Section 7: Functions of Two Variables

Example

The temperature T at a point on the Earth's surface at a given time depends on the latitude x and the longitude y. We think of T being a function of the variables x, y and write $T = f(x, y)$.

In general

A function of two variables is a mapping f that assigns a unique real number $z = f(x, y)$ to each pair of real numbers (x, y) in some subset D of the xy plane $\mathbb{R}^2$. We also write

$$f : D \to \mathbb{R}$$

where D is called the domain of f.

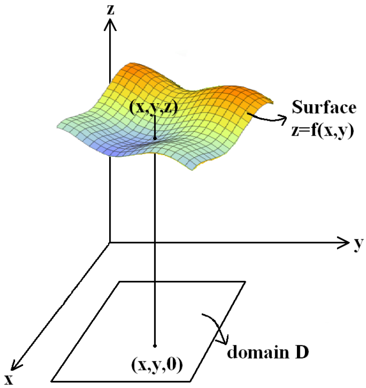
Example

If $f(x, y) = x^2 + y^3$ then $f(2, 1) = 4 + 1 = 5$.

We can represent the function f by its graph in $\mathbb{R}^3$. The graph of f is:

$$\left\{(x, y, z) \in \mathbb{R}^3 : (x, y) \in D \text{ and } z = f(x, y)\right\}.$$

This is a surface lying directly above the domain D. The x and y axes lie in the horizontal plane and the z axis is vertical.



Equations of a Plane

The Cartesian equation of a plane has the form

$$ax + by + cz = d$$

where a, b, c, d are real constants.

$\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is a normal vector to the plane.

In fact, the plane passing through a point (x_0, y_0, z_0) with a normal vector (a, b, c) consists of the points (x, y, z) such that (a, b, c) is perpendicular to $(x - x_0, y - y_0, z - z_0)$ and thus has equation

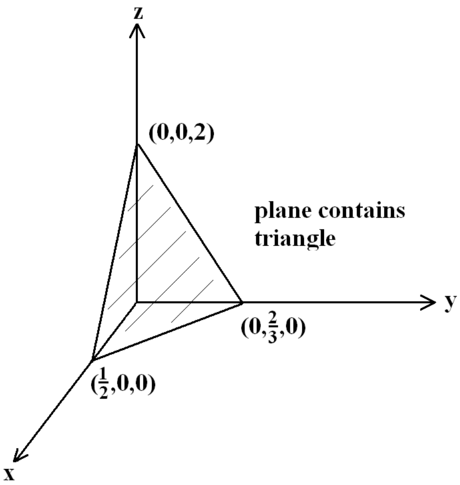
$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0,$$

that is,

$$ax + by + cz = ax_0 + by_0 + cz_0.$$

Example 7.1: The plane $4x + 3y + z = 2$ can be written as $z = 2 - 4x - 3y$, so is the graph of the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = 2 - 4x - 3y$. Sketch the plane.

Solution:


z
(0,0,2)
plane contains
triangle
y
(0,$\frac{2}{3}$,0)
($\frac{1}{2}$,0,0)
x

Level Curves

A curve on the surface $z = f(x, y)$ for which z is a constant is a contour.

The same curve drawn in the xy plane is a level curve.

So a level curve of f has the form

$$\{(x, y) : f(x, y) = c\}$$

where $c \in \mathbb{R}$ is a constant.

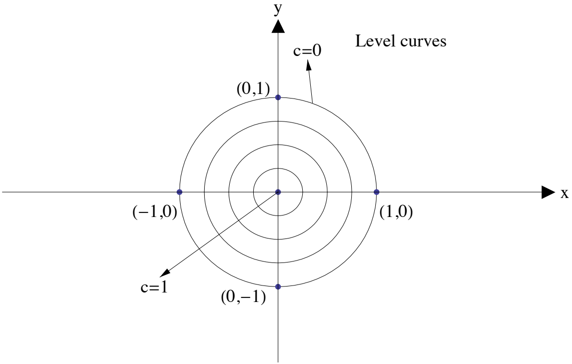
Sketching Functions of Two Variables

The key steps in drawing a graph of a function of two variables $z = f(x, y)$ are:

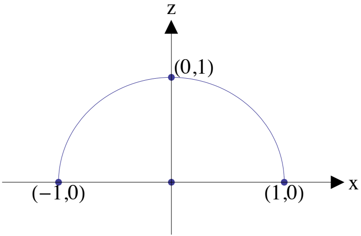
1. Draw the x, y, z axes.
   For right handed axes: the positive x axis is towards you, the positive y axis points to the right, and the positive z axis points upward.
2. Draw the $y - z$ cross section.
3. Draw some level curves and their contours.
4. Draw the $x - z$ cross section.
5. Label any x, y, z intercepts and key points.

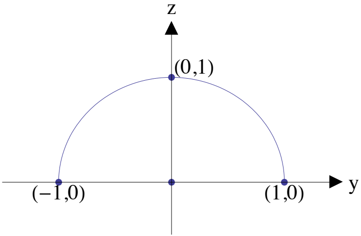
Example 7.2: Find the level curves of $z = \sqrt{1 - x^2 - y^2}$. Hence sketch the surface and identify it.

Solution:

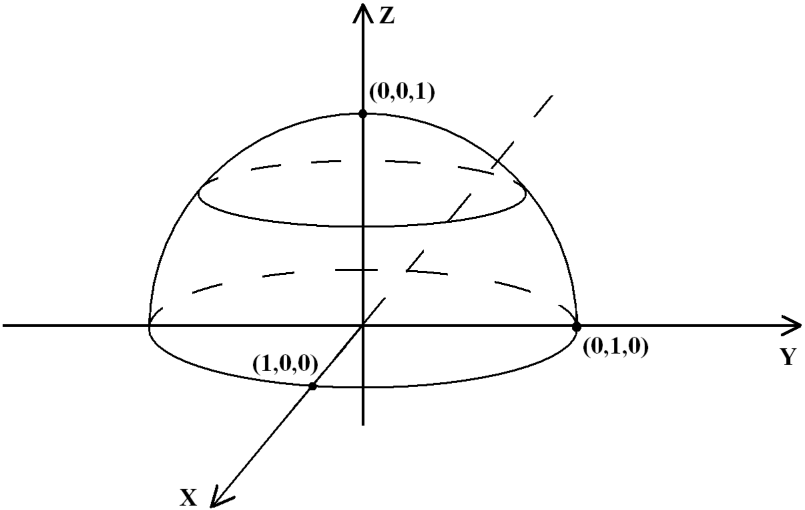

y
Level curves
c=0
(0,1)
x
(−1,0)
(1,0)
c=1
(0,−1)

Consider cross sections (slices) to help sketch graph.



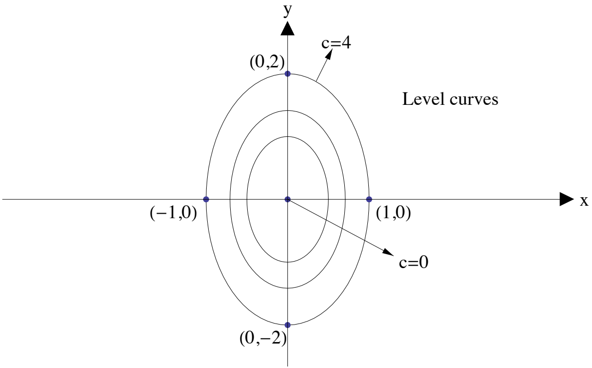


Surface is a hemisphere radius 1, centre at $(0,0,0)$ for $z \geq 0$.

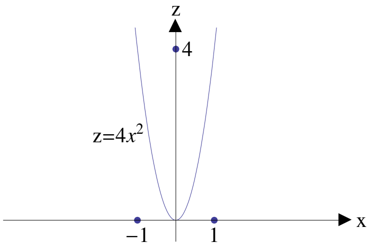


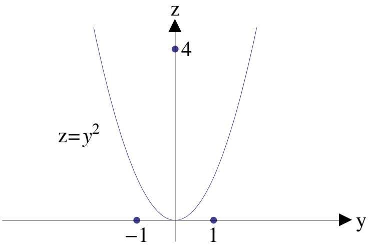
Example 7.3: Sketch the graph of $z = 4x^2 + y^2$.

Solution:

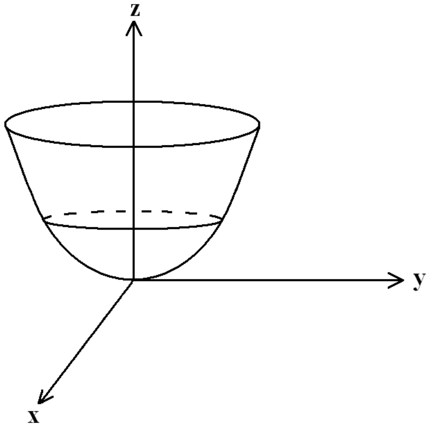


Consider cross sections (slices) to help sketch graph.



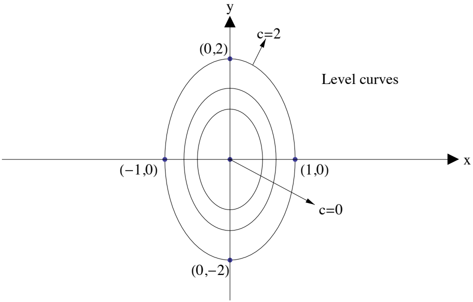


The surface is an elliptic paraboloid (parabolic bowl).

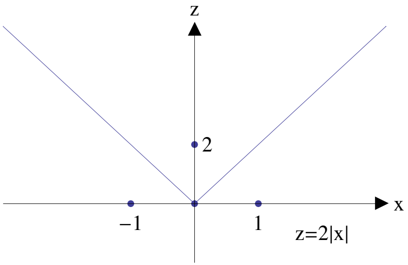


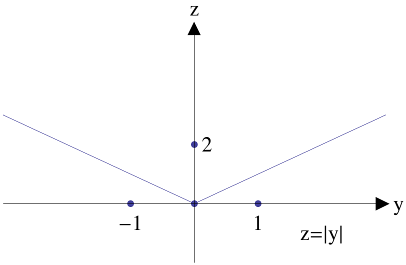
Example 7.4: Sketch the graph of $z = \sqrt{4x^2 + y^2}$.

Solution:

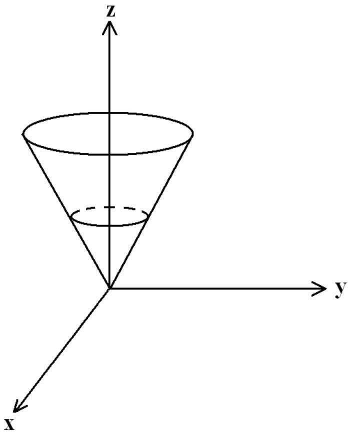


Cross sections





The surface is an elliptic cone.



Limits

Let $f : D \to \mathbb{R}$ be a real-valued function, where $D \subseteq \mathbb{R}^2$.

We say f has the limit L as (x, y) approaches (x_0, y_0)

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$

if when (x, y) approaches (x_0, y_0) along ANY path in the domain, $f(x, y)$ gets arbitrarily close to L.

Note:

1. L must be finite.
2. The limit can exist if f is undefined at (x_0, y_0).
3. The usual limit laws apply.

Continuity

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a real-valued function.

f is continuous at $(x, y) = (x_0, y_0)$ if

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$$

same

Note:

The continuity theorems for functions of one variable can be generalised to functions of two variables.

Example 7.5: Let $f(x, y) = x^2 + y^2$. For which values of x and y is f continuous?

Solution:

Example 7.6: Evaluate $\lim_{(x,y)\to(2,1)} \log(1 + 2x^2 + 3y^2)$.

Solution:

First Order Partial Derivatives

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a real-valued function. The first order partial derivatives of f with respect to the variables x and y are defined by the limits:

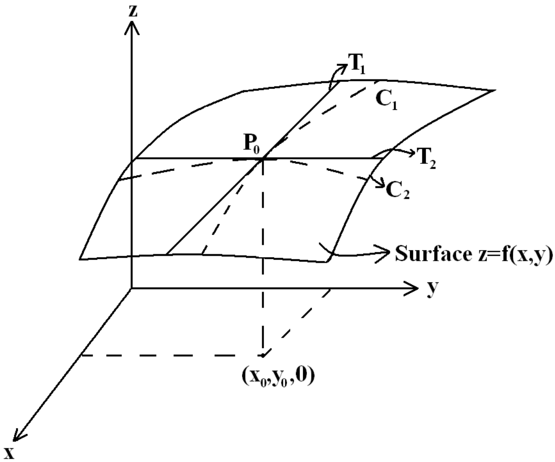
$$f_x = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_y = \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Note:

- $\dfrac{\partial f}{\partial x}$ measures the rate of change of f with respect to x when y is held constant.
- $\dfrac{\partial f}{\partial y}$ measures the rate of change of f with respect to y when x is held constant.

Geometric Interpretation of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$



Let C_1 be the curve where the vertical plane $y = y_0$ intersects the surface. Then $\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}$ gives the slope of the tangent to C_1 at (x_0, y_0, z_0).

Let C_2 be the curve where the vertical plane $x = x_0$ intersects the surface. The $\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)}$ gives the slope of the tangent to C_2 at (x_0, y_0, z_0).

- T_1 and T_2 are the tangent lines to C_1 and C_2.

Example 7.7: Let $f(x,y) = xy^2$. Find $\dfrac{\partial f}{\partial y}$ from first principles.

Solution:

Example 7.8: Let $f(x,y) = 3x^3y^2 + 3xy^4$. Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

Solution:

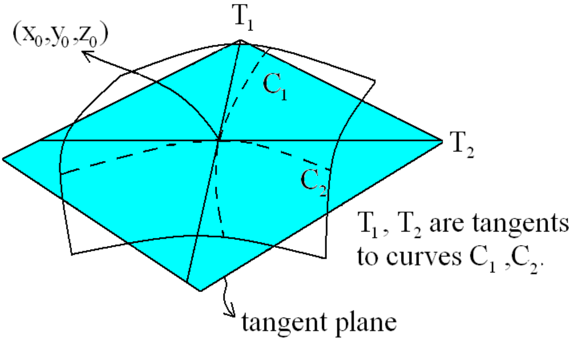
Example 7.9: Let $f(x, y) = y \log x + x \tanh(3y)$. Find f_x, f_y at $(1, 0)$.

Solution:

Tangent Planes and Differentiability

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a real-valued function. We say that f is differentiable at (x_0, y_0) if the tangent lines to all curves on the surface $z = f(x, y)$ passing through (x_0, y_0, z_0) form a plane, called the tangent plane.

This holds if f_x and f_y exist and are continuous near (x_0, y_0).



The tangent line T_1 has equation ($y = y_0$ fixed):

$$z - z_0 = \frac{\partial f}{\partial x}\Big|_{(x_0,y_0)}(x - x_0)$$

The tangent line T_2 has equation ($x = x_0$ fixed):

$$z - z_0 = \frac{\partial f}{\partial y}\Big|_{(x_0,y_0)}(y - y_0)$$

Since a plane passing through (x_0, y_0, z_0) has the form

$$z - z_0 = \alpha(x - x_0) + \beta(y - y_0)$$

the tangent plane has equation

$$\boxed{z - z_0 = \frac{\partial f}{\partial x}\Big|_{(x_0,y_0)}(x - x_0) + \frac{\partial f}{\partial y}\Big|_{(x_0,y_0)}(y - y_0).}$$

Example 7.10: Find the equation of the tangent plane to the surface $z = f(x, y) = 2x^2 + y^2$ at $(1, 1, 3)$.

Solution:

Linear Approximations

If f is differentiable at (x_0, y_0), we can approximate $z = f(x, y)$ by its tangent plane at (x_0, y_0, z_0).

This linear approximation of f near (x_0, y_0) is:

$$\boxed{f(x,y) \approx f(x_0,y_0) + \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}(x - x_0) + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)}(y - y_0)}$$

Let $\Delta x = x - x_0, \quad \Delta y = y - y_0, \quad \Delta f = z - z_0 = f(x, y) - f(x_0, y_0)$.

Then the approximate change in f near (x_0, y_0), for given small changes in x and y, is:

$$\boxed{\Delta f \approx \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}\Delta x + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)}\Delta y}$$

Example 7.11: Let $z = f(x, y) = x^2 + 3xy - y^2$. If x changes from 2 to 2.05 and y changes from 3 to 2.96, estimate the change in z.

Solution:

Note:

The actual change in f is

$$\begin{aligned}\Delta f &= f(2.05, 2.96) - f(2, 3) \\ &= 13.6449 - 13 \\ &= 0.6449\end{aligned}$$

Example 7.12: Find the linear approximation of $f(x, y) = xe^{xy}$ at $(1, 0)$. Hence, approximate $f(1.1, -0.1)$.

Solution:

Note:

The actual value is

$$(1.1)e^{-0.11} \approx 0.98542$$

Second Order Partial Derivatives

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a real-valued function. The second order partial derivatives of f with respect to x and y are defined by:

- $f_{xx} = \left(f_x\right)_x = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial^2 f}{\partial x^2}$
- $f_{yy} = \left(f_y\right)_y = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial^2 f}{\partial y^2}$
- $f_{xy} = \left(f_x\right)_y = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial^2 f}{\partial y \partial x}$
- $f_{yx} = \left(f_y\right)_x = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial^2 f}{\partial x \partial y}$

Theorem:

If the second order partial derivatives of f exist and are continuous then $f_{xy} = f_{yx}$.

Example 7.13: Find the second order partial derivatives of $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = x \sin(x + 2y)$.

Solution:

- $f_{xy}(x,y) = \dfrac{\partial}{\partial y}\left[\sin(x+2y) + x\cos(x+2y)\right]$

$$= 2\cos(x+2y) - 2x\sin(x+2y)$$

- $f_{yx}(x,y) = \dfrac{\partial}{\partial x}\left[2x\cos(x+2y)\right]$

$$= 2\cos(x+2y) - 2x\sin(x+2y)$$

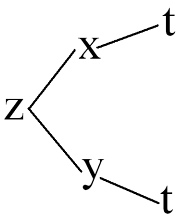
Note:

$f_{xy} = f_{yx}$ as expected since trigonometric functions and polynomials are continuous for all $(x,y) \in \mathbb{R}^2$.

Chain Rule

1. If $z = f(x, y)$ and $x = g(t)$, $y = h(t)$ are differentiable functions, then $z = f(g(t), h(t))$ is a function of t, and

$$\boxed{\frac{dz}{dt} = \overbrace{\frac{\partial z}{\partial x}\frac{dx}{dt}}^{\text{chain rule for first variable}} + \underbrace{\frac{\partial z}{\partial y}\frac{dy}{dt}}_{\text{chain rule for second variable}}}$$



hyperboloid

Example 7.14: If $z = x^2 - y^2$, $x = \sin t$, $y = \cos t$. Find $\frac{dz}{dt}$ at $t = \frac{\pi}{6}$.

Solution:

$$\frac{\partial z}{\partial x} = 2x \; ; \; \frac{\partial z}{\partial y} = -2y \; ; \; \frac{\partial x}{\partial t} = \cos(t) \; ; \; \frac{\partial y}{\partial t} = -\sin(t)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$= (2x)(\cos(t)) + (-2y)(-\sin(t))$$

At $t = \frac{\pi}{6}$; $x = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$; $y = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

$$\Rightarrow \left.\frac{\partial z}{\partial t}\right|_{\pi/6} = 2 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} + \left(-2 \cdot \frac{\sqrt{3}}{2}\right) \cdot \left(-\frac{1}{2}\right) = \sqrt{3}$$

- check by substitution:

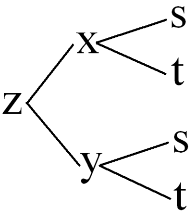
$$z = x^2 - y^2 = \sin^2(t) - \cos^2(t) = -\cos(2t)$$

$$\Rightarrow \frac{dz}{dt} = 2\sin(2t)$$

$$\Rightarrow \left.\frac{dz}{dt}\right|_{\pi/6} = 2\underbrace{\sin\left(\frac{\pi}{3}\right)}_{\frac{\sqrt{3}}{2}} = \sqrt{3}$$

2. If $z = f(x,y)$ and $x = g(s,t)$, $y = h(s,t)$ are differentiable functions, then z is a function of s and t with

$$\boxed{\begin{aligned} \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \\[2ex] \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \end{aligned}}$$



Example 7.15: If $z = e^x \sinh y$, $x = st^2$, $y = s^2 t$.

Find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$.

Solution: $\dfrac{\partial z}{\partial x} = e^x \sinh(y)$; $\dfrac{\partial z}{\partial y} = e^x \cosh(y)$

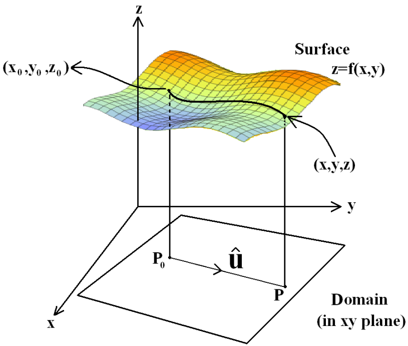
- $$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$
$$= (e^x \sinh(y))\, t^2 + (e^x \cosh(y))(2st)$$
$$= t^2 e^{st^2} \sinh(s^2 t) + 2st\, e^{st^2} \cosh(s^2 t)$$

- $$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$
$$= (e^x \sinh(y))(2st) + (e^x \cosh(y))(s^2)$$
$$= 2st\, e^{st^2} \sinh(s^2 t) + s^2 e^{st^2} \cosh(s^2 t)$$

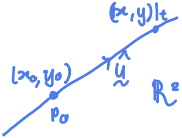
Directional Derivatives

Let $\hat{\mathbf{u}} = (u_1, u_2)$ be a unit vector in the xy-plane (so $u_1^2 + u_2^2 = 1$). The rate of change of f at $P_0 = (x_0, y_0)$ in the direction $\hat{\mathbf{u}}$ is the directional derivative $D_{\hat{\mathbf{u}}} f \Big|_{P_0}$. $= D_{\hat{u}} f(x_0, y_0)$

Geometrically this represents the slope of the surface $z = f(x, y)$ above the point P_0 in the direction $\hat{\mathbf{u}}$.



The straight line starting at $P_0 = (x_0, y_0)$ with velocity $\hat{\mathbf{u}} = (u_1, u_2)$ has parametric equations:



$$x = x_0 + tu_1, \quad y = y_0 + tu_2.$$

Hence,

$$\begin{aligned}
D_{\hat{\mathbf{u}}} f \Big|_{P_0} &= \text{rate of change of } f \text{ along the straight line at } t = 0 \\
&= \text{value of } \frac{d}{dt} f(x_0 + tu_1, y_0 + tu_2) \text{ at } t = 0 \\
&= f_x(x_0, y_0) x'(0) + f_y(x_0, y_0) y'(0) \qquad \text{by the chain rule} \\
&= f_x(x_0, y_0)\, u_1 + f_y(x_0, y_0)\, u_2.
\end{aligned}$$

We can also write this as a dot product (scalar / inner product)

$$D_{\hat{\mathbf{u}}} f \Big|_{P_0} = \left(\frac{\partial f}{\partial x}\bigg|_{P_0}, \frac{\partial f}{\partial y}\bigg|_{P_0} \right) \cdot (u_1, u_2).$$

Gradient Vectors

If $f : \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function, we can define the gradient of f to be the vector

$$\boxed{\text{grad}\, f = \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)}$$

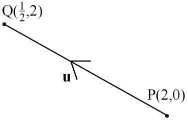
Then the directional derivative of f at the point P_0 in the direction $\hat{\mathbf{u}}$ is the dot product

$$\boxed{D_{\hat{\mathbf{u}}} f \,\Big|_{P_0} = \nabla f\Big|_{P_0} \cdot \hat{\mathbf{u}}}$$

Example 7.16: Find the directional derivative of $f(x,y) = xe^y$ at $\underbrace{(2,0)}_{P_0}$ in the direction from $(2,0)$ towards $\left(\frac{1}{2}, 2\right) = Q$.

Solution:

- direction $\hat{\mathbf{u}}$



$$\underset{\sim}{u} = \overrightarrow{PQ} = \left(\frac{1}{2}, 2\right) - (2, 0)$$

$$= \left(-\frac{3}{2}, 2\right)$$

$$|\underset{\sim}{u}| = \sqrt{\left(-\frac{3}{2}\right)^2 + (2)^2} = \frac{5}{2}$$

$$\Rightarrow \hat{\underset{\sim}{u}} = \frac{\underset{\sim}{u}}{|\underset{\sim}{u}|} = \frac{2}{5}\left(-\frac{3}{2}, 2\right)$$

$$= \left(-\frac{3}{5}, \frac{4}{5}\right) \text{ (direction)}$$

- ∇f at $(2,0) = P$

$$\nabla f = \frac{\partial f}{\partial x}\,\underset{\sim}{i} + \frac{\partial f}{\partial y}\,\underset{\sim}{j} = e^{y}\,\underset{\sim}{i} + xe^{y}\,\underset{\sim}{j}$$

$$\Rightarrow \nabla f(2,0) = e^{0}\,\underset{\sim}{i} + 2e^{0}\,\underset{\sim}{j} = \underset{\sim}{i} + 2\underset{\sim}{j}$$

- $\nabla_{\hat{\underset{\sim}{u}}} f$ at $(2,0) = P$

$$D_{\hat{\underset{\sim}{u}}} f(2,0) = \nabla f(2,0) \cdot \hat{\underset{\sim}{u}}$$

$$= (1,2) \cdot \left(-\frac{3}{5}, \frac{4}{5}\right)$$

$$= 1 \cdot \left(-\frac{3}{5}\right) + 2 \cdot \left(\frac{4}{5}\right)$$

$$= -\frac{3}{5} + \frac{8}{5}$$

$$= 1 \,/\!/$$

Example 7.17: Find the directional derivative of $f(x,y) = \arcsin\left(\dfrac{x}{y}\right)$ at $\underbrace{(1,2)}_{P_0}$ in the direction $\underbrace{\dfrac{\pi}{4}}_{\varphi}$ anticlockwise from the positive x axis.

Solution:

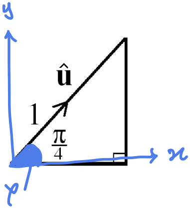
- direction $\hat{\mathbf{u}} = (\cos(\varphi), \sin(\varphi)) = (u_1, u_2)$

$$u_1 = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$u_2 = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \underset{\sim}{\hat{u}} = \frac{1}{\sqrt{2}}\underset{\sim}{i} + \frac{1}{\sqrt{2}}\underset{\sim}{j}$$

$$= \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$



• ∇f at $(1,2)$:

$$\frac{\partial f}{\partial x} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \frac{1}{y} \ ; \quad \frac{\partial f}{\partial y} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \left(-\frac{x}{y^2}\right)$$

At $(1,2)$: $\sqrt{1-\left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$

$$\Rightarrow \frac{\partial f}{\partial x}(1,2) = \frac{1}{\sqrt{3}/2} \frac{1}{2} = \frac{1}{\sqrt{3}}$$

$$\frac{\partial f}{\partial y}(1,2) = \frac{1}{\sqrt{3}/2}\left(-\frac{1}{2^2}\right) = \frac{-1}{2\sqrt{3}}$$

$$\nabla f(1,2) = \left(\frac{\partial f}{\partial x}(1,2), \frac{\partial f}{\partial y}(1,2)\right) = \left(\frac{1}{\sqrt{3}}, -\frac{1}{2\sqrt{3}}\right)$$

• $D_{\hat{u}} f$ at $(1,2)$:

$$D_{\hat{u}} f(1,2) = \nabla f(1,2) \cdot \hat{u}$$

$$= \left(\frac{1}{\sqrt{3}}, -\frac{1}{2\sqrt{3}}\right) \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{2}} + \left(\frac{-1}{2\sqrt{3}}\right)\left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{6}} - \frac{1}{2\sqrt{6}}$$

$$= \frac{1}{2\sqrt{6}} \quad //$$

Properties of ∇f and $D_{\hat{\mathbf{u}}} f$

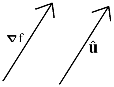
The directional derivative of f is

$$
\begin{aligned}
D_{\hat{\mathbf{u}}} f &= \nabla f \cdot \hat{\mathbf{u}} \\
&= |\nabla f|\,|\hat{\mathbf{u}}| \cos\theta \\
\boxed{D_{\vec{u}} f} &\boxed{= |\nabla f| \cos\theta}
\end{aligned}
$$

where θ is the angle between ∇f and $\hat{\mathbf{u}}$, and $|\mathbf{v}|$ denotes the length of a vector $\mathbf{v}$.

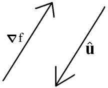
So for fixed ∇f:

- $D_{\hat{\mathbf{u}}}f$ is maximum when $\cos\theta = 1$ so $\theta = 0$



$\Rightarrow f$ increases most rapidly along ∇f.

- $D_{\hat{\mathbf{u}}}f$ is minimum when $\cos\theta = -1$ so $\theta = \pi$



$\Rightarrow f$ decreases most rapidly along $-\nabla f$.

- $D_{\hat{\mathbf{u}}}f = 0$ when $\cos\theta = 0$ so $\theta = \dfrac{\pi}{2}$ and $\nabla f \perp \hat{\mathbf{u}}$.

But $D_{\hat{\mathbf{u}}}f = 0$, whenever $\hat{\mathbf{u}}$ is tangent to a level curve of f (where f = constant).

$$\Rightarrow \nabla f \perp \text{ level curves of } f$$

Example 7.18: Let $f(x,y) = 4x^2 + y^2$.
(a) Find ∇f at $(1,0)$ and $(0,2)$.
(b) Show that ∇f is perpendicular to the level curves, by sketching ∇f at these points and the level curves of f.

Solution:

$$\boxed{\nabla f = \frac{\partial f}{\partial x}\underset{\sim}{i} + \frac{\partial f}{\partial y}\underset{\sim}{j} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)}$$

- $\frac{\partial f}{\partial x} = 8x \Rightarrow \left.\frac{\partial f}{\partial x}\right|_{(1,0)} = 8(1) = 8 \Rightarrow \left.\frac{\partial f}{\partial x}\right|_{(0,2)} = 8(0) = 0$

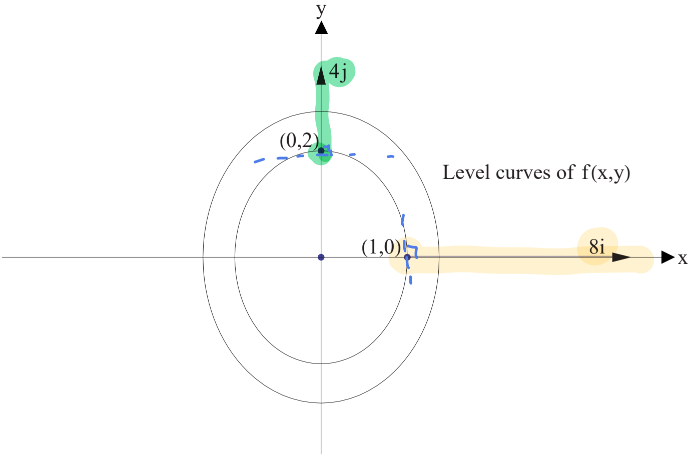
- $\frac{\partial f}{\partial y} = 2y \Rightarrow \left.\frac{\partial f}{\partial y}\right|_{(1,0)} = 2(0) = 0 \Rightarrow \left.\frac{\partial f}{\partial y}\right|_{(0,2)} = 2(2) = 4$

$\nabla f = 8x\,\underset{\sim}{i} + 2y\,\underset{\sim}{j} = (8x, 2y)$

At $(1,0) = 8\underset{\sim}{i} + 0\underset{\sim}{j} = (8, 0)$

At $(0,2) = 0\underset{\sim}{i} + 4\underset{\sim}{j} = (0,4)$

(b)



Example 7.19: In what direction does $f(x, y) = xe^y$

(a) increase (b) decrease

most rapidly at $(2, 0)$? Express direction as a unit vector.

Solution:

From Example 7.16

$$\nabla f(\overset{P_0}{2, 0}) = \mathbf{i} + 2\mathbf{j} = \left.\frac{\partial f}{\partial x}\right|_{(2,0)} \underset{\sim}{i} + \left.\frac{\partial f}{\partial y}\right|_{(2,0)} j$$

$\Rightarrow |\nabla f(2,0)| = \sqrt{1^2 + 2^2} = \sqrt{5}$

$\Rightarrow$ The unit vector in the direction of $\nabla(2,0)$ is $\underset{\sim}{\hat{u}} = \frac{1}{\sqrt{5}}(i + 2j)$

The direction of most rapid

a) increase is $\hat{\underset{\sim}{u}} = \frac{1}{\sqrt{5}}(\underset{\sim}{i} + 2\underset{\sim}{j}) = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$

b) decrease is $\hat{\underset{\sim}{u}} = -\frac{1}{\sqrt{5}}(\underset{\sim}{i} + 2\underset{\sim}{j}) = \left(-\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right)$

Stationary Points

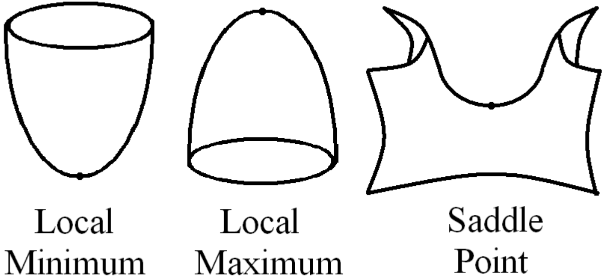
A stationary point of f is a point (x_0, y_0) at which

$$\boxed{\nabla f = \mathbf{0}} = (0,0)$$

So $\boxed{\dfrac{\partial f}{\partial x} = 0 \text{ and } \dfrac{\partial f}{\partial y} = 0}$ simultaneously at (x_0, y_0).

Geometrically, this means that the tangent plane to the graph $z = f(x, y)$ at (x_0, y_0) is horizontal, i.e. parallel to the xy-plane.

Three important types of stationary points are



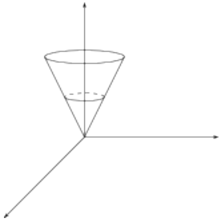
A function f has a

1. local maximum at (x_0, y_0) if $f(x,y) \leq f(x_0, y_0)$ for all (x,y) in some disk centred at (x_0, y_0),
2. local minimum at (x_0, y_0) if $f(x,y) \geq f(x_0, y_0)$ for all (x,y) in some disk centred at (x_0, y_0),
3. saddle point at (x_0, y_0) if (x_0, y_0) is a stationary point, and there are points near (x_0, y_0) with $f(x,y) > f(x_0, y_0)$ and other points near (x_0, y_0) with $f(x,y) < f(x_0, y_0)$.

Any local maximum or minimum of f will occur at a critical point (x_0, y_0) such that

1. $\nabla f(x_0, y_0) = \mathbf{0}$ or

2. $\dfrac{\partial f}{\partial x}$ and/or $\dfrac{\partial f}{\partial y}$ do not exist at (x_0, y_0).



$z = \sqrt{x^2 + y^2}$. Minimum at $(0, 0)$ BUT ∇f does not exist at $(0, 0)$.

Second Derivative Test

If $\nabla f(x_0, y_0) = \mathbf{0}$ and the second partial derivatives of f are continuous on an open disk centred at (x_0, y_0), consider the Hessian function

$$H(x, y) = f_{xx} f_{yy} - (f_{xy})^2$$

evaluated at (x_0, y_0).

Then (x_0, y_0) is a

1. local minimum if $H(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) > 0$.
2. local maximum if $H(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) < 0$.
3. saddle point if $H(x_0, y_0) < 0$.

Note: Test is inconclusive if $H(x_0, y_0) = 0$.

Example 7.20: Find and classify the stationary points of $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = x^3 + y^3 + 3x^2 - 3y^2 - 8$.

Solution:

$$\frac{\partial f}{\partial x}(x,y) = 3x^2 + 6x = 0 \Rightarrow 3x(x+2) = 0$$

$$\Rightarrow x = 0 \qquad x = -2 \qquad (1)$$

$$\frac{\partial f}{\partial y}(x,y) = 3y^2 - 6y = 0 \Rightarrow 3y(y-2) = 0$$

$$\Rightarrow y = 0 \qquad y = 2 \qquad (2)$$

Combining (1) and (2) yields 4 points:

$(0,0)$; $(-2,0)$; $(0,2)$; $(-2,2)$

- CLASSIFY THE stationary points
  - $f_{xx}(x,y) = 6x+6$
  - $f_{yy}(x,y) = 6y-6$
  - $f_{xy}(x,y) = 0$
  - $H(x,y) = (6x+6)(6y-6) - (0)^2$

$$= 36(x+1)(y-1)$$

- $H(0,0) = 36(0+1)(0-1) = -36 < 0$

  $\Rightarrow (0,0)$ is a saddle point

- $H(-2,0) = 36(-2+1)(0-1) = 36 > 0$

  $f_{xx}(-2,0) = 6\cdot(-2)+6 = -6 < 0$

  $\Rightarrow (-2,0)$ is a local maximum

- $H(-2,2) = 36(-2+1)(2-1) = -36 < 0$

  $\Rightarrow (-2,2)$ is a saddle point

- $H(0,2) = 36(0+1)(2-1) = 36 > 0$

  $f_{xx}(0,2) = 6\cdot(0) + 6 = 6 > 0$

  $\Rightarrow (0,2)$ is a local minimum

Example 7.21: Find and classify the stationary points of $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = y \sin x$.

Solution:

① FIND STATIONARY POINTS

- Find x/y vals when $\frac{\partial f}{\partial x} \overset{(1)}{=} 0$ and $\frac{\partial f}{\partial y} \overset{(2)}{=} 0$

  - $\frac{\partial f}{\partial x} = y \cos x = 0$

    $y = 0$ $\quad$ $\cos x = 0 \Rightarrow \underline{x = (m + \frac{1}{2})\pi \quad (m \in \mathbb{Z})}$

  - $\frac{\partial f}{\partial y} = \sin x = 0 \Rightarrow \underline{x = n\pi \quad (n \in \mathbb{Z})}$

- Combining (1) and (2) yields ...

  - $x = (m + \frac{1}{2})\pi$ cannot yield a stationary point as it cannot satisfy both equations
  - The stationary points are at $(n\pi, 0)$

② CLASSIFYING STATIONARY POINTS

$$\boxed{H(x,y) = f_{xx} f_{yy} - (f_{xy})^2}$$

$$f_{xx}(x,y) = -y\sin(x)$$

$$f_{yy}(x,y) = 0$$

$$f_{xy}(x,y) = f_{yx}(x,y) = \cos(x)$$

$$\Rightarrow H(x,y) = -y\sin(x)\cdot 0 - (\cos(x))^2$$

$$= -(\cos^2(x))$$

$$H(n\pi, 0) = -(\cos^2(n\pi)) = -1 < 0$$

$\Rightarrow$ $(n\pi, 0)$ with $n \in \mathbb{Z}$ are saddle points.

Partial Integration

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function over a domain D in $\mathbb{R}^2$.

The partial indefinite integrals of f with respect to the first and second variables (say x and y) are denoted by:

$\int f(x,y)\,dx$ and $\int f(x,y)\,dy$.

- $\int f(x,y)\,dx$ is evaluated by holding y fixed and integrating with respect to x.
- $\int f(x,y)\,dy$ is evaluated by holding x fixed and integrating with respect to y.

Example 7.22: Evaluate $\int (3x^2y + 12y^2x^3)\, dx$.

Solution:

Hold y fixed and integrate with respect to x

$$\int (3x^2y + 12y^2x^3)\, dx = 3y \int x^2\, dx + 12y^2 \int x^3\, dx$$

$$= 3y\left(\frac{1}{3}x^3 + c_1(y)\right) + 12y^2\left(\frac{1}{4}x^4 + c_2(y)\right)$$

$$= yx^3 + 3y^2x^4 + c(y), \quad c(y) = 3y\, c_1(y) + 12y^2 c_2(y),$$

a constant of integration that may depend on y

Note:

Example 7.23: Evaluate $\int_0^1 (3x^2y + 12y^2x^3)\,dy$.

Solution:

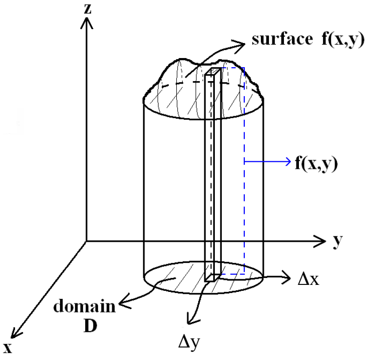
Double Integrals

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function over a domain D in $\mathbb{R}^2$.

We can evaluate the double integral:

$$\boxed{\iint_D f(x,y)\,dA = \iint_D f(x,y)\,dx\,dy}$$

$\iint_D f(x,y)\,dA$ is the volume under the surface $z = f(x,y)$ that lies above the domain D in the xy plane, if $f(x,y) \geq 0$ in D.



$$\text{Volume of thin rod} \quad = \quad \underbrace{(\text{Area base})}_{\substack{\| \\ \Delta x \Delta y}} \cdot \underbrace{(\text{height})}_{\substack{\| \\ f(x,y)}}$$

The double integral is defined as the limit of sums of the volumes of the rods:

$$\begin{aligned}\iint_D f(x,y)\,dA &= \iint_D f(x,y)\,dx\,dy \\ &= \lim_{\Delta x\to 0}\lim_{\Delta y\to 0}\sum_{i=1}^{n}[f(x,y)\Delta x\Delta y]_i\end{aligned}$$

Note:

If $f(x,y)=1$ then

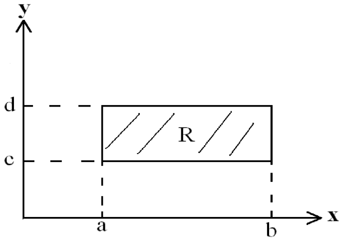
$$\iint_D dA = \iint_D dx\,dy$$

gives the area of the domain D.

Double Integrals Over Rectangular Domains

Definitions

1. $R = [a, b] \times [c, d]$ is a rectangular domain defined by
   $a \leq x \leq b, \quad c \leq y \leq d$.



2. $\int_c^d \int_a^b f(x, y)\, dx\, dy = \int_c^d \left[\int_a^b f(x, y)\, dx \right] dy$ means integrate with respect to x first and then integrate with respect to y.

Fubini's Theorem:

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function over the domain $R = [a, b] \times [c, d]$. Then

$$\iint_R f(x,y)dA = \int_c^d \int_a^b f(x,y)\,dx\,dy$$
$$= \int_a^b \int_c^d f(x,y)\,dy\,dx$$

So order of integration is not important.

Example 7.24: Evaluate $\iint_R (x^2+y^2)\,dx\,dy$ if $R = [-1,1]\times[0,1]$.

Solution:

In R: $-1 \le x \le 1$; $0 \le y \le 1$

- Integrate with respect to x:

$$\iint_R (x^2+y^2)\,dx\,dy = \int_0^1 \int_{-1}^1 (x^2+y^2)\,dx\,dy$$

$$= \int_0^1 \left[\frac{1}{3}x^3 + y^2 x\right]_{-1}^1 dy$$

$$= \int_0^1 \left[\left(\frac{1}{3}(1)^3 + y^2(1)\right) - \left(\frac{1}{3}(-1)^3 + y^2(-1)\right)\right] dy$$

$$= \int_0^1 \left(\frac{2}{3} + 2y^2\right) dy$$

$$= \left[\frac{2}{3}y + \frac{2}{3}y^3\right]_0^1$$

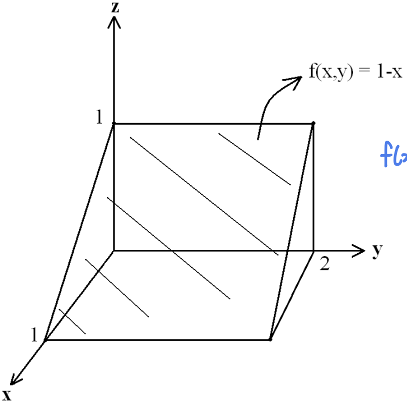
$$= \left(\frac{2}{3}(1) + \frac{2}{3}(1)^3\right) - \left(\frac{2}{3}(0) + \frac{2}{3}(0)^3\right)$$

$$= \frac{4}{3} \,//$$

Note:

As expected, the order of integration is not important since polynomials are continuous for all $(x, y) \in \mathbb{R}^2$.

Example 7.25: Using double integrals, find the volume of the wedge shown below.



$f(x,y) = 1-x$

Solution:

The domain in the x-y plane is:

$R = [0,1] \times [0,2] \iff 0 \leq x \leq 1$

$0 \leq y \leq 2$

$$\text{Volume} = \iint_R f(x,y)\,dx\,dy$$

$$= \int_0^2 \int_0^1 1 - x \quad dx \quad dy$$

$$= \int_0^2 \left[1 \cdot x - \frac{1}{2}x^2\right]_0^1 dy$$

$$= \int_0^2 \left[(1) - \frac{1}{2}(1)^2\right] dy$$

$$= \int_0^2 1 - \frac{1}{2} \, dy$$

$$= \left[y - \frac{1}{2}y\right]_0^2$$

$$= [2) - \frac{1}{2}(2) = \underline{1 \quad (\text{length})^3} \; //$$