent

FUNCTIONS THAT ARE LINEARLY INDEPENDENT

x^m, x^n with $m \neq n \Rightarrow x^m$ and x^n are linearly independent

$\cos(x), \sin(x)$ are linearly independent

$\cos(x), \cos(2x)$ "

$\cosh(x), \cosh(2x)$ "

e^x, e^{2x} "

Aside - A connection with Linear Algebra*

Why are these ODEs called linear?

Because they can be written in terms of a linear transformation. The ODE

$$\frac{d^2y}{dx^2} + \mathcal{P}(x)\frac{dy}{dx} + \mathcal{Q}(x)y = \mathcal{R}(x)$$

can be written as

$$T(y) = \mathcal{R}(x)$$

with λ, μ be constant
and y_1, y_2 be functions
$T(\lambda y_1 + r y_2)$
$=$
$\lambda T(y_1) + r T(y_2)$

where

$$T(y) = \frac{d^2y}{dx^2} + \mathcal{P}(x)\frac{dy}{dx} + \mathcal{Q}(x)y$$

is a linear transformation on a vector space of functions.

These concepts are covered in MAST10007 Linear Algebra.

** This slide is not examinable in MAST10006.*

Homogeneous 2^{nd} Order Linear ODEs with Constant Coefficients

General form: $$ay'' + by' + cy = 0$$

where a, b, c are constants. ; $a \neq 0$

To solve for $y(x)$:

λ is a constant to be fixed

$$\begin{aligned} \text{Try } y(x) &= e^{\lambda x} \\ \Rightarrow y'(x) &= \lambda e^{\lambda x}, \quad y''(x) = \lambda^2 e^{\lambda x} \end{aligned}$$

$$\text{so } \left(a\lambda^2 + b\lambda + c\right) \underbrace{e^{\lambda x}}_{\neq 0} = 0$$

$$\Rightarrow \boxed{a\lambda^2 + b\lambda + c = 0}$$ Characteristic Equation

$$\Rightarrow \lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Case 1: $b^2 - 4ac > 0$

- 2 distinct real values λ_1, λ_2

- 2 linearly independent solutions

$$e^{\lambda_1 x}, \quad e^{\lambda_2 x}$$

- General Solution:

$$\boxed{y(x) = Ae^{\lambda_1 x} + Be^{\lambda_2 x}}$$

for $A, B \in \mathbb{R}$ constants

PROVING LINEAR INDEPENDENCE

$\boxed{c_1 y_1(x) + c_2 y_2(x) = 0 \ \forall x \in \mathbb{R} \text{ and } c_1 = c_2 = 0 \text{ THEN } y_1(x), y_2(x) \text{ are linearly independent}}$

$$\underbrace{c_1 e^{\lambda_1 x}}_{y_1(x)} + \underbrace{c_2 e^{\lambda_2 x}}_{y_2(x)} = 0 \ ; \forall x \in \mathbb{R}$$

$x = 0$: $c_1 + c_2 = 0 \Rightarrow c_1 = -c_2$ ①

$x = 1$: $e^{\lambda_1} c_1 + e^{\lambda_2} c_2 = 0$ ②

① → ②:

$$e^{\lambda_1} c_1 - e^{\lambda_2} c_1 = 0$$

$$\underbrace{(e^{\lambda_1} - e^{\lambda_2})}_{\neq 0} c_1 = 0 \quad \Rightarrow c_1 = 0$$

$c_1 = 0 \to$ ①:

$$0 = -c_2$$

$$\Rightarrow c_2 = 0$$

$\Rightarrow$ Since $c_1 = c_2 = 0$ when $c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} = 0 \ \forall x \in \mathbb{R}$ then $e^{\lambda_1 x}$ and $e^{\lambda_2 x}$ are linearly independent

Example 6.2: Solve $\overset{a=1}{y''} + \overset{b}{\underbrace{7}}y' + \overset{c}{\underbrace{12}}y = 0$ for $y(x)$.

Solution: Try $y(x) = e^{\lambda x}$, λ is constant to be fixed

$\Rightarrow y'(x) = \lambda e^{\lambda x}$, $y''(x) = \lambda^2 e^{\lambda x}$

$\Rightarrow (\lambda^2 + 7\lambda + 12)\underbrace{e^{\lambda x}}_{\neq 0} = 0$

$\Rightarrow \lambda^2 + 7\lambda + 12 = 0$ (characteristic eq$\underline{^n}$)

$\Rightarrow \lambda = \dfrac{-7 \pm \sqrt{7^2 - 4(1)(12)}}{2(1)} = \dfrac{-7 \pm \sqrt{49-48}}{2} = \dfrac{-7 \pm 1}{2}$

$\Rightarrow \lambda_1 = -4$ and $\lambda_2 = -3$

Case 1: $b^2 - 4ac > 0$

$y_1(x) = e^{-4x}$ and $y_2(x) = e^{-3x}$ are 2 linearly independent solutions

$\Rightarrow$ General Solution is:

$y(x) = Ae^{-4x} + Be^{-3x}$ with $A, B \in \mathbb{R}$ constants and for all $x \in \mathbb{R}$

Case 2: $b^2 - 4ac = 0$

- 1 real value $\lambda = \dfrac{-b}{2a}$

- 1 solution is $e^{\lambda x}$

- 2^{nd} linearly independent solution is $xe^{\lambda x}$ (found using variation of parameters — not in syllabus).

- General Solution:

$$\boxed{y(x) = Ae^{\lambda x} + Bxe^{\lambda x}} = (A + Bx)e^{\lambda x}$$

with $A, B \in \mathbb{R}$ constants

We now verify that $xe^{\lambda x}$ is a solution:

If $y(x) = xe^{\lambda x}$, then

$$\begin{aligned} y'(x) &= (\lambda x + 1)\, e^{\lambda x}, \quad = \left(e^{\lambda x} + e \lambda e^{\lambda x}\right) \text{ (Product Rule)} \\ y''(x) &= \left(\lambda^2 x + 2\lambda\right) e^{\lambda x}. = \left(\lambda \cdot e^{\lambda x} + (\lambda x + 1) \cdot \lambda e^{\lambda x}\right) \text{ (Product Rule)} \end{aligned}$$

So $ay'' + by' + cy = 0$

$$\begin{aligned} ay'' + by' + cy &= a\left(\lambda^2 x + 2\lambda\right) e^{\lambda x} + b\,(\lambda x + 1)\, e^{\lambda x} + cxe^{\lambda x} \\ &= xe^{\lambda x} \underbrace{\left(a\lambda^2 + b\lambda + c\right)}_{=0} + \underbrace{(2\lambda a + b)}_{=0} e^{\lambda x} \\ &= 0 \end{aligned}$$

So $y(x) = xe^{\lambda x}$ is a solution.

Example 6.3: Solve $y'' + 2y' + y = 0$ for $y(x)$.

Solution:

Try $y(x) = e^{\lambda x}$

$\Rightarrow y'(x) = \lambda e^{\lambda x}$; $y''(x) = \lambda^2 e^{\lambda x}$

$$\Rightarrow (\lambda^2 + 2\lambda + 1)\underbrace{e^{\lambda x}}_{\neq 0} = 0$$

$$\Rightarrow (\lambda^2 + 2\lambda + 1) = 0$$

$$\Rightarrow (\lambda + 1)^2 = 0$$

$$\Rightarrow \lambda = -1$$

Case 2: $b^2 - 4ac = 0$:

So $y_1(x) = e^{-x}$ is one solution of the ODE.

A second linearly independent sol$^{\underline{n}}$ is $y_2(x) = xe^{-x}$

$\Rightarrow$ General sol$^{\underline{n}}$

$$y(x) = Ae^{-x} + Bxe^{-x} = (A + Bx)e^{-x}$$

with $A, B \in \mathbb{R}$ constants and for all $x \in \mathbb{R}$

Case 3: $b^2 - 4ac < 0$

- 2 complex conjugate values

$$\lambda_1 = \alpha + i\beta, \quad \lambda_2 = \alpha - i\beta \quad ; \quad \alpha = -\frac{b}{2a}, \quad \beta = \frac{\sqrt{4ac-b^2}}{2a}$$

- 2 complex linearly independent solutions

$$e^{(\alpha+i\beta)x}, \quad e^{(\alpha-i\beta)x}$$

- Complex General Solution:

$$\begin{aligned} y(x) &= C_1 \overbrace{e^{(\alpha+i\beta)x}}^{e^{\alpha x}\cdot e^{i\beta x}} + C_2 \overbrace{e^{(\alpha-i\beta)x}}^{e^{\alpha x}\cdot e^{i\beta x}} \qquad \text{where } C_1, C_2 \in \mathbb{C} \\ &= C_1 e^{\alpha x}\left(\cos(\beta x) + i\sin(\beta x)\right) + C_2 e^{\alpha x}\left(\cos(\beta x) - i\sin(\beta x)\right) \\ &= \underbrace{(C_1 + C_2)}_{A}\underbrace{e^{\alpha x}\cos(\beta x)}_{\in\mathbb{R}} + \underbrace{(C_1 i - C_2 i)}_{B}\underbrace{e^{\alpha x}\sin(\beta x)}_{\in\mathbb{R}} \end{aligned}$$

Put $A = C_1 + C_2$ and $B = (C_1 - C_2)i$. If $C_1 = \overline{C_2}$, then $A, B \in \mathbb{R}$.

Note:

initial value / boundary value

Imposing real conditions on the ODE will always lead to real coefficients A and B.

- 2 real linearly independent solutions

$$e^{\alpha x}\cos(\beta x), \quad e^{\alpha x}\sin(\beta x)$$

Real General Solution:

$$\boxed{y(x) = Ae^{\alpha x}\cos(\beta x) + Be^{\alpha x}\sin(\beta x)}$$

with $A, B \in \mathbb{R}$ constants and for all $x \in \mathbb{R}$

$a=1 \quad b=-4 \quad c=13$

Example 6.4: Solve $y'' - 4y' + 13y = 0$ for $y(x)$ if $y(0) = 1$ and $y'(0) = 6$.

initial value condition

Solution:

Try $y(x) = e^{\lambda x}$

$\Rightarrow y'(x) = \lambda e^{\lambda x}$

$\Rightarrow y''(x) = \lambda^2 e^{\lambda x}$

$\Rightarrow (\lambda^2 - 4\lambda + 13)\underbrace{e^{\lambda x}}_{\neq 0} = 0$

16
-52

$4\cancel{5}2$
-16
36

$4\overline{)36}$ 9

$\Rightarrow \lambda^2 - 4\lambda + 13 = 0$

$\Rightarrow \lambda = \dfrac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)} = \dfrac{4 \pm \sqrt{16-52}}{2} = \dfrac{4 \pm \sqrt{-36}}{2} = 2 \pm \sqrt{-9}$

$= 2 \pm 3\sqrt{-1}$

$= 2 \pm 3i$

$\Rightarrow \lambda = 2 + 3i$ and $\lambda = 2 - 3i$

Case 3: $b^2 - 4ac < 0$

so $y_1(x) = e^{(2+3i)x}$ and $y_2(x) = e^{(2-3i)x}$ are linearly independent $\text{sol}^{\underline{n}}\text{s}$ of the ODE.

Complex general solution:

$$y(x) = C_1 e^{(2+3i)x} + C_2 e^{(2-3i)x}$$

with $C_1, C_2 \in \mathbb{C}$ constant and for all $x \in \mathbb{R}$

Real general $\text{sol}^{\underline{n}}$:

$$y(x) = A e^{2x} \cos(3x) + B e^{2x} \sin(3x)$$

with $A, B \in \mathbb{R}$ constant and for all $x \in \mathbb{R}$

- $y(0) = 1 \Rightarrow y(0) = A e^0 \cos(3(0)) + B e^0 \sin(3(0))$

$A = 0$

$y'(x) = 2Ae^{2x}\cos(3x) - 3Ae^{2x}\sin(3x) + 2Be^{2x}\sin(3x) + 3Be^{2x}\cos(3x)$

• $y'(0) = 6 \Rightarrow 2A + 3B = 6 \overset{A=1}{\Longrightarrow} B = \frac{4}{3}$

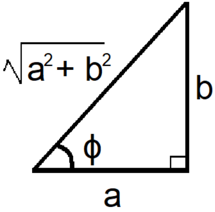
$\Rightarrow$ The solution that satisfies the initial conditions is :

$$y(x) = e^{2x}\cos(3x) + \frac{4}{3}e^{2x}\sin(3x) \text{ with } x \in \mathbb{R}$$

This solution has the form $y(x) = e^{2x}(a\cos\theta + b\sin\theta)$.

In general, for $a, b > 0$,

$$a\cos\theta + b\sin\theta = \sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}}\cos\theta + \frac{b}{\sqrt{a^2+b^2}}\sin\theta\right)$$



$\phi = \arctan\left(\frac{b}{a}\right)$

when $\frac{b}{a} > 0$: $\phi \in (0, \frac{\pi}{2})$

$\frac{b}{a} < 0$: $\phi \in (-\frac{\pi}{2}, 0)$

$$= \sqrt{a^2+b^2}\left(\cos\phi\cos\theta + \sin\phi\sin\theta\right)$$

$$= \sqrt{a^2+b^2}\cos(\theta - \phi).$$

Hence we can rewrite the solution from Example 6.4 as

From 6.4 : $y(x) = e^{2x}\cos(3x) + \frac{4}{3}e^{2x}\sin(3x)$ with $x \in \mathbb{R}$

$$= e^{2x}\left(\cos(3x) + \frac{4}{3}\sin(3x)\right)$$

$$= e^{2x}\sqrt{1^2+\left(\frac{4}{3}\right)^2}\cos\left(3x - \arctan\left(\frac{4}{3}\right)\right)$$

$$= \frac{5}{3}e^{2x}\cos(3x - \phi) \text{ with } \phi = \arctan\left(\frac{4}{3}\right)$$

oscillations — easier to see phase shift

This form is sometimes preferable for graphing or further manipulation.

Inhomogeneous 2^{nd} Order Linear ODEs

Theorem:

The general solution of

$$y'' + \mathcal{P}(x)y' + \mathcal{Q}(x)y = \mathcal{R}(x)$$

is the function y given by

$$y(x) = \underbrace{y_{\mathcal{H}}(x)}_{\text{GS(H)}} + \underbrace{y_{\mathcal{P}}(x)}_{\text{PS(IH)}}$$

where

- $y_{\mathcal{H}}(x) = c_1 y_1(x) + c_2 y_2(x)$ is the general solution of the homogeneous ODE (called the homogeneous solution, GS(H)),
- $y_{\mathcal{P}}(x)$ is a solution of the inhomogeneous ODE (called a particular solution, PS(IH)), will always exist

not unique

no constants

Inhomogeneous 2^{nd} Order Linear ODEs with Constant Coefficients

General form:

$$ay'' + by' + cy = \mathcal{R}(x)$$

where a, b, c are constants.

Example 6.5: Solve $y'' + 2y' - 8y = \mathcal{R}(x)$ where

(a) $\mathcal{R}(x) = 1 - 8x^2$ polynomial

(b) $\mathcal{R}(x) = e^{3x}$ exponential

(c) $\mathcal{R}(x) = 85 \cos x$ trigonometric

(d) $\mathcal{R}(x) = 3 - 24x^2 + 7e^{3x}$. polynomial + exponential

Solution:

homogeneous $\quad a=1,\ b=2,\ c=-8$

Step 1: Find the general solution of $y'' + 2y' - 8y = 0$.

Try $y_H(x) = e^{\lambda x}$

$\Rightarrow y_H'(x) = \lambda e^{\lambda x}, \quad y_H''(x) = \lambda^2 e^{\lambda x}$

$\Rightarrow (\lambda^2 + 2\lambda - 8)\underbrace{e^{\lambda x}}_{\neq 0} = 0$

$\Rightarrow \lambda^2 + 2\lambda - 8 = 0$ (Characteristic Equation)

$\Rightarrow \lambda = -1 \pm \sqrt{(-1)^2 - (-8)} = -1 \pm 3$

$\Rightarrow \lambda_1 = -4$ and $\lambda_2 = 2$

case 1: $b^2 - 4ac > 0$

gS(H): $y_H(x) = Ae^{-4x} + Be^{2x}$, with $A, B \in \mathbb{R}$ constant and for all $x \in \mathbb{R}$

Step 2: Find a particular solution of $y'' + 2y' - 8y = \mathcal{R}(x)$.

(a) $\underbrace{\mathcal{R}(x) = 1 - 8x^2}_{\text{polynomial}}$: $\quad y'' + 2y' - 8y = \underbrace{1 - 8x^2}_{\text{polynomial}}$ ①

also polynomial

Try $y_p(x) = \overbrace{ax^2 + bx + c}$, with $a, b, c \in \mathbb{R}$ constants

$\Rightarrow y_p'(x) = 2ax + b$ ② $\Rightarrow y_p''(x) = 2a$ ③

②, ③ → ①:

$\Rightarrow 2a + 2(2ax + b) - 8(ax^2 + bx + c) = 1 - 8x^2$

$(2a + 2b - 8c) + (4a - 8b)x - 8ax^2 = 1 - 8x^2 \quad \forall x \in \mathbb{R}$

Equate the coefficients:

x^2: $-8a = -8 \Rightarrow a = 1$

x: $4a - 8b = 0 \Rightarrow 4(1) - 8b = 0 \Rightarrow b = \frac{1}{2}$

constant: $2a + 2b - 8c = 1 \Rightarrow 2(1) + 2(\frac{1}{2}) - 8c = 1 \Rightarrow c = \frac{1}{4}$

$2 + 1 - 8c = 1$

$3 - 8c = 1$

$8c = 2$

$c = \frac{1}{4}$

$\Rightarrow y_p(x) = x^2 + \frac{1}{2}x + \frac{1}{4}$ is a PS (IH)

$gs(IH) = y(x) = \underbrace{Ae^{-4x} + Be^{2x}}_{y_H(x)} + \underbrace{x^2 + \frac{1}{2}x + \frac{1}{4}}_{y_p(x)}$, with $A, B \in \mathbb{R}$ constants and $x \in \mathbb{R}$

(b) $\mathcal{R}(x) = e^{3x}$: $\quad y'' + 2y' - 8y = e^{3x}$

$\left(e^{3x}\right.$ is NOT part of $\left.GS(H)\right)$

Try $y_p(x) = ae^{3x}$ with $a \in \mathbb{R}$ a constant

$\Rightarrow y'_p(x) = 3ae^{3x}$, $y''(x) = 9ae^{3x}$

$\Rightarrow 9a + 2(3a) - 8 \cdot a) e^{3x} = e^{3x}$

$\Rightarrow 9a + 6a - 8a = 1$ -

$\Rightarrow a = \frac{1}{7}$

$\Rightarrow y_p(x) = \frac{1}{7} e^{3x}$ is a $PS(IH)$

$$GS(IH) = \underbrace{Ae^{-4x} + Be^{2x}}_{\substack{y_H(x) \\ (\text{FROM } (a))}} + \underbrace{\frac{1}{7} e^{3x}}_{y_p(x)}$$

, with $A, B \in \mathbb{R}$ constants and for all $x \in \mathbb{R}$

(c) $\mathcal{R}(x) = 85\cos x$: $\quad y'' + 2y' - 8y = 85\cos x \; + 0 \cdot \sin(x)$

Try $y_p(x) = a\cos(x) + b\sin(x)$, with $a, b \in \mathbb{R}$ constants

$\Rightarrow y_p'(x) = -a\sin(x) + b\cos(x)$, $y_p''(x) = -a\cos(x) - b\sin(x)$

$\Rightarrow -a\cos(x) - b\sin(x) + 2(-a\sin(x) + b\cos(x)) - 8(a\cos(x) + b\sin(x)) = 85\cos x$

$\Rightarrow (-a + 2b - 8a)\cos(x) + (-b - 2a - 8b)\sin(x) = 85\cos(x) + 0 \cdot \sin(x)$, for all $x \in \mathbb{R}$

equate coefficients:

$\cos(x)$: $2b - 9a = 85$ ①

$\sin(x)$: $-2a - 9b = 0 \Rightarrow a = -\frac{9}{2}b$ ②

② → ①:

$2b - 9\left(-\frac{9}{2}b\right) = 85$

$\frac{4b}{2} + \frac{81b}{2} = 85$

$\frac{85b}{2} = 85$

$\Rightarrow b = 2$

$b = 2 \rightarrow$ ②:

$a = -\frac{9}{2}(2)$

$\Rightarrow a = -9$

$$\begin{array}{r} 81 \\ -4 \\ \hline 77 \end{array}$$

$\Rightarrow y_p(x) = -9\cos(x) + 2\sin(x)$ is a PS (IH)

$\Rightarrow$ GS(IH) $= y(x) = \underbrace{Ae^{-4x} + Be^{2x}}_{\text{from }(\alpha)} - 9\cos(x) + 2\sin(x)$, with $A, B \in \mathbb{R}$ and $x \in \mathbb{R}$

Superposition of Particular Solutions

Theorem:

A particular solution of

$$ay'' + by' + cy = \alpha\,\mathcal{R}_1(x) + \beta\,\mathcal{R}_2(x)$$

is

$$y_{\mathcal{P}}(x) = \alpha y_1(x) + \beta y_2(x)$$

where

- $y_1(x)$ is a particular solution of $ay'' + by' + cy = \mathcal{R}_1(x)$,
- $y_2(x)$ is a particular solution of $ay'' + by' + cy = \mathcal{R}_2(x)$,
- a, b, c, α, β are constants.

Example 6.5 (d): $\mathcal{R}(x) = 3 - 24x^2 + 7e^{3x}$.

Solution:

Split into $\mathcal{R}(x) = \underbrace{3(1-8x^2)}_{\alpha R_1(x)} + \underbrace{7(e^{3x})}_{\beta R_2(x)}$

From (a) and (b), the particular soln is:

$$y_p(x) = \underbrace{3\left(x^2+\frac{1}{2}x+\frac{1}{4}\right)}_{\alpha y_1(x)} + \underbrace{7\left(\frac{1}{7}e^{3x}\right)}_{\beta y_2(x)}$$

$$\text{PS(IH)}: y_p(x) = 3x^2 + \frac{3}{2}x + \frac{3}{4} + e^{3x}$$

$$\Rightarrow \text{GS (IH)}: y(x) = Ae^{-4x} + Be^{2x} + 3x^2 + \frac{3}{2}x + \frac{3}{4} + e^{3x}$$

with $A, B \in \mathbb{R}$ and for all $x \in \mathbb{R}$

Example 6.6: Solve $y'' - y = e^x$.

Solution:

$GS(H): y_{\mathcal{H}}(x) = Ae^x + Be^{-x}$

multiply by ax

e^x is a part of $GS(H)$

Try $y_p(x) = axe^x$

$\Rightarrow y_p'(x) = ae^x + axe^x = a(x+1)e^x$

$\Rightarrow y_p''(x) = ae^x + a(x+1)e^x = a(x+2)e^x$

$\Rightarrow a(x+2)e^x - axe^x = e^x$

$2ae^x = e^x \Rightarrow a = \frac{1}{2}$

$\Rightarrow y_p(x) = \frac{1}{2}xe^x$ $PS(IH)$

$\Rightarrow GS(IH): y(x) = Ae^x + Be^{-x} + \frac{1}{2}xe^x$ with $A, B \in \mathbb{R}$ constants and $x \in \mathbb{R}$

Example 6.7: Solve $y'' + 2y' + y = e^{-x}$.

Solution:

$GS(H): \; y_{\mathcal{H}}(x) = (A + Bx)\, e^{-x}$

Since xe^{-x} and e^{-x} are part of $GS(H)$, try

$$y_p(x) = ax^2e^{-x}$$

$$\Rightarrow y_p'(x) = (2ax - ax^2)e^{-x} \Rightarrow y_p''(x) = (2a - 4ax + ax^2)e^{-x}$$

Substitute y_p, y_p', y_p'' into ODE

$$(2a \cancel{-4ax} \cancel{+ax^2})e^{-x} + \cancel{2(2ax - ax^2)e^{-x}} + \cancel{ax^2e^{-x}} = e^{-x}$$

$$2ae^{-x} = e^{-x} \quad \text{for all } x \in \mathbb{R}$$

$$\Rightarrow a = \tfrac{1}{2}$$

$\Rightarrow y_p(x) = \frac{1}{2}x^2e^{-x}$ PS(IH)

$\Rightarrow GS(IH) = (A + Bx)e^{-x} + \frac{1}{2}x^2e^{-x}$, with $A, B \in \mathbb{R}$ and for all $x \in \mathbb{R}$

Example 6.8: Solve $y'' + 49y = 28\sin(7t)$.

Solution:

$GS(H): y_{\mathcal{H}}(t) = A\cos(7t) + B\sin(7t)$

Since $\sin(7t)$ is part of $GS(H)$, try

$y_p(t) = at\cos(7t) + bt\sin(7t)$ - with $a, b \in \mathbb{R}$ constants

$\Rightarrow y'_p(t) = (a + 7bt)\cos(7t) + (-7at + b)\sin(7t)$

$\Rightarrow y''_p(t) = (14b - 49at)\cos(7t) + (-14a - 49bt)\sin(7t)$

Substitute into ODE:

$(14b - 49at)\cos(7t) + (-14a - 49bt)\sin(7t) + 49(at\cos(7t) + bt\sin(7t))$
$= 28\sin(7t)$

$\Rightarrow 14b\cos(7t) - 14a\sin(7t) = 28\sin(7t)$, for all $t \in \mathbb{R}$

Equate coefficients:

$\sin(7t): -14a = 28 \Rightarrow a = -2$

$\cos(7t): 14b = 0 \Rightarrow b = 0$

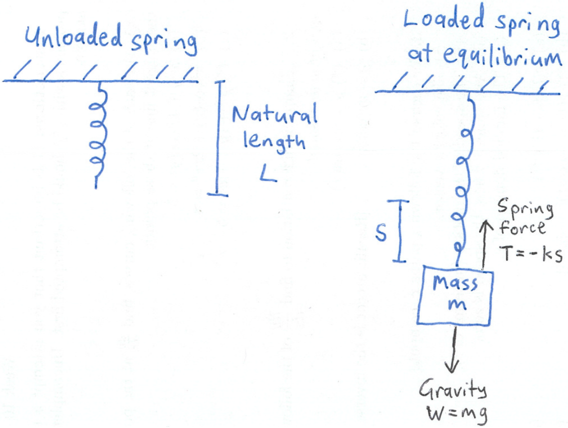
$\Rightarrow y_p(t) = -2t\cos(7t)$ PS(IH)

$\Rightarrow$ GS(IH): $y(t) = A\cos(7t) + B\sin(7t) - 2t\cos(7t)$ with $A, B \in \mathbb{R}$ constants and for all $t \in \mathbb{R}$

Springs - Free Vibrations

Consider an object (of mass m kg) attached to a spring hanging from a fixed support.

Suppose that the natural length of the spring when unloaded is L m, and that when the object is attached, it causes the spring to stretch by s m downwards at equilibrium.



The forces are:

- gravitational force, given by the gravitational law:

$$W = mg \qquad (g = 9.8m/s^2 \text{ on Earth})$$

- restoring force in spring, given by Hooke's Law:

$$T = \underbrace{-k}_{\text{spring constant}} \cdot \text{extension} \qquad (k > 0)$$

Note:

The negative sign in Hooke's law is because the direction of the force T is opposite to the direction of extension:

- if the spring is stretched downwards, then it pulls up
- if the spring is compressed upwards, then it pushes down.

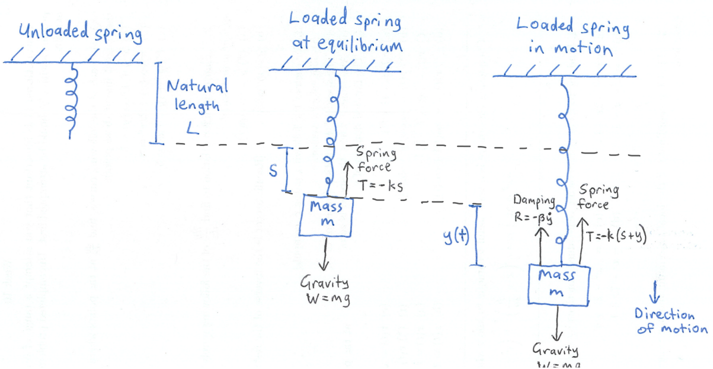
At equilibrium, the net force is zero, so

$$\text{net force} = W + T = mg - ks = 0$$

$$\Rightarrow mg = ks \iff s = \frac{mg}{k}$$

Suppose the object is set in motion.

Let $y(t)$ be the displacement of the object below the equilibrium position ($y = 0$) at time t seconds.



Note:

Since displacement y is measured below the equilibrium position, downwards is the positive direction.

$\frac{dy}{dt} = \dot{y}$

When the object is moving, there is an extra force acting:

- damping force is proportional to velocity

$$R = \underset{\substack{\uparrow \\ \text{damping constant}}}{-\beta}\dot{y} \qquad (\beta \geq 0)$$

stoke's Law

Note:

The negative sign is because the direction of the damping force is opposite the direction of motion.

Derive the equation of motion

$$\frac{d^2y}{dt^2} = \ddot{y}$$

Newton's Law states that

$$\text{net force} = \text{mass} \cdot \text{acceleration}$$

$$F = \quad m \quad \cdot \quad \ddot{y}$$

Apply Newton's Law to the object:

$$m\ddot{y} = W + T + R$$

$$\Rightarrow m\ddot{y} = mg - k(s+y) - \beta\dot{y}$$

(Equilibrium position) $s = \frac{mg}{k}$

$$\Rightarrow m\ddot{y} + \beta\dot{y} + ky = 0$$

Equation of motion for a spring

To solve, try $y(t) = e^{\lambda t}$

$$\Rightarrow \quad m\lambda^2 + \beta\lambda + k = 0$$

$$\Rightarrow \quad \lambda = \frac{-\beta \pm \sqrt{\beta^2 - 4mk}}{2m}$$

- If $\beta = 0$: $\quad \lambda = \pm ib$ simple harmonic motion with $b = \sqrt{\frac{k}{m}}$
- If $0 < \beta < 2\sqrt{mk}$: $\quad \lambda = a \pm ib$ underdamped, weak damping
  $a = -\frac{\beta}{2m}$; $b = \sqrt{\left|\frac{\beta^2}{4m} - k\right|}$
- If $\beta = 2\sqrt{mk}$: $\quad \lambda = a, a$ critical damping
  $a = -\frac{\beta}{2m}$
- If $\beta > 2\sqrt{mk}$: $\quad \lambda = a, b$ overdamped, strong damping
  $a = -\frac{\beta}{2m}$; $b = \sqrt{\frac{\beta^2}{4m} - k}$

Example 6.9: A $\frac{40}{49}$ kg mass stretches a spring hanging from a fixed support by 0.2m. The mass is released from the equilibrium position with a downward velocity of 3m/s. Find the position of the mass y below equilibrium at any time t, if the damping constant β is:

(a) 0 (b) $\frac{160}{49}$ (c) $\frac{80}{7}$ (d) $\frac{2000}{49}$

"released from equilibrium" "velocity of 3m/s"

Solution: $m = \frac{40}{49}$ [kg] ; $s = 0.2$ [m] ; $y(0) = 0$; $\dot{y}(0) = 3$

$$|T| = ks = W = mg \Rightarrow k = \frac{mg}{s} = \frac{\frac{40}{49} \times 9.8}{0.2}$$

$$= 40$$

The spring constant is $40 \frac{N}{m}$.

The equation of motion is

$$m\ddot{y} + \beta\dot{y} + ky = 0$$

$$\Rightarrow \frac{40}{49}\ddot{y} + \beta\dot{y} + 40y = 0$$

$$\Rightarrow \ddot{y} + \frac{49}{40}\dot{y} + 49y = 0$$

The initial conditions are:

$$y(0) = 0 \quad ; \quad \dot{y}(0) = 3$$

(a) $\beta = 0: \quad \ddot{y} + 49y = 0$

Try $y(t) = e^{\lambda t}$

$\Rightarrow y'(t) = \lambda e^{\lambda t} \Rightarrow y''(t) = \lambda^2 e^{\lambda t}$

$\Rightarrow (\lambda^2 + 49)\underbrace{e^{\lambda t}}_{\neq 0} = 0$

$\Rightarrow \lambda^2 + 49 = 0$ (characteristic Equation)

$\Rightarrow (\lambda + 7i)(\lambda - 7i) = 0$

$\Rightarrow \lambda = \pm 7i$

Case 3: $b^2 - 4ac < 0$

GS(H): $y(t) = A\cos(7t) + B\sin(7t)$; $A, B \in \mathbb{R}$ constants

$\Rightarrow \dot{y}(t) = -7A\sin(7t) + 7B\cos(7t)$

$y(0) = 0 \Rightarrow A = 0$

$\dot{y}(0) = 3 \Rightarrow 7B = 3 \Rightarrow B = \frac{3}{7}$

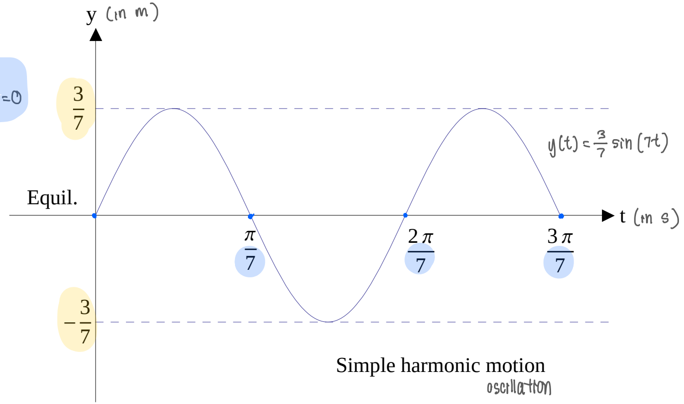
solution that satisfies initial conditions:

$\Rightarrow y(t) = \frac{3}{7}\sin(7t)$

DRAWING simple harmonic motion graph:

- [ ] ① axes
  - └ y (in m)
  - └ t (in s)
- [ ] ② amplitude
- [ ] ③ frequency
  - └ label when $y(t)=0$
- [ ] ④ $y(t) = \ldots$
- [ ] ⑤ At least 1-2 oscillations

$$y(t) = \underbrace{\frac{3}{7}}_{\text{amplitude}} \sin(\underbrace{7}_{\text{frequency}} t)$$



Simple harmonic motion
oscillation

(b) $\beta = \frac{160}{49}$: $\quad \ddot{y} + 4\dot{y} + 49y = 0$ $\quad ; \frac{49}{40}\beta = 4$

Try $y(t) = e^{\lambda t}$

$\Rightarrow \lambda^2 + 4\lambda + 49 = 0$

$\Rightarrow \lambda = -2 \pm \sqrt{4-49} = -2 \pm \sqrt{-45} = -2 \pm 3\sqrt{5}\, i$

Case 3: $b^2 - 4ac < 0$

GS(H): $y(t) = Ae^{-2t}\cos(3\sqrt{5}t) + Be^{-2t}\sin(3\sqrt{5}t)$; $A, B \in \mathbb{R}$ constant

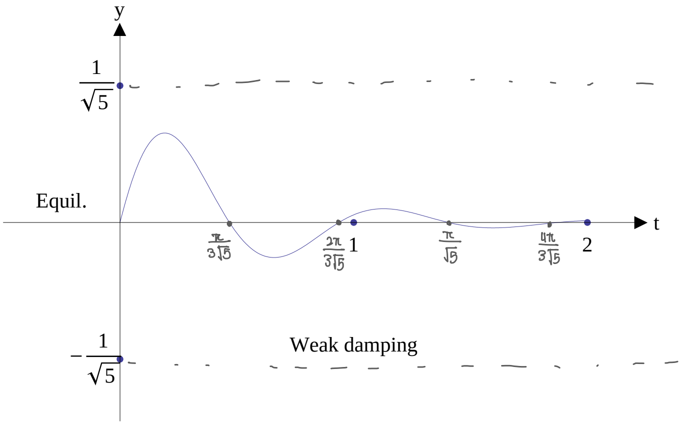
$\Rightarrow \dot{y}(t) = -2Ae^{-2t}\cos(3\sqrt{5}t) - 3\sqrt{5}Ae^{-2t}\sin(3\sqrt{5}t) - 2Be^{-2t}\sin(3\sqrt{5}t)$
$\qquad + 3\sqrt{5}Be^{-2t}\cos(3\sqrt{5}t)$

$y(0) = 0 \Rightarrow A = 0$

$\dot{y}(0) = 3 \Rightarrow 3\sqrt{5}B = 3 \Rightarrow B = \frac{1}{\sqrt{5}}$

solution that satisfies initial conditions:

$\Rightarrow y(t) = \frac{1}{\sqrt{5}} e^{-2t}\sin(3\sqrt{5}t)$


y
$\frac{1}{\sqrt{5}}$
Equil.
t
$\frac{\pi}{3\sqrt{5}}$
$\frac{2\pi}{3\sqrt{5}}$
1
$\frac{\pi}{\sqrt{5}}$
$\frac{4\pi}{3\sqrt{5}}$
2
$-\frac{1}{\sqrt{5}}$
Weak damping

(c) $\beta = \frac{80}{7}$: $\quad \ddot{y} + 14\dot{y} + 49y = 0$ $\quad \frac{80}{7} \cdot \frac{49}{40} = \frac{\beta}{m} = 14$

Try $y(t) = e^{\lambda t} \Rightarrow \lambda^2 + 14\lambda + 49 = 0$

$\Rightarrow \lambda = -7 \pm \sqrt{49-49} = -7$

GS(H) : $y(t) = (A + Bt)\, e^{-7t}$

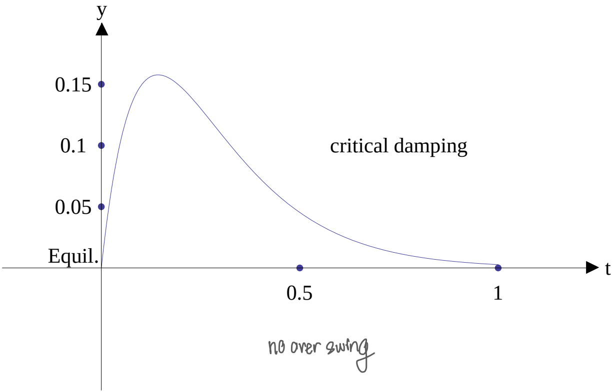
$\Rightarrow \dot{y}(t) = B e^{-7t} - 7(A + Bt) e^{-7t}$

$y(0) = 0 \Rightarrow A = 0$

$\dot{y}(0) = 3 \Rightarrow B = 3$

Solution that satisfies initial condition:

$\Rightarrow y(t) = 3te^{-7t}$



no over swing

(d) $\beta = \frac{2000}{49}$: $\quad \ddot{y} + 50\dot{y} + 49y = 0$ $\quad ; \frac{B}{m} = \frac{2000}{49} \cdot \frac{49}{40} = 50$

Try $y(t) = e^{\lambda t}$

$\Rightarrow \lambda^2 + 50\lambda + 49 = 0$

$\Rightarrow \lambda = -25 \pm \sqrt{625 - 49}$

$\Rightarrow \lambda = -1 \; ; \; \lambda = -49$

GS(H): $y(t) = Ae^{-t} + Be^{-49t}$; $A, B \in \mathbb{R}$ constant

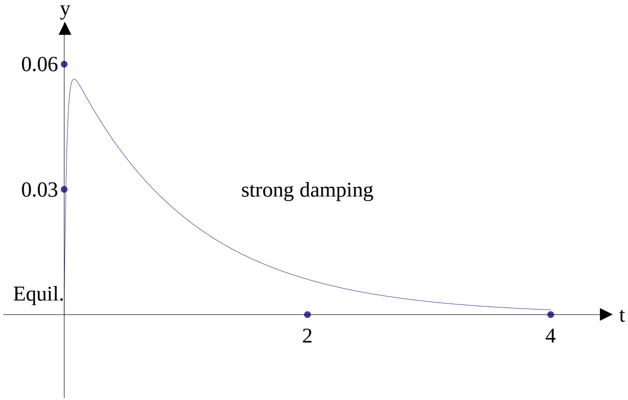
$\Rightarrow \dot{y}(t) = -Ae^{-t} - 49Be^{-49t}$

$$\left.\begin{array}{l} y(0) = 0 \Rightarrow A + B = 0 \\ y(0) = 3 \Rightarrow -A - 49B = 3 \end{array}\right\} \Rightarrow -48B = 3$$

$\Rightarrow B = -\frac{1}{16}$

Solution that satisfies conditions: $\Rightarrow A = \frac{1}{16}$

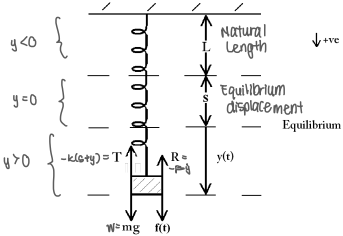
$\Rightarrow y(t) = \frac{1}{16}\left(e^{-t} - e^{-49t}\right)$



y
0.06
0.03
strong damping
Equil.
t
2
4

Springs - Forced Vibrations

If an external downwards force f is applied to the spring-mass system at time t, the forces acting on the mass are:



EQUATION OF MOTION

$$m\ddot{y} = W + T + R + f$$

$$= mg - k(s+y) - \beta\dot{y} + f$$

Equilibrium displacement: $s = \frac{mg}{k}$

$$\Rightarrow m\ddot{y} + \beta\dot{y} + k\dot{y} = f$$

Example 6.10: Apply an external downwards force $f(t) = \frac{160}{7}\sin(7t)$ in Example 6.9.

Solution: $m = \frac{40}{49}$; $s = 0.2m$; $y(0) = 0$; $\dot{y}(0) = 3$; $k = 40$ $\Big\}$ From Example 6.9

$$GS(H):\ \ddot{y} + \frac{49}{40}\beta\dot{y} + 49y = 0$$

The equation of motion:

$$\frac{40}{49}\ddot{y} + \beta\dot{y} + 40y = \frac{160}{7}\sin(7t)$$

$$\Rightarrow \ddot{y} + \frac{49}{40}\beta\dot{y} + 49y = 28\sin(7t)$$

Initial conditions:

$$y(0) = 0 \ ; \ \dot{y}(0) = 3$$

$(a) \quad \beta = \frac{80}{7}: \quad \ddot{y} + 14\dot{y} + 49y = 28\sin(7t)$ (critical damping)

$$GS(IH): \quad y(t) = \underbrace{(A + Bt)\,e^{-7t}}_{GS(H)} \underbrace{- \tfrac{2}{7}\cos(7t)}_{PS(IH)}$$

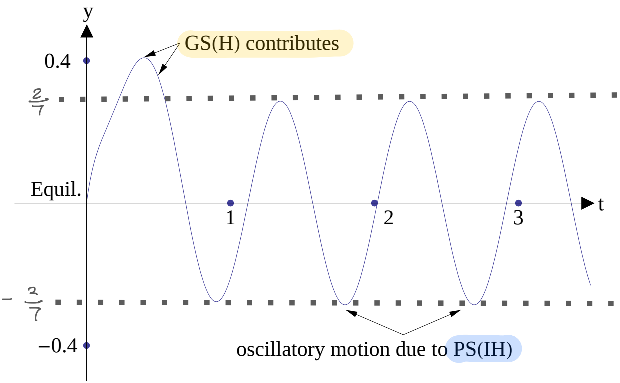
$$\Rightarrow \dot{y}(t) = Be^{-7t} - 7(A+Bt)e^{-7t} + 2\sin(7t)$$

$$y(0) = 0 \Rightarrow A - \frac{2}{7} = 0 \Rightarrow A = \frac{2}{7}$$

$$\dot{y}(0) = 3 \Rightarrow B - 7\left(\frac{2}{7}\right) = 3 \Rightarrow B = 5$$

Solution that satisfies initial conditions

$$\Rightarrow y(t) = \underbrace{\left(\frac{2}{7} + 5t\right)e^{-7t}}_{\text{transient term}} \underbrace{- \frac{2}{7}\cos(7t)}_{\text{steady state term}}$$


y
GS(H) contributes
0.4
2/7
Equil.
1
2
3
t
-3/7
-0.4
oscillatory motion due to PS(IH)

$(b) \quad \beta = 0: \quad \ddot{y} + 49y = 28\sin(7t)$ (No damping)

$$GS(IH): \quad y(t) = \underbrace{A\cos(7t) + B\sin(7t)}_{GS(H)} \underbrace{- 2t\cos(7t)}_{PS(IH)}$$

$$\Rightarrow \dot{y}(t) = -7A\sin(7t) + 7\cos(7t) - 2\cos(7t) + 14t\sin(7t)$$

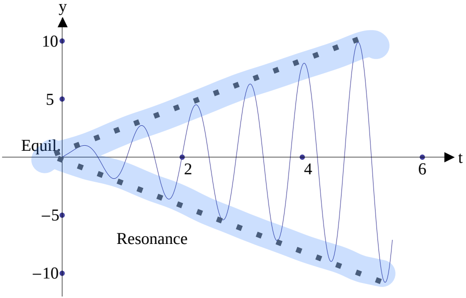
Initial conditions:

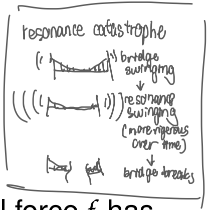
$y(0) = 0 \Rightarrow A = 0$

$\dot{y}(0) = 3 \Rightarrow 7B - 2 = 3 \Rightarrow B = \frac{5}{7}$

Solution that satisfies initial conditions:

$$\Rightarrow y(t) = \frac{5}{7}\sin(7t) - 2t\cos(7t)$$





Definition

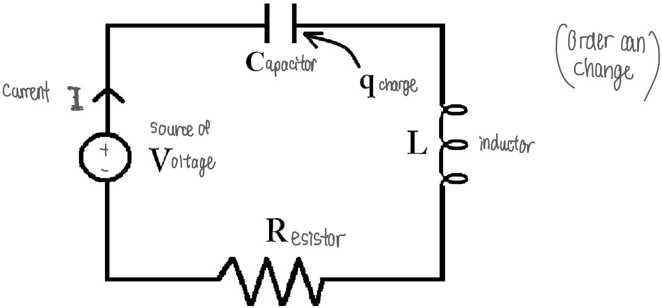
Resonance: Resonance occurs when the external force f has the same form as one of the terms in the $GS(H)$.

If $\beta = 0$, then the $PS(IH)$ will grow without bound as $t \to \infty$.

RLC series electric circuit

Resistor Inductor Capacitor

An RLC series electric circuit is an electric circuit with 4 components connected sequentially in a loop:

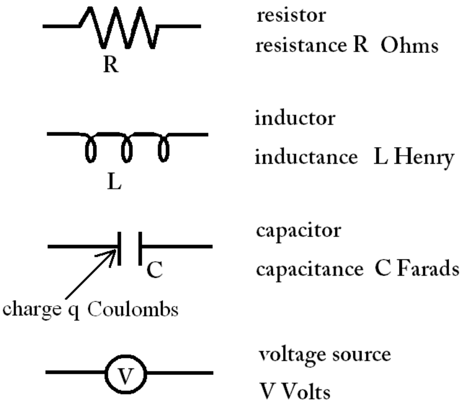


Circuits such as this are common in radio communications/

everything electrical

RLC series electric circuit

The circuit components are



electrical current I [A, Ampere]

RLC series electric circuit

Let $q(t)$ be the charge on the capacitor (measured in Coulomb) at time t seconds.

The charge satisfies the second-order ODE

Kirchhoff's Voltage Law
(Energy conservation)

$V = V_L + V_R + V_c$

$$\underbrace{L\frac{d^2q}{dt^2}}_{V_L} + \underbrace{R\frac{dq}{dt}}_{V_R} + \underbrace{\frac{q}{C}}_{V_c} = V$$

This equation has the same form as the equation of motion for a spring with external driving force, and can exhibit all the same solution types as the spring system.

RLC series electrical circuit:

In terms of electrical current (I [A, Ampere])

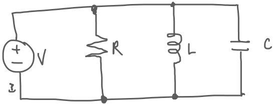
★ May appear in assignment / exam

$$I = \frac{dq}{dt}$$

$$L \frac{d^3q}{dt} + R \frac{d^2q}{dt^2} + \frac{1}{C}\frac{dq}{dt} = \frac{dv}{dt}$$

$$\Rightarrow L \frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C} I = \frac{dv}{dt}$$

RLC <u>parallel</u> electrical circuit:



Kirchoff's current Law

(Electrical charge conservation)

$$I = I_R + I_L + I_C$$

$$\Rightarrow \frac{dI}{dt} = \frac{1}{R}\frac{dv}{dt} + \frac{V}{L} + C\frac{d^2V}{dt^2}$$

$$\Rightarrow C\frac{d^2V}{dt^2} + \frac{1}{R}\frac{dV}{dt} + \frac{V}{L} = \frac{dI}{dt}$$

Section 7: Functions of Two Variables

Example

The temperature T at a point on the Earth's surface at a given time depends on the latitude x and the longitude y. We think of T being a function of the variables x, y and write $T = f(x, y)$.

In general

A function of two variables is a mapping f that assigns a unique real number $z = f(x, y)$ to each pair of real numbers (x, y) in some subset D of the xy plane $\mathbb{R}^2$. We also write

$$f : D \to \mathbb{R}$$

where D is called the domain of f.

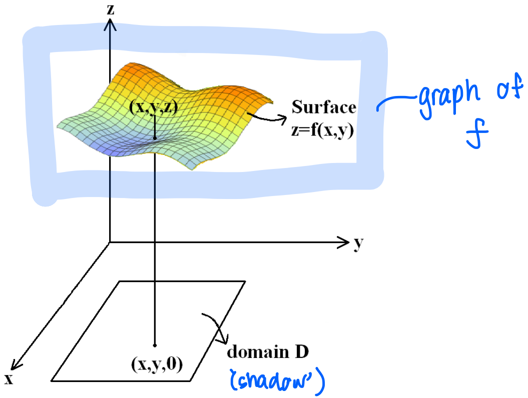
Example

If $f(x, y) = x^2 + y^3$ then $f(\overset{x}{2}, \overset{y}{1}) = \overset{x^2}{4} + \overset{y^3}{1} = 5$.

We can represent the function f by its graph in $\mathbb{R}^3$. The graph of f is:

$$\left\{(x,y,z) \in \mathbb{R}^3 : (x,y) \in D \text{ and } z = f(x,y)\right\}.$$

This is a surface lying directly above the domain D. The x and y axes lie in the horizontal plane and the z axis is vertical.



Equations of a Plane

The Cartesian equation of a plane has the form

posⁿ of the plane

$$ax + by + cz = d \qquad \text{(Linear)}$$

alignment of the plane

where a, b, c, d are real constants.

(1) $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is a normal vector to the plane.

$\underset{\sim}{n} = (a, b, c)$

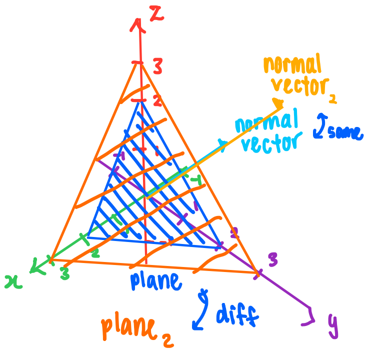
In fact, the plane passing through a point (2) (x_0, y_0, z_0) with a normal vector (a, b, c) consists of the points (x, y, z) such that (a, b, c) is perpendicular to $(x - x_0, y - y_0, z - z_0)$ (3) and thus has equation

$\Delta \underset{\sim}{r}$

$$\underset{\sim}{n} \cdot \Delta \underset{\sim}{r} \quad a(x - x_0) + b(y - y_0) + c(z - z_0) = 0,$$

that is,

$$ax + by + cz = \overbrace{ax_0 + by_0 + cz_0}^{d}.$$



$\underset{\sim}{n} = (1, 1, 1) = (1, 1, 1) = \underset{\sim}{n}$

$x + y + z = 2$

$x + y + z = 3$

Example 7.1: The plane $4x + 3y + z = 2$ can be written as $z = 2 - 4x - 3y$, so is the graph of the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = 2 - 4x - 3y$. Sketch the plane.

Solution: $\underset{\sim}{n} = (4, 3, 1)$

x-intercept $\iff y = z = 0 \Rightarrow 0 = 2 - 4x - 3 \cdot 0$

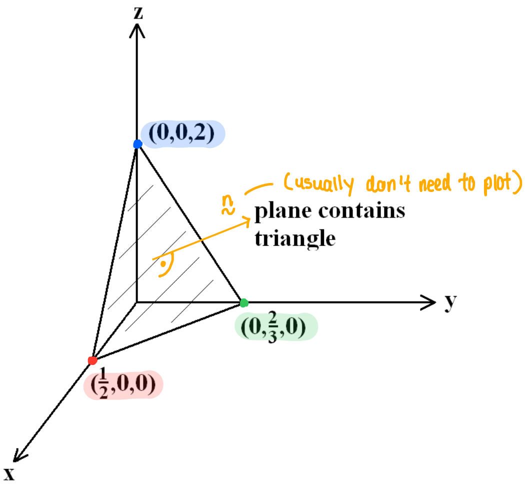
$\Rightarrow x = \frac{1}{2} \Rightarrow (\frac{1}{2}, 0, 0)$

y-intercept $\iff x = z = 0 \Rightarrow 0 = 2 - 4 \cdot 0 - 3y$

$\Rightarrow y = \frac{2}{3} \Rightarrow (0, \frac{2}{3}, 0)$

z-intercept $\iff x = y = 0 \Rightarrow z = 2 - 4 \cdot 0 - 3 \cdot 0$

$\Rightarrow z = 2 \quad (0, 0, 2)$


z
(0,0,2)
(usually don't need to plot)
n
plane contains
triangle
y
(0,2/3,0)
(1/2,0,0)
x

Level Curves

A curve on the surface $z = f(x,y)$ for which z is a constant is a contour.

$\{(x, y, z) : z = f(x, y) = \text{const.}\}$ (3 dimensions)

The same curve drawn in the xy plane is a level curve.

So a level curve of f has the form

$$\{(x,y) : f(x,y) = \underset{\uparrow \atop \text{const.}}{c}\}$$

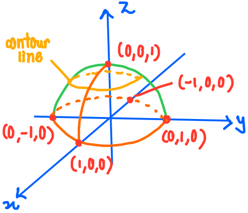
(2 dimensions)

where $c \in \mathbb{R}$ is a constant.

Sketching Functions of Two Variables

The key steps in drawing a graph of a function of two variables $z = f(x,y)$ are:

1. Draw the x, y, z axes.
   For right handed axes: the positive x axis is towards you, the positive y axis points to the right, and the positive z axis points upward.
2. Draw the $y - z$ cross section.
3. Draw some level curves and their contours.
4. Draw the $x - z$ cross section.
5. Label any x, y, z intercepts and key points.



Example 7.2: Find the level curves of $z = \sqrt{1 - x^2 - y^2}$. $= f(x, y)$
Hence sketch the surface and identify it.

Solution:

Let $z = c \in \mathbb{R} \Rightarrow c = \sqrt{1 - x^2 - y^2} \geqslant 0 \Rightarrow c \geqslant 0$

$\Rightarrow c^2 = 1 - x^2 - y^2$

$\Rightarrow x^2 + y^2 = 1 - c^2 \geqslant 0 \Rightarrow c \leqslant 1$

(an eq$^{\underline{n}}$ for a circle)

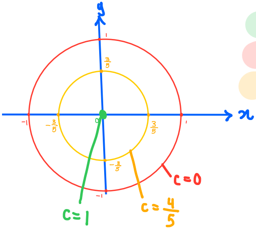
$\Rightarrow c \in [0, 1] \Leftarrow$

Choose $0 \leq c \leq 1$ being constant

$x^2+y^2 = 1-c^2 \Rightarrow$ All level curves are circles centred at $(0,0)$ with radius

$= r^2$ $\qquad 0 \leq r \leq 1$

$r = \sqrt{1-c^2}$

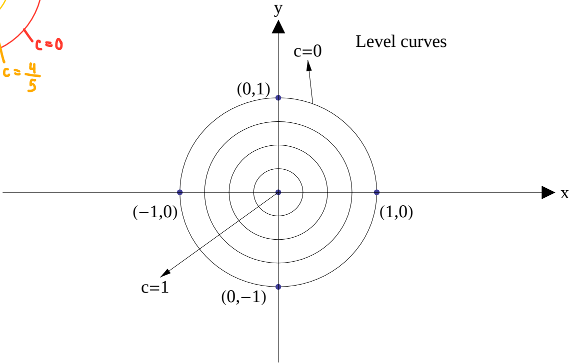


$c=1 \Rightarrow x^2+y^2=0$ $\sim r=0$

$c=0 \Rightarrow x^2+y^2=1$ $\sim r=1$

$c=\frac{4}{5} \Rightarrow x^2+y^2=\frac{9}{25}$ $\sim r=\frac{3}{5}$

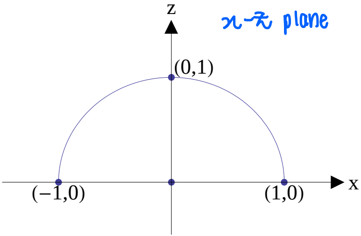
$\vdots$

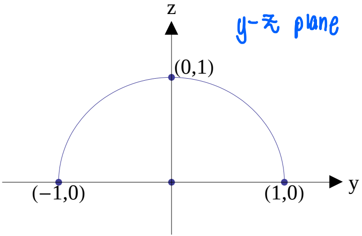


Consider cross sections (slices) to help sketch graph.

- If $y=0$ (x-z plane) $\Rightarrow z = \sqrt{1-x^2}$

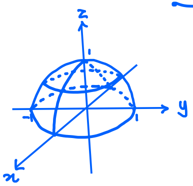
- If $x=0$ (y-z plane) $\Rightarrow z = \sqrt{1-y^2}$

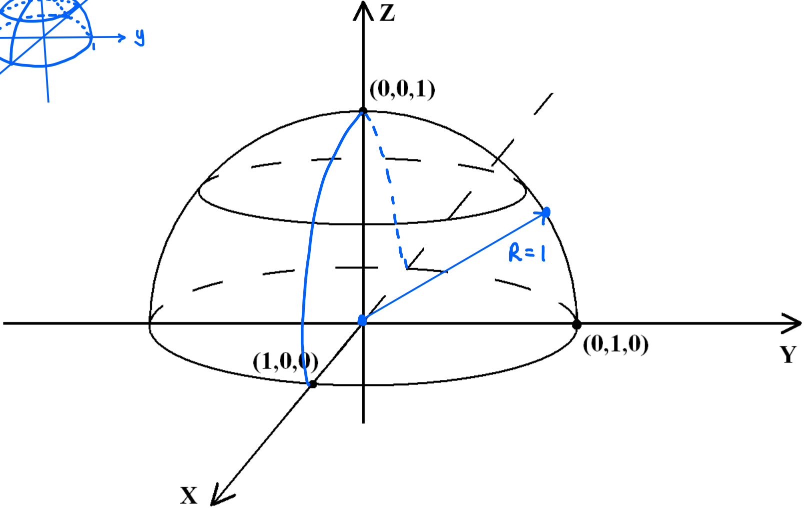




△ state – b/c the graph ain't clear enough

Surface is a hemisphere radius 1, centre at $(0,0,0)$ for $z \geq 0$.





Example 7.3: Sketch the graph of $z = 4x^2 + y^2$.

Solution: Let $z = c \in \mathbb{R} \Rightarrow 4x^2 + y^2 = c \geq 0$

$\Rightarrow$ level curves are ellipses

$c = 0 \Rightarrow 4x^2 + y^2 = 0$

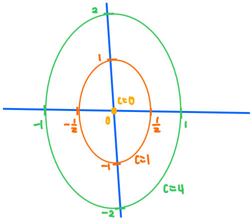
$c = 1 \Rightarrow 4x^2 + y^2 = 1$

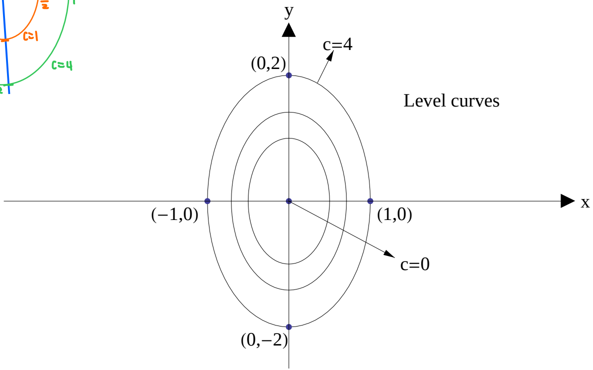
$c = 4 \Rightarrow 4x^2 + y^2 = 4$

$4x^2 = 1$

$x^2 = \frac{1}{4}$

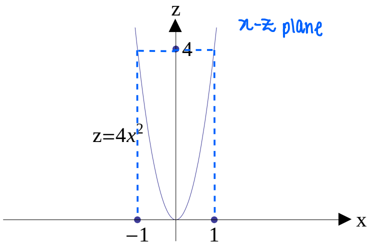
$x = \pm\frac{1}{2}$

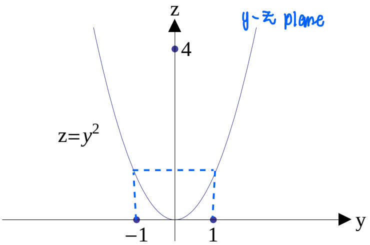




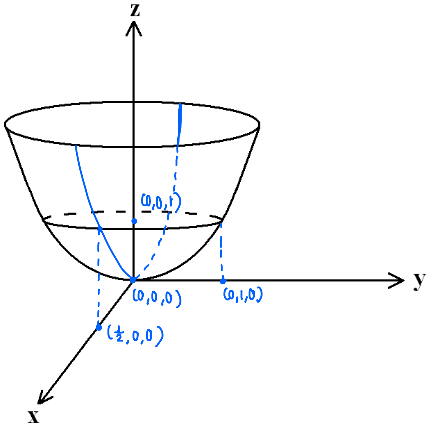
Consider cross sections (slices) to help sketch graph.

- If $y=0$ (x-z plane) $\Rightarrow$ $z = 4x^2$ ⟨parabola⟩
- If $x=0$ (y-z plane) $\Rightarrow$ $z = y^2$ ⟨parabola⟩





The surface is an elliptic paraboloid (parabolic bowl).



Example 7.4: Sketch the graph of $z = \sqrt{4x^2 + y^2}$.

Solution: Let $z = c \in \mathbb{R} \Rightarrow \sqrt{4x^2+y^2} = c \geq 0$

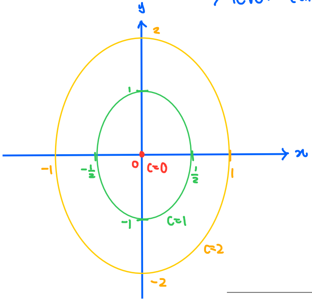
$\Rightarrow 4x^2+y^2 = c^2$

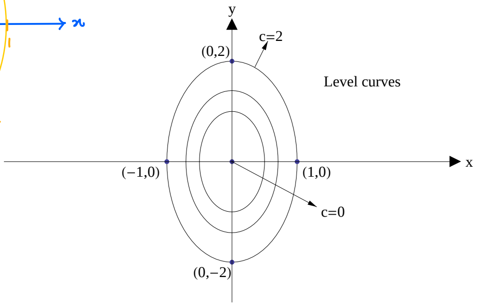
$\Rightarrow$ level curves are ellipses

$c=0 \Rightarrow 4x^2+y^2=0$

$c=1 \Rightarrow 4x^2+y^2=1$

$c=2 \Rightarrow 4x^2+y^2=2$

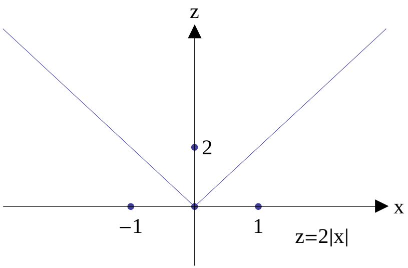


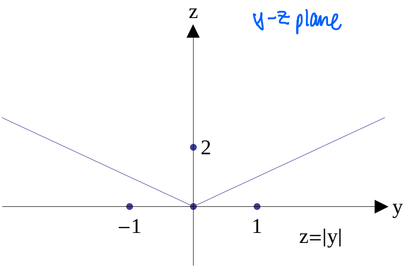


Cross sections

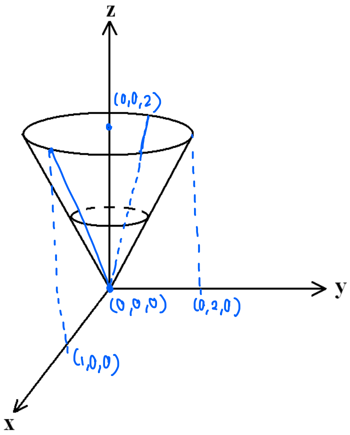
- If $y=0$ (x-z plane) $z = \sqrt{4x^2} = 2|x|$
- If $x=0$ (y-z plane) $z = \sqrt{y^2} = |y|$

x-z plane





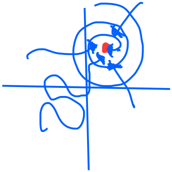
The surface is an elliptic cone.



Limits

Let $f : D \to \mathbb{R}$ be a real-valued function, where $D \subseteq \mathbb{R}^2$.

We say f has the limit L as (x, y) approaches (x_0, y_0)



$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$

if when (x, y) approaches (x_0, y_0) along ANY path in the domain, $f(x, y)$ gets arbitrarily close to L.

Note:

$f(x,y) = x+y$ for $(x,y) \neq (0,0)$

1. L must be finite. & unique
2. The limit can exist if f is undefined at (x_0, y_0).
3. The usual limit laws apply.

Continuity

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a real-valued function.

f is continuous at $(x, y) = (x_0, y_0)$ if

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0) = L$$

Note:

The continuity theorems for functions of one variable can be generalised to functions of two variables.

Example 7.5: Let $f(x, y) = x^2 + y^2$. For which values of x and y is f continuous?

$\therefore$ continuous $\quad \underbrace{\text{continuous} \oplus \text{continuous}}$

Solution:

f is a sum of a polynomial in x and one in y and polynomials are continuous on $\mathbb{R}$, it implies f is continuous for all $(x, y) \in \mathbb{R}$

Example 7.6: Evaluate $\lim_{(x,y)\to(2,1)} \log(1 + 2x^2 + 3y^2)$.

always write justification

more is better than less

Solution:

The logarithm, $\log(z)$ is continuous for all $z > 0$ and

$1 + 2x^2 + 3y^2 > 0$ at $(x, y) = (2, 1)$

$$\Rightarrow \lim_{(x,y)\to(2,1)} \log(1+2x^2+3y^2) = \log\left[\lim_{(x,y)\to(2,1)}(1+2x^2+3y^2)\right]$$

(sum of three continuous polynomials)

$$= \log\left[1 + 2\cdot 2^2 + 3\cdot 1\right]$$

$$= \log\left[1+8+3\right]$$

$$= \log(12)$$

First Order Partial Derivatives

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a real-valued function. The first order partial derivatives of f with respect to the variables x and y are defined by the limits:

⚠ must be curly ∂
/
can lose notation marks

$$f_x = \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

derivative in x direction

$$f_y = \frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

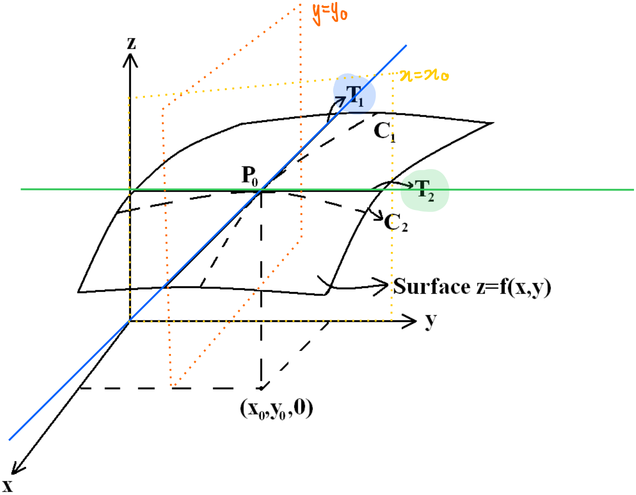
derivative in y direction

why curly ?
→ indicates that x/y are held constant

Note:

- $\dfrac{\partial f}{\partial x}$ measures the rate of change of f with respect to x when y is held constant.
- $\dfrac{\partial f}{\partial y}$ measures the rate of change of f with respect to y when x is held constant.

Geometric Interpretation of $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$



Let C_1 be the curve where the vertical plane $y = y_0$ intersects the surface. Then $\left.\dfrac{\partial f}{\partial x}\right|_{(x_0,y_0)}$ gives the slope of the tangent to C_1 at (x_0, y_0, z_0).

Let C_2 be the curve where the vertical plane $x = x_0$ intersects the surface. The $\left.\dfrac{\partial f}{\partial y}\right|_{(x_0,y_0)}$ gives the slope of the tangent to C_2 at (x_0, y_0, z_0).

- T_1 and T_2 are the tangent lines to C_1 and C_2.

Example 7.7: Let $f(x,y) = xy^2$. Find $\dfrac{\partial f}{\partial y}$ from first principles.

Solution:

$$f_y = \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

$$= \lim_{h \to 0} \frac{x(y+h)^2 - xy^2}{h}$$

$$= \lim_{h \to 0} \frac{x(\cancel{y^2} + 2hy + h^2) - \cancel{xy^2}}{h}$$

$$= \lim_{h \to 0} \frac{2hxy + h^2x}{h}$$

$$= \lim_{h \to 0} (2xy + hx)$$

$$= 2xy$$

Example 7.8: Let $f(x, y) = 3x^3y^2 + 3xy^4$. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Solution:

- $\frac{\partial f}{\partial x} = 3y^2 \underbrace{\frac{\partial x^3}{\partial x}}_{3x^2} + 3y^4 \underbrace{\frac{\partial x}{\partial x}}_{1}$

  $= 9x^2y^2 + 3y^4$

- $\frac{\partial f}{\partial y} = 3x^3 \underbrace{\frac{\partial y^2}{\partial y}}_{2y} + 3x \underbrace{\frac{\partial y^4}{\partial y}}_{4y^3}$

  $= 6x^3y + 12xy^3$

Example 7.9: Let $f(x,y) = y \log x + x \tanh(3y)$. Find f_x, f_y at $(1,0)$.

Solution:

- $f_x(x,y) = y \dfrac{\partial \log x}{x} + \tanh(3y)\dfrac{\partial x}{\partial x}$

$$= \frac{y}{x} + \tanh(3y)$$

- $f_y(x,y) = \log x \underbrace{\dfrac{\partial y}{\partial y}}_{1} + x \underbrace{\dfrac{\partial \tanh(3y)}{\partial y}}_{3\operatorname{sech}^3(3y)}$

$$= \log(x) + 3x \operatorname{sech}^2(3y)$$

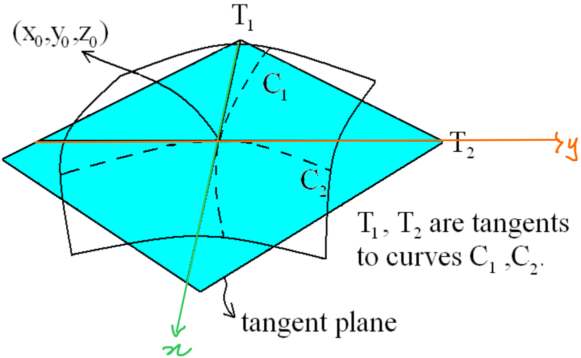
Evaluate the partial derivatives at $(1,0)$:

- $f_x(1,0) = \frac{0}{1} + \tan(3 \cdot 0) = 0$
- $f_y(1,0) = \log(1) + 3(1)\operatorname{sech}^2(3(0)) = 3$

Tangent Planes and Differentiability

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a real-valued function. We say that f is differentiable at (x_0, y_0) if the tangent lines to all curves on the surface $z = f(x, y)$ passing through (x_0, y_0, z_0) form a plane, called the tangent plane.

This holds if f_x and f_y exist and are continuous near (x_0, y_0). necessary condition



The tangent line T_1 has equation ($y = y_0$ fixed):

$$z - z_0 = \frac{\partial f}{\partial x}\bigg|_{(x_0,y_0)}(x - x_0)$$

The tangent line T_2 has equation ($x = x_0$ fixed):

$$z - z_0 = \frac{\partial f}{\partial y}\bigg|_{(x_0,y_0)}(y - y_0)$$

Since a plane passing through (x_0, y_0, z_0) has the form

$$z - z_0 = \alpha(x - x_0) + \beta(y - y_0)$$

$\alpha(x - x_0) + \beta(y - y_0) - (z - z_0) = 0$; $\underset{\sim}{n} = (\alpha, \beta, -1)$

the tangent plane has equation

$$\boxed{z - z_0 = \frac{\partial f}{\partial x}\bigg|_{(x_0,y_0)}(x - x_0) + \frac{\partial f}{\partial y}\bigg|_{(x_0,y_0)}(y - y_0).}$$

Example 7.10: Find the equation of the tangent plane to the surface $z = f(x, y) = 2x^2 + y^2$ at $(1, 1, 3)$.

Solution: $\bullet\ (x_0, y_0, z_0) = (1, 1, 3)$

$$\bullet\ \frac{\partial f}{\partial x} = 2\frac{\partial x^2}{\partial x} + y^2 \frac{\partial 1}{\partial x} = 4x \qquad (y \text{ is fixed})$$

$$\Rightarrow \frac{\partial f}{\partial x}\bigg|_{(1,1)} = 4 \cdot 1 = 4$$

$$\bullet\ \frac{\partial f}{\partial y} = 2x^2 \frac{\partial 1}{\partial y} + \frac{\partial y^2}{\partial y} = 2y$$

$$\Rightarrow \frac{\partial f}{\partial y}\bigg|_{(1,1)} = 2 \cdot 1 = 2$$

THEREFORE

The eq$^{\underline{n}}$ of the tangent plane is $z - 3 = \frac{\partial f}{\partial x}\Big|_{(1,1)}(x-1) + \frac{\partial f}{\partial y}\Big|_{(1,1)}(y-1)$

$$\Rightarrow z - 3 = 4(x-1) + 2(y-1) \Rightarrow z = 4x + 2y - 3 \Rightarrow 4x + 2y - z = 3 \Rightarrow \underset{\sim}{n} = (4, 2, -1)$$

(Standard Eq$^{\underline{n}}$)

Linear Approximations

If f is differentiable at (x_0, y_0), we can approximate $z = f(x, y)$ by its tangent plane at (x_0, y_0, z_0).

This linear approximation of f near (x_0, y_0) is:

$$\boxed{f(x,y) \approx f(x_0,y_0) + \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}(x - x_0) + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)}(y - y_0)}$$

Let $\Delta x = x - x_0$, $\Delta y = y - y_0$, $\Delta f = z - z_0 = f(x,y) - f(x_0,y_0)$.

Then the approximate change in f near (x_0, y_0), for given small changes in x and y, is:

$$\boxed{\Delta f \approx \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}\Delta x + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)}\Delta y}$$

Example 7.11: Let $z = f(x, y) = x^2 + 3xy - y^2$. If x changes from 2 to 2.05 and y changes from 3 to 2.96, estimate the change in z.

Δf

Solution:

- $(x_0, y_0) = 2, 3$
- $(x_0 + \Delta x, y_0 + \Delta y) = 2.05, 2.96$
- $(\Delta x, \Delta y) = (2.05 - 2, 2.96 - 3) = (0.05, -0.04)$
- $f_x(x, y) = 2x + 3y$; $f_y(x, y) = 3x - 2y$

$\Rightarrow f_x(2, 3) = 2(2) + 3(3) = 13$

$\Rightarrow f_y(2, 3) = 3(2) - 2(3) = 0$

$$\boxed{\Delta f \approx \left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)} \Delta x + \left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0)} \Delta y}$$

$\Rightarrow \Delta f \approx f_x(2,3)\, \Delta x + f_y(2,3)\, \Delta y$

$= 13 \times 0.05 + 0 \times -0.04$

$= 0.65$

Note:

The actual change in f is

$$\begin{aligned} \Delta f &= f(2.05, 2.96) - f(2,3) \\ &= 13.6449 - 13 \\ &= 0.6449 \end{aligned}$$

Example 7.12: Find the linear approximation of $f(x,y) = xe^{xy}$ at $(1,0)$. Hence, approximate $f(1.1,-0.1)$.

Solution:

- $f(1,0) = 1 \cdot e^{1 \cdot 0} = 1$
- $f_x(x,y) = e^{x \cdot y} + xye^{xy} \Rightarrow f_x(1,0) = e^0 + 1 \cdot 0 \cdot e^0$
  $\Rightarrow f_x(1,0) = 1$
- $f_y(x,y) = x^2 e^{xy} \Rightarrow f_y(1,0) = 1^2 e^0$
  $f_y(1,0) = 1$

Linear Approximation of f near $(1,0)$ is

$$f(x,y) \approx f(1,0) + \left.\frac{\partial f}{\partial x}\right|_{(1,0)} (x-1) + \left.\frac{\partial f}{\partial y}\right|_{(1,0)} (y-0)$$

$$\Rightarrow f(x,y) \approx 1 + 1 \cdot (x-1) + 1\,(y-0)$$

$$\Rightarrow f(x,y) \approx x + y$$

$\Rightarrow f(1.1, -0.1) \approx 1.1 + (-0.1) = 1$

Note:

The actual value is

$$(1.1)e^{-0.11} \approx 0.98542$$

Second Order Partial Derivatives

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a real-valued function. The second order partial derivatives of f with respect to x and y are defined by:

$$\begin{tikzcd}
 & & f_{xx} \\
 & f_x \arrow[ur, "\frac{\partial}{\partial x}"] \arrow[dr, "\frac{\partial}{\partial y}"'] & \\
f \arrow[ur, "\frac{\partial}{\partial x}"] \arrow[dr, "\frac{\partial}{\partial y}"'] & & f_{xy} \\
 & f_y \arrow[r, "\frac{\partial}{\partial x}"] \arrow[dr, "\frac{\partial}{\partial y}"'] & f_{yx} \\
 & & f_{yy}
\end{tikzcd}$$

- $f_{xx} = \left(f_x\right)_x = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial^2 f}{\partial x^2}$
- $f_{yy} = \left(f_y\right)_y = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial^2 f}{\partial y^2}$
- $f_{xy} = \left(f_x\right)_y = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial^2 f}{\partial y \partial x}$
- $f_{yx} = \left(f_y\right)_x = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial^2 f}{\partial x \partial y}$

Theorem:

If the second order partial derivatives of f exist and are continuous then $f_{xy} = f_{yx}$.

continous in
all directions

Example 7.13: Find the second order partial derivatives of $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = x \sin(x + 2y)$.

Solution:

- $f_x(x, y) = \sin(x+2y) + x\cos(x+2y)$
- $f_y(x, y) = 2x\cos(x+2y)$
- $f_{xx}(x, y) = \frac{\partial}{\partial x}\left[\sin(x+2y) + x\cos(x+2y)\right]$

$= 2\cos(x+2y) - x\sin(x+2y)$

- $f_{yy}(x, y) = \frac{\partial}{\partial y}\left[2x\cos(x+2y)\right]$

$= 4x\sin(x+2y)$

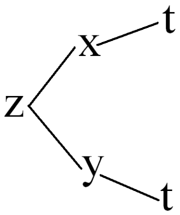
Note:

$f_{xy} = f_{yx}$ as expected since trigonometric functions and polynomials are continuous for all $(x, y) \in \mathbb{R}^2$.

Chain Rule

1. If $z = f(x, y)$ and $x = g(t)$, $y = h(t)$ are differentiable functions, then $z = f(g(t), h(t))$ is a function of t, and

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}}$$

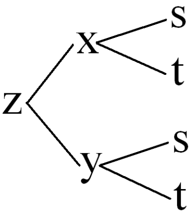


Example 7.14: If $z = x^2 - y^2$, $x = \sin t$, $y = \cos t$. Find $\dfrac{dz}{dt}$ at $t = \dfrac{\pi}{6}$.

Solution:

2. If $z = f(x, y)$ and $x = g(s, t)$, $y = h(s, t)$ are differentiable functions, then z is a function of s and t with

$$\boxed{\begin{aligned} \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \\[2ex] \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \end{aligned}}$$



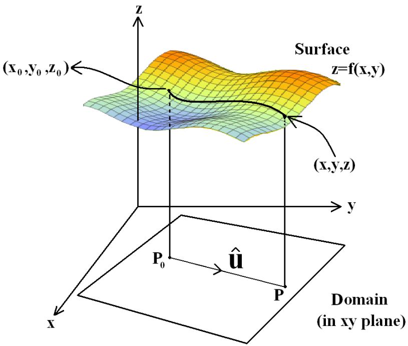
Example 7.15: If $z = e^x \sinh y$, $x = st^2$, $y = s^2 t$.
Find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$.

Solution:

Directional Derivatives

Let $\hat{\mathbf{u}} = (u_1, u_2)$ be a unit vector in the xy-plane (so $u_1^2 + u_2^2 = 1$). The rate of change of f at $P_0 = (x_0, y_0)$ in the direction $\hat{\mathbf{u}}$ is the directional derivative $D_{\hat{\mathbf{u}}} f \Big|_{P_0}$.

Geometrically this represents the slope of the surface $z = f(x, y)$ above the point P_0 in the direction $\hat{\mathbf{u}}$.



The straight line starting at $P_0 = (x_0, y_0)$ with velocity $\hat{\mathbf{u}} = (u_1, u_2)$ has parametric equations:

$$x = x_0 + tu_1, \quad y = y_0 + tu_2.$$

Hence,

$$\begin{aligned}
D_{\hat{\mathbf{u}}} f \Big|_{P_0} &= \text{rate of change of } f \text{ along the straight line at } t = 0 \\
&= \text{value of } \frac{d}{dt} f(x_0 + tu_1, y_0 + tu_2) \text{ at } t = 0 \\
&= f_x(x_0, y_0) x'(0) + f_y(x_0, y_0) y'(0) \qquad \text{by the chain rule} \\
&= f_x(x_0, y_0)\, u_1 + f_y(x_0, y_0)\, u_2.
\end{aligned}$$

We can also write this as a dot product

$$D_{\hat{\mathbf{u}}} f \Big|_{P_0} = \left(\frac{\partial f}{\partial x}\bigg|_{P_0}, \frac{\partial f}{\partial y}\bigg|_{P_0} \right) \cdot (u_1, u_2).$$

Gradient Vectors

If $f : \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function, we can define the gradient of f to be the vector

$$\boxed{\text{grad } f = \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)}$$

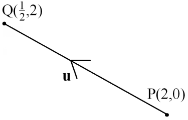
Then the directional derivative of f at the point P_0 in the direction $\hat{\mathbf{u}}$ is the dot product

$$\boxed{D_{\hat{\mathbf{u}}} f \Big|_{P_0} = \nabla f \Big|_{P_0} \cdot \hat{\mathbf{u}}}$$

Example 7.16: Find the directional derivative of $f(x, y) = xe^y$ at $(2, 0)$ in the direction from $(2, 0)$ towards $\left(\frac{1}{2}, 2\right)$.

Solution:

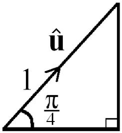
- direction $\hat{\mathbf{u}}$



Example 7.17: Find the directional derivative of $f(x, y) = \arcsin\left(\frac{x}{y}\right)$ at $(1, 2)$ in the direction $\frac{\pi}{4}$ anticlockwise from the positive x axis.

Solution:

- direction $\hat{\mathbf{u}}$



Properties of ∇f and $D_{\hat{\mathbf{u}}} f$

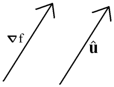
The directional derivative of f is

$$\begin{aligned} D_{\hat{\mathbf{u}}} f &= \nabla f \cdot \hat{\mathbf{u}} \\ &= |\nabla f|\,|\hat{\mathbf{u}}| \cos\theta \\ &= |\nabla f| \cos\theta \end{aligned}$$

where θ is the angle between ∇f and $\hat{\mathbf{u}}$, and $|\mathbf{v}|$ denotes the length of a vector $\mathbf{v}$.

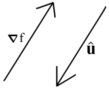
So for fixed ∇f:

- $D_{\hat{\mathbf{u}}} f$ is maximum when $\cos\theta = 1$ so $\theta = 0$



$\Rightarrow f$ increases most rapidly along ∇f.

- $D_{\hat{\mathbf{u}}} f$ is minimum when $\cos\theta = -1$ so $\theta = \pi$



$\Rightarrow f$ decreases most rapidly along $-\nabla f$.

- $D_{\hat{\mathbf{u}}}f = 0$ when $\cos\theta = 0$ so $\theta = \dfrac{\pi}{2}$ and $\nabla f \perp \hat{\mathbf{u}}$.

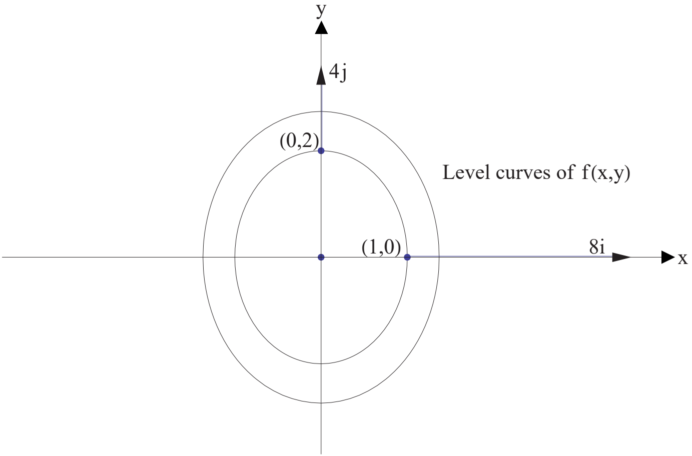
But $D_{\hat{\mathbf{u}}}f = 0$, whenever $\hat{\mathbf{u}}$ is tangent to a level curve of f (where f = constant).

$$\Rightarrow \nabla f \perp \text{ level curves of } f$$

Example 7.18: Let $f(x, y) = 4x^2 + y^2$.
(a) Find ∇f at $(1, 0)$ and $(0, 2)$.
(b) Show that ∇f is perpendicular to the level curves, by sketching ∇f at these points and the level curves of f.

Solution:

(b)



Example 7.19: In what direction does $f(x, y) = xe^y$

(a) increase (b) decrease

most rapidly at $(2, 0)$? Express direction as a unit vector.

Solution:

From Example 7.16

$$\nabla f(2, 0) = \mathbf{i} + 2\mathbf{j}$$

Stationary Points

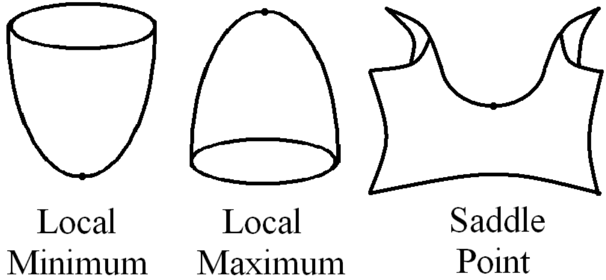
A stationary point of f is a point (x_0, y_0) at which

$$\nabla f = \mathbf{0}$$

So $\dfrac{\partial f}{\partial x} = 0$ and $\dfrac{\partial f}{\partial y} = 0$ simultaneously at (x_0, y_0).

Geometrically, this means that the tangent plane to the graph $z = f(x, y)$ at (x_0, y_0) is horizontal, i.e. parallel to the xy-plane.

Three important types of stationary points are



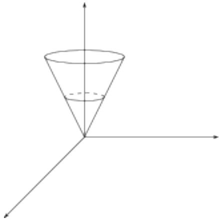
A function f has a

1. local maximum at (x_0, y_0) if $f(x, y) \leq f(x_0, y_0)$ for all (x, y) in some disk centred at (x_0, y_0),
2. local minimum at (x_0, y_0) if $f(x, y) \geq f(x_0, y_0)$ for all (x, y) in some disk centred at (x_0, y_0),
3. saddle point at (x_0, y_0) if (x_0, y_0) is a stationary point, and there are points near (x_0, y_0) with $f(x, y) > f(x_0, y_0)$ and other points near (x_0, y_0) with $f(x, y) < f(x_0, y_0)$.

Any local maximum or minimum of f will occur at a critical point (x_0, y_0) such that

1. $\nabla f(x_0, y_0) = \mathbf{0}$ <u>or</u>

2. $\dfrac{\partial f}{\partial x}$ and/or $\dfrac{\partial f}{\partial y}$ do not exist at (x_0, y_0).



$z = \sqrt{x^2 + y^2}$. Minimum at $(0, 0)$ BUT ∇f does not exist at $(0, 0)$.

Second Derivative Test

If $\nabla f(x_0, y_0) = \mathbf{0}$ and the second partial derivatives of f are continuous on an open disk centred at (x_0, y_0), consider the Hessian function

$$H(x, y) = f_{xx}f_{yy} - (f_{xy})^2$$

evaluated at (x_0, y_0).

Then (x_0, y_0) is a

1. local minimum if $H(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) > 0$.
2. local maximum if $H(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) < 0$.
3. saddle point if $H(x_0, y_0) < 0$.

Note: Test is inconclusive if $H(x_0, y_0) = 0$.

Example 7.20: Find and classify the stationary points of $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = x^3 + y^3 + 3x^2 - 3y^2 - 8$.

Solution:

Example 7.21: Find and classify the stationary points of $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = y \sin x$.

Solution:

Partial Integration

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function over a domain D in $\mathbb{R}^2$.

The partial indefinite integrals of f with respect to the first and second variables (say x and y) are denoted by:

$\int f(x,y)\,dx$ and $\int f(x,y)\,dy$.

- $\int f(x,y)\,dx$ is evaluated by holding y fixed and integrating with respect to x.

- $\int f(x,y)\,dy$ is evaluated by holding x fixed and integrating with respect to y.

Example 7.22: Evaluate $\int (3x^2y + 12y^2x^3)\, dx$.

Solution:

Note:

Example 7.23: Evaluate $\int_0^1 (3x^2y + 12y^2x^3)\, dy$.

Solution:

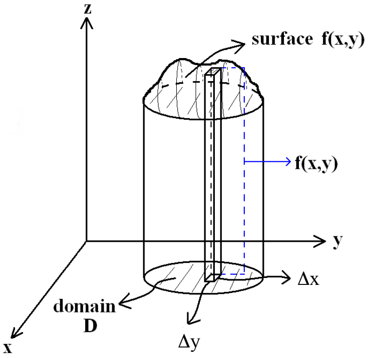
Double Integrals

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function over a domain D in $\mathbb{R}^2$.

We can evaluate the double integral:

$$\boxed{\iint_D f(x,y)\,dA = \iint_D f(x,y)\,dx\,dy}$$

$\iint_D f(x,y)\,dA$ is the volume under the surface $z = f(x,y)$ that lies above the domain D in the xy plane, if $f(x,y) \geq 0$ in D.



$$\text{Volume of thin rod} \quad = \quad \underbrace{(\text{Area base})}_{\substack{\| \\ \Delta x \Delta y}} \cdot \underbrace{(\text{height})}_{\substack{\| \\ f(x,y)}}$$

The double integral is defined as the limit of sums of the volumes of the rods:

$$\begin{aligned}\iint_D f(x,y)\,dA &= \iint_D f(x,y)\,dx\,dy \\ &= \lim_{\Delta x\to 0}\lim_{\Delta y\to 0}\sum_{i=1}^{n}\left[f(x,y)\Delta x\Delta y\right]_i\end{aligned}$$

Note:

If $f(x,y) = 1$ then

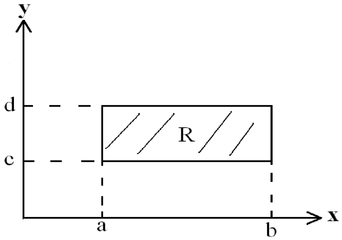
$$\iint_D dA = \iint_D dx\,dy$$

gives the area of the domain D.

Double Integrals Over Rectangular Domains

Definitions

1. $R = [a, b] \times [c, d]$ is a rectangular domain defined by
   $a \le x \le b, \quad c \le y \le d$.



2. $\int_c^d \int_a^b f(x, y)\,dx\,dy = \int_c^d \left[\int_a^b f(x, y)\,dx \right] dy$ means integrate with respect to x first and then integrate with respect to y.

Fubini's Theorem:

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function over the domain $R = [a, b] \times [c, d]$. Then

$$\begin{aligned} \iint_R f(x,y) dA &= \int_c^d \int_a^b f(x,y)\, dx\, dy \\ &= \int_a^b \int_c^d f(x,y)\, dy\, dx \end{aligned}$$

So order of integration is not important.

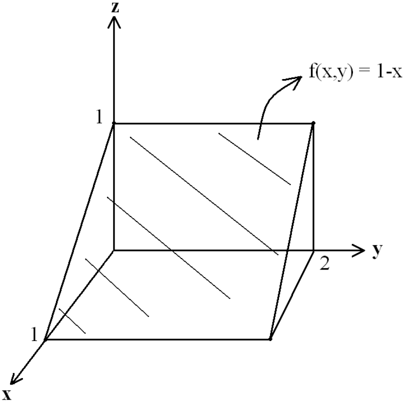
Example 7.24: Evaluate $\iint_R (x^2 + y^2)\,dx\,dy$ if $R = [-1, 1] \times [0, 1]$.

Solution:

Note:

As expected, the order of integration is not important since polynomials are continuous for all $(x, y) \in \mathbb{R}^2$.

Example 7.25: Using double integrals, find the volume of the wedge shown below.



Solution: