

Federated Learning with Convex Global and Local Constraints

Chuan He*

Le Peng*

Ju Sun*

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Abstract

In practice, many machine learning (ML) problems come with constraints, and their applied domains involve distributed sensitive data that cannot be shared with others, e.g., in healthcare. Collaborative learning in such practical scenarios entails federated learning (FL) for ML problems with constraints, or *FL with constraints* for short. Despite the extensive developments of FL techniques in recent years, these techniques only deal with unconstrained FL problems. To fill this gap, we take the first step toward building a general algorithmic framework for solving FL problems with constraints. In particular, we propose a new FL algorithm for constrained ML problems based on the proximal augmented Lagrangian (AL) method. Assuming convex objective and convex constraints plus other mild conditions, we establish the worst-case complexity of the proposed algorithm. Our numerical experiments show the effectiveness of our algorithm in performing Neyman-Pearson classification and fairness-aware learning with nonconvex constraints, in an FL setting.

Keywords: Federated learning, constrained machine learning, augmented Lagrangian, complexity analysis, imbalanced classification, fairness-aware learning

Mathematics Subject Classification: 65Y20 68W15 90C60

1 Introduction

Federated learning (FL) has emerged as a prominent distributed machine learning (ML) paradigm that respects data privacy by design and has found extensive applications in diverse domains [23]. In FL, ML models are trained without centralized training data: local clients hold their local data and never directly share them with other clients or the central server. Given a global ML model to train, typical FL strategies consist of repeated local computation and central aggregation: in each round, each local client performs local computation of quantities of interest (e.g., local model parameters or derivatives) based on the local data, and then the central server collects and aggregates the local results and updates the parameters of the global ML model. Since the shared local results are usually highly nonlinear functions of local data, making reverse engineering of local data unlikely, data privacy is naturally protected.

1.1 Federated learning for constrained machine learning problems

However, existing FL techniques are developed almost exclusively for unconstrained ML problems, despite the growing list of ML problems with constraints—where constraints typically encode prior knowledge and desired properties, e.g., robustness evaluation [16], fairness-aware learning [1], learning with imbalanced data [55], neural architecture search [70], topology optimization [10], physics-informed machine learning [40]. Here, we sketch two quick examples.

*Department of Computer Science and Engineering, University of Minnesota, USA (email: he000233@umn.edu; peng0347@umn.edu; jusun@umn.edu).

Neyman-Pearson classification, or optimizing the false-positive rate with a controlled false-negative rate Conventional binary classification assumes equal importance in both classes, so predictive errors in both classes are counted equally. In numerous applications, such as medical diagnosis, misclassifying one class (i.e., the priority class) is much more costly than misclassifying the other. The Neyman-Pearson classification framework addresses this asymmetry in misclassification cost by explicitly controlling the error rate in the priority class while optimizing that in the other [60, 57, 53, 59]:

$$\min_{\theta} \frac{1}{n_0} \sum_{i=1}^{n_0} \varphi(f_{\theta}, z_{i,0}) \quad \text{s.t.} \quad \frac{1}{n_1} \sum_{i=1}^{n_1} \varphi(f_{\theta}, z_{i,1}) \leq r, \quad (1)$$

where f_{θ} is the trainable binary classifier parameterized by θ , φ is the loss function serving as a proxy to classification error, and $\{z_{i,0}\}_{i=1}^{n_0}$ and $\{z_{i,1}\}_{i=1}^{n_1}$ are the training data from class 0 and 1, respectively. The constraint imposes an upper bound on the error rate for class 1.

Fairness-aware learning Typical ML models are known to have biases toward the majority subgroups of the input space [1, 9, 42]. For example, a disease diagnostic model that is trained on a male-dominant dataset tends to predict much more accurately on the male subgroup than on the female subgroup. To counteract such potential model biases, a natural way is to enforce fairness constraints to ensure that the performance of the model on different subgroups is comparable [1, 9, 42]. A possible formulation for two-subgroup problems is

$$\min_{\theta} \frac{1}{n'} \sum_{i=1}^{n'} \varphi(f_{\theta}, z_i) \quad \text{s.t.} \quad -\delta \leq \frac{1}{|\mathcal{S}_0|} \sum_{i \in \mathcal{S}_0} \varphi(f_{\theta}, z_i) - \frac{1}{|\mathcal{S}_1|} \sum_{i \in \mathcal{S}_1} \varphi(f_{\theta}, z_i) \leq \delta, \quad (2)$$

where f_{θ} is the ML model parameterized by θ , $\{z_i\}$ is the training set, and φ is the proxy loss function, similar to the setup in Eq. (1). With \mathcal{S}_0 and \mathcal{S}_1 denoting the two subgroups of interest, the constraint imposes that the performance disparity of f_{θ} on \mathcal{S}_0 and \mathcal{S}_1 should not be larger than $\delta > 0$, which is usually set close to 0.

Both examples are particularly relevant to biomedical problems, where class imbalance and subgroup imbalance are prevalent. Moreover, there are strict regulations on the distribution and centralization of biomedical data for research, e.g., the famous Health Insurance Portability and Accountability Act (HIPAA) protection of patient privacy [47]. Together, these underscore the importance of developing FL techniques for constrained ML problems, which is largely lacking: a small number of papers have tried to mitigate class imbalance [58] and improve model fairness [13, 11, 14] through constrained optimization in FL settings. However, these developments are specialized to their particular use cases and lack computational guarantees for the feasibility and optimality of their solutions.

In this paper, we take the first step toward a general and rigorous FL framework for constrained ML problems. Consider the constrained ML problems in Eqs. (1) and (2) in an FL setting with n local clients, where the i^{th} client holds local data Z_i and so the whole training set is the union $Z_1 \cup \dots \cup Z_n$. Since the objectives and constraints in Eqs. (1) and (2) are in the finite-sum form, both examples are special cases of the following finite-sum constrained optimization problem:

$$\min_{\theta} \left\{ \sum_{i=1}^n f_i(\theta; Z_i) + h(\theta) \right\} \quad \text{s.t.} \quad \underbrace{\sum_{i=1}^n \tilde{c}_i(\theta; Z_i)}_{\text{local data coupled}} \leq 0. \quad (3)$$

Note that inside the constraint, the local data are coupled which necessarily lead to communication between the local clients and the central server to allow the evaluation of the constraint function. To

try to reduce such communication so that we can have more flexibility in algorithm design, we introduce decoupling variables $\{s_i\}$, leading to an equivalent formulation:

$$\min_{\theta, s_i} \left\{ \sum_{i=1}^n f_i(\theta; Z_i) + h(\theta) \right\} \quad \text{s.t.} \quad \underbrace{\sum_{i=1}^n s_i \leq 0}_{\text{no local data}}, \quad \underbrace{\tilde{c}_i(\theta; Z_i) \leq s_i, \quad 1 \leq i \leq n}_{\text{local data decoupled}}, \quad (4)$$

which decouples the local data into n local constraints.

In this paper, we consider the following setup for FL with global and local constraints, a strict generalization of Eq. (4):

$$\min_w \left\{ \sum_{i=1}^n f_i(w; Z_i) + h(w) \right\} \quad \text{s.t.} \quad \underbrace{c_0(w; Z_0) \leq 0}_{\text{global constraint}}, \quad \underbrace{c_i(w; Z_i) \leq 0, \quad 1 \leq i \leq n}_{\text{local constraints}}. \quad (5)$$

Here, we assume n local clients, each with a local objective $f_i(w; Z_i)$ and a set of local constraints $c_i(w; Z_i) \leq 0$ (i.e., scalar constraints are vectorized) for $i = 1, \dots, n$. To stress the FL setting, we spell out the dependency of these local objectives and local constraints on the local data Z_i 's: *these Z_i 's are only accessible to their respective local clients and should never be shared with other local clients or the central server*. Henceforth, we omit Z_i 's in local objectives and constraints when no confusion arises. To allow flexibility in modeling, we also include the global constraint $c_0(w; Z_0) \leq 0$, with central data Z_0 that is only accessible by the central server. To facilitate the theoretical study of the algorithm that we develop, we further assume that all objective and scalar constraint functions are convex, but we also verify the applicability of our FL algorithm to classification problems with nonconvex fairness constraints in Section 5.2. To summarize, our standing assumptions on top of Eq. (5) include

Assumption 0. *We make the following assumption throughout this paper:*

- (a) *The objective functions $f_i : \mathbb{R}^d \rightarrow \mathbb{R}$, $1 \leq i \leq n$, and the m_i scalar constraint functions inside $c_i : \mathbb{R}^d \rightarrow \mathbb{R}^{m_i}$, $0 \leq i \leq n$, are convex and continuously differentiable, and $h : \mathbb{R}^d \rightarrow (-\infty, \infty]$ is a simple closed convex function.*
- (b) *For each $1 \leq i \leq n$, only the local objective f_i and local constraint c_i have access to the local data Z_i , which are never shared with other local clients and the central server. Only the central server has access to the global data Z_0 ,*

1.2 Our contributions

This paper tackles Eq. (5) by adopting the sequential penalization approach, which involves solving a sequence of unconstrained subproblems that combine the objective function with penalization of constraint violations. In particular, we propose an FL algorithm based on the proximal augmented Lagrangian (AL) method. In each iteration, the unconstrained subproblem is solved by an inexact solver based on the alternating direction method of multipliers (ADMM) in a federated manner. We study the worst-case complexity of the proposed algorithm assuming *locally* Lipschitz continuous gradients. Our main contributions are highlighted below.

- We propose an FL algorithm (Algorithm 1) for solving Eq. (5) based on the proximal AL method. To the best of our knowledge, the proposed algorithm is the first in solving general constrained ML problems in an FL setting. Assuming *locally* Lipschitz continuous gradients and other mild conditions, we establish its worst-case complexity to find an approximate optimal solution of Eq. (5). The complexity results are entirely new in the literature.

- We propose an ADMM-based inexact solver in [Algorithm 2](#) to solve the unconstrained subproblems arising in [Algorithm 1](#). We equip this inexact solver with a newly introduced verifiable termination criterion and establish its global linear convergence assuming strong convexity and *locally* Lipschitz continuous gradients.
- We perform numerical experiments to compare our proposed FL algorithm ([Algorithm 1](#)) with the centralized proximal AL method ([Algorithm 3](#)) on binary Neyman-Pearson classification and classification with nonconvex fairness constraints using real-world datasets ([Section 5](#)). Our numerical results demonstrate that our FL algorithm can achieve solution quality comparable to that of the centralized proximal AL method.

1.3 Related work

FL algorithms for unconstrained optimization FL has emerged as a cornerstone for privacy-preserved learning since Google’s seminal work [41], and has found applications in numerous domains where the protection of data privacy precludes centralized learning, including healthcare [52, 49, 50], finance [36], Internet of things [43], and transportation [35]. FedAvg [41] is the first and also the most popular FL algorithm to date. After FedAvg, numerous FL algorithms have been proposed to improve performance and address practical issues, such as data heterogeneity [24, 31, 68], system heterogeneity [30, 63, 15], fairness [29], communication efficiency [56, 25], and incentives [61]. More FL algorithms and their applications can be found in the survey [28]. Despite the intensive research on FL, existing algorithms focus primarily on unconstrained ML problems, versus constrained ML problems considered in this paper.

Centralized algorithms for constrained optimization Recent decades have seen fruitful algorithm developments for centralized constrained optimization in numerical optimization. In particular, there has been a rich literature on AL methods for solving convex constrained optimization problems [4, 44, 48, 65, 27, 39, 38]. In addition, variants of AL methods have been developed to solve nonconvex constrained optimization problems [21, 17, 6, 26, 32, 19, 20, 37]. Besides AL methods and their variants, sequential quadratic programming methods [7, 12], trust-region methods [8, 51], interior point methods [62], and extra-point methods [22] have been proposed to solve centralized constrained optimization problems.

Distributed algorithms for constrained optimization Developing distributed algorithms for constrained optimization has started relatively recently. To handle simple local constraints in distributed optimization, [45, 33, 64] study distributed projected subgradient methods. For complicated conic local constraints, [2, 3] develop distributed primal-dual algorithms. For distributed optimization with global and local constraints, [69, 67] develop primal-dual projected subgradient algorithms. For an overview of distributed constrained optimization, see [66]. However, these distributed algorithms are only applicable to problems where global constraints or local constraints are simple, and therefore cannot be applied directly to our setup [Eq. \(5\)](#) with general global and local constraints. The algorithm development in this paper follows a distinct trajectory.

FL algorithms for constrained ML applications A small number of papers have developed FL algorithms for particular constrained ML applications, such as learning with class imbalance and fairness-aware ML. For example, [58, 11] propose FL algorithms to address class imbalance and subgroup imbalance, respectively, by optimizing the Lagrangian function. [13] applies quadratic penalty method to deal with the constraint in fairness-aware ML. In addition, [14] proposes an FL algorithm to tackle fairness-aware

ML based on optimizing the AL function. However, these developments are tailored to specific applications and lack rigorous computational guarantees regarding the feasibility and optimality of their solutions. In contrast, this paper focuses on developing algorithms with theoretical guarantees for FL with convex global and local constraints. To the best of our knowledge, this work provides the first general FL framework for constrained ML problems.

2 Notation and preliminaries

Throughout this paper, we let \mathbb{R}^d and \mathbb{R}_+^d denote the d -dimensional Euclidean space and its nonnegative orthant, respectively. We use $\langle \cdot, \cdot \rangle$ to denote the standard inner product, $\|\cdot\|$ to denote the Euclidean norm of a vector or the spectral norm of a matrix, and $\|\cdot\|_\infty$ to denote the ℓ_∞ -norm of a vector. For any vector $v \in \mathbb{R}^d$, $[v]_+ \in \mathbb{R}^d$ is its nonnegative part (i.e., with all negative values set to zero). We adopt the standard big-O notation $\mathcal{O}(\cdot)$ to present complexity results; $\tilde{\mathcal{O}}(\cdot)$ represents $\mathcal{O}(\cdot)$ with logarithmic terms omitted.

Given a closed convex function $h : \mathbb{R}^d \rightarrow (-\infty, \infty]$, ∂h and $\text{dom}(h)$ denote the subdifferential and domain of h , respectively. The proximal operator associated with h is denoted by prox_h , that is, $\text{prox}_h(u) = \arg \min_w \{\|w - u\|^2/2 + h(w)\}$ for all $u \in \mathbb{R}^d$. Given a continuously differentiable mapping $\phi : \mathbb{R}^d \rightarrow \mathbb{R}^p$, we write the transpose of its Jacobian as $\nabla \phi(w) = [\nabla \phi_1(w) \ \cdots \ \nabla \phi_p(w)] \in \mathbb{R}^{d \times p}$. We say that $\nabla \phi$ is L -Lipschitz continuous on a set Ω for some $L > 0$ if $\|\nabla \phi(u) - \nabla \phi(v)\| \leq L\|u - v\|$ for all $u, v \in \Omega$. In addition, we say that $\nabla \phi$ is locally Lipschitz continuous on Ω if for any $w \in \Omega$, there exist $L_w > 0$ and an open set \mathcal{U}_w containing w such that $\nabla \phi$ is L_w -Lipschitz continuous on \mathcal{U}_w .

Given a nonempty closed convex set $\mathcal{C} \subseteq \mathbb{R}^d$ and any point $u \in \mathbb{R}^d$, $\text{dist}(u, \mathcal{C})$ and $\text{dist}_\infty(u, \mathcal{C})$ stand for the Euclidean distance and the Chebyshev distance from u to \mathcal{C} , respectively. That is, $\text{dist}(u, \mathcal{C}) = \min_{v \in \mathcal{C}} \|u - v\|$ and $\text{dist}_\infty(u, \mathcal{C}) = \min_{v \in \mathcal{C}} \|u - v\|_\infty$. The normal cone of \mathcal{C} at $u \in \mathcal{C}$ is denoted by $\mathcal{N}_{\mathcal{C}}(u)$. The Minkowski sum of two sets \mathcal{B} and \mathcal{C} is defined as $\mathcal{B} + \mathcal{C} := \{b + c : b \in \mathcal{B}, c \in \mathcal{C}\}$.

For ease of presentation, we let $m := \sum_{i=0}^n m_i$ and adopt the following notations throughout this paper:

$$f(w) = \sum_{i=1}^n f_i(w), \quad c(w) = \begin{bmatrix} c_0(w) \\ \vdots \\ c_n(w) \end{bmatrix} \in \mathbb{R}^m, \quad \mu = \begin{bmatrix} \mu_0 \\ \vdots \\ \mu_n \end{bmatrix} \in \mathbb{R}^m. \quad (6)$$

Assumption 1. *Throughout this paper, we assume that the strong duality holds for Eq. (5) and its dual problem*

$$\sup_{\mu \geq 0} \inf_w \{f(w) + h(w) + \langle \mu, c(w) \rangle\}. \quad (7)$$

That is, both problems have optimal solutions and, moreover, their optimal values coincide.

Under Assumption 1, it is known that $(w, \mu) \in \text{dom}(h) \times \mathbb{R}_+^m$ is a pair of optimal solutions of Eq. (5) and Eq. (7) if and only if it satisfies (see, e.g., [39])

$$0 \in \begin{pmatrix} \nabla f(w) + \partial h(w) + \nabla c(w)\mu \\ c(w) - \mathcal{N}_{\mathbb{R}_+^m}(\mu) \end{pmatrix}. \quad (8)$$

In general, it is hard to find an exact optimal solution of Eq. (5) and Eq. (7). Thus, we are instead interested in seeking an approximate optimal solution of Eq. (5) and Eq. (7) defined as follows.

Definition 1. Given any $\epsilon_1, \epsilon_2 > 0$, we say $(w, \mu) \in \text{dom}(h) \times \mathbb{R}_+^m$ is an (ϵ_1, ϵ_2) -optimal solution of Eq. (5) and Eq. (7) if $\text{dist}_\infty(0, \nabla f(w) + \partial h(w) + \nabla c(w)\mu) \leq \epsilon_1$ and $\text{dist}_\infty(c(w), \mathcal{N}_{\mathbb{R}_+^m}(\mu)) \leq \epsilon_2$.

Here, the two different tolerances ϵ_1, ϵ_2 are used for measuring stationarity and feasibility violation, respectively. This definition is consistent with the ϵ -KKT solution considered in [39] except that Definition 1 uses the Chebyshev distance rather than the Euclidean distance.

3 A proximal AL based FL algorithm for solving Eq. (5)

In this section, we propose an FL algorithm for solving Eq. (5) based on the proximal AL method. Specifically, we describe this algorithm in Section 3.1, and then analyze its complexity results in Section 3.2.

Assumption 2. *Throughout this section, we assume that*

- (a) *The proximal operator for h can be exactly evaluated.*
- (b) *The gradients ∇f_i , $1 \leq i \leq n$, and the transposed Jacobians ∇c_i , $0 \leq i \leq n$, are locally Lipschitz continuous on \mathbb{R}^d .*

Assumption 2(b) clearly holds if all ∇f_i 's and ∇c_i 's are globally Lipschitz continuous on \mathbb{R}^d , but this assumption holds for a broad class of problems without global Lipschitz continuity on ∇f_i 's and ∇c_i 's. For example, the quadratic penalty function of $c(w) \leq 0$, namely $\|[c(w)]_+\|^2$, only has a locally Lipschitz continuous gradient even if ∇c is globally Lipschitz continuous on \mathbb{R}^d (see Remark 6). In addition, the gradient of a convex high-degree polynomial, such as $\|w\|^4$ with $w \in \mathbb{R}^d$, is locally Lipschitz continuous but not globally Lipschitz continuous on \mathbb{R}^d .

3.1 Algorithm description

In this subsection, we describe a proximal AL-based FL algorithm (Algorithm 1) for finding an (ϵ_1, ϵ_2) -optimal solution of Eq. (5) for prescribed $\epsilon_1, \epsilon_2 \in (0, 1)$. This algorithm follows a framework similar to a centralized proximal AL method (see Appendix D). At each iteration, it applies an inexact ADMM solver (Algorithm 2) to find an approximate solution w^{k+1} to the proximal AL subproblem associated with Eq. (5):

$$\min_w \left\{ \ell_k(w) := \sum_{i=1}^n f_i(w) + h(w) + \frac{1}{2\beta} \sum_{i=0}^n \left(\|[\mu_i^k + \beta c_i(w)]_+\|^2 - \|\mu_i^k\|^2 \right) + \frac{1}{2\beta} \|w - w^k\|^2 \right\}. \quad (9)$$

Then, the multiplier estimates are updated according to the classical scheme:

$$\mu_i^{k+1} = [\mu_i^k + \beta c_i(w^{k+1})]_+, \quad 0 \leq i \leq n.$$

Notice that the subproblem in Eq. (9) can be rewritten as

$$\min_w \left\{ \ell_k(w) := \sum_{i=0}^n P_{i,k}(w) + h(w) \right\}, \quad (12)$$

where $P_{i,k}$, $0 \leq i \leq n$, are defined as

$$P_{0,k}(w) := \frac{1}{2\beta} \left(\|[\mu_0^k + \beta c_0(w)]_+\|^2 - \|\mu_0^k\|^2 \right) + \frac{1}{2(n+1)\beta} \|w - w^k\|^2, \quad (13)$$

$$P_{i,k}(w) := f_i(w) + \frac{1}{2\beta} \left(\|[\mu_i^k + \beta c_i(w)]_+\|^2 - \|\mu_i^k\|^2 \right) + \frac{1}{2(n+1)\beta} \|w - w^k\|^2, \quad \forall 1 \leq i \leq n. \quad (14)$$

When Algorithm 2 (see Section 4) is applied to solve Eq. (12), the local merit function $P_{i,k}$, constructed from the local objective f_i and local constraint c_i , is handled by the respective local client i , while the

Algorithm 1 A proximal AL based FL algorithm for solving Eq. (5)

Input: tolerances $\epsilon_1, \epsilon_2 \in (0, 1)$, $w^0 \in \text{dom}(h)$, $\mu_i^0 \geq 0$ for $0 \leq i \leq n$, $\bar{s} > 0$, and $\beta > 0$.

for $k = 0, 1, 2, \dots$ **do**

Set $\tau_k = \bar{s}/(k+1)^2$.

Call Algorithm 2 (see Section 4 below) with $(\tau, \tilde{w}^0) = (\tau_k, w^k)$ to find an approximate solution w^{k+1} to Eq. (12) in a federated manner such that

$$\text{dist}_\infty(0, \partial \ell_k(w^{k+1})) \leq \tau_k. \quad (10)$$

Server update: The central server updates $\mu_0^{k+1} = [\mu_0^k + \beta c_0(w^{k+1})]_+$.

Communication (broadcast): Each local client i , $1 \leq i \leq n$, receives w^{k+1} from the central server.

Client update (local): Each local client i , $1 \leq i \leq n$, updates $\mu_i^{k+1} = [\mu_i^k + \beta c_i(w^{k+1})]_+$.

Communication: Each local client i , $1 \leq i \leq n$, sends $\|\mu_i^{k+1} - \mu_i^k\|_\infty$ to the central server.

Termination (server side): Output (w^{k+1}, μ^{k+1}) and terminate the algorithm if

$$\|w^{k+1} - w^k\|_\infty + \beta \tau_k \leq \beta \epsilon_1, \quad \max_{0 \leq i \leq n} \{\|\mu_i^{k+1} - \mu_i^k\|_\infty\} \leq \beta \epsilon_2. \quad (11)$$

end for

merit function $P_{0,k}$ is handled by the central server. We observe that Algorithm 1 with the subproblem in Eq. (12) solved by Algorithm 2 meets the basic FL requirement: since local objective f_i 's and local constraint c_i 's are handled by their respective local clients and the central server only performs aggregation and handles the global constraint c_0 , no raw data are shared between the local clients and the central server, i.e., Assumption 0(b) is obeyed.

For ease of later reference, we refer to the update from w^k to w^{k+1} as one outer iteration of Algorithm 1, and call one iteration of Algorithm 2 for solving Eq. (9) one inner iteration of Algorithm 1. In the rest of this section, we study the following measures of complexity for Algorithm 1.

- *Outer iteration complexity*, which measures the number of outer iterations of Algorithm 1;
- *Total inner iteration complexity*, which measures the total number of iterations of Algorithm 2 that are performed in Algorithm 1.

The following theorem concerns the output of Algorithm 1, whose proof is deferred to Appendix A.1.

Theorem 2 (output of Algorithm 1). *If Algorithm 1 successfully terminates, its output (w^{k+1}, μ^{k+1}) is an (ϵ_1, ϵ_2) -optimal solution of Eq. (5).*

3.2 Complexity analysis

In this subsection, we establish the outer and total inner iteration complexity for Algorithm 1. To proceed, we let (w^*, μ^*) be any pair of optimal solutions of Eq. (5) and Eq. (7). First, we establish a lemma to show that all iterates generated by Algorithm 1 are bounded. Its proof can be found in Appendix A.2.

Lemma 3 (bounded iterates of Algorithm 1). *Suppose that Assumptions 0 to 2 hold. Let $\{w^k\}_{k \geq 0}$ be all the iterates generated by Algorithm 1. Then we have $w^k \in \mathcal{Q}_1$ for all $k \geq 0$, where*

$$\mathcal{Q}_1 := \{w \in \mathbb{R}^d : \|w - w^*\| \leq r_0 + 2\sqrt{n\bar{s}\beta}\} \quad \text{with } r_0 := \|(w^0, \mu^0) - (w^*, \mu^*)\|, \quad (15)$$

and w^0, μ^0, \bar{s} , and β are inputs of Algorithm 1.

The following theorem states the worst-case complexity results of [Algorithm 1](#), whose proof is relegated to [Appendix A.3](#).

Theorem 4 (complexity results of [Algorithm 1](#)). *Suppose that Assumptions 0 to 2 hold. Then,*

- (a) *the number of outer iteration of [Algorithm 1](#) is at most $\mathcal{O}(\max\{\epsilon_1^{-2}, \epsilon_2^{-2}\})$; and*
- (b) *the total number of inner iterations of [Algorithm 1](#) is at most $\tilde{\mathcal{O}}(\max\{\epsilon_1^{-2}, \epsilon_2^{-2}\})$.*

To the best of our knowledge, [Theorem 4](#) provides the first worst-case complexity results for finding an approximate optimal solution of [Eq. \(5\)](#) in an FL framework.

3.3 Communication overheads

In the outer loop of [Algorithm 1](#), a single communication round occurs after solving a proximal AL subproblem. During this round, the central server sends the current weights w^{k+1} to all local clients, and each client sends back the maximum change in their respective multipliers, measured by $\|\mu_i^{k+1} - \mu_i^k\|_\infty$, to the central server. The communication overheads of the inner solver [Algorithm 2](#) are discussed in [Section 4.3](#).

4 An inexact ADMM for FL

In this section, we propose an inexact ADMM-based FL algorithm to solve the subproblem in [Eq. \(12\)](#) (the same as [Eq. \(9\)](#)) for [Algorithm 1](#). Before proceeding, we show that $\nabla P_{i,k}$, $0 \leq i \leq n$, are locally Lipschitz continuous on \mathbb{R}^d , whose proof is deferred to [Appendix B.1](#).

Lemma 5 (local Lipschitz continuity of $\nabla P_{i,k}$). *Suppose that Assumptions 0 to 2 hold. Then the gradients $\nabla P_{i,k}$, $0 \leq i \leq n$, are locally Lipschitz continuous on \mathbb{R}^d .*

Remark 6. It is worth noting that $\nabla P_{i,k}$, $0 \leq i \leq n$, are typically not globally Lipschitz continuous on \mathbb{R}^d even if ∇f_i , $1 \leq i \leq n$, and ∇c_i , $0 \leq i \leq n$, are globally Lipschitz continuous on \mathbb{R}^d . For example, consider $c_0(w) = \|w\|^2 - 1$. By [Eq. \(13\)](#), one has that

$$\nabla P_{0,k}(w) = 2[\mu_0^k + \beta(\|w\|^2 - 1)]_+ w + \frac{1}{(n+1)\beta}(w - w^k).$$

In this case, it is not hard to verify that ∇c_0 is globally Lipschitz continuous on \mathbb{R}^d , but $\nabla P_{0,k}$ is not.

Moreover, it is easy to see that $P_{i,k}$ are strongly convex with the modulus $1/[(n+1)\beta]$ for all $0 \leq i \leq n$ and all $k \geq 0$.

Since both the local Lipschitz and the strong convexity (including its modulus) properties hold for all $k \geq 0$, and we need to solve the subproblem of the same form each k , below we drop k and focus on solving the following model problem in an FL manner:

$$\min_w \left\{ \ell(w) := \sum_{i=0}^n P_i(w; Z_i) + h(w) \right\}, \quad (16)$$

where the data Z_i 's are only accessible to their corresponding local/global functions P_i 's, necessitating FL. We will drop Z_i 's henceforth for simplicity. The model problem in [Eq. \(16\)](#) satisfies:

1. The functions P_i , $0 \leq i \leq n$, are continuously differentiable, and moreover, ∇P_i , $0 \leq i \leq n$, are locally Lipschitz continuous on \mathbb{R}^d ;
2. The functions P_i , $0 \leq i \leq n$, are strongly convex with a modulus $\sigma > 0$ on \mathbb{R}^d , that is,

$$\langle \nabla P_i(u) - \nabla P_i(v), u - v \rangle \geq \sigma \|u - v\|^2, \quad \forall u, v \in \mathbb{R}^d, \quad 0 \leq i \leq n. \quad (17)$$

4.1 Algorithm description

Algorithm 2 An inexact ADMM based FL algorithm for solving Eq. (16)

Input: tolerance $\tau \in (0, 1]$, $q \in (0, 1)$, $\tilde{w}^0 \in \text{dom}(h)$, and $\rho_i > 0$ for $1 \leq i \leq n$;

Set $w^0 = \tilde{w}^0$, and $(u_i^0, \lambda_i^0, \tilde{u}_i^0) = (\tilde{w}^0, -\nabla P_i(\tilde{w}^0), \tilde{w}^0 - \nabla P_i(\tilde{w}^0)/\rho_i)$ for $1 \leq i \leq n$.

for $t = 0, 1, 2, \dots$ **do**

Set $\varepsilon_{t+1} = q^t$;

Server update: The central server finds an approximate solution w^{t+1} to

$$\min_w \left\{ \varphi_{0,t}(w) := P_0(w) + h(w) + \sum_{i=1}^n \left[\frac{\rho_i}{2} \|\tilde{u}_i^t - w\|^2 \right] \right\} \quad (18)$$

such that $\text{dist}_\infty(0, \partial\varphi_{0,t}(w^{t+1})) \leq \varepsilon_{t+1}$.

Communication (broadcast): Each local client i , $1 \leq i \leq n$, receives w^{t+1} from the server.

Client update (local): Each local client i , $1 \leq i \leq n$, finds an approximate solution u_i^{t+1} to

$$\min_{u_i} \left\{ \varphi_{i,t}(u_i) := P_i(u_i) + \langle \lambda_i^t, u_i - w^{t+1} \rangle + \frac{\rho_i}{2} \|u_i - w^{t+1}\|^2 \right\} \quad (19)$$

such that $\|\nabla\varphi_{i,t}(u_i^{t+1})\|_\infty \leq \varepsilon_{t+1}$, and then updates

$$\lambda_i^{t+1} = \lambda_i^t + \rho_i(u_i^{t+1} - w^{t+1}), \quad (20)$$

$$\tilde{u}_i^{t+1} = u_i^{t+1} + \lambda_i^{t+1}/\rho_i, \quad (21)$$

$$\tilde{\varepsilon}_{i,t+1} = \|\nabla\varphi_{i,t}(w^{t+1}) - \rho_i(w^{t+1} - u_i^t)\|_\infty. \quad (22)$$

Communication: Each local client i , $1 \leq i \leq n$, sends $(\tilde{u}_i^{t+1}, \tilde{\varepsilon}_{i,t+1})$ back to the central server.

Termination (server side): Output w^{t+1} and terminate this algorithm if

$$\varepsilon_{t+1} + \sum_{i=1}^n \tilde{\varepsilon}_{i,t+1} \leq \tau. \quad (23)$$

end for

In this subsection, we propose an inexact ADMM-based FL algorithm (Algorithm 2) for solving Eq. (16). To make each participating client i handle their local objective P_i independently (see Section 3.1), we introduce decoupling variables u_i 's and obtain the following equivalent consensus reformulation for Eq. (16):

$$\min_{w, u_i} \left\{ \sum_{i=1}^n P_i(u_i) + P_0(w) + h(w) \right\} \quad \text{s.t.} \quad u_i = w, \quad 1 \leq i \leq n, \quad (24)$$

which allows each local client i to handle the local variable u_i and the local objective function P_i while imposing consensus constraints that force clients' local parameters u_i equal to the global parameter w . This reformulation enables the applicability of an inexact ADMM that solves Eq. (24) in a federated manner. At each iteration, an ADMM solver optimizes the AL function associated with Eq. (24):

$$\mathcal{L}_P(w, u, \lambda) := \sum_{i=1}^n \left[P_i(u_i) + \langle \lambda_i, u_i - w \rangle + \frac{\rho_i}{2} \|u_i - w\|^2 \right] + P_0(w) + h(w) \quad (25)$$

with respect to the variables w , u , and λ alternately, where $u = [u_1^T, \dots, u_n^T]^T$ and $[\lambda_1^T, \dots, \lambda_n^T]^T$ collect all the local parameters and the multipliers associated with the consensus constraints, respectively. Specifically,

in iteration t , one performs

$$w^{t+1} \approx \arg \min_w \mathcal{L}_P(w, u^t, \lambda^t), \quad (26)$$

$$u^{t+1} \approx \arg \min_u \mathcal{L}_P(w^{t+1}, u, \lambda^t), \quad (27)$$

$$\lambda_i^{t+1} = \lambda_i^t + \rho_i(u_i^{t+1} - w^{t+1}), \quad \forall 1 \leq i \leq n. \quad (28)$$

By the definition of \mathcal{L}_P in Eq. (25), one can verify that the step in Eq. (26) is equivalent to Eq. (18), and also the step in Eq. (27) can be computed in parallel, which corresponds to Eq. (19). Therefore, the ADMM updates naturally suit the FL framework, as the separable structure in Eq. (25) over the pairs $\{(u_i, \lambda_i)\}$ enables the local update of (u_i, λ_i) at each client i while w is updated by the central server.

Since the subproblems in Eq. (18) and Eq. (19) are strongly convex, their approximate solutions w^{t+1} and u_i^{t+1} , $1 \leq i \leq n$, can be found using a gradient-based algorithm with a global linear convergence rate [46]. Furthermore, the value $\tilde{\varepsilon}_{i,t+1}$ in Eq. (22) serves as a measure of local optimality and consensus for client i . By summing up $\tilde{\varepsilon}_{i,t+1}$ for $1 \leq i \leq n$ and including ε_{t+1} , one can obtain a stationarity measure for the current iterate (see (Eq. (23))), as presented in the following theorem. Its proof can be found in Appendix B.2.

Theorem 7 (output of Algorithm 2). *If Algorithm 2 terminates at some iteration $T \geq 0$, then its output w^{T+1} satisfies $\text{dist}_\infty(0, \partial \ell(w^{T+1})) \leq \tau$.*

Theorem 7 states that Algorithm 2 outputs a point that approximately satisfies the first-order optimality condition of Eq. (5). In addition, it follows from Theorem 7 that Algorithm 2 with $(\tau, \tilde{w}^0) = (\tau_k, w^k)$ finds an approximate solution w^{k+1} to Eq. (12) such that Eq. (10) holds.

4.2 Complexity analysis

In this subsection, we establish the iteration complexity for the inexact ADMM, namely, Algorithm 2. Recall from Eq. (17) that Eq. (16) is strongly convex and thus has a unique optimal solution. We refer to this optimal solution of Eq. (16) as \tilde{w}^* throughout this section. The following lemma shows that all the iterates generated by Algorithm 2 lie in a compact set. Its proof can be found in Appendix B.3.

Lemma 8 (bounded iterates of Algorithm 2). *Suppose that Assumptions 0 to 2 hold and let $\{u_i^{t+1}\}_{1 \leq i \leq n, t \geq 0}$ and $\{w^{t+1}\}_{t \geq 0}$ be all the iterates generated by Algorithm 2. Then it holds that all these iterates stay in a compact set \mathcal{Q} , where*

$$\mathcal{Q} := \left\{ v : \|v - \tilde{w}^*\|^2 \leq \frac{n+1}{\sigma^2(1-q^2)} + \frac{1}{\sigma} \sum_{i=1}^n \left(\rho_i \|\tilde{w}^* - \tilde{w}^0\|^2 + \frac{1}{\rho_i} \|\nabla P_i(\tilde{w}^*) - \nabla P_i(\tilde{w}^0)\|^2 \right) \right\}. \quad (29)$$

The iteration complexity of Algorithm 2 is established in the following theorem, whose proof is relegated to Appendix B.4.

Theorem 9 (iteration complexity of Algorithm 2). *Suppose that Assumptions 0 to 2 hold. Then Algorithm 2 terminates in at most $\mathcal{O}(|\log \tau|)$ iterations.*

In other words, Algorithm 2 enjoys a global linear convergence rate when solving the problem in Eq. (16). The result generalizes classical convergence results for ADMM in the literature, which typically require a strongly convex objective with *globally* Lipschitz continuous gradient (e.g., see [34]). In contrast, our result is the first to establish a global linear convergence of an inexact ADMM assuming a strongly convex objective with only a *locally* Lipschitz continuous gradient.

4.3 Communication overheads

In each iteration of [Algorithm 2](#), a single communication round happens between the clients and the central server. During this round, the central server transmits the global weight w^{t+1} to all clients, and subsequently each local client performs multiple local updates to solve a local subproblem and then sends the updated local weights \tilde{u}_i^{t+1} and a local stationarity measure $\tilde{\varepsilon}_{i,t+1}$ back to the central server.

5 Numerical experiments

Here, we conduct numerical experiments to evaluate the performance of our proposed FL algorithm ([Algorithm 1](#)). Specifically, we benchmark our algorithm against a centralized proximal AL method (cProx-AL, described in [Algorithm 3](#)) on a convex Neyman-Pearson classification problem ([Section 5.1](#)) and a fair-aware learning problem ([Section 5.2](#)) with real-world datasets, and further on linear-equality-constrained quadratic programming problems with simulated data ([Appendix E.2](#)). All experiments are carried out on a Windows system with an AMD EPYC 7763 64-core processor, and all algorithms are implemented in Python. The code to implement the proposed algorithm on these numerical examples is available at https://github.com/PL97/Constr_FL.

Table 1: Numerical results for solving [Eq. \(30\)](#) using our algorithm vs. using cProx-AL. Inside the parentheses are the respective standard deviations over 10 random trials. For feasibility, we include the mean and maximum losses for class 1 among all local clients.

dataset	n	objective value (loss for class 0)			feasibility (loss for class 1 (≤ 0.2))			
		Algorithm 1	cProx-AL	relative difference	Algorithm 1	cProx-AL		
					mean	max	mean	max
breast-cancer-wisc	1	0.27 (1.52e-04)	0.27 (3.02e-05)	7.09e-04 (2.02e-04)	0.20 (1.80e-07)	0.20 (1.80e-07)	0.20 (1.84e-08)	0.20 (1.84e-08)
	5	0.34 (4.50e-02)	0.33 (4.55e-02)	1.15e-02 (5.17e-03)	0.19 (7.33e-06)	0.20 (1.08e-06)	0.19 (1.13e-04)	0.20 (1.72e-05)
	10	0.37 (1.08e-01)	0.37 (1.08e-01)	3.92e-04 (2.76e-04)	0.17 (1.15e-05)	0.20 (6.05e-09)	0.17 (1.14e-05)	0.20 (2.95e-08)
	20	0.46 (2.12e-01)	0.45 (2.12e-01)	3.43e-02 (2.91e-02)	0.16 (3.52e-05)	0.20 (3.76e-06)	0.16 (7.03e-06)	0.20 (7.70e-08)
adult-a	1	0.73 (2.19e-04)	0.73 (1.25e-04)	2.24e-04 (3.46e-04)	0.20 (6.30e-07)	0.20 (6.30e-07)	0.20 (1.73e-06)	0.20 (1.73e-06)
	5	0.74 (1.03e-02)	0.74 (1.03e-02)	4.25e-03 (7.44e-04)	0.20 (2.14e-04)	0.20 (2.80e-04)	0.20 (1.21e-05)	0.20 (2.28e-06)
	10	0.77 (1.98e-02)	0.77 (1.98e-02)	2.69e-03 (3.24e-03)	0.19 (6.41e-05)	0.20 (9.76e-05)	0.19 (2.00e-05)	0.20 (1.23e-05)
	20	0.78 (2.86e-02)	0.79 (2.81e-02)	1.13e-02 (4.11e-03)	0.18 (6.40e-04)	0.20 (6.59e-05)	0.18 (1.96e-05)	0.20 (3.19e-06)
monks-1	1	1.58 (7.61e-05)	1.58 (7.50e-05)	1.39e-05 (1.09e-05)	0.20 (1.09e-07)	0.20 (1.09e-07)	0.20 (3.01e-07)	0.20 (3.01e-07)
	5	1.65 (8.39e-02)	1.65 (8.41e-02)	2.08e-04 (1.84e-04)	0.19 (6.39e-05)	0.20 (5.39e-05)	0.19 (5.04e-06)	0.20 (5.60e-07)
	10	1.71 (1.18e-01)	1.71 (1.18e-01)	4.59e-04 (3.32e-04)	0.18 (3.98e-05)	0.20 (4.46e-05)	0.18 (6.44e-06)	0.20 (1.60e-06)
	20	1.81 (1.49e-01)	1.79 (1.60e-01)	1.78e-02 (1.38e-02)	0.17 (1.68e-04)	0.20 (2.24e-04)	0.17 (4.60e-06)	0.20 (1.62e-06)

5.1 Neyman-Pearson classification

In this subsection, we consider the Neyman-Pearson classification problem:

$$\min_w \frac{1}{n} \sum_{i=1}^n \frac{1}{m_{i0}} \sum_{j=1}^{m_{i0}} \phi(w; (x_j^{(i0)}, 0)) \quad \text{s.t.} \quad \frac{1}{m_{i1}} \sum_{j=1}^{m_{i1}} \phi(w; (x_j^{(i1)}, 1)) \leq r_i, \quad 1 \leq i \leq n, \quad (30)$$

where $\{x_j^{(i0)}\}_{1 \leq j \leq m_{i0}}$ and $\{x_j^{(i1)}\}_{1 \leq j \leq m_{i1}}$ are the sets of samples at client i associated with labels 0 and 1, respectively, and ϕ is the binary logistic loss [\[18\]](#)

$$\phi(w; (x, y)) = -yw^T x + \log(1 + e^{w^T x}), \quad y \in \{0, 1\}. \quad (31)$$

Then, both the objective and the constraints in [Eq. \(30\)](#) are convex. We consider three real-world datasets, namely ‘breast-cancer-wisc’, ‘adult-a’, and ‘monks-1’, from the UCI repository¹ and described

¹see <https://archive.ics.uci.edu/datasets>

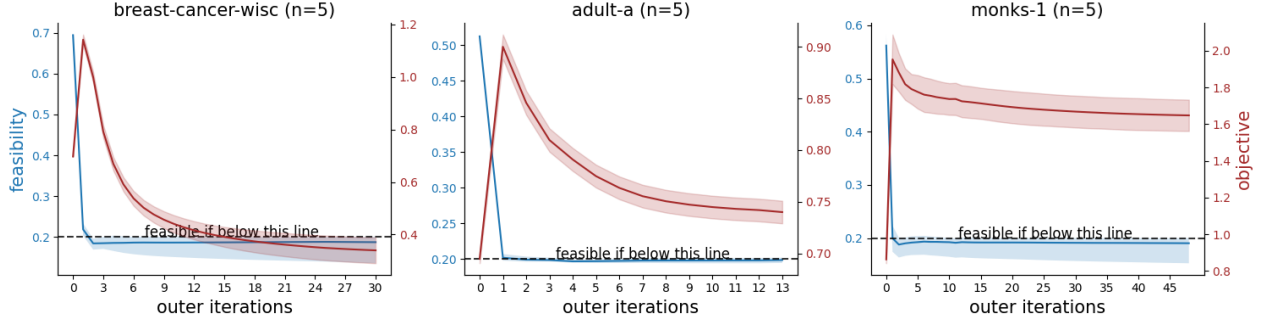


Figure 1: Convergence behavior of local objective and local feasibility in one random trial over the outer iterations of Algorithm 1 on three real-world datasets. The solid blue and brown lines indicate the mean local objective and the mean local feasibility over all clients, respectively. The blue and the brown areas indicate the cross-client variations of local objectives and local feasibility, respectively. The dashed black line indicates the feasibility threshold.

in Appendix E.1. For each dataset, we perform the Neyman-Pearson classification that minimizes the loss of classification for class 0 (majority) while ensuring that the loss for class 1 (minority) is less than a threshold $r_i = 0.2$. To simulate the FL setting, we divide each dataset into n folds, mimicking local clients, each holding the same amount of data with equal ratios of the two classes.

We apply Algorithm 1 and cProx-AL (Algorithm 3) to find a $(10^{-3}, 10^{-3})$ -optimal solution of Eq. (30). We run 10 trials of Algorithm 1 and cProx-AL. For each run, both algorithms have the same initial point w^0 , randomly chosen from the unit Euclidean sphere. We set the other parameters for Algorithm 1 and cProx-AL as $\mu_i^0 = (0, \dots, 0)^T \forall 0 \leq i \leq n$, $\bar{s} = 0.001$ and $\beta = 300$. We also set $\rho_i = 0.01 \forall 1 \leq i \leq n$ for Algorithm 2.

Comparing the objective value and feasibility of solutions achieved by Algorithm 1 and cProx-AL in Table 1, we see that both algorithms can yield solutions of similar quality. Given the small standard deviations, we observe that the convergence behavior of Algorithm 1 remains stable across 10 trial runs. These observations demonstrate the ability of Algorithm 1 to reliably solve Eq. (30) in the FL setting without compromising solution quality. From Fig. 1, we observe that Algorithm 1 consistently achieves feasibility for all local constraints while also minimizing all the local objectives.

5.2 Classification with fairness constraints

In this subsection, we consider fairness-aware learning with global and local fairness constraints:

$$\min_w \frac{1}{n} \sum_{i=1}^n \frac{1}{m_i} \sum_{j=1}^{m_i} \phi(w; z_j^{(i)}) \quad \text{s.t.} \quad -r_i \leq \frac{1}{\tilde{m}_i} \sum_{j=1}^{\tilde{m}_i} \phi(w; \tilde{z}_j^{(i)}) - \frac{1}{\hat{m}_i} \sum_{j=1}^{\hat{m}_i} \phi(w; \hat{z}_j^{(i)}) \leq r_i, \quad 0 \leq i \leq n. \quad (32)$$

Here, $\{z_j^{(i)} = (x_j^{(i)}, y_j^{(i)}) \in \mathbb{R}^d \times \{0, 1\} : i = 0, \dots, n, j = 1, \dots, m_i\}$ is the training set, where i indexes the central server/local clients. For each $i = 0, \dots, n$, the dataset $\{z_j^{(i)}\}_{1 \leq j \leq m_i}$ is further divided into two subgroups $\{\tilde{z}_j^{(i)}\}_{1 \leq j \leq \tilde{m}_i}$ and $\{\hat{z}_j^{(i)}\}_{1 \leq j \leq \hat{m}_i}$ based on certain subgroup attributes. The constraints with $i = 1, \dots, n$ refer to local constraints at client i , while the constraints with $i = 0$ refer to global constraints at the central server.

We choose ϕ as the binary logistic loss defined in Eq. (31), leading to nonconvex constraints in Eq. (32). For the real-world dataset, we consider ‘adult-b’²: each sample in this dataset has 39 features and one

²This dataset can be found in <https://github.com/heyaudace/ml-bias-fairness/tree/master/data/adult>.

Table 2: Numerical results for Eq. (32) using our algorithm vs. using cProx-AL. Inside the parentheses are the respective standard deviations over 10 random trials. For feasibility, we include the mean and maximum loss disparities (absolute difference between losses for two subgroups) among all clients and the central server.

n	objective value			feasibility (loss disparity (≤ 0.1))			
	Algorithm 1	cProx-AL	relative difference	Algorithm 1	cProx-AL		
				mean	max	mean	max
1	0.37 (9.83e-05)	0.37 (4.14e-05)	1.97e-03 (2.53e-04)	0.10 (1.14e-04)	0.10 (1.36e-04)	0.10 (3.69e-06)	0.10 (5.38e-06)
5	0.37 (3.99e-03)	0.37 (4.05e-03)	1.86e-03 (4.69e-04)	0.09 (5.34e-05)	0.10 (7.51e-05)	0.09 (3.68e-05)	0.10 (4.36e-06)
10	0.37 (6.39e-03)	0.37 (6.52e-03)	2.39e-03 (8.40e-04)	0.08 (1.68e-04)	0.10 (2.15e-05)	0.08 (1.52e-04)	0.10 (6.56e-06)
20	0.38 (9.46e-03)	0.37 (9.86e-03)	4.61e-03 (2.43e-03)	0.08 (9.75e-05)	0.10 (1.01e-04)	0.08 (4.90e-05)	0.10 (6.06e-06)

binary label. To simulate the FL setting, we divide the 22,654 training samples from the ‘adult-b’ dataset into n folds and distribute them to n local clients. The central server holds the 5,659 test samples from the ‘adult-b’ dataset. Note that although we have taken both the “training” and “test” samples from the ‘adult-b’ dataset here, these samples are used to simulate our local samples and central samples, respectively. The focus here is to test optimization performance, not generalization—we do not have a test step, unlike in typical supervised learning.

We apply Algorithm 1 and cProx-AL (Algorithm 3) to find a $(10^{-3}, 10^{-3})$ -optimal solution of Eq. (32). We run 10 trials of Algorithm 1 and cProx-AL. For each run, both algorithms have the same initial point w^0 , randomly chosen from the unit Euclidean sphere. We set the other parameters for Algorithm 1 and cProx-AL as $\mu_i^0 = (0, \dots, 0)^T \forall 0 \leq i \leq n$, $\bar{s} = 0.001$ and $\beta = 10$. We also set $\rho_i = 10^8 \forall 1 \leq i \leq n$ for Algorithm 2.

Comparing the objective value and feasibility of solutions achieved by Algorithm 1 and cProx-AL in Table 2 reveals that Algorithm 1 and cProx-AL can produce solutions of similar quality. Given the small standard deviations, we observe that the convergence behavior of Algorithm 1 remains stable across 10 trial runs. These observations demonstrate the ability of Algorithm 1 to reliably solve Eq. (32) in the FL setting without compromising solution quality. It also suggests the potential of our algorithm in solving FL problems with nonconvex constraints. From Fig. 2, we see that our proposed method consistently achieves feasibility for all local and global constraints while also minimizing all the local objectives.

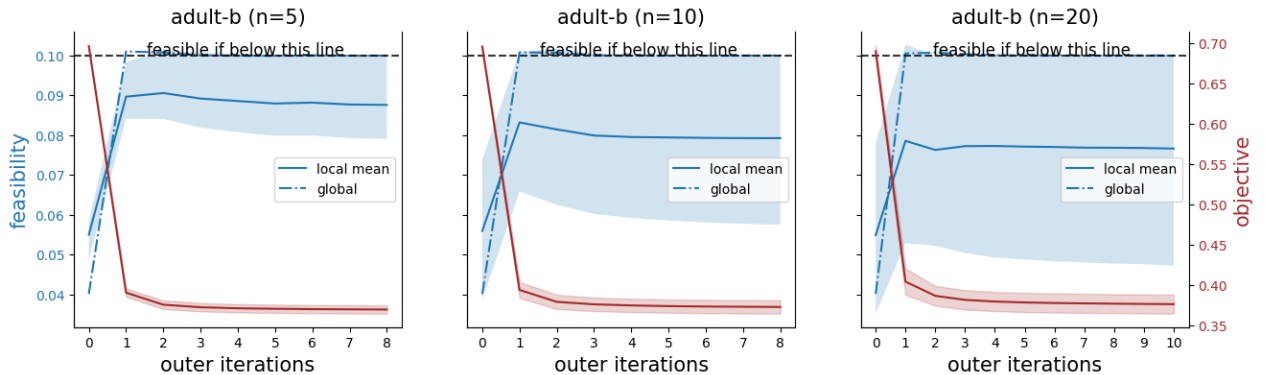


Figure 2: Convergence of local objective, local feasibility, and the feasibility for global constraints in one random trial over the outer iterations of Algorithm 1. The solid blue and brown lines indicate the mean local objective and the mean local feasibility over all clients, respectively. The blue and brown areas indicate the cross-client variations of local objectives and local feasibility, respectively. The dashdot blue line indicates the feasibility for global constraints. The dashed black line indicates the feasibility threshold.

6 Concluding remarks

In this paper, we propose an FL algorithm for solving general constrained ML problems based on the proximal AL method. We analyze the worst-case iteration complexity of the proposed algorithm, assuming convex objective and convex constraints with locally Lipschitz continuous gradients. Finally, we perform numerical experiments to assess the performance of the proposed algorithm for constrained classification problems, using real-world datasets. The numerical results clearly demonstrate the practical efficacy of our proposed algorithm. Since our work is the first of its kind, there are numerous possible future directions. For example, one could try to extend our FL algorithms to allow partial client participation and stochastic solvers at local clients. In addition, developing FL algorithms for general constrained ML with convergence guarantees in nonconvex settings remains largely open.

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References

- [1] Alekh Agarwal, Alina Beygelzimer, Miroslav Dudík, John Langford, and Hanna Wallach. A reductions approach to fair classification. In *International Conference on Machine Learning*, pages 60–69. PMLR, 2018.
- [2] Necdet S Aybat and Erfan Y Hamedani. A primal-dual method for conic constrained distributed optimization problems. *Advances in Neural Information Processing Systems*, 29, 2016.
- [3] Necdet S Aybat and Erfan Y Hamedani. A distributed ADMM-like method for resource sharing over time-varying networks. *SIAM Journal on Optimization*, 29(4):3036–3068, 2019.
- [4] Necdet S Aybat and Garud Iyengar. An augmented Lagrangian method for conic convex programming. *arXiv preprint arXiv:1302.6322*, 2013.
- [5] Heinz H Bauschke and Patrick L Combettes. *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*. Springer, 2nd edition, 2017.
- [6] Ernesto G Birgin and José M Martínez. Complexity and performance of an augmented Lagrangian algorithm. *Optimization Methods and Software*, 35(5):885–920, 2020.
- [7] Paul T Boggs and Jon W Tolle. Sequential quadratic programming. *Acta Numerica*, 4:1–51, 1995.
- [8] Richard H Byrd, Robert B Schnabel, and Gerald A Shultz. A trust region algorithm for nonlinearly constrained optimization. *SIAM Journal on Numerical Analysis*, 24(5):1152–1170, 1987.
- [9] Elisa Celis, Lingxiao Huang, Vijay Keswani, and Nisheeth K Vishnoi. Classification with fairness constraints: A meta-algorithm with provable guarantees. In *Proceedings of the Conference on Fairness, Accountability, and Transparency*, pages 319–328, 2019.

- [10] Peter W Christensen and Anders Klarbring. *An Introduction to Structural Optimization*, volume 153. Springer Science & Business Media, 2008.
- [11] Lingyang Chu, Lanjun Wang, Yanjie Dong, Jian Pei, Zirui Zhou, and Yong Zhang. FedFair: Training fair models in cross-silo federated learning. *arXiv preprint arXiv:2109.05662*, 2021.
- [12] Frank E Curtis and Michael L Overton. A sequential quadratic programming algorithm for nonconvex, nonsmooth constrained optimization. *SIAM Journal on Optimization*, 22(2):474–500, 2012.
- [13] Wei Du, Depeng Xu, Xintao Wu, and Hanghang Tong. Fairness-aware agnostic federated learning. In *Proceedings of the 2021 SIAM International Conference on Data Mining (SDM)*, pages 181–189. SIAM, 2021.
- [14] Borja R Gálvez, Filip Granqvist, Rogier van Dalen, and Matt Seigel. Enforcing fairness in private federated learning via the modified method of differential multipliers. In *NeurIPS 2021 Workshop Privacy in Machine Learning*, 2021.
- [15] Yonghai Gong, Yichuan Li, and Nikolaos M Freris. FedADMM: A robust federated deep learning framework with adaptivity to system heterogeneity. In *2022 IEEE 38th International Conference on Data Engineering (ICDE)*, pages 2575–2587. IEEE, 2022.
- [16] Ian J Goodfellow, Jonathon Shlens, and Christian Szegedy. Explaining and harnessing adversarial examples. *arXiv preprint arXiv:1412.6572*, 2014.
- [17] Geovani N Grapiglia and Ya-xiang Yuan. On the complexity of an augmented Lagrangian method for nonconvex optimization. *IMA Journal of Numerical Analysis*, 41(2):1546–1568, 2021.
- [18] Trevor Hastie, Robert Tibshirani, and Jerome H Friedman. *The Elements of Statistical Learning: Data Mining, Inference, and Prediction*. Springer, New York, 2nd edition, 2009.
- [19] Chuan He, Heng Huang, and Zhaosong Lu. A Newton-CG based barrier-augmented Lagrangian method for general nonconvex conic optimization. *arXiv preprint arXiv:2301.04204*, 2023.
- [20] Chuan He, Zhaosong Lu, and Ting Kei Pong. A Newton-CG based augmented Lagrangian method for finding a second-order stationary point of nonconvex equality constrained optimization with complexity guarantees. *SIAM Journal on Optimization*, 33(3):1734–1766, 2023.
- [21] Mingyi Hong, Davood Hajinezhad, and Ming-Min Zhao. Prox-PDA: The proximal primal-dual algorithm for fast distributed nonconvex optimization and learning over networks. In *International Conference on Machine Learning*, pages 1529–1538. PMLR, 2017.
- [22] Kevin Huang, Nuozhou Wang, and Shuzhong Zhang. An accelerated variance reduced extra-point approach to finite-sum VI and optimization. *arXiv preprint arXiv:2211.03269*, 2022.
- [23] Peter Kairouz, H Brendan McMahan, Brendan Avent, Aurélien Bellet, Mehdi Bennis, Arjun Nitin Bhagoji, Kallista Bonawitz, Zachary Charles, Graham Cormode, Rachel Cummings, et al. Advances and open problems in federated learning. *Foundations and Trends® in Machine Learning*, 14(1–2):1–210, 2021.
- [24] Sai P Karimireddy, Satyen Kale, Mehryar Mohri, Sashank Reddi, Sebastian Stich, and Ananda T Suresh. Scaffold: Stochastic controlled averaging for federated learning. In *International Conference on Machine Learning*, pages 5132–5143. PMLR, 2020.

- [25] Jakub Konečný, H Brendan McMahan, Felix X Yu, Peter Richtárik, Ananda T Suresh, and Dave Bacon. Federated learning: Strategies for improving communication efficiency. *arXiv preprint arXiv:1610.05492*, 2016.
- [26] Weiwei Kong, Jefferson G Melo, and Renato DC Monteiro. Iteration complexity of an inner accelerated inexact proximal augmented Lagrangian method based on the classical Lagrangian function. *SIAM Journal on Optimization*, 33(1):181–210, 2023.
- [27] Guanghui Lan and Renato DC Monteiro. Iteration-complexity of first-order augmented Lagrangian methods for convex programming. *Mathematical Programming*, 155(1-2):511–547, 2016.
- [28] Qinbin Li, Zeyi Wen, Zhaomin Wu, Sixu Hu, Naibo Wang, Yuan Li, Xu Liu, and Bingsheng He. A survey on federated learning systems: Vision, hype and reality for data privacy and protection. *IEEE Transactions on Knowledge and Data Engineering*, 2021.
- [29] Tian Li, Shengyuan Hu, Ahmad Beirami, and Virginia Smith. Ditto: Fair and robust federated learning through personalization. In *International Conference on Machine Learning*, pages 6357–6368. PMLR, 2021.
- [30] Tian Li, Anit K Sahu, Manzil Zaheer, Maziar Sanjabi, Ameet Talwalkar, and Virginia Smith. Federated optimization in heterogeneous networks. *Proceedings of Machine Learning and Systems*, 2:429–450, 2020.
- [31] Xiaoxiao Li, Meirui Jiang, Xiaofei Zhang, Michael Kamp, and Qi Dou. FedBN: Federated learning on non-iid features via local batch normalization. *arXiv preprint arXiv:2102.07623*, 2021.
- [32] Zichong Li, Pin-Yu Chen, Sijia Liu, Songtao Lu, and Yangyang Xu. Rate-improved inexact augmented Lagrangian method for constrained nonconvex optimization. In *International Conference on Artificial Intelligence and Statistics*, pages 2170–2178. PMLR, 2021.
- [33] Peng Lin, Wei Ren, and Yongduan Song. Distributed multi-agent optimization subject to nonidentical constraints and communication delays. *Automatica*, 65:120–131, 2016.
- [34] Tianyi Lin, Shiqian Ma, and Shuzhong Zhang. On the global linear convergence of the ADMM with multiblock variables. *SIAM Journal on Optimization*, 25(3):1478–1497, 2015.
- [35] Yi Liu, JQ James, Jiawen Kang, Dusit Niyato, and Shuyu Zhang. Privacy-preserving traffic flow prediction: A federated learning approach. *IEEE Internet of Things Journal*, 7(8):7751–7763, 2020.
- [36] Guodong Long, Yue Tan, Jing Jiang, and Chengqi Zhang. Federated learning for open banking. In *Federated Learning: Privacy and Incentive*, pages 240–254. Springer, 2020.
- [37] Songtao Lu. A single-loop gradient descent and perturbed ascent algorithm for nonconvex functional constrained optimization. In *International Conference on Machine Learning*, pages 14315–14357. PMLR, 2022.
- [38] Zhaosong Lu and Sanyou Mei. Accelerated first-order methods for convex optimization with locally Lipschitz continuous gradient. *SIAM Journal on Optimization*, 33(3):2275–2310, 2023.
- [39] Zhaosong Lu and Zirui Zhou. Iteration-complexity of first-order augmented Lagrangian methods for convex conic programming. *SIAM Journal on Optimization*, 33(2):1159–1190, 2023.

- [40] Levi McClenny and Ulisses Braga-Neto. Self-adaptive physics-informed neural networks using a soft attention mechanism. *arXiv preprint arXiv:2009.04544*, 2020.
- [41] Brendan McMahan, Eider Moore, Daniel Ramage, Seth Hampson, and Blaise Agüera y Arcas. Communication-efficient learning of deep networks from decentralized data. In *Artificial Intelligence and Statistics*, pages 1273–1282. PMLR, 2017.
- [42] Ninareh Mehrabi, Fred Morstatter, Nripsuta Saxena, Kristina Lerman, and Aram Galstyan. A survey on bias and fairness in machine learning. *ACM Computing Surveys (CSUR)*, 54(6):1–35, 2021.
- [43] Jed Mills, Jia Hu, and Geyong Min. Communication-efficient federated learning for wireless edge intelligence in IoT. *IEEE Internet of Things Journal*, 7(7):5986–5994, 2019.
- [44] Ion Necoara, Andrei Patrascu, and Francois Glineur. Complexity of first-order inexact Lagrangian and penalty methods for conic convex programming. *Optimization Methods and Software*, 34(2):305–335, 2019.
- [45] Angelia Nedic, Asuman Ozdaglar, and Pablo A Parrilo. Constrained consensus and optimization in multi-agent networks. *IEEE Transactions on Automatic Control*, 55(4):922–938, 2010.
- [46] Yurii Nesterov et al. *Lectures on convex optimization*, volume 137. Springer, 2018.
- [47] Office for Civil Rights (OCR). Hipaa home, Aug 2023.
- [48] Andrei Patrascu, Ion Necoara, and Quoc Tran-Dinh. Adaptive inexact fast augmented Lagrangian methods for constrained convex optimization. *Optimization Letters*, 11:609–626, 2017.
- [49] Le Peng, Gaoxiang Luo, Andrew Walker, Zachary Zaiman, Emma K Jones, Hemant Gupta, Kristopher Kersten, John L Burns, Christopher A Harle, Tanja Magoc, Benjamin Shickel, Scott D Steenburg, Tyler Loftus, Genevieve B Melton, Judy W Gichoya, Ju Sun, and Christopher J Tignanelli. Evaluation of federated learning variations for COVID-19 diagnosis using chest radiographs from 42 US and European hospitals. *Journal of the American Medical Informatics Association*, 30(1):54–63, 2023.
- [50] Le Peng, Sicheng Zhou, Jiandong Chen, Rui Zhang, Ziyue Xu, and Ju Sun. A systematic evaluation of federated learning on biomedical natural language processing. In *International Workshop on Federated Learning for Distributed Data Mining*, 2023.
- [51] MJD Powell and Ya-xiang Yuan. A trust region algorithm for equality constrained optimization. *Mathematical Programming*, 49(1):189–211, 1991.
- [52] Nicola Rieke, Jonny Hancox, Wenqi Li, Fausto Milletari, Holger R Roth, Shadi Albarqouni, Spyridon Bakas, Mathieu N Galtier, Bennett A Landman, Klaus Maier-Hein, Sébastien Ourselin, Micah Sheller, Ronald M Summers, Andrew Trask, Daguang Xu, Maximilian Baust, and M Jorge Cardoso. The future of digital health with federated learning. *NPJ Digital Medicine*, 3(1):119, 2020.
- [53] Philippe Rigollet and Xin Tong. Neyman-Pearson classification, convexity and stochastic constraints. *Journal of Machine Learning Research*, 2011.
- [54] R Tyrrell Rockafellar. Augmented Lagrangians and applications of the proximal point algorithm in convex programming. *Mathematics of Operations Research*, 1(2):97–116, 1976.
- [55] Takaya Saito and Marc Rehmsmeier. The precision-recall plot is more informative than the ROC plot when evaluating binary classifiers on imbalanced datasets. *PloS one*, 10(3):e0118432, 2015.

- [56] Felix Sattler, Simon Wiedemann, Klaus-Robert Müller, and Wojciech Samek. Robust and communication-efficient federated learning from non-IID data. *IEEE Transactions on Neural Networks and Learning Systems*, 31(9):3400–3413, 2019.
- [57] Clayton Scott. Performance measures for Neyman-Pearson classification. *IEEE Transactions on Information Theory*, 53(8):2852–2863, 2007.
- [58] Zebang Shen, Juan Cervino, Hamed Hassani, and Alejandro Ribeiro. An agnostic approach to federated learning with class imbalance. In *International Conference on Learning Representations*, 2021.
- [59] Xin Tong, Yang Feng, and Jingyi Jessica Li. Neyman-Pearson classification algorithms and NP receiver operating characteristics. *Science Advances*, 4(2):eaao1659, 2018.
- [60] Xin Tong, Yang Feng, and Anqi Zhao. A survey on Neyman-Pearson classification and suggestions for future research. *Wiley Interdisciplinary Reviews: Computational Statistics*, 8(2):64–81, 2016.
- [61] Yash Travadi, Le Peng, Xuan Bi, Ju Sun, and Mochen Yang. Welfare and fairness dynamics in federated learning: A client selection perspective. *Statistics and Its Interface*, 2023.
- [62] Andreas Wächter and Lorenz T Biegler. On the implementation of an interior-point filter line-search algorithm for large-scale nonlinear programming. *Mathematical Programming*, 106:25–57, 2006.
- [63] Jianyu Wang, Qinghua Liu, Hao Liang, Gauri Joshi, and H Vincent Poor. Tackling the objective inconsistency problem in heterogeneous federated optimization. *Advances in Neural Information Processing Systems*, 33:7611–7623, 2020.
- [64] Peng Wang, Peng Lin, Wei Ren, and Yongduan Song. Distributed subgradient-based multiagent optimization with more general step sizes. *IEEE Transactions on Automatic Control*, 63(7):2295–2302, 2017.
- [65] Yangyang Xu. Iteration complexity of inexact augmented Lagrangian methods for constrained convex programming. *Mathematical Programming*, 185:199–244, 2021.
- [66] Tao Yang, Xinlei Yi, Junfeng Wu, Ye Yuan, Di Wu, Ziyang Meng, Yiguang Hong, Hong Wang, Zongli Lin, and Karl H Johansson. A survey of distributed optimization. *Annual Reviews in Control*, 47:278–305, 2019.
- [67] Deming Yuan, Shengyuan Xu, and Huanyu Zhao. Distributed primal-dual subgradient method for multiagent optimization via consensus algorithms. *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)*, 41(6):1715–1724, 2011.
- [68] Xinwei Zhang, Mingyi Hong, Sairaj Dhople, Wotao Yin, and Yang Liu. FedPD: A federated learning framework with adaptivity to non-iid data. *IEEE Transactions on Signal Processing*, 69:6055–6070, 2021.
- [69] Minghui Zhu and Sonia Martínez. On distributed convex optimization under inequality and equality constraints. *IEEE Transactions on Automatic Control*, 57(1):151–164, 2011.
- [70] Barret Zoph, Vijay Vasudevan, Jonathon Shlens, and Quoc V Le. Learning transferable architectures for scalable image recognition. In *Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition*, pages 8697–8710, 2018.

Appendix

In [Appendices A to C](#), we provide proofs of the main results in [Sections 3 and 4](#). [Appendix D](#) presents a proximal AL method for centralized constrained optimization. In [Appendix E](#), we include some extra numerical results.

A Proofs of [Theorem 2](#), [Lemma 3](#), and [Theorem 4\(a\)](#)

First, we set up the technical tools necessary for the proof, following [\[39\]](#). With the abbreviations in [Eq. \(6\)](#), we define the Lagrangian function associated with [Eqs. \(5\) and \(7\)](#) as

$$l(w, \mu) = \begin{cases} f(w) + h(w) + \langle \mu, c(w) \rangle & \text{if } w \in \text{dom}(h) \text{ and } \mu \geq 0, \\ -\infty & \text{if } w \in \text{dom}(h) \text{ and } \mu \not\geq 0, \\ \infty & \text{if } w \notin \text{dom}(h), \end{cases}$$

Then, one can verify that

$$\partial l(w, \mu) = \begin{cases} \begin{pmatrix} \nabla f(w) + \partial h(w) + \nabla c(w)\mu \\ c(w) - \mathcal{N}_{\mathbb{R}_+^m}(\mu) \end{pmatrix} & \text{if } w \in \text{dom}(h) \text{ and } \mu \geq 0, \\ \emptyset & \text{otherwise.} \end{cases} \quad (33)$$

We also define a set-valued operator \mathcal{T} associated with [Eqs. \(5\) and \(7\)](#):

$$\mathcal{T} : (w, \mu) \rightarrow \{(u, \nu) \in \mathbb{R}^d \times \mathbb{R}^m : (u, -\nu) \in \partial l(w, \mu)\}, \quad \forall (w, \mu) \in \mathbb{R}^d \times \mathbb{R}^m, \quad (34)$$

which is maximally monotone (see, e.g., Section 2 of [\[54\]](#)). Finding a KKT solution of [Eq. \(5\)](#) can be viewed as solving the monotone inclusion problem [\[54\]](#):

$$\text{Find } (w, \mu) \in \mathbb{R}^d \times \mathbb{R}^m \text{ such that } (0, 0) \in \mathcal{T}(w, \mu). \quad (35)$$

Furthermore, applying the proximal AL method to solve [Eq. \(5\)](#) is equivalent to applying the proximal point algorithm (PPA) to solve this monotone inclusion problem [\[54, 5\]](#), that is,

$$w^{k+1} = \arg \min_w \ell_k(w), \quad \mu^{k+1} = [\mu^k + \beta c(w^{k+1})]_+, \quad \iff (w^{k+1}, \mu^{k+1}) = \mathcal{J}(w^k, \mu^k), \quad \forall k \geq 0, \quad (36)$$

where $(w^0, \mu^0) \in \text{dom}(h) \times \mathbb{R}_+^m$ and \mathcal{J} is the resolvent of \mathcal{T} defined as

$$\mathcal{J} := (\mathcal{I} + \beta \mathcal{T})^{-1} \quad (37)$$

with \mathcal{I} being the identity operator. When the $\arg \min_w \ell_k(w)$ subproblem is only solved up to approximate stationarity, that is, $\text{dist}_\infty(0, \partial \ell_k(w^{k+1})) \leq \tau_k$ as in our [Algorithm 1](#), the error τ_k will propagate to the next iterate that we obtain. This is quantitatively captured by the following result.

Lemma 10 (adaptation of Lemma 5 of [\[39\]](#)). *Suppose that Assumptions 0 to 2 hold. Let $\{(w^k, \mu^k)\}_{k \geq 0}$ be generated by [Algorithm 1](#). Then for any $k \geq 0$, we have*

$$\|(w^{k+1}, \mu^{k+1}) - \mathcal{J}(w^k, \mu^k)\| \leq \beta \sqrt{n} \tau_k,$$

where \mathcal{J} is the resolvent of \mathcal{T} defined in [Eq. \(37\)](#).

Proof. Notice from [Eq. \(10\)](#) that $\text{dist}(0, \partial \ell_k(w^{k+1})) \leq \sqrt{n} \text{dist}_\infty(0, \partial \ell_k(w^{k+1})) \leq \sqrt{n} \tau_k$. By this and Lemma 5 of [\[39\]](#), the conclusion of this lemma holds. \square

A.1 Proof of Theorem 2

Proof of Theorem 2. Notice from Eqs. (6) and (9) that

$$\ell_k(w) = f(w) + h(w) + \frac{1}{2\beta} \left(\|\mu^k + \beta c(w)\|_+^2 - \|\mu^k\|^2 \right) + \frac{1}{2\beta} \|w - w^k\|^2.$$

By this, Eq. (33), and the fact that $\mu^{k+1} = [\mu^k + \beta c(w^{k+1})]_+$, one has

$$\begin{aligned} \partial \ell_k(w^{k+1}) - \frac{1}{\beta}(w^{k+1} - w^k) &= \nabla f(w^{k+1}) + \partial h(w^{k+1}) + \nabla c(w^{k+1})[\mu^k + \beta c(w^{k+1})]_+ \\ &= \nabla f(w^{k+1}) + \partial h(w^{k+1}) + \nabla c(w^{k+1})\mu^{k+1} = \partial_w l(w^{k+1}, \mu^{k+1}). \end{aligned} \quad (38)$$

Notice that

$$\mu^{k+1} = [\mu^k + \beta c(w^{k+1})]_+ = \arg \min_{\mu \in \mathbb{R}_+^m} \frac{1}{2} \|\mu - (\mu^k + \beta c(w^{k+1}))\|^2.$$

By the optimality condition of this projection, we have

$$0 \in \mu^{k+1} - (\mu^k + \beta c(w^{k+1})) + \mathcal{N}_{\mathbb{R}_+^m}(\mu^{k+1}),$$

which together with Eq. (33) implies that

$$\frac{1}{\beta}(\mu^{k+1} - \mu^k) \in \partial_\mu l(w^{k+1}, \mu^{k+1}). \quad (39)$$

In view of this, Eqs. (10), (11) and (38), we can see that

$$\begin{aligned} \text{dist}_\infty(0, \partial_w l(w^{k+1}, \mu^{k+1})) &\stackrel{\text{Eq. (38)}}{\leq} \text{dist}_\infty(0, \partial \ell_k(w^{k+1})) + \frac{1}{\beta} \|w^{k+1} - w^k\|_\infty \\ &\stackrel{\text{Eq. (10)}}{\leq} \tau_k + \frac{1}{\beta} \|w^{k+1} - w^k\|_\infty \stackrel{\text{Eq. (11)}}{\leq} \epsilon_1, \\ \text{dist}_\infty(0, \partial_\mu l(w^{k+1}, \mu^{k+1})) &\stackrel{\text{Eq. (39)}}{\leq} \frac{1}{\beta} \|\mu^{k+1} - \mu^k\|_\infty \stackrel{\text{Eq. (11)}}{\leq} \epsilon_2. \end{aligned}$$

These along with Eq. (33) and Definition 1 imply that (w^{k+1}, μ^{k+1}) is an (ϵ_1, ϵ_2) -KKT solution of Eq. (5), which proves this theorem as desired. \square

A.2 Proof of Lemma 3

Define

$$w_*^k := \arg \min_w \ell_k(w), \quad \mu_*^k := [\mu^k + \beta c(w_*^k)]_+, \quad \forall k \geq 0, \quad (40)$$

which, by Eq. (36), is equivalent to

$$(w_*^k, \mu_*^k) = \mathcal{J}(w^k, \mu^k). \quad (41)$$

Recall that (w^*, μ^*) is assumed to be any pair of optimal solutions to Eq. (5) and Eq. (7). Toward the proof, we first present an intermediate result, which mostly follows the fact that \mathcal{J} is firmly nonexpansive.

Lemma 11. *Suppose that Assumptions 0 to 2 hold. Let $\{(w^k, \mu^k)\}_{k \geq 0}$ be generated by Algorithm 1. Let (w_*^k, μ_*^k) be defined in Eq. (40) for all $k \geq 0$. Then the following relations hold.*

$$\|(w^k, \mu^k) - (w_*^k, \mu_*^k)\|^2 + \|(w_*^k, \mu_*^k) - (w^*, \mu^*)\|^2 \leq \|(w^k, \mu^k) - (w^*, \mu^*)\|^2, \quad \forall k \geq 0, \quad (42)$$

$$\|(w^k, \mu^k) - (w^*, \mu^*)\| \leq \|(w^0, \mu^0) - (w^*, \mu^*)\| + \beta \sqrt{n} \sum_{j=0}^{k-1} \tau_j, \quad \forall k \geq 0. \quad (43)$$

Proof. Since (w^*, μ^*) is a solution to the monotone inclusion problem [Eq. \(35\)](#), we have

$$(0, 0) \in \mathcal{T}(w^*, \mu^*), \quad \text{and} \quad (w^*, \mu^*) = \mathcal{J}(w^*, \mu^*). \quad (44)$$

Moreover, since \mathcal{T} is maximally monotone, its resolvent \mathcal{J} is firmly nonexpansive (see, e.g., Corollary 23.9 of [\[5\]](#)), that is, $\|\mathcal{J}(w, \mu) - \mathcal{J}(w', \mu')\|^2 + \|(\mathcal{I} - \mathcal{J})(w, \mu) - (\mathcal{I} - \mathcal{J})(w', \mu')\|^2 \leq \|(w, \mu) - (w', \mu')\|^2$ for any feasible pairs (w, μ) and (w', μ') . Using [Eqs. \(41\)](#) and [\(44\)](#), we obtain that

$$\begin{aligned} & \|(w^k, \mu^k) - (w_*, \mu_*)\|^2 + \|(w_*, \mu_*) - (w^*, \mu^*)\|^2 \\ & \stackrel{\text{Eqs. (41) and (44)}}{=} \|(\mathcal{I} - \mathcal{J})(w^k, \mu^k) - (\mathcal{I} - \mathcal{J})(w^*, \mu^*)\|^2 + \|\mathcal{J}(w^k, \mu^k) - \mathcal{J}(w^*, \mu^*)\|^2 \\ & \leq \|(w^k, \mu^k) - (w^*, \mu^*)\|^2. \quad (\text{firm nonexpansiveness of } \mathcal{J}) \end{aligned}$$

Hence, [Eq. \(42\)](#) holds as desired.

Now we prove [Eq. \(43\)](#). It suffices to consider the case where $k \geq 1$. We have

$$\begin{aligned} \|(w^k, \mu^k) - (w^*, \mu^*)\| & \leq \|(w^k, \mu^k) - \mathcal{J}(w^{k-1}, \mu^{k-1})\| + \|\mathcal{J}(w^{k-1}, \mu^{k-1}) - \mathcal{J}(w^*, \mu^*)\| \\ & \leq \beta\sqrt{n}\tau_{k-1} + \|(w^{k-1}, \mu^{k-1}) - (w^*, \mu^*)\|, \end{aligned} \quad (45)$$

where we have invoked [Lemma 10](#) and the nonexpansiveness of \mathcal{J} to obtain the final upper bound. Repeatedly applying [Eq. \(45\)](#) for iterates (w^1, μ^1) through (w^{k-1}, μ^{k-1}) , we have

$$\beta\sqrt{n}\tau_{k-1} + \|(w^{k-1}, \mu^{k-1}) - (w^*, \mu^*)\| \leq \beta\sqrt{n} \sum_{j=0}^{k-1} \tau_j + \|(w^0, \mu^0) - (w^*, \mu^*)\|, \quad (46)$$

completing the proof. \square

Proof of Lemma 3. Notice from [Algorithm 1](#) that $\tau_k = \bar{s}/(k+1)^2$ for all $k \geq 0$. Therefore, one has $\sum_{j=0}^{\infty} \tau_j \leq 2\bar{s}$. In view of this, [Eq. \(15\)](#), and [Lemma 11](#), we observe that

$$\max\{\|w^k - w^*\|, \|\mu^k - \mu^*\|, \|w^k - w_*^k\|, \|w_*^k - w^*\|\} \leq r_0 + 2\sqrt{n}\bar{s}\beta, \quad \forall k \geq 0. \quad (47)$$

where r_0 is defined in [Eq. \(15\)](#), and β and \bar{s} are inputs of [Algorithm 1](#). [Eq. \(47\)](#) implies that $w^k \in \mathcal{Q}_1$ for all $k \geq 0$, completing the proof. \square

A.3 Proof of [Theorem 4\(a\)](#)

To prove [Theorem 4\(a\)](#), we first present a general technical lemma on the convergence rate of an inexact PPA applied to monotone inclusion problems.

Lemma 12 (restatement of Lemma 3 of [\[39\]](#)). *Let $\tilde{\mathcal{T}} : \mathbb{R}^p \rightrightarrows \mathbb{R}^q$ be a maximally monotone operator and $z^* \in \mathbb{R}^p$ such that $0 \in \tilde{\mathcal{T}}(z^*)$. Let $\{z^k\}$ be a sequence generated by an inexact PPA, starting with z^0 and obtaining z^{k+1} by approximately evaluating $\tilde{\mathcal{J}}(z^k)$ such that*

$$\|z^{k+1} - \tilde{\mathcal{J}}(z^k)\| \leq e_k$$

for some $\beta > 0$ and $e_k \geq 0$, where $\tilde{\mathcal{J}} := (\mathcal{I} + \beta\tilde{\mathcal{T}})^{-1}$ and \mathcal{I} is the identity operator. Then, for any $K \geq 1$, we have

$$\min_{K \leq k \leq 2K} \|z^{k+1} - z^k\| \leq \frac{\sqrt{2} \left(\|z^0 - z^*\| + 2 \sum_{k=0}^{2K} e_k \right)}{\sqrt{K+1}}.$$

Proof of Theorem 4(a). Observe that Algorithm 1 terminates when two consecutive iterates (w^{k+1}, μ^{k+1}) and (w^k, μ^k) are close. We use this observation and Lemmas 10 and 12 to derive the maximum number of outer iterations of Algorithm 1.

Recall that $\sum_{j=0}^{\infty} \tau_j \leq 2\bar{s}$. It follows from Lemmas 10 and 12 that

$$\begin{aligned} \min_{K \leq k \leq 2K} \frac{1}{\beta} \|(w^{k+1}, \mu^{k+1}) - (w^k, \mu^k)\| &\leq \frac{\sqrt{2} \left(\|(w^0, \mu^0) - (w^*, \mu^*)\| + 2\sqrt{n}\beta \sum_{j=0}^{\infty} \tau_j \right)}{\beta\sqrt{K+1}} \\ &\leq \frac{\sqrt{2} \left(\|(w^0, \mu^0) - (w^*, \mu^*)\| + 4\sqrt{n}\bar{s}\beta \right)}{\beta\sqrt{K+1}} = \frac{\sqrt{2}(r_0 + 4\sqrt{n}\bar{s}\beta)}{\beta\sqrt{K+1}}, \end{aligned}$$

which then implies that

$$\begin{aligned} \min_{K \leq k \leq 2K} \left\{ \tau_k + \frac{1}{\beta} \|w^{k+1} - w^k\|_{\infty} \right\} &\leq \frac{\bar{s}}{(K+1)^2} + \frac{\sqrt{2}(r_0 + 4\sqrt{n}\bar{s}\beta)}{\beta\sqrt{K+1}} \leq \left[\bar{s} + \frac{\sqrt{2}(r_0 + 4\sqrt{n}\bar{s}\beta)}{\beta} \right] \frac{1}{\sqrt{K+1}}, \\ \min_{K \leq k \leq 2K} \frac{1}{\beta} \|\mu^{k+1} - \mu^k\|_{\infty} &\leq \frac{\sqrt{2}(r_0 + 4\sqrt{n}\bar{s}\beta)}{\beta\sqrt{K+1}}. \end{aligned}$$

We see from these and the termination criterion in Eq. (11) that the number of outer iterations of Algorithm 1 is at most

$$K_{\epsilon_1, \epsilon_2} := \left[\bar{s} + \frac{\sqrt{2}(r_0 + 4\sqrt{n}\bar{s}\beta)}{\beta} \right]^2 \max\{\epsilon_1^{-2}, \epsilon_2^{-2}\} = \mathcal{O}(\max\{\epsilon_1^{-2}, \epsilon_2^{-2}\}).$$

Hence, Theorem 4(a) holds as desired. \square

B Proofs of the main results in Section 4

Throughout this section, we let (\tilde{w}^*, u^*) be the optimal solution of Eq. (24), and λ^* be the associated Lagrangian multiplier. Recall from the definition of \tilde{u}_i^0 in Algorithm 2 and Eq. (21) that

$$\tilde{u}_i^t = u_i^t + \lambda_i^t / \rho_i, \quad \forall 1 \leq i \leq n, t \geq 0. \quad (48)$$

B.1 Proof of Lemma 5

For notational convenience, write $f_0(w) \equiv 0$. Then, by Eqs. (13) and (14), one can verify that

$$\nabla P_{i,k}(w) = \nabla f_i(w) + \nabla c_i(w)[\mu_i^k + \beta c_i(w)]_+ + \frac{1}{(n+1)\beta}(w - w^k), \quad \forall 0 \leq i \leq n. \quad (49)$$

Proof of Lemma 5. Fix an arbitrary $w \in \mathbb{R}^d$ and a bounded open set \mathcal{U}_w containing w . We suppose that ∇f_i is $L_{w,1}$ -Lipschitz continuous on \mathcal{U}_w , and ∇c_i is $L_{w,2}$ -Lipschitz continuous on \mathcal{U}_w . Also, let $U_{w,1} = \sup_{w \in \mathcal{U}_w} \|c_i(w)\|$ and $U_{w,2} = \sup_{w \in \mathcal{U}_w} \|\nabla c_i(w)\|$. By Eqs. (13), (14) and (49) one has for each $0 \leq i \leq n$ and $u, v \in \mathcal{U}_w$ that

$$\begin{aligned} \|\nabla P_{i,k}(u) - \nabla P_{i,k}(v)\| &\stackrel{\text{Eq. (49)}}{\leq} \|\nabla f_i(u) - \nabla f_i(v)\| + \|\nabla c_i(u) - \nabla c_i(v)\| \|[\mu_i^k + \beta c_i(u)]_+\| \\ &\quad + \|[\mu_i^k + \beta c_i(u)]_+ - [\mu_i^k + \beta c_i(v)]_+\| \|\nabla c_i(v)\| + \frac{1}{(n+1)\beta} \|u - v\| \\ &\leq L_{w,1} \|u - v\| + (\|\mu_i^k\| + \beta U_{w,1}) L_{w,2} \|u - v\| \\ &\quad + \beta \|c_i(u) - c_i(v)\| \|\nabla c_i(v)\| + \frac{1}{(n+1)\beta} \|u - v\| \\ &\leq \left[L_{w,1} + (\|\mu_i^k\| + \beta U_{w,1}) L_{w,2} + \beta U_{w,2}^2 + \frac{1}{(n+1)\beta} \right] \|u - v\|. \end{aligned}$$

Therefore, $\nabla P_{i,k}(u)$ is locally Lipschitz continuous on \mathbb{R}^d , and the conclusion holds as desired. \square

B.2 Proof of Theorem 7

Proof of Theorem 7. In view of the termination criterion Eq. (23), it suffices to show that

$$\text{dist}_\infty(0, \partial\ell(w^{T+1})) \leq \varepsilon_{T+1} + \sum_{i=1}^n \tilde{\varepsilon}_{i,T+1}.$$

By the definition of ℓ in Eq. (16), one has that

$$\partial\ell(w^{T+1}) = \sum_{i=0}^n \nabla P_i(w^{T+1}) + \partial h(w^{T+1}). \quad (50)$$

In addition, notice from Eqs. (18), (19) and (48) that

$$\begin{aligned} \partial\varphi_{0,T}(w^{T+1}) &= \nabla P_0(w^{T+1}) + \sum_{i=1}^n \rho_i(w^{T+1} - \tilde{u}_i^T) + \partial h(w^{T+1}) \\ &= \nabla P_0(w^{T+1}) + \sum_{i=1}^n [\rho_i(w^{T+1} - u_i^T) - \lambda_i^T] + \partial h(w^{T+1}), \\ \nabla\varphi_{i,T}(w^{T+1}) &= \nabla P_i(w^{T+1}) + \lambda_i^T, \quad \forall 1 \leq i \leq n. \end{aligned}$$

Combining these with Eq. (50), we obtain that

$$\partial\ell(w^{T+1}) = \partial\varphi_{0,T}(w^{T+1}) + \sum_{i=1}^n [\nabla\varphi_{i,T}(w^{T+1}) - \rho_i(w^{T+1} - u_i^T)],$$

which together with $\text{dist}_\infty(0, \partial\varphi_{0,T}(w^{T+1})) \leq \varepsilon_{T+1}$ (see Algorithm 2 and Eq. (22)) implies that

$$\begin{aligned} \text{dist}_\infty(0, \partial\ell(w^{T+1})) &\leq \text{dist}_\infty(0, \partial\varphi_{0,T}(w^{T+1})) + \sum_{i=1}^n \|\nabla\varphi_{i,T}(w^{T+1}) - \rho_i(w^{T+1} - u_i^T)\|_\infty \\ &\leq \varepsilon_{T+1} + \sum_{i=1}^n \tilde{\varepsilon}_{i,T+1}, \end{aligned}$$

as desired. \square

B.3 Proof of Lemma 8

To prove Lemma 8, we use convergence analysis techniques for ADMM to show that the distances between iterates $\{u_i^k\}_{1 \leq i \leq n}$ and w^k and the optimal solution \tilde{w}^* are controlled by the distance between the initial iterate \tilde{w}^0 and \tilde{w}^* . To the best of our knowledge, such boundedness results without assuming global Lipschitz continuity are entirely new in the literature on ADMM.

Proof of Lemma 8. From the optimality conditions and stopping criteria for Eq. (18) and Eq. (19), there exist e_i^{t+1} 's for $0 \leq i \leq n$ with $\|e_i^{t+1}\|_\infty \leq \varepsilon_{t+1}$ and $h^{t+1} \in \partial h(w^{t+1})$ so that:

$$\begin{aligned} e_0^{t+1} &= \nabla P_0(w^{t+1}) + h^{t+1} + \sum_{i=1}^n \rho_i(w^{t+1} - \tilde{u}_i^t) \stackrel{\text{Eq. (48)}}{=} \nabla P_0(w^{t+1}) + h^{t+1} + \sum_{i=1}^n [\rho_i(w^{t+1} - u_i^t) - \lambda_i^t] \\ &\stackrel{\text{Eq. (20)}}{=} \nabla P_0(w^{t+1}) + h^{t+1} + \sum_{i=1}^n [\rho_i(u_i^{t+1} - u_i^t) - \lambda_i^{t+1}] \end{aligned} \quad (51)$$

and

$$e_i^{t+1} = \nabla \varphi_{i,t}(u_i^{t+1}) \stackrel{\text{Eq. (19)}}{=} \nabla P_i(u_i^{t+1}) + \lambda_i^t + \rho_i(u_i^{t+1} - w^{t+1}) \stackrel{\text{Eq. (20)}}{=} \nabla P_i(u_i^{t+1}) + \lambda_i^{t+1}, \quad \forall 1 \leq i \leq n. \quad (52)$$

Moreover, since \tilde{w}^* and u^* are the optimal solution of Eq. (24) with the associated Lagrangian multiplier $\lambda^* \in \mathbb{R}^m$, we have by the optimality condition that there exists $h^* \in \partial h(\tilde{w}^*)$ such that

$$\nabla P_i(u_i^*) + \lambda_i^* = 0, \quad \nabla P_0(\tilde{w}^*) + h^* - \sum_{i=1}^n \lambda_i^* = 0, \quad u_i^* = \tilde{w}^*, \quad \forall 1 \leq i \leq n. \quad (53)$$

Recall that P_i , $0 \leq i \leq n$, are strongly convex with the modulus $\sigma > 0$, we have

$$\begin{aligned} \sigma \|u_i^{t+1} - \tilde{w}^*\|^2 &\leq \langle u_i^{t+1} - \tilde{w}^*, \nabla P_i(u_i^{t+1}) - \nabla P_i(\tilde{w}^*) \rangle \quad (\text{strong convexity of } P_i) \\ &= \langle u_i^{t+1} - \tilde{w}^*, -\lambda_i^{t+1} + \lambda_i^* + e_i^{t+1} \rangle \quad (\tilde{w}^* = u_i^*, \nabla P_i(u_i^*) = \lambda_i^*, \text{ and Eq. (52)}) \\ &\leq \langle u_i^{t+1} - \tilde{w}^*, -\lambda_i^{t+1} + \lambda_i^* \rangle + \frac{\sigma}{2} \|u_i^{t+1} - \tilde{w}^*\|^2 + \frac{1}{2\sigma} \|e_i^{t+1}\|^2, \\ &\quad (\langle a, b \rangle \leq t/2 \|a\|^2 + 1/(2t) \|b\|^2 \text{ for all } a, b \in \mathbb{R}^d \text{ and } t > 0), \end{aligned}$$

and

$$\begin{aligned} \sigma \|w^{t+1} - \tilde{w}^*\|^2 &\leq \langle w^{t+1} - \tilde{w}^*, \nabla P_0(w^{t+1}) + h^{t+1} - \nabla P_0(\tilde{w}^*) - h^* \rangle \quad (\text{strong convexity of } P_0 + h) \\ &= \langle w^{t+1} - \tilde{w}^*, \sum_{i=1}^n [\lambda_i^{t+1} - \lambda_i^* - \rho_i(u_i^{t+1} - u_i^t)] + e_0^{t+1} \rangle, \quad (\text{Eq. (51) and Eq. (53)}) \\ &\leq \langle w^{t+1} - \tilde{w}^*, \sum_{i=1}^n [\lambda_i^{t+1} - \lambda_i^* - \rho_i(u_i^{t+1} - u_i^t)] \rangle + \frac{\sigma}{2} \|w^{t+1} - \tilde{w}^*\|^2 + \frac{1}{2\sigma} \|e_0^{t+1}\|^2, \\ &\quad (\langle a, b \rangle \leq t/2 \|a\|^2 + 1/(2t) \|b\|^2 \text{ for all } a, b \in \mathbb{R}^d \text{ and } t > 0). \end{aligned}$$

Summing up these inequalities and rearranging the terms, we obtain that

$$\begin{aligned} &\frac{\sigma}{2} (\|w^{t+1} - \tilde{w}^*\|^2 + \sum_{i=1}^n \|u_i^{t+1} - \tilde{w}^*\|^2) \\ &\leq \sum_{i=1}^n \langle w^{t+1} - \tilde{w}^*, \lambda_i^{t+1} - \lambda_i^* - \rho_i(u_i^{t+1} - u_i^t) \rangle + \frac{1}{2\sigma} \|e_0^{t+1}\|^2 + \sum_{i=1}^n (\langle u_i^{t+1} - \tilde{w}^*, -\lambda_i^{t+1} + \lambda_i^* \rangle + \frac{1}{2\sigma} \|e_i^{t+1}\|^2) \\ &\leq \sum_{i=1}^n \langle w^{t+1} - u_i^{t+1}, \lambda_i^{t+1} - \lambda_i^* \rangle + \sum_{i=1}^n \rho_i \langle w^{t+1} - \tilde{w}^*, u_i^t - u_i^{t+1} \rangle + \frac{n+1}{2\sigma} \varepsilon_{t+1}^2 \\ &\quad (\|e_i^{t+1}\| \leq \varepsilon_{t+1} \text{ for all } 0 \leq i \leq n \text{ and } t \geq 0) \\ &\stackrel{\text{Eq. (20)}}{=} \sum_{i=1}^n \frac{1}{\rho_i} \langle \lambda_i^t - \lambda_i^{t+1}, \lambda_i^{t+1} - \lambda_i^* \rangle + \sum_{i=1}^n \rho_i \langle w^{t+1} - \tilde{w}^*, u_i^t - u_i^{t+1} \rangle + \frac{n+1}{2\sigma} \varepsilon_{t+1}^2, \end{aligned} \quad (54)$$

Notice that the following well-known identities hold:

$$\langle w^{t+1} - \tilde{w}^*, u_i^t - u_i^{t+1} \rangle = \frac{1}{2} (\|w^{t+1} - u_i^{t+1}\|^2 - \|w^{t+1} - u_i^t\|^2 + \|\tilde{w}^* - u_i^t\|^2 - \|\tilde{w}^* - u_i^{t+1}\|^2), \quad (55)$$

$$\langle \lambda_i^t - \lambda_i^{t+1}, \lambda_i^{t+1} - \lambda_i^* \rangle = \frac{1}{2} (\|\lambda_i^* - \lambda_i^t\|^2 - \|\lambda_i^* - \lambda_i^{t+1}\|^2 - \|\lambda_i^t - \lambda_i^{t+1}\|^2). \quad (56)$$

These along with Eqs. (20) and (54) imply that

$$\begin{aligned}
& \frac{\sigma}{2} (\|w^{t+1} - \tilde{w}^*\|^2 + \sum_{i=1}^n \|u_i^{t+1} - \tilde{w}^*\|^2) + \sum_{i=1}^n \frac{\rho_i}{2} \|w^{t+1} - u_i^t\|^2 - \frac{n+1}{2\sigma} \varepsilon_{t+1}^2 \\
& \stackrel{\text{Eq. (54)}}{\leq} \sum_{i=1}^n \frac{1}{\rho_i} \langle \lambda_i^t - \lambda_i^{t+1}, \lambda_i^{t+1} - \lambda_i^* \rangle + \sum_{i=1}^n \rho_i \langle w^{t+1} - \tilde{w}^*, u_i^t - u_i^{t+1} \rangle + \sum_{i=1}^n \frac{\rho_i}{2} \|w^{t+1} - u_i^t\|^2 \\
& \stackrel{\text{Eq. (55)}}{\leq} \sum_{i=1}^n \frac{1}{\rho_i} \langle \lambda_i^t - \lambda_i^{t+1}, \lambda_i^{t+1} - \lambda_i^* \rangle + \sum_{i=1}^n \frac{\rho_i}{2} (\|\tilde{w}^* - u_i^t\|^2 - \|\tilde{w}^* - u_i^{t+1}\|^2 + \|w^{t+1} - u_i^{t+1}\|^2) \\
& \stackrel{\text{Eq. (20)}}{=} \sum_{i=1}^n \frac{1}{\rho_i} \langle \lambda_i^t - \lambda_i^{t+1}, \lambda_i^{t+1} - \lambda_i^* \rangle + \sum_{i=1}^n \frac{1}{2\rho_i} \|\lambda_i^{t+1} - \lambda_i^t\|^2 + \sum_{i=1}^n \frac{\rho_i}{2} (\|\tilde{w}^* - u_i^t\|^2 - \|\tilde{w}^* - u_i^{t+1}\|^2) \\
& \stackrel{\text{Eq. (56)}}{=} \sum_{i=1}^n \frac{1}{2\rho_i} (\|\lambda_i^* - \lambda_i^t\|^2 - \|\lambda_i^* - \lambda_i^{t+1}\|^2) + \sum_{i=1}^n \frac{\rho_i}{2} (\|\tilde{w}^* - u_i^t\|^2 - \|\tilde{w}^* - u_i^{t+1}\|^2) \\
& = \sum_{i=1}^n \left[\left(\frac{\rho_i}{2} \|\tilde{w}^* - u_i^t\|^2 + \frac{1}{2\rho_i} \|\lambda_i^* - \lambda_i^t\|^2 \right) - \left(\frac{\rho_i}{2} \|\tilde{w}^* - u_i^{t+1}\|^2 + \frac{1}{2\rho_i} \|\lambda_i^* - \lambda_i^{t+1}\|^2 \right) \right]. \tag{57}
\end{aligned}$$

Summing up this inequality over $t = 0, \dots, T$ for any $T \geq 0$, we obtain that

$$\begin{aligned}
& \sum_{t=0}^T \left[\frac{\sigma}{2} \left(\|w^{t+1} - \tilde{w}^*\|^2 + \sum_{i=1}^n \|u_i^{t+1} - \tilde{w}^*\|^2 \right) + \sum_{i=1}^n \frac{\rho_i}{2} \|w^{t+1} - u_i^t\|^2 - \frac{n+1}{2\sigma} \varepsilon_{t+1}^2 \right] \\
& \leq \sum_{i=1}^n \left[\left(\frac{\rho_i}{2} \|\tilde{w}^* - u_i^0\|^2 + \frac{1}{2\rho_i} \|\lambda_i^* - \lambda_i^0\|^2 \right) - \left(\frac{\rho_i}{2} \|\tilde{w}^* - u_i^{T+1}\|^2 + \frac{1}{2\rho_i} \|\lambda_i^* - \lambda_i^{T+1}\|^2 \right) \right]. \tag{58}
\end{aligned}$$

Recall from Algorithm 2 that $\varepsilon_{t+1} = q^t$, $u_i^0 = \tilde{w}^0$, and $\lambda_i^0 = -\nabla P_i(\tilde{w}^0)$. Notice from Eq. (53) that $\tilde{w}^* = u_i^*$ and $\lambda_i^* = -\nabla P_i(u_i^*)$. By these and Eq. (58), one can deduce that

$$\begin{aligned}
& \frac{\sigma}{2} (\|w^{t+1} - \tilde{w}^*\|^2 + \sum_{i=1}^n \|u_i^{t+1} - \tilde{w}^*\|^2) \leq \frac{n+1}{2\sigma} \sum_{t=0}^{\infty} q^{2t} + \sum_{i=1}^n \left(\frac{\rho_i}{2} \|\tilde{w}^* - u_i^0\|^2 + \frac{1}{2\rho_i} \|\lambda_i^* - \lambda_i^0\|^2 \right) \\
& \leq \frac{n+1}{2\sigma(1-q^2)} + \sum_{i=1}^n \left(\frac{\rho_i}{2} \|\tilde{w}^* - u_i^0\|^2 + \frac{1}{2\rho_i} \|\lambda_i^* - \lambda_i^0\|^2 \right) \\
& = \frac{n+1}{2\sigma(1-q^2)} + \sum_{i=1}^n \left(\frac{\rho_i}{2} \|\tilde{w}^* - \tilde{w}^0\|^2 + \frac{1}{2\rho_i} \|\nabla P_i(\tilde{w}^*) - \nabla P_i(\tilde{w}^0)\|^2 \right).
\end{aligned}$$

In view of this and the definition of \mathcal{Q} in Eq. (29), we can observe that $w^{t+1} \in \mathcal{Q}$ and $u_i^{t+1} \in \mathcal{Q}$ for all $t \geq 0$ and $1 \leq i \leq n$. Hence, the conclusion of this lemma holds as desired. \square

B.4 Proof of Theorem 9

We first prove an auxiliary recurrence result that will be used later.

Lemma 13. Assume that $r, c > 0$ and $q \in (0, 1)$. Let $\{a_t\}_{t \geq 0}$ be a sequence satisfying

$$(1+r)a_{t+1} \leq a_t + cq^{2t}, \quad \forall t \geq 0. \tag{59}$$

Then we have

$$a_{t+1} \leq \max \left\{ q, \frac{1}{1+r} \right\}^{t+1} \left(a_0 + \frac{c}{1-q} \right), \quad \forall t \geq 0. \tag{60}$$

Proof. It follows [Eq. \(59\)](#) that

$$\begin{aligned}
a_{t+1} &\leq \frac{1}{1+r}a_t + \frac{1}{1+r}cq^{2t} \leq \frac{1}{(1+r)^2}a_{t-1} + \frac{cq^{2(t-1)}}{(1+r)^2} + \frac{cq^{2t}}{1+r} \\
&\leq \cdots \leq \frac{1}{(1+r)^{t+1}}a_0 + \sum_{i=0}^t \frac{cq^{2i}}{(1+r)^{t+1-i}} = \frac{1}{(1+r)^{t+1}}a_0 + c \sum_{i=0}^t \frac{q^i}{(1+r)^{t+1-i}} q^i \\
&\leq \frac{1}{(1+r)^{t+1}}a_0 + c \max \left\{ q, \frac{1}{1+r} \right\}^{t+1} \sum_{i=0}^t q^i \\
&\quad (q^i \leq \max\{q, 1/(1+r)\}^i \text{ and } 1/(1+r)^{t+1-i} \leq \max\{q, 1/(1+r)\}^{t+1-i}) \\
&\leq \frac{1}{(1+r)^{t+1}}a_0 + \frac{c}{1-q} \max \left\{ q, \frac{1}{1+r} \right\}^{t+1} \\
&\leq \max \left\{ q, \frac{1}{1+r} \right\}^{t+1} \left(a_0 + \frac{c}{1-q} \right).
\end{aligned}$$

Hence, [Eq. \(60\)](#) holds as desired. \square

The following lemma proves the Lipschitz continuity of ∇P_i on \mathcal{Q} .

Lemma 14. *Let \mathcal{Q} be defined in [Eq. \(29\)](#). Then there exists some $L_{\nabla P} > 0$ such that*

$$\|\nabla P_i(u) - \nabla P_i(v)\| \leq L_{\nabla P} \|u - v\|, \quad \forall u, v \in \mathcal{Q}, 0 \leq i \leq n. \quad (61)$$

Proof. Notice from [Eq. \(29\)](#) that the set \mathcal{Q} is convex and compact. By this and the local Lipschitz continuity of ∇P_i on \mathbb{R}^d , one can verify that there exists some constant $L_{\nabla P} > 0$ such that [Eq. \(61\)](#) holds (see also Lemma 1 in [\[38\]](#)). \square

We introduce a potential function S_t to measure the convergence of [Algorithm 2](#):

$$S_t := \sum_{i=1}^n \left(\frac{\rho_i}{2} \|\tilde{w}^* - u_i^t\|^2 + \frac{1}{2\rho_i} \|\lambda_i^* - \lambda_i^t\|^2 \right), \quad \forall t \geq 0. \quad (62)$$

The following lemma gives a recursive result of S_t , which will play a key role on establishing the global convergence rate for [Algorithm 2](#) in [Theorem 9](#).

Lemma 15. *Suppose that Assumptions 0 to 2 hold. Let $\{w^{t+1}\}_{t \geq 0}$ and $\{u_i^{t+1}\}_{1 \leq i \leq n, t \geq 0}$ be all the iterates generated by [Algorithm 2](#). Then we have*

$$S_t \leq q_r^t \left[S_0 + \frac{1}{1-q} \left(\frac{n+1}{2\sigma} + \sum_{i=1}^n \frac{\sigma}{\rho_i^2 + 2L_{\nabla P}^2} \right) \right], \quad \forall t \geq 0, \quad (63)$$

where σ and $L_{\nabla P}$ are given in [Eq. \(17\)](#) and [Lemma 14](#), respectively, q and ρ_i , $1 \leq i \leq n$, are inputs of [Algorithm 2](#), and

$$q_r := \max \left\{ q, \frac{1}{1+r} \right\}, \quad r := \min_{1 \leq i \leq n} \left\{ \frac{\sigma \rho_i}{\rho_i^2 + 2L_{\nabla P}^2} \right\}. \quad (64)$$

Proof. Recall from Eq. (57) that

$$\begin{aligned}
S_t &= \sum_{i=1}^n \left(\frac{\rho_i}{2} \|\tilde{w}^* - u_i^t\|^2 + \frac{1}{2\rho_i} \|\lambda_i^* - \lambda_i^t\|^2 \right) \\
&\geq \sum_{i=1}^n \left(\frac{\rho_i + \sigma}{2} \|\tilde{w}^* - u_i^{t+1}\|^2 + \frac{1}{2\rho_i} \|\lambda_i^* - \lambda_i^{t+1}\|^2 + \frac{\rho_i}{2} \|w^{t+1} - u_i^t\|^2 \right) + \frac{\sigma}{2} \|w^{t+1} - \tilde{w}^*\|^2 - \frac{n+1}{2\sigma} \varepsilon_{t+1}^2 \\
&\geq \sum_{i=1}^n \left(\frac{\rho_i + \sigma}{2} \|\tilde{w}^* - u_i^{t+1}\|^2 + \frac{1}{2\rho_i} \|\lambda_i^* - \lambda_i^{t+1}\|^2 \right) - \frac{n+1}{2\sigma} \varepsilon_{t+1}^2.
\end{aligned} \tag{65}$$

Also, notice from Eqs. (52), (53) and (61) that

$$\|\lambda_i^* - \lambda_i^{t+1}\|^2 \stackrel{\text{Eqs. (52) and (53)}}{\leq} (\|\nabla P_i(\tilde{w}^*) - \nabla P_i(u_i^{t+1})\| + \|e_i^{t+1}\|)^2 \stackrel{\text{Eq. (61)}}{\leq} 2L_{\nabla P}^2 \|\tilde{w}^* - u_i^{t+1}\|^2 + 2\varepsilon_{t+1}^2,$$

which implies that

$$\|\tilde{w}^* - u_i^{t+1}\|^2 \geq \frac{2\rho_i}{\rho_i^2 + 2L_{\nabla P}^2} \left(\frac{\rho_i}{2} \|\tilde{w}^* - u_i^{t+1}\|^2 + \frac{1}{2\rho_i} \|\lambda_i^* - \lambda_i^{t+1}\|^2 \right) - \frac{2\varepsilon_{t+1}^2}{\rho_i^2 + 2L_{\nabla P}^2}. \tag{66}$$

Plugging this into Eq. (65), we have

$$\begin{aligned}
S_t &\stackrel{\text{Eq. (66)}}{\geq} \sum_{i=1}^n \left(1 + \frac{\sigma\rho_i}{\rho_i^2 + 2L_{\nabla P}^2} \right) \left(\frac{\rho_i}{2} \|\tilde{w}^* - u_i^{t+1}\|^2 + \frac{1}{2\rho_i} \|\lambda_i^* - \lambda_i^{t+1}\|^2 \right) - \frac{n+1}{2\sigma} q^{2t} - \sum_{i=1}^n \frac{\sigma}{\rho_i^2 + 2L_{\nabla P}^2} q^{2t} \\
&= (1+r)S_{t+1} - \left(\frac{n+1}{2\sigma} + \sum_{i=1}^n \frac{\sigma}{\rho_i^2 + 2L_{\nabla P}^2} \right) q^{2t} \quad (r := \min_{1 \leq i \leq n} \sigma\rho_i/(\rho_i^2 + 2L_{\nabla P}^2)).
\end{aligned}$$

When $t = 0$, Eq. (63) holds clearly. When $t \geq 1$, by the above inequality, Eq. (64), and Lemma 13 with $(a_t, c) = (S_t, \frac{n+1}{2\sigma} + \sum_{i=1}^n \frac{\sigma}{\rho_i^2 + 2L_{\nabla P}^2})$, we obtain that

$$\begin{aligned}
S_t &\leq \max \left\{ q, \frac{1}{1+r} \right\}^t \left[S_0 + \frac{1}{1-q} \left(\frac{n+1}{2\sigma} + \sum_{i=1}^n \frac{\sigma}{\rho_i^2 + 2L_{\nabla P}^2} \right) \right] \\
&= q_r^t \left[S_0 + \frac{1}{1-q} \left(\frac{n+1}{2\sigma} + \sum_{i=1}^n \frac{\sigma}{\rho_i^2 + 2L_{\nabla P}^2} \right) \right].
\end{aligned}$$

Therefore, the conclusion of this lemma is true as desired. \square

Proof of Theorem 9. Notice that Algorithm 2 terminates when $\varepsilon_{t+1} + \sum_{i=1}^n \tilde{\varepsilon}_{i,t+1}$ is small. Next, we show that this quantity is bounded by S_t defined in Eq. (62) plus other small quantities, and then use Lemma 15 to bound the maximum number of iterations of Algorithm 2.

By Eq. (22), and the fact that $\|\nabla \varphi_{i,t}(u_i^{t+1})\|_\infty \leq \varepsilon_{t+1}$ (see Algorithm 2), one can obtain that

$$\begin{aligned}
\varepsilon_{t+1} + \sum_{i=1}^n \tilde{\varepsilon}_{i,t+1} &\stackrel{\text{Eq. (22)}}{=} \varepsilon_{t+1} + \sum_{i=1}^n \|\nabla \varphi_{i,t}(w^{t+1}) - \rho_i(w^{t+1} - u_i^t)\|_\infty \\
&\leq \varepsilon_{t+1} + \sum_{i=1}^n \|\nabla \varphi_{i,t}(u_i^{t+1})\|_\infty + \sum_{i=1}^n \|\nabla \varphi_{i,t}(w^{t+1}) - \nabla \varphi_{i,t}(u_i^{t+1})\| + \sum_{i=1}^n \rho_i \|w^{t+1} - u_i^t\| \\
&\quad (\|u\|_\infty \leq \|u\| \text{ for all } u \in \mathbb{R}^d \text{ and the triangle inequality}) \\
&\leq (n+1)\varepsilon_{t+1} + \sum_{i=1}^n (L_{\nabla P} + \rho_i) \|w^{t+1} - u_i^{t+1}\| + \sum_{i=1}^n \rho_i \|w^{t+1} - u_i^t\|,
\end{aligned} \tag{67}$$

where the second inequality follows from

$$\begin{aligned} \|\nabla\varphi_{i,t}(w^{t+1}) - \nabla\varphi_{i,t}(u_i^{t+1})\| &\stackrel{\text{Eq. (19)}}{\leq} \|\nabla P_i(w^{t+1}) - \nabla P_i(u_i^{t+1})\| + \rho_i \|w^{t+1} - u_i^{t+1}\| \\ &\stackrel{\text{Eq. (61)}}{\leq} (L_{\nabla P} + \rho_i) \|w^{t+1} - u_i^{t+1}\|, \quad \forall 1 \leq i \leq n. \end{aligned}$$

Next, we derive upper bounds for $\|w^{t+1} - u_i^{t+1}\|$ and $\rho_i \|w^{t+1} - u_i^t\|$, respectively. First, by Eqs. (57), (62) and (63), we have

$$\begin{aligned} \frac{\sigma}{4} \|w^{t+1} - u_i^{t+1}\|^2 &\leq \frac{\sigma}{2} \|w^{t+1} - \tilde{w}^*\|^2 + \frac{\sigma}{2} \|u_i^{t+1} - \tilde{w}^*\|^2 \\ &\stackrel{\text{Eq. (57)}}{\leq} \sum_{i=1}^n \left(\frac{\rho_i}{2} \|\tilde{w}^* - u_i^t\|^2 + \frac{1}{2\rho_i} \|\lambda_i^* - \lambda_i^t\|^2 \right) + \frac{n+1}{2\sigma} \varepsilon_{t+1}^2 \\ &= S_t + \frac{n+1}{2\sigma} q^{2t} \quad (\text{the definition of } S_t \text{ in Eq. (62) and } \varepsilon_{t+1} = q^t \text{ for all } t \geq 0) \\ &\stackrel{\text{Eq. (63)}}{\leq} q_r^t \left[S_0 + \frac{1}{1-q} \left(\frac{n+1}{2\sigma} + \sum_{i=1}^n \frac{\sigma}{\rho_i^2 + 2L_{\nabla P}^2} \right) \right] + \frac{n+1}{2\sigma} q^{2t} \\ &\leq \left\{ q_r^{t/2} \left[S_0 + \frac{1}{1-q} \left(\frac{n+1}{2\sigma} + \sum_{i=1}^n \frac{\sigma}{\rho_i^2 + 2L_{\nabla P}^2} \right) \right]^{1/2} + \sqrt{\frac{n+1}{2\sigma}} q^t \right\}^2 \\ &\quad (a^2 + b^2 \leq (a+b)^2 \text{ for all } a, b \geq 0). \end{aligned} \tag{68}$$

Using again Eqs. (57), (62) and (63), we obtain that

$$\begin{aligned} \frac{1}{2} \left(\sum_{i=1}^n \rho_i \|w^{t+1} - u_i^t\| \right)^2 &\leq \left(\sum_{i=1}^n \rho_i \right) \left(\sum_{i=1}^n \frac{\rho_i}{2} \|w^{t+1} - u_i^t\|^2 \right) \quad (\text{Cauchy-Schwarz inequality}) \\ &\stackrel{\text{Eq. (57)}}{\leq} \left(\sum_{i=1}^n \rho_i \right) \sum_{i=1}^n \left(\frac{\rho_i}{2} \|\tilde{w}^* - u_i^t\|^2 + \frac{1}{2\rho_i} \|\lambda_i^* - \lambda_i^t\|^2 \right) + \left(\sum_{i=1}^n \rho_i \right) \frac{n+1}{2\sigma} \varepsilon_{t+1}^2 \\ &= \left(\sum_{i=1}^n \rho_i \right) \left(S_t + \frac{n+1}{2\sigma} q^{2t} \right) \quad (\text{the definition of } S_t \text{ in Eq. (62) and } \varepsilon_{t+1} = q^t \text{ for all } t \geq 0) \\ &\stackrel{\text{Eq. (63)}}{\leq} \left(\sum_{i=1}^n \rho_i \right) \left\{ \left[S_0 + \frac{1}{1-q} \left(\frac{n+1}{2\sigma} + \sum_{i=1}^n \frac{\sigma}{\rho_i^2 + 2L_{\nabla P}^2} \right) \right] q_r^t + \frac{n+1}{2\sigma} q^{2t} \right\} \\ &\leq \left(\sum_{i=1}^n \rho_i \right) \left\{ q_r^{t/2} \left[S_0 + \frac{1}{1-q} \left(\frac{n+1}{2\sigma} + \sum_{i=1}^n \frac{\sigma}{\rho_i^2 + 2L_{\nabla P}^2} \right) \right]^{1/2} + \sqrt{\frac{n+1}{2\sigma}} q^t \right\}^2 \\ &\quad (a^2 + b^2 \leq (a+b)^2 \text{ for all } a, b \geq 0). \end{aligned} \tag{70}$$

Combining Eqs. (67), (69) and (70), we obtain that

$$\begin{aligned}
\varepsilon_{t+1} + \sum_{i=1}^n \tilde{\varepsilon}_{i,t+1} &\leq (n+1)q^t + \left(\frac{2}{\sqrt{\sigma}} \sum_{i=1}^n (L_{\nabla P} + \rho_i) + \sqrt{2 \sum_{i=1}^n \rho_i} \right) \\
&\quad \cdot \left\{ \left[S_0 + \frac{1}{1-q} \left(\frac{n+1}{2\sigma} + \sum_{i=1}^n \frac{\sigma}{\rho_i^2 + 2L_{\nabla P}^2} \right) \right]^{1/2} q_r^{t/2} + \sqrt{\frac{n+1}{2\sigma}} q^t \right\} \\
&\leq (n+1)q_r^{t/2} + \left(\frac{2}{\sqrt{\sigma}} \sum_{i=1}^n (L_{\nabla P} + \rho_i) + \sqrt{2 \sum_{i=1}^n \rho_i} \right) \\
&\quad \cdot \left\{ \left[S_0 + \frac{1}{1-q} \left(\frac{n+1}{2\sigma} + \sum_{i=1}^n \frac{\sigma}{\rho_i^2 + 2L_{\nabla P}^2} \right) \right]^{1/2} + \sqrt{\frac{n+1}{2\sigma}} \right\} q_r^{t/2} \\
&\quad (q \leq q_r \leq q_r^{1/2} < 1). \tag{71}
\end{aligned}$$

Recall from Algorithm 2 and Eq. (53) that $(u_i^0, \lambda_i^0) = (\tilde{w}^0, -\nabla P_i(\tilde{w}^0))$ and $\lambda_i^* = -\nabla P_i(\tilde{w}^*)$. By these and Eq. (62), one has

$$S_0 = \sum_{i=1}^n \left(\frac{\rho_i}{2} \|\tilde{w}^* - \tilde{w}^0\|^2 + \frac{1}{2\rho_i} \|\nabla P_i(\tilde{w}^*) - \nabla P_i(\tilde{w}^0)\|^2 \right). \tag{72}$$

For convenience, denote

$$\begin{aligned}
b &:= \left(\frac{2}{\sqrt{\sigma}} \sum_{i=1}^n (L_{\nabla P} + \rho_i) + \sqrt{2 \sum_{i=1}^n \rho_i} \right) \\
&\quad \cdot \left\{ \left[\sum_{i=1}^n \left(\frac{\rho_i}{2} \|\tilde{w}^* - \tilde{w}^0\|^2 + \frac{1}{2\rho_i} \|\nabla P_i(\tilde{w}^*) - \nabla P_i(\tilde{w}^0)\|^2 \right) + \frac{1}{1-q} \left(\frac{n+1}{2\sigma} + \sum_{i=1}^n \frac{\sigma}{\rho_i^2 + 2L_{\nabla P}^2} \right) \right]^{1/2} \right. \\
&\quad \left. + \sqrt{\frac{n+1}{2\sigma}} \right\}.
\end{aligned}$$

Using this, Eqs. (71) and (72), we obtain that

$$\varepsilon_{t+1} + \sum_{i=1}^n \tilde{\varepsilon}_{i,t+1} \leq (n+1+b)q_r^{t/2}.$$

This along with the termination criterion in Eq. (23) implies that the number of iterations of Algorithm 2 is bounded above by

$$\left\lceil \frac{2 \log(\tau/(n+1+b))}{\log q_r} \right\rceil + 1 = \mathcal{O}(|\log \tau|). \tag{73}$$

Hence, the conclusion of this theorem holds as desired. \square

We observe from the proof of Theorem 9 that under Assumptions 0 to 2, the number of iterations of Algorithm 2 is bounded by the quantity in Eq. (73).

C Proof of Theorem 4(b)

To establish the total inner-iteration complexity, we first show that all the inner iterates produced by Algorithm 2 for solving all the subproblems of form Eq. (12) within Algorithm 1 are in a compact set (i.e., \mathcal{Q}_2 later), and then estimate the Lipschitz modulus of $\nabla P_{i,k}$ for all $0 \leq i \leq n$ and all k over \mathcal{Q}_2 . Then we can bound the inner-iteration complexity based on the size of \mathcal{Q}_2 and the Lipschitz modulus.

Boundedness of all inner iterates Recall from Eq. (15) that \mathcal{Q}_1 is a compact set. Let

$$U_{\nabla f} := \sup_{w \in \mathcal{Q}_1} \max_{1 \leq i \leq n} \|\nabla f_i(w)\|, \quad U_{\nabla c} := \sup_{w \in \mathcal{Q}_1} \max_{0 \leq i \leq n} \|\nabla c_i(w)\|, \quad U_c := \sup_{w \in \mathcal{Q}_1} \max_{0 \leq i \leq n} \|c_i(w)\|. \quad (74)$$

Since ∇f_i for all $1 \leq i \leq n$ are locally Lipschitz, they are Lipschitz on \mathcal{Q}_1 with some $L_{\nabla f} > 0$. Similarly, ∇c_i for all $0 \leq i \leq n$ are Lipschitz on \mathcal{Q}_1 with some modulus $L_{\nabla c} > 0$. Hence, $\nabla P_{i,k}$ for all $0 \leq i \leq n$ and all k are Lipschitz continuous on \mathcal{Q}_1 :

$$\begin{aligned} \|\nabla P_{i,k}(u) - \nabla P_{i,k}(v)\| &\stackrel{\text{Eq. (49)}}{\leq} \|\nabla f_i(u) - \nabla f_i(v)\| + \|[\mu_i^k + \beta c_i(u)]_+\| \|\nabla c_i(u) - \nabla c_i(v)\| \\ &\quad + \|[\mu_i^k + \beta c_i(u)]_+ - [\mu_i^k + \beta c_i(v)]_+\| \|\nabla c_i(v)\| + \frac{1}{(n+1)\beta} \|u - v\| \\ &\stackrel{\text{Eq. (80)}}{\leq} L_{\nabla f} \|u - v\| + (\|\mu_i^*\| + \|\mu_i^k - \mu_i^*\| + \beta U_c) L_{\nabla c} \|u - v\| \\ &\quad + \beta U_{\nabla c}^2 \|u - v\| + \frac{1}{(n+1)\beta} \|u - v\| \\ &\stackrel{\text{Eq. (47)}}{\leq} L_{\nabla P,1} \|u - v\| \end{aligned} \quad (75)$$

where

$$L_{\nabla P,1} := L_{\nabla f} + (\|\mu^*\| + r_0 + 2\sqrt{n}\bar{s}\beta + \beta U_c) L_{\nabla c} + \beta U_{\nabla c}^2 + \frac{1}{(n+1)\beta}. \quad (76)$$

The next lemma says that all the inner iterates generated by Algorithm 2 stay in a compact set.

Lemma 16. *Suppose that Assumptions 0 to 2 hold and let $\{w^{k,t+1}\}_{t \geq 0}$ and $\{u_i^{k,t+1}\}_{1 \leq i \leq n, t \geq 0}$ be all the inner iterates generated by Algorithm 2 for solving the subproblems of form Eq. (12) in Algorithm 1. Then it holds that all these iterates stay in a compact set \mathcal{Q}_2 , where*

$$\mathcal{Q}_2 := \left\{ v : \|v - u\|^2 \leq \frac{(n+1)^3 \beta^2}{(1-q^2)} + (n+1)\beta \sum_{i=1}^n \left[\left(\rho_i + \frac{L_{\nabla P,1}^2}{\rho_i} \right) (r_0 + 2\sqrt{n}\bar{s}\beta)^2 \right], u \in \mathcal{Q}_1 \right\}, \quad (77)$$

and $L_{\nabla P,1}$ and \mathcal{Q}_1 are as defined in Eq. (76) and Eq. (15), respectively.

Proof. Recall that for any $k \geq 0$, the subproblem in Eq. (12) has an optimal solution w_*^k (see Eq. (40)), the initial iterate of Algorithm 2 for solving Eq. (12) is w^k , and $P_{i,k}$, $0 \leq i \leq n$, are strongly convex with modulus $1/[(n+1)\beta]$. By Lemma 8 with $(P_i, \tilde{w}^*, \tilde{w}^0, \sigma) = (P_{i,k}, w_*^k, w^k, 1/[(n+1)\beta])$, we obtain that $\{w^{k,t+1}\}_{t \geq 0}$ and $\{u_i^{k,t+1}\}_{t \geq 0, 1 \leq i \leq n}$ stay in a set $\tilde{\mathcal{Q}}$ defined as

$$\tilde{\mathcal{Q}} := \left\{ v : \|v - w_*^k\|^2 \leq \frac{(n+1)^3 \beta^2}{(1-q^2)} + (n+1)\beta \sum_{i=1}^n \left(\rho_i \|w_*^k - w^k\|^2 + \frac{1}{\rho_i} \|\nabla P_{i,k}(w_*^k) - \nabla P_{i,k}(w^k)\|^2 \right) \right\}.$$

Thus, to show the boundedness of all the inner iterates, it suffices to derive upper bounds for $\|w_*^k - w^k\|$ and $\|\nabla P_{i,k}(w_*^k) - \nabla P_{i,k}(w^k)\|$ that are independent of k . Since $w^k, w_*^k \in \mathcal{Q}_1$, we have

$$\|\nabla P_{i,k}(w_*^k) - \nabla P_{i,k}(w^k)\| \leq L_{\nabla P,1} \|w_*^k - w^k\| \quad (78)$$

due to Eq. (75), where we note that $L_{\nabla P,1}$ is independent of k . Moreover, $\|w_*^k - w^k\| \leq r_0 + 2\sqrt{n}\bar{s}\beta$ from Eq. (47) provides a k -independent upper bound for $\|w_*^k - w^k\|$. Thus, we have

$$\sum_{i=1}^n \left(\rho_i \|w_*^k - w^k\|^2 + \frac{1}{\rho_i} \|\nabla P_{i,k}(w_*^k) - \nabla P_{i,k}(w^k)\|^2 \right) \leq \sum_{i=1}^n \left[\left(\rho_i + \frac{L_{\nabla P,1}^2}{\rho_i} \right) (r_0 + 2\sqrt{n}\bar{s}\beta)^2 \right]. \quad (79)$$

Finally, combining the above results and noting that $w_*^k \in \mathcal{Q}_1$ completes the proof. \square

Lipschitz modulus of $\nabla P_{i,k}$ for all $0 \leq i \leq n$ and all k over \mathcal{Q}_2 Let $L_{\nabla f,2}$ be the Lipschitz constant of ∇f_i , $1 \leq i \leq n$, on \mathcal{Q}_2 , and $L_{\nabla c,2}$ be the Lipschitz constant of ∇c_i , $0 \leq i \leq n$, on \mathcal{Q}_2 . Also, define

$$U_{\nabla c,2} := \sup_{w \in \mathcal{Q}_2} \max_{0 \leq i \leq n} \|\nabla c_i(w)\|, \quad U_{c,2} := \sup_{w \in \mathcal{Q}_2} \max_{0 \leq i \leq n} \|c_i(w)\|. \quad (80)$$

Using similar arguments as for deriving $L_{\nabla P,1}$ in Eq. (76), we can see that $\nabla P_{i,k}$, $0 \leq i \leq n$, are Lipschitz continuous on \mathcal{Q}_2 with modulus $L_{\nabla P,2}$ defined as

$$L_{\nabla P,2} := L_{\nabla f,2} + (\|\mu^*\| + r_0 + 2\sqrt{n}\bar{s}\beta + \beta U_{c,2})L_{\nabla c,2} + \beta U_{\nabla c,2}^2 + \frac{1}{(n+1)\beta}. \quad (81)$$

Proof of Theorem 4(b) Recall that Theorem 9 has established the number of iterations of Algorithm 2 for solving each subproblem of Algorithm 1. In the rest of this proof, we derive an upper bound for the total number of inner iterations for solving all subproblems of Algorithm 1.

We see from Lemma 16 that all iterates generated by Algorithm 2 for solving Eq. (12) lie in \mathcal{Q}_2 . Also, $\nabla P_{i,k}$, $1 \leq i \leq n$, are $L_{\nabla P,2}$ -Lipschitz continuous on \mathcal{Q}_2 . Therefore, by Theorem 9 with $(\tau, P_i, \sigma, L_{\nabla P}, \tilde{w}^*, \tilde{w}^0) = (\tau_k, P_{i,k}, 1/[(n+1)\beta], L_{\nabla P,2}, w_*^k, w^k)$ and the discussion at the end of Appendix B.4, one has that the number of iterations of Algorithm 2 for solving Eq. (12) during the k -th outer loop is no more than

$$T_k := \left\lceil \frac{2 \log(\tau_k / (n+1+b_k))}{\log \tilde{q}_r} \right\rceil + 1 \quad (82)$$

where

$$\begin{aligned} \tilde{q}_r &:= \max \left\{ q, \frac{1}{1+\tilde{r}} \right\}, \quad \tilde{r} := \min_{1 \leq i \leq n} \left\{ \frac{\rho_i}{(n+1)\beta(\rho_i^2 + 2L_{\nabla P,2}^2)} \right\}, \\ b_k &:= \left(2\sqrt{(n+1)\beta} \sum_{i=1}^n (L_{\nabla P,2} + \rho_i) + \sqrt{2 \left(\sum_{i=1}^n \rho_i \right)} \right) \\ &\quad \cdot \left\{ \left[\sum_{i=1}^n \left(\frac{\rho_i}{2} \|w_*^k - w^k\|^2 + \frac{1}{2\rho_i} \|\nabla P_{i,k}(w_*^k) - \nabla P_{i,k}(w^k)\|^2 \right) \right. \right. \\ &\quad \left. \left. + \frac{1}{1-q} \left(\frac{(n+1)^2\beta}{2} + \frac{1}{(n+1)\beta} \sum_{i=1}^n \frac{1}{\rho_i^2 + 2L_{\nabla P,2}^2} \right) \right]^{1/2} + (n+1)\sqrt{\frac{\beta}{2}} \right\}. \end{aligned}$$

Plugging Eq. (79) into \bar{b} , we have that $b_k \leq \bar{b}$, where

$$\begin{aligned} \bar{b} &:= \left(2\sqrt{(n+1)\beta} \sum_{i=1}^n (L_{\nabla P,2} + \rho_i) + \sqrt{2 \left(\sum_{i=1}^n \rho_i \right)} \right) \\ &\quad \cdot \left\{ \left[\sum_{i=1}^n \frac{\rho_i^2 + L_{\nabla P,1}^2}{2\rho_i} (r_0 + 2\sqrt{n}\bar{s}\beta)^2 + \frac{1}{1-q} \left(\frac{(n+1)^2\beta}{2} + \frac{1}{(n+1)\beta} \sum_{i=1}^n \frac{1}{\rho_i^2 + 2L_{\nabla P,2}^2} \right) \right]^{1/2} + (n+1)\sqrt{\frac{\beta}{2}} \right\}, \end{aligned}$$

which is independent of k . By $b_k \leq \bar{b}$, $\tau_k = \bar{s}/(k+1)^2$, $k \leq K_{\epsilon_1, \epsilon_2}$ where $K_{\epsilon_1, \epsilon_2}$ is the upper bound for the number of outer iterations as defined in Appendix A.3, and Eq. (82), one has that

$$T_k \leq \left\lceil \frac{2 \log((n+1+\bar{b})(K_{\epsilon_1, \epsilon_2} + 1)^2 / \bar{s})}{\log(\tilde{q}_r^{-1})} \right\rceil + 1.$$

Therefore, by $K_{\epsilon_1, \epsilon_2} = \mathcal{O}(\max\{\epsilon_1^{-2}, \epsilon_2^{-2}\})$, one can see that the total number of inner iterations of Algorithm 1 is at most

$$\sum_{k=0}^{K_{\epsilon_1, \epsilon_2}} T_k \leq (K_{\epsilon_1, \epsilon_2} + 1) \left(\left\lceil \frac{2 \log((n+1+\bar{b})(K_{\epsilon_1, \epsilon_2} + 1)^2/\bar{s})}{\log(\tilde{q}_r^{-1})} \right\rceil + 1 \right) = \tilde{\mathcal{O}}(\max\{\epsilon_1^{-2}, \epsilon_2^{-2}\}).$$

Hence, Theorem 4(b) holds as desired.

Algorithm 3 A centralized proximal AL method for solving Eq. (83)

Input: tolerances $\epsilon_1, \epsilon_2 \in (0, 1)$, $w^0 \in \text{dom}(h)$, $\mu^0 \geq 0$, nondecreasing positive $\{\tau_k\}_{k \geq 0}$, and $\beta > 0$.

for $k = 0, 1, 2, \dots$ **do**

 Apply a centralized solver to find an approximate solution w^{k+1} to:

$$\min_w \left\{ \ell_k(w) = f(w) + h(w) + \frac{1}{2\beta} \left(\|\mu^k + \beta c(w)\|_+^2 - \|\mu^k\|^2 \right) + \frac{1}{2\beta} \|w - w^k\|^2 \right\}$$

 such that

$$\text{dist}_\infty(0, \partial \ell_k(w^{k+1})) \leq \tau_k.$$

 Update the Lagrangian multiplier:

$$\mu^{k+1} = [\mu^k + \beta c(w^{k+1})]_+.$$

 Output (w^{k+1}, μ^{k+1}) and terminate the algorithm if

$$\|w^{k+1} - w^k\|_\infty + \beta \tau_k \leq \beta \epsilon_1, \quad \|\mu^{k+1} - \mu^k\|_\infty \leq \beta \epsilon_2.$$

end for

D A centralized proximal AL method

In this part, we present a centralized proximal AL method (adapted from Algorithm 2 of [39]) for solving the convex constrained optimization problem:

$$\min_w f(w) + h(w) \quad \text{s.t.} \quad c(w) \leq 0, \quad (83)$$

where the function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ and the mapping $c : \mathbb{R}^d \rightarrow \mathbb{R}^m$ are continuous differentiable and convex, and h is closed convex.

E Extra Numerical Results

E.1 Dataset description for Neyman-Pearson classification

In this part, we describe the datasets for Neyman-Pearson classification in Section 5.1. ‘breast-cancer-wisc’, ‘adult-a’, and ‘monks-1’ are three binary classification datasets. We present the total number of samples for class 0 and class 1 and the number of features.

Table 3: Datasets for Neyman-Pearson classification

dataset	class 0/class 1	feature dimension
breast-cancer-wisc	455/240	20
adult-a	24715/7840	21
monks-1	275/275	21

Table 4: Numerical results for Eq. (84) using our algorithm vs. using cProx-AL. Inside the parentheses are the respective standard deviations over 10 random trials.

n	d	\tilde{m}	objective value			feasibility violation	
			Algorithm 1	cProx-AL	relative difference	Algorithm 1	cProx-AL
1	100	1	-0.23 (4.65e-6)	-0.23 (2.38e-5)	1.63e-3 (1.01e-4)	3.33e-4 (1.14e-5)	5.68e-4 (2.82e-5)
	300	3	-0.37 (2.74e-6)	-0.37 (1.32e-6)	1.01e-3 (3.51e-5)	3.52e-4 (1.44e-5)	4.45e-4 (1.70e-5)
	500	5	-0.30 (1.36e-5)	-0.30 (7.54e-6)	1.34e-3 (4.62e-5)	4.38e-4 (5.36e-5)	3.85e-4 (1.05e-5)
5	100	1	9.81 (7.18e-5)	9.80 (1.46e-5)	1.09e-3 (7.96e-6)	1.34e-4 (9.02e-6)	8.03e-4 (1.57e-6)
	300	3	8.47 (8.12e-5)	8.45 (1.30e-5)	1.36e-3 (9.62e-6)	1.09e-4 (1.31e-5)	8.28e-4 (1.98e-6)
	500	5	9.92 (4.43e-5)	9.91 (4.87e-6)	8.26e-4 (4.27e-6)	1.33e-4 (9.68e-6)	3.73e-4 (2.43e-7)
10	100	1	49.40 (9.02e-5)	49.37 (5.82e-6)	5.59e-4 (1.67e-5)	7.31e-5 (7.54e-6)	5.88e-4 (1.34e-7)
	300	3	41.49 (7.04e-5)	41.44 (5.48e-6)	1.14e-3 (1.77e-6)	8.56e-5 (2.27e-7)	8.73e-4 (7.26e-6)
	500	5	41.45 (2.25e-5)	41.41 (5.30e-6)	9.39e-4 (4.94e-7)	9.29e-4 (2.55e-6)	7.66e-4 (1.37e-7)

E.2 Linear equality constrained quadratic programming

In this subsection, we consider the linear equality constrained quadratic programming:

$$\min_w \sum_{i=1}^n \left(\frac{1}{2} w^T A_i w + b_i^T w \right) \quad \text{s.t.} \quad C_i w + d_i = 0, \quad 0 \leq i \leq n, \quad (84)$$

where $A_i \in \mathbb{R}^{d \times d}$, $1 \leq i \leq n$, are positive semidefinite, $b_i \in \mathbb{R}^d$, $1 \leq i \leq n$, $C_i \in \mathbb{R}^{\tilde{m} \times d}$, $0 \leq i \leq n$, and $d_i \in \mathbb{R}^{\tilde{m}}$, $0 \leq i \leq n$.

For each (d, n, \tilde{m}) , we randomly generate an instance of Eq. (84). In particular, for each $1 \leq i \leq n$, we first generate a random matrix A_i by letting $A_i = U_i D_i U_i^T$, where $D_i \in \mathbb{R}^{d \times d}$ is a diagonal matrix. The diagonal entries of D_i are generated randomly from a uniform distribution over $[0.5, 1]$, and $U_i \in \mathbb{R}^{d \times d}$ is a randomly generated orthogonal matrix. We then randomly generate C_i , $0 \leq i \leq n$, with all entries drawn from a normal distribution with mean zero and a standard deviation of $1/\sqrt{d}$. Finally, we generate b_i , $1 \leq i \leq n$ and d_i , $0 \leq i \leq n$ as random vectors uniformly selected from the unit Euclidean sphere.

We apply Algorithm 1 and cProx-AL (Algorithm 3) to find a $(10^{-3}, 10^{-3})$ -optimal solution of Eq. (84), and compare their solution quality. In particular, when implementing Algorithm 1, we exactly solve the quadratic subproblems in Eqs. (18) and (19). We run 10 trials of Algorithm 1 and cProx-AL, where for each run, both algorithms share the same initial point w^0 , randomly chosen from the unit Euclidean sphere. We set the other parameters for Algorithm 1 and cProx-AL as $\mu_i^0 = (0, \dots, 0)^T \forall 0 \leq i \leq n$, $\bar{s} = 0.1$ and $\beta = 10$. We also set $\rho_i = 1$ for each $1 \leq i \leq n$ for Algorithm 2.

We observe that Algorithm 1 and cProx-AL are capable of finding nearly feasible solutions, and achieve similar objective value. In view of the small standard deviations, we observe that the convergence behavior of Algorithm 1 remains stable across 10 trial runs.