Exercises on orthogonal projections

April 10, 2024

Exercise 1. Show that the linear map

$$f : \mathbb{R}^2 \to \mathbb{R}^2$$
$$(x_1, x_2) \mapsto (x_1 + x_2, -x_2)$$

is orthogonal with respect to the inner product $\varphi\left((x_1, x_2), (y_1, y_2)\right) = 2x_1y_1 + x_2y_1 + x_1y_2 + 2x_2y_2$.

Remark. What is the projection matrix to a given vector $N = (a, b, c)$ with respect to **the standard inner product**? $\mathrm{pr}_{\mathbb{R}N}(x) = \frac{<N,x>}{\|N\|^2}N = \frac{1}{\|N\|^2}N({}^tN \cdot x) = \frac{1}{{}^tN \cdot N}(N \cdot {}^tN)x$ by the associativity of matrices multiplication. So the formula for the projection matrix to a vector N is $\frac{1}{{}^tN \cdot N}(N \cdot {}^tN)$.

Solution. By definition, we need to check if $\varphi\left(f(x_1, x_2), f(y_1, y_2)\right) = \varphi\left((x_1, x_2), (y_1, y_2)\right)$ is satisfied. We have $\varphi\left(f(x_1, x_2), f(y_1, y_2)\right) = \varphi\left((x_1 + x_2, -x_2), (y_1 + y_2, -y_2)\right) = 2(x_1+x_2)(y_1+y_2)+(-x_2)(y_1+y_2)+(x_1+x_2)(-y_2)+2(-x_2)(-y_2) = 2x_1y_1 + x_1y_2 + x_2y_1 + 2x_2y_2$, which equals $\varphi\left((x_1, x_2), (y_1, y_2)\right)$.

Method by using matrix: The matrix associated with φ is $M_\varphi = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, the matrix associated with f is $M_f = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ with respect to the standard basis. Then $\varphi\left(f(x_1, x_2), f(y_1, y_2)\right)$ in terms of matrix multiplication is :
$(x_1\ x_2)\left({}^t\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\right)\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = (x_1\ x_2)\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ which is $\varphi\left((x_1, x_2), (y_1, y_2)\right)$. ■

Exercise 2. Endow $\mathbb{R}^3$ with the standard inner product, and find the matrix of the orthogonal projection $p_W : \mathbb{R}^3 \to \mathbb{R}$ where

- $W = \{x + 2y + 3z = 0\}$, or
- $W = \mathrm{Span}\{(1, 1, 1), (1, -1, 0)\}$.

Solution.

- For $W = \{x + 2y + 3z = 0\}$: Since W is 2-dimensional subspace of $\mathbb{R}^3$, we know $W^\perp = \mathrm{Span}(N)$ where $N = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. We use Proposition 3.13 in Chapter 2 $\mathrm{pr}_W = \mathrm{Id} - \mathrm{pr}_{W^\perp} = \mathrm{Id} - \mathrm{pr}_{\mathrm{Span}(N)}$. For any $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, one has $\mathrm{pr}_{W^\perp}(x) = \frac{<N,x>}{\|N\|^2}N = \frac{1}{14}(x_1 + 2x_2 + 3x_3)\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. And $\mathrm{pr}_W(x) = \mathrm{Id}(x) - \mathrm{pr}_{W^\perp}(x)$:

$$
\begin{aligned}
&= \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} - \frac{1}{14}(x_1+2x_2+3x_3)\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \\
&= \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} - \frac{1}{14}\begin{pmatrix} x_1+2x_2+3x_3 \\ 2x_1+4x_2+6x_3 \\ 3x_1+6x_2+9x_3 \end{pmatrix} \\
&= \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} - \begin{pmatrix} \frac{1}{14}(x_1+2x_2+3x_3) \\ \frac{1}{7}(x_1+2x_2+3x_3) \\ \frac{3}{14}(x_1+2x_2+3x_3) \end{pmatrix} \\
&= \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} - \begin{pmatrix} \frac{1}{14} & \frac{1}{7} & \frac{3}{14} \\ \frac{1}{7} & \frac{2}{7} & \frac{3}{7} \\ \frac{3}{14} & \frac{3}{7} & \frac{9}{14} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\
&= \begin{pmatrix} \frac{13}{14} & -\frac{1}{7} & -\frac{3}{14} \\ -\frac{1}{7} & \frac{5}{7} & -\frac{3}{7} \\ -\frac{3}{14} & -\frac{3}{7} & \frac{5}{14} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}
\end{aligned}
\tag{1}
$$

Remark. We may firstly find a basis of W, then use Gram-Schmidt process to orthonormalize it, and apply Proposition 3.15 in Chapter 2.

- For $W = \text{Span}\{(1,1,1),(1,-1,0)\}$. The method of previous question still applies (and maybe easier), but to avoid repetition:

Given vectors: $\mathbf{v}_1 = (1,1,1), \quad \mathbf{v}_2 = (1,-1,0)$

Step 1:

Normalize the first vector $\mathbf{v}_1$:

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{(1,1,1)}{\sqrt{3}} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

Step 2:

Compute the projection of $\mathbf{v}_2$ onto $\mathbf{u}_1$ and subtract it from $\mathbf{v}_2$ to get the second orthonormal vector $\mathbf{u}_2$:

We find these two vectors are orthogonal:$\mathbf{v}_2 \cdot \mathbf{u}_1 = (1,-1,0) \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}} + 0 = 0$

$\mathbf{v}_2 - ({}^t\mathbf{v}_2 \cdot \mathbf{u}_1)\mathbf{u}_1 = v_2 = (1,-1,0)$

$\|\mathbf{v}_2 - ({}^t\mathbf{v}_2 \cdot \mathbf{u}_1)\mathbf{u}_1\| = \|\mathbf{v}_2\| = \sqrt{1^2 + (-1)^2 + 0^2} = \sqrt{2}$

$\mathbf{u}_2 = \dfrac{(1,-1,0)}{\sqrt{2}} = \left(\dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}}, 0\right)$

Final Result:

$$\mathbf{u}_1 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \quad \mathbf{u}_2 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right) \tag{2}$$

We remind the readers that vectors written like (x_1, x_2, x_3) means column vector $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, and ${}^t(x_1, x_2, x_3)$ means row vector

$(x_1\ x_2\ x_3)$.

Projection matrix onto $\mathbf{u_1}$:

$$P_1 = \mathbf{u_1}^t\mathbf{u_1} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

Projection matrix onto $\mathbf{u_2}$: (3)

$$P_2 = \mathbf{u_2}^t\mathbf{u_2} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

Projection matrix onto $\mathbf{u_1}$:

$$P_1 = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

Projection matrix onto $\mathbf{u_2}$:

$$P_2 = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4}$$

Sum of P_1 and P_2 :

$$P_1 + P_2 = \begin{pmatrix} \frac{5}{6} & -\frac{1}{6} & \frac{1}{3} \\ -\frac{1}{6} & \frac{5}{6} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

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Exercise 3. In each case, use the Gram-Schmidt algorithm to find an orthogonal basis of the subspace U, and find the vector in U closest to the vector x.

- $U = \text{Span}\{(1,1,1),(0,1,1)\}, x = (1,2,1)$;
- $U = \text{Span}\{(1,1,0),(1,0,1)\}, x = (2,1,0)$;
- $U = \text{Span}\{(1,0,1,0),(1,1,1,0),(1,1,0,0)\}, x = (2,0,1,3)$.

Solution.

- For $U = \text{Span}\{(1,1,1),(0,1,1)\}, x = (1,2,1)$:

Step 1:

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \begin{bmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \end{bmatrix}.$$

Step 2:

$$\mathbf{u_2'} = \mathbf{v_2} - \text{pr}_{\mathbf{u}_1}(\mathbf{v_2}) = \begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}.$$

$$\mathbf{u}_2 = \frac{\mathbf{u_2'}}{\|\mathbf{u_2'}\|} = \begin{bmatrix} -\frac{\sqrt{6}}{3} \\ \frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{6} \end{bmatrix}.$$

By definition 3.16 and proposition 3.17, we know the vector in U closest to x is $\text{pr}_U(x)$ and by $\text{pr}_U = \text{pr}_{u_1} + \text{pr}_{u_2}$. P_1 is already computed before.

So, the result of $P_1 + P_2$ is the matrix:

$$P_1 + P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$\mathrm{pr}_U(x) = (1, 3/2, 3/2)$.

- For $U = \mathrm{Span}\{(1,1,0),(1,0,1)\}, x = (2,1,0)$:
  Step 1:
  $\ldots \mathbf{u_1} = \frac{\mathbf{v_1}}{\|\mathbf{v_1}\|} = \begin{bmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \end{bmatrix}$.
  Step 2:
  $\mathbf{u_2'} = \mathbf{v_2} - \mathrm{pr}_{\mathbf{u}_1}(\mathbf{v_2}) = \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$.
  $\mathbf{u}_2 = \frac{\mathbf{u_2'}}{\|\mathbf{u_2'}\|} = \begin{bmatrix} \frac{\sqrt{6}}{6} \\ -\frac{\sqrt{6}}{3} \\ \frac{\sqrt{6}}{6} \end{bmatrix}$.
  By definition 3.16 and proposition 3.17, we know the vector in U closest to x is $\mathrm{pr}_U(x)$ and $\mathrm{pr}_U = \mathrm{pr}_{\mathbf{u_1}} + \mathrm{pr}_{\mathbf{u_2}}$. Let P_1, P_2 denote the corresponding matrices, respectively. (P_1 is already calculated in the previous exercise.)
  $$P_2 = \begin{pmatrix} \frac{\sqrt{6}}{6} \\ -\frac{\sqrt{6}}{3} \\ \frac{\sqrt{6}}{6} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{6}}{6} & -\frac{\sqrt{6}}{3} & \frac{\sqrt{6}}{6} \end{pmatrix} = \begin{pmatrix} \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{pmatrix}$$
  So, the result of $P_1 + P_2$ is the matrix:
  $$P_1 + P_2 = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$
  $\mathrm{pr}_U(x) = (1,1,1)$.
- For $U = \mathrm{Span}\{(1,0,1,0),(1,1,1,0),(1,1,0,0)\}, x = (2,0,1,3)$:
  - **Step 1:**
    $$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$
    $$\mathbf{u}_1 = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}$$
  - **Step 2:**
    $$\mathbf{u}_2' = \mathbf{v}_2 - \mathrm{pr}_{\mathbf{u}_1}(\mathbf{v}_2) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \frac{\mathbf{u}_2'}{\|\mathbf{u}_2'\|} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$
  - **Step 3:**
    $$\mathbf{u}_3' = \mathbf{v}_3 - \mathrm{pr}_{\mathbf{u}_1}(\mathbf{v}_3) - \mathrm{pr}_{\mathbf{u}_2}(\mathbf{v}_3) = \begin{bmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \\ 0 \end{bmatrix}$$
    $$\mathbf{u}_3 = \frac{\mathbf{u}_3'}{\|\mathbf{u}_3'\|} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ -\frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}$$

By definition 3.16 and proposition 3.17, we know the vector in U closest to x is $\mathrm{pr}_U(x)$ and $\mathrm{pr}_U = \mathrm{pr}_{\mathbf{u}_1} + \mathrm{pr}_{\mathbf{u}_2} + \mathrm{pr}_{\mathbf{u}_3}$.

Projection Matrices:

$$P_{\mathbf{u}_1} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$P_{\mathbf{u}_2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$P_{\mathbf{u}_3} = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Sum of Projection Matrices:

$$P_{\mathbf{u}_1} + P_{\mathbf{u}_2} + P_{\mathbf{u}_3} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\mathrm{pr}_U(x) = (2, 0, 1, 0)$

Remark. An easier way: in fact $U = \mathrm{Span}((0,0,0,1))^{\perp}$, so $\mathrm{pr}_U = \mathrm{Id} - \mathrm{pr}_{(0,0,0,1)}$.

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Exercise 4. Consider the following matrices :

$$A_1 := \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; \quad A_2 := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix};$$

$$A_3 := \frac{1}{3}\begin{pmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \end{pmatrix}; \quad A_4 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -\sqrt{3}/2 \\ 0 & \sqrt{3}/2 & 1/2 \end{pmatrix}.$$

For $i \in \{1,2,3,4\}$, prove that A_i is orthogonal and describe the endomorphism represented by A_i.

Solution. $\det A_1 = 1$ so $A_1 \in \mathrm{SO}_3(\mathbb{R})$; $\operatorname{tr} A_1 = 0$ so $\theta = 2/3\pi$ by Proposition 2.12; $A_1(1,1,1) = (1,1,1)$, combined with intuition we know A_1 is rotation with respect to $(1,1,1)$ with an (positive) angle $2/3\pi$. We exclude the possibility that A_1 is rotation of $2/3\pi$ with respect to $-(1,1,1)$ by looking at Ae_1.

$\det A_2 = -1$, $\operatorname{tr} A_2 = 0$ so $\theta = \frac{\pi}{3}$ by Proposition 2.13. $A_1(1,-1,-1) = -(1,-1,-1)$, A_2 is rotation of $1/3\pi$ with respect to $(1,-1,-1)$ then make a reflection (*a.k.a* orthogonal symmetry) with respect to $\{x - y - z = 0\}$-plane. (To exclude the other possibility which is rotation of $\pi/3$ with respect to $(-1,1,1)$, look at the image of e_2.)

$\det A_3 = 1$ so $A_3 \in \mathrm{SO}_3(\mathbb{R})$; $\operatorname{tr} A_1 = 2$ so $\theta = 1/3\pi$ by Proposition 2.12; $A_1(1,1,1) = (1,1,1)$, combined with intuition we know A_3 is rotation with respect to $(1,1,1)$ with an (positive) angle $1/3\pi$. We exclude the possibility that A_1 is rotation of $1/3\pi$ with respect to $-(1,1,1)$ by looking at Ae_1.

$\det A_4 = 1$ so $A_4 \in \mathrm{SO}_3(\mathbb{R})$; $\operatorname{tr} A_4 = 2$ so $\theta = 1/3\pi$ by Proposition 2.12; $A_1(1,0,0) = (1,0,0)$, we know A_4 is rotation with respect to $(1,0,0)$ with an (positive) angle $1/3\pi$. ■

Exercise 5. Suppose that $\mathbb{R}^3$ is endowed with the standard inner product. Let C be the unit cube $[0,1]\times [0,1] \times [0,1]$. Find an orthogonal projection such that the image of the cube C is a regular hexagon. Proceeed as follows:

- Draw the cube and guess the line and the plane defining the orthogonal projection.
- Write a formula of the projection and the coordinates of the images of the vertices of the cube.
- Compute the distance between the image of the vertices and compute the angles between the images of the edges. To this end, remember that if θ is the angle between two vectors v_1, v_2 in $\mathbb{R}^n$ endowed with the standard inner product $\langle , \rangle$, *then*

$$\cos(\theta) = \frac{\langle v_1, v_2\rangle}{\|v_1\| \cdot \|v_2\|}$$

Solution. We guess by symmetry, the projection is with respect to hyperplane $W = N^{\perp}$, with $N = (1,1,1)$. From Exercise 2, the projection matrix to $\mathrm{Span}(N)$ is

$$\begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$$

So $\mathrm{pr}_W = \mathrm{id} - \mathrm{pr}_N$ is

$$\begin{pmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{pmatrix}$$

The eight vertices of the cube are:

$$\begin{array}{l} P_0\ (0,0,0) \\ P_1\ (1,0,0) \\ P_2\ (1,1,0) \\ P_3\ (0,1,0) \\ P_4\ (0,1,1) \\ P_5\ (0,0,1) \\ P_6\ (1,0,1) \\ P_7\ (1,1,1) \end{array}$$

$$A \cdot P_0 = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$A \cdot P_1 = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \end{pmatrix}$$

$$A \cdot P_2 = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{pmatrix}$$

$$A \cdot P_3 = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}$$

$$A \cdot P_4 = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

$$A \cdot P_5 = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$$

$$A \cdot P_6 = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$$

$$A \cdot P_7 = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Now we compute $v_i = AP_i - AP_{i+1}$ for $i = 1, 2, \ldots, 5$ and $v_6 := AP_6 - AP_1$.

$$\begin{aligned} v_1 &= (1/3, -2/3, 1/3) \\ v_2 &= (2/3, -1/3, -1/3) \\ v_3 &= (1/3, 1/3, -2/3) \\ v_4 &= (-1/3, 2/3, -1/3) \\ v_5 &= (-2/3, 1/3, 1/3) \\ v_6 &= (-1/3, -1/3, 2/3) \end{aligned}$$

Clearly we have $\|v_i\| = \frac{\sqrt{6}}{3}$. For example,

$$\cos(\theta) = \frac{\langle v_1, v_2 \rangle}{\|v_1\| \cdot \|v_2\|} = 1/2$$

so $\theta = \pi/3 = \frac{2\pi}{6}$. We can find 5 other such angles similarly. We conclude the projection is a regular hexagon.

In the attached txt file, you can find the codes (sagemath) to illustrate this example in `https://cocalc.com/features/sage`. ■

Exercise 6. Let $n \in \mathbb{N}^*$. Prove that for any orthogonal matrix $A = \left(a_{ij}\right)_{1 \leq i,j \leq n} \in O_n(\mathbb{R})$ one has

$$\left| \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij} \right| \leqslant n \leqslant \sum_{i=1}^{n} \sum_{j=1}^{n} \left| m_{ij} \right| \leqslant n^{\frac{3}{2}}$$

Solution. See exercise 19 of chapter 3. ■