Diffie–Hellman Key Exchange from Commutativity to Group Laws

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Outline

- Review the classical Diffie–Hellman key exchange.

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- Propose our group action-based key exchange framework.

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- Propose our group action-based key exchange framework.
- Instantiate the framework by linear code equivalence problems.

What is key exchange?

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- Key exchange: a public-key protocol allowing two parties to establish a shared secret over an insecure channel.
  - The shared secret is computed from the combination of a public key and one's private key.
  - An adversary can eavesdrop on all transmitted messages.
- Application: HTTPS, VPN, and messaging services.

Diffie–Hellman key exchange protocol

Alice

Bob

Diffie–Hellman key exchange protocol

Alice

pk : prime p and generator γ of a cyclic group C_p

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$a \xleftarrow{\$} \mathbb{Z}_p^*$

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$a \xleftarrow{\$} \mathbb{Z}_p^*$

$A = \gamma^a$

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$a \xleftarrow{\$} \mathbb{Z}_p^*$

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$\mathbf{key} = B^a$

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Correctness: $A^b = \gamma^{ab} = \gamma^{ba} = B^a$.

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$a \xleftarrow{\$} \mathbb{Z}_p^*$

$A = \gamma^a$

Given $\gamma, \gamma^a,$ it's hard to solve a!

$b \xleftarrow{\$} \mathbb{Z}_p^*$

$B = \gamma^b$

A →

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$\mathbf{key} = B^a$

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Correctness: $A^b = \gamma^{ab} = \gamma^{ba} = B^a$.

Diffie–Hellman key exchange protocol

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Discrete Log Assumption

Given γ, γ^a, it's hard to solve a!

$a \xleftarrow{\$} \mathbb{Z}_p^*$

$A = \gamma^a$

$b \xleftarrow{\$} \mathbb{Z}_p^*$

$B = \gamma^b$

A →

← B

$\mathbf{key} = B^a$

$\mathbf{key} = A^b$

Correctness: $A^b = \gamma^{ab} = \gamma^{ba} = B^a$.

Diffie–Hellman key exchange protocol from group action view

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$\mathsf{pk} : s \in S$

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Diffie–Hellman key exchange protocol from group action view

Alice

$\mathsf{pk} : s \in S$

Bob

$g \xleftarrow{\$} G$

$A = s * g$

Diffie–Hellman key exchange protocol from group action view

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$\mathsf{pk} : s \in S$

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$g \xleftarrow{\$} G$

$A = s * g$

$h \xleftarrow{\$} G$

$B = s * h$

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$g \xleftarrow{\$} G$

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Correctness: $B * g = s * hg = s * gh = A * h.$

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$g \xleftarrow{\$} G$

$A = s * g$

Given $s, s * g$ it's hard to solve g!

$h \xleftarrow{\$} G$

$B = s * h$

A →

← B

$\mathbf{key} = B * g$

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Correctness: $B * g = s * hg = s * gh = A * h.$

Diffie–Hellman key exchange protocol from group action view

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One-way Hardness Assumption[1]

Given $s, s * g$ it's hard to solve g!

$g \xleftarrow{\$} G$

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A →

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$\mathbf{key} = B * g$

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[1] [Brassard-Yung, *Crypto*, 90]

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Good candidate for instantiation

Correctness: $B * g = s * hg = s * gh = A * h.$

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Good candidate for instantiation

$S = C_p \setminus \mathrm{id}$

$G = \mathrm{Aut}(C_p)$

Correctness: $B * g = s * hg = s * gh = A * h.$

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Good candidate for instantiation

$S = C_p \setminus \mathrm{id}$

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$\mathrm{Aut}(C_p) \cong \mathbb{Z}_p^*$

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Beyond commutativity?

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Motivation

Hidden Subgroup Problems

Correctness: $B * g = s * hg = s * gh = A * h.$

Beyond commutativity?

Diffie–Hellman key exchange protocol from group action view

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$g \xleftarrow{\$} G$

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Motivation

Hidden Subgroup Problems over Abelian Groups

Shor's techniques

Correctness: $B * g = s * hg = s * gh = A * h.$

Beyond commutativity?

Diffie–Hellman key exchange protocol from group action view

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$g \xleftarrow{\$} G$

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$\mathbf{key} = B * g$

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Correctness: $B * g = s * hg = s * gh = A * h.$

Beyond commutativity?

Motivation

Hidden Subgroup Problems over Non-Abelian Groups ✓

↑

[Hallgren-Moore-Rötteler-Russell-Sen, *JACM*, 10]

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Correctness: $B * g = s * hg = s * gh = A * h.$

Idea: treating this as a law in a group!

Law in a group

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A *law* in a group G is an equation that is satisfied by any assignments of variables by group elements in G.

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  $[a,b][c,d] = [c,d][a,b]$

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- $u(a, b, c, \ldots) = v(a, b, c, \ldots)$ is a law in a group.

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  word

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  ↓

  word: e.g., $a^2b^3a^{-5}c^2b^7$

Key exchange protocol for actions of groups with laws

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Key exchange protocol for actions of groups with laws

Alice

$\mathsf{pk} : s_0 \in S$

Bob

Key exchange protocol for actions of groups with laws

Alice

$\mathsf{pk} : s_0 \in S$

$$y^{b_1}x^{a_1}\dots y^{b_k}x^{a_k} = x^{c_1}y^{d_1}\dots x^{c_\ell}y^{d_\ell}$$
for any $x, y \in G$

Bob

Key exchange protocol for actions of groups with laws

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Bob

$g \xleftarrow{\$} G$

$h \xleftarrow{\$} G$

Key exchange protocol for actions of groups with laws

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for any $x, y \in G$

$g \xleftarrow{\$} G$

$h \xleftarrow{\$} G$

$t_1 = s_0 * h^{b_1}$

Key exchange protocol for actions of groups with laws

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Bob

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$y^{b_1}x^{a_1}\dots y^{b_k}x^{a_k} = x^{c_1}y^{d_1}\dots x^{c_\ell}y^{d_\ell}$
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$g \xleftarrow{\$} G$

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t_1 ←

$s_1 = t_1 * g^{a_1}$

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t_1

s_1

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$h \xleftarrow{\$} G$

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t_1 ←

$s_1 = t_1 * g^{a_1}$

→ s_1

$t_2 = s_1 * h^{b_2}$

Key exchange protocol for actions of groups with laws

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for any $x, y \in G$

$g \xleftarrow{\$} G$

$h \xleftarrow{\$} G$

$t_1 = s_0 * h^{b_1}$

t_1

$s_1 = t_1 * g^{a_1}$

s_1

$t_2 = s_1 * h^{b_2}$

t_2

Key exchange protocol for actions of groups with laws

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for any $x, y \in G$

$g \xleftarrow{\$} G$

$h \xleftarrow{\$} G$

$t_1 = s_0 * h^{b_1}$

t_1 ←

$s_1 = t_1 * g^{a_1}$

s_1 →

$t_2 = s_1 * h^{b_2}$

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$s_2 = t_2 * g^{a_2}$

Key exchange protocol for actions of groups with laws

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$t_1 = s_0 * h^{b_1}$

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$s_1 = t_1 * g^{a_1}$

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⋮

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t_2

$s_2 = t_2 * g^{a_2}$

$\vdots$

$t_k = s_{k-1} * h^{b_k}$

Key exchange protocol for actions of groups with laws

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t_1

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$t_k = s_{k-1} * h^{b_k}$

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$s_2 = t_2 * g^{a_2}$

⋮

$t_k = s_{k-1} * h^{b_k}$

t_k ←

key : $s_k = t_k * g^{a_k}$

Key exchange protocol for actions of groups with laws

Alice

Bob

$\mathsf{pk} : s_0 \in S$

$y^{b_1}x^{a_1}\dots y^{b_k}x^{a_k} = x^{c_1}y^{d_1}\dots x^{c_\ell}y^{d_\ell}$
for any $x, y \in G$

$g \xleftarrow{\$} G$

$h \xleftarrow{\$} G$

$t_1 = s_0 * h^{b_1}$

t_1 ←

$s_1 = t_1 * g^{a_1}$

s_1 →

$t_2 = s_1 * h^{b_2}$

t_2 ←

$s_2 = t_2 * g^{a_2}$

$\vdots$

$t_k = s_{k-1} * h^{b_k}$

t_k ←

$\mathbf{key} : s_k = t_k * g^{a_k} = s_0 * h^{b_1}g^{a_1}\dots h^{b_k}g^{a_k}$

Key exchange protocol for actions of groups with laws

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$$y^{b_1}x^{a_1}\dots y^{b_k}x^{a_k} = x^{c_1}y^{d_1}\dots x^{c_\ell}y^{d_\ell}$$
$$\text{for any } x, y \in G$$

Bob

$s_1' = s_0 * g^{c_1}$

Key exchange protocol for actions of groups with laws

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s_1' →

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for any $x, y \in G$

$s'_1 = s_0 * g^{c_1}$

s'_1 →

$t'_1 = s'_1 * h^{d_1}$

Key exchange protocol for actions of groups with laws

Alice

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$$y^{b_1}x^{a_1}\dots y^{b_k}x^{a_k} = x^{c_1}y^{d_1}\dots x^{c_\ell}y^{d_\ell}$$
for any $x, y \in G$

Bob

$s_1' = s_0 * g^{c_1}$

s_1'

t_1'

$t_1' = s_1' * h^{d_1}$

Key exchange protocol for actions of groups with laws

Alice

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$$y^{b_1}x^{a_1}\dots y^{b_k}x^{a_k} = x^{c_1}y^{d_1}\dots x^{c_\ell}y^{d_\ell}$$
for any $x, y \in G$

Bob

$s_1' = s_0 * g^{c_1}$

s_1' →

← t_1'

$t_1' = s_1' * h^{d_1}$

$s_2' = t_1' * g^{c_2}$

Key exchange protocol for actions of groups with laws

Alice

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$$y^{b_1}x^{a_1}\dots y^{b_k}x^{a_k} = x^{c_1}y^{d_1}\dots x^{c_\ell}y^{d_\ell}$$
for any $x, y \in G$

Bob

$s'_1 = s_0 * g^{c_1}$

s'_1

$t'_1 = s'_1 * h^{d_1}$

t'_1

$s'_2 = t'_1 * g^{c_2}$

s'_2

Key exchange protocol for actions of groups with laws

Alice

pk : $s_0 \in S$

$$y^{b_1}x^{a_1}\dots y^{b_k}x^{a_k} = x^{c_1}y^{d_1}\dots x^{c_\ell}y^{d_\ell}$$
for any $x, y \in G$

Bob

$s_1' = s_0 * g^{c_1}$

s_1' →

$t_1' = s_1' * h^{d_1}$

← t_1'

$s_2' = t_1' * g^{c_2}$

s_2' →

$t_2' = s_2' * h^{d_2}$

Key exchange protocol for actions of groups with laws

Alice

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$$y^{b_1}x^{a_1}\dots y^{b_k}x^{a_k} = x^{c_1}y^{d_1}\dots x^{c_\ell}y^{d_\ell}$$
for any $x, y \in G$

Bob

$s'_1 = s_0 * g^{c_1}$

s'_1 →

$t_1 = s'_1 * h^{d_1}$

← t'_1

$s'_2 = t'_1 * g^{c_2}$

s'_2 →

$t'_2 = s'_2 * h^{d_2}$

⋮

Key exchange protocol for actions of groups with laws

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$s'_1 = s_0 * g^{c_1}$

s'_1 →

$t'_1 = s'_1 * h^{d_1}$

t'_1 ←

$s'_2 = t'_1 * g^{c_2}$

s'_2 →

$t'_2 = s'_2 * h^{d_2}$

⋮

$s'_\ell = t'_{\ell-1} * g^{d_\ell}$

Key exchange protocol for actions of groups with laws

Alice

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$$y^{b_1}x^{a_1}\dots y^{b_k}x^{a_k} = x^{c_1}y^{d_1}\dots x^{c_\ell}y^{d_\ell}$$
for any $x, y \in G$

Bob

$s'_1 = s_0 * g^{c_1}$

s'_1 →

$t'_1 = s'_1 * h^{d_1}$

t'_1 ←

$s'_2 = t'_1 * g^{c_2}$

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s'_ℓ →

Key exchange protocol for actions of groups with laws

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for any $x, y \in G$

$s'_1 = s_0 * g^{c_1}$

s'_1 →

$t'_1 = s'_1 * h^{d_1}$

← t'_1

$s'_2 = t'_1 * g^{c_2}$

s'_2 →

$t'_2 = s'_2 * h^{d_2}$

$\vdots$

$s'_\ell = t'_{\ell-1} * g^{d_\ell}$

s'_ℓ →

$\mathbf{key} : t'_\ell = s'_\ell * h^{d_\ell}$

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$s'_1 = s_0 * g^{c_1}$

s'_1 →

$t'_1 = s'_1 * h^{d_1}$

t'_1 ←

$s'_2 = t'_1 * g^{c_2}$

s'_2 →

$t'_2 = s'_2 * h^{d_2}$

$\vdots$

$s'_\ell = t'_{\ell-1} * g^{d_\ell}$

s'_ℓ →

$\mathbf{key} : t'_\ell = s'_\ell * h^{d_\ell} = s_0 * g^{c_1}h^{d_1}\dots g^{c_\ell}h^{d_\ell}$

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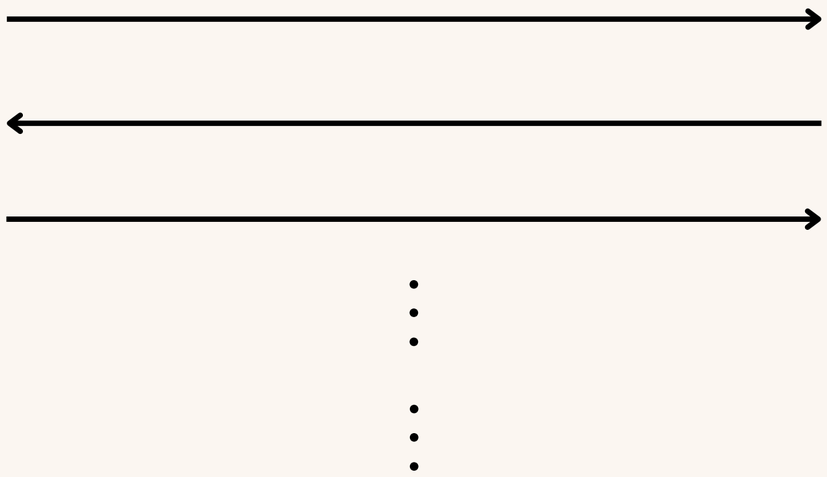
for any $x, y \in G$

$g \xleftarrow{\$} G$

$h \xleftarrow{\$} G$



$\mathbf{key} : s_k = t_k * g^{a_k} = s_0 * h^{b_1}g^{a_1}\dots h^{b_k}g^{a_k}$

$\mathbf{key} : t'_\ell = s'_\ell * h^{d_\ell} = s_0 * g^{c_1}h^{d_1}\dots g^{c_\ell}h^{d_\ell}$

Key exchange protocol for actions of groups with laws

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Bob

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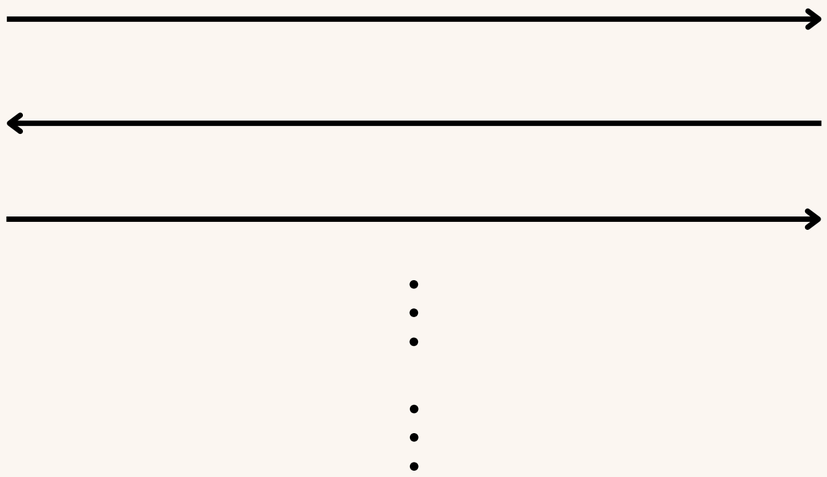
$$y^{b_1}x^{a_1}\dots y^{b_k}x^{a_k} = x^{c_1}y^{d_1}\dots x^{c_\ell}y^{d_\ell}$$
for any $x, y \in G$

$g \xleftarrow{\$} G$

$h \xleftarrow{\$} G$



This can be generalised to the multi-variable case!

$\mathbf{key} : s_k = t_k * g^{a_k} = s_0 * h^{b_1}g^{a_1}\dots h^{b_k}g^{a_k}$

$\mathbf{key} : t'_\ell = s'_\ell * h^{d_\ell} = s_0 * g^{c_1}h^{d_1}\dots g^{c_\ell}h^{d_\ell}$

Key exchange protocol for actions of groups with laws

Alice

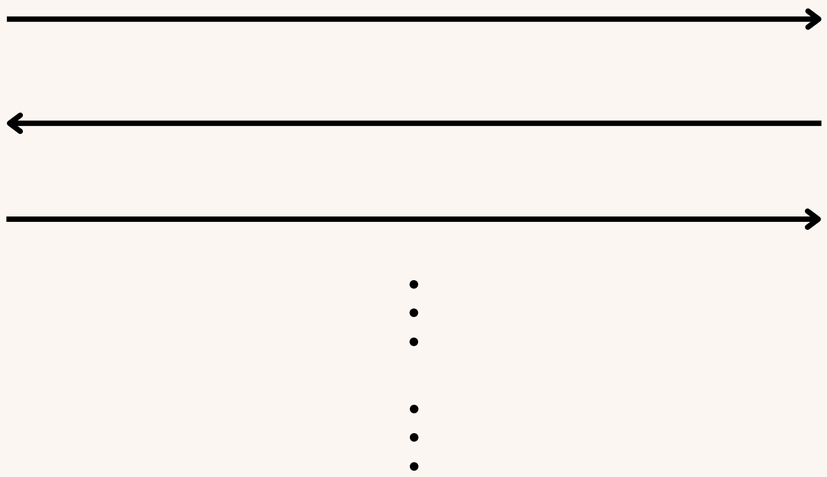
Bob

$\mathsf{pk} : s_0 \in S$

$y^{b_1}x^{a_1}\dots y^{b_k}x^{a_k} = x^{c_1}y^{d_1}\dots x^{c_\ell}y^{d_\ell}$
for any $x, y \in G$

$g \xleftarrow{\$} G$

$h \xleftarrow{\$} G$



$\mathbf{key} : s_k = t_k * g^{a_k} = s_0 * h^{b_1}g^{a_1}\dots h^{b_k}g^{a_k}$

$\mathbf{key} : t'_\ell = s'_\ell * h^{d_\ell} = s_0 * g^{c_1}h^{d_1}\dots g^{c_\ell}h^{d_\ell}$

Key exchange protocol for actions of groups with laws

Alice

$g \xleftarrow{\$} G$

$\mathsf{pk} : s_0 \in S$

$$y^{b_1}x^{a_1}\dots y^{b_k}x^{a_k} = x^{c_1}y^{d_1}\dots x^{c_\ell}y^{d_\ell}$$
for any $x, y \in G$

Bob

$h \xleftarrow{\$} G$

length of a law

$$\sum_{i=1}^{k}(|a_i|+|b_i|)+\sum_{i=1}^{\ell}(|c_i|+|d_i|)$$

$\mathbf{key} : s_k = t_k * g^{a_k} = s_0 * h^{b_1}g^{a_1}\dots h^{b_k}g^{a_k}$

$\mathbf{key} : t'_\ell = s'_\ell * h^{d_\ell} = s_0 * g^{c_1}h^{d_1}\dots g^{c_\ell}h^{d_\ell}$

Key exchange protocol for actions of groups with laws

Alice

Bob

$\mathsf{pk} : s_0 \in S$

$$y^{b_1}x^{a_1}\dots y^{b_k}x^{a_k} = x^{c_1}y^{d_1}\dots x^{c_\ell}y^{d_\ell}$$
for any $x, y \in G$

length of a law
$\sum_{i=1}^{k}(|a_i|+|b_i|)+\sum_{i=1}^{\ell}(|c_i|+|d_i|)$

$g \xleftarrow{\$} G$

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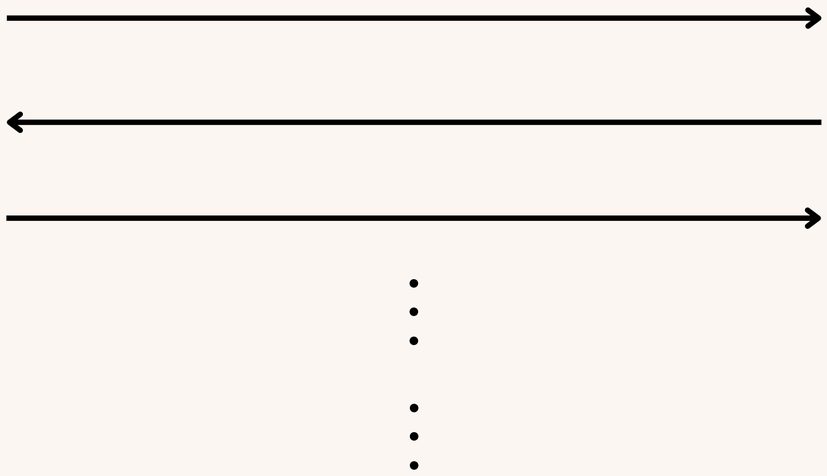

Multi-variable laws can be turned into 2-variable laws by a polynomial blow-up in length [Bradford-Thom, *TAMS*, 19]

key : $s_k = t_k * g^{a_k} = s_0 * h^{b_1}g^{a_1}\dots h^{b_k}g^{a_k}$

key : $t'_\ell = s'_\ell * h^{d_\ell} = s_0 * g^{c_1}h^{d_1}\dots g^{c_\ell}h^{d_\ell}$

Key exchange protocol for actions of groups with laws

Alice

$\mathsf{pk} : s_0 \in S$

$$y^{b_1}x^{a_1}\dots y^{b_k}x^{a_k} = x^{c_1}y^{d_1}\dots x^{c_\ell}y^{d_\ell}$$
for any $x, y \in G$

Bob

length of a law
$\sum_{i=1}^{k}(|a_i| + |b_i|) + \sum_{i=1}^{\ell}(|c_i| + |d_i|)$

What kind of (non-abelian) Group should we choose?

Key exchange protocol for actions of groups with laws

Alice

$\mathsf{pk} : s_0 \in S$

$$y^{b_1}x^{a_1}\dots y^{b_k}x^{a_k} = x^{c_1}y^{d_1}\dots x^{c_\ell}y^{d_\ell}$$
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Bob

length of a law
$\sum_{i=1}^{k}(|a_i| + |b_i|) + \sum_{i=1}^{\ell}(|c_i| + |d_i|)$

$g \xleftarrow{\$} G$ $h \xleftarrow{\$} G$

What kind of (non-abelian) Group should we choose?

1. One-way hardness (with multiple copies)

$\mathbf{key} : s_k = t_k * g^{a_k} = s_0 * h^{b_1}g^{a_1}\dots h^{b_k}g^{a_k}$ $\mathbf{key} : t'_\ell = s'_\ell * h^{d_\ell} = s_0 * g^{c_1}h^{d_1}\dots g^{c_\ell}h^{d_\ell}$

Key exchange protocol for actions of groups with laws

Alice

$\mathsf{pk} : s_0 \in S$

$$y^{b_1}x^{a_1}\dots y^{b_k}x^{a_k} = x^{c_1}y^{d_1}\dots x^{c_\ell}y^{d_\ell}$$
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Bob

length of a law
$\sum_{i=1}^{k}(|a_i| + |b_i|) + \sum_{i=1}^{\ell}(|c_i| + |d_i|)$

What kind of (non-abelian) Group should we choose?

1. One-way hardness (with multiple copies)
2. With a law whose length is as short as possible

Key exchange protocol for actions of groups with laws

Alice

$\mathsf{pk} : s_0 \in S$

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Metabelian groups

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1. One-way hardness (with multiple copies)
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Highly non-abelian groups,
e.g., general linear groups

Key exchange protocol for actions of groups with laws

Alice

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Highly non-abelian groups,
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[Bradford-Thom, *JEMS*, 24]
The length could be exponentially long!

Key exchange protocol for actions of groups with laws

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What kind of (non-abelian) Group should we choose?

1. One-way hardness (with multiple copies)
2. There is a short law with high probability

Highly non-abelian groups,
e.g., symmetric groups

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length of a law
$\sum_{i=1}^{k}(|a_i|+|b_i|)+\sum_{i=1}^{\ell}(|c_i|+|d_i|)$

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1. One-way hardness (with multiple copies)
2. There is a short law with high probability

$(xy)^{\lceil n/2\rceil} = (y^{-1}x^{-1})^{\lfloor n/2\rfloor}$
i.e., $(xy)^n = \mathrm{id}$

Highly non-abelian groups,
e.g., symmetric groups

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$(xy)^{\lceil n/2\rceil} = (y^{-1}x^{-1})^{\lfloor n/2\rfloor}$
i.e., $(xy)^n = \mathrm{id}$

Highly non-abelian groups,
e.g., symmetric groups

Key exchange protocol for actions of groups with laws

Alice

pk : $s_0 \in S$

$$y^{b_1}x^{a_1}\dots y^{b_k}x^{a_k} = x^{c_1}y^{d_1}\dots x^{c_\ell}y^{d_\ell}$$
for any $x, y \in G$

Bob

length of a law
$\sum_{i=1}^{k}(|a_i|+|b_i|)+\sum_{i=1}^{\ell}(|c_i|+|d_i|)$

What kind of (non-abelian) Group should we choose?

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2. There is a short law with high probability

$(xy)^{\lceil n/2\rceil} = (y^{-1}x^{-1})^{\lfloor n/2\rfloor}$
i.e., $(xy)^n = \mathrm{id}$
↓
with probability $1/n$

Highly non-abelian groups,
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Instantiation by Linear Code Equivalence

Notation:

- $\mathrm{M}(k \times n, \mathbb{F})$: the set of all $k \times n$ matrices over $\mathbb{F}$.

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Problem (Linear Code Equivalence)
For two generator matrices $C_1, C_2 \in \mathrm{M}(k \times n, \mathbb{F}_q)$, determine if there is $A \in \mathrm{GL}(k, \mathbb{F}_q)$ and $M \in \mathrm{Mon}(n, \mathbb{F}_q)$ such that $C_1 = AC_2M$.

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- View 2 : $\mathrm{Mon}(n, \mathbb{F}_q)$ acts on the set of all k-dimensional codes in $\mathbb{F}_q^n$.

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- Key property : $[C]_\sim P = [CP]_\sim$ for any $P \in \mathrm{S}_n$.

Instantiation by Linear Code Equivalence

$$G = \mathrm{S}_n$$

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- $[C]_\sim := \{ACD : A \in \mathrm{GL}(k, \mathbb{F}_q), D \in \mathrm{D}(n, \mathbb{F}_q)\}$.
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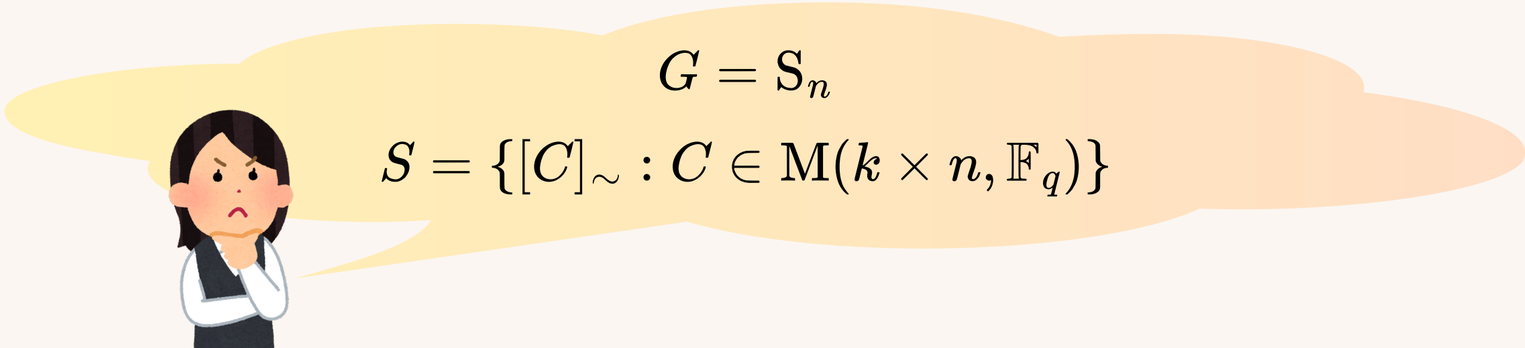
- We give a canonical form algorithm to efficiently compute a representative in $[C]_\sim$.



Instantiation by Linear Code Equivalence

Alice

Bob

Instantiation by Linear Code Equivalence

Alice

$\mathsf{pk} : C_0 \in \mathrm{M}(k \times n, \mathbb{F}_q)$

Bob

Instantiation by Linear Code Equivalence

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$(xy)^{\lceil n/2 \rceil} = (y^{-1}x^{-1})^{\lfloor n/2 \rfloor}$
for a large proportion of $x, y \in G$

Bob

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C_2 →

⋮

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for a large proportion of $x, y \in G$

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Instantiation by Linear Code Equivalence

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$\longrightarrow$

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C_nP and $C'_{n-2}Q^{-1}$ are in the same equivalence class, so they have the same canonical form!

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We also run a Blackbox from [Fischlin-Günther-Schmidt-Warinschi, *S&P*, 16]. If it doesn't satisfy the key confirmation, we will repeat the key exchange protocol by choosing a new pair of P and Q.

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Problem (Diagonal-masked Linear Code Equivalence)
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- There is a similar assumption that has been broken in polynomial time[1] :

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Problem (2-sample Monomial Code Equivalence)
For generator matrices $\{C_i : i \in [4]\} \in \mathrm{M}(k \times n, \mathbb{F}_q)$, determine if there exist $A_1, A_2 \in \mathrm{GL}(k, \mathbb{F}_q), D \in \mathrm{D}(n, \mathbb{F}_q)$ and $P \in \mathrm{S}_n$ such that $A_1 C_1 D P = C_2$ and $A_2 C_3 D P = C_4$. If yes, compute such a permutation P.

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Question and Answer

Thank you so much!