Notes on Kalman Filter

data: $Y_t, Y_{t-1}, \cdots, Y_1$

observed at time

$t, t-1, \cdots, 1$

assume Y_t depends on an unobserable θ_t (state):

$$Y_t = F_t \theta_t + \nu_t$$

where F_t is known, but ν_t is noise / unknown, and

$$\nu_t \sim N(0, V_t)$$

assume $\theta_t = G_t \theta_{t-1} + \omega_t$ and

$$\omega_t \sim N(0, W_t)$$

V_t, W_t, F_t, G_t are all known

at time 0:

$$\theta_0 \sim N(\hat{\theta}_0, \Sigma_0)$$

at time $t-1$, given $Y_{t-1}, Y_{t-2}, \ldots, Y_1$

$$\theta_{t-1} \sim N(\hat{\theta}_{t-1}, \Sigma_{t-1})$$

assume we know $\hat{\theta}_{t-1}$ and Σ_{t-1}

problem is to compute $\hat{\theta}_t$ and Σ_t, i.e.

given Y_t (observed) and distribution of $\theta_{t-1} \sim N(\hat{\theta}_{t-1}, \Sigma_{t-1})$

compute distribution of $\theta_t | Y_t \sim N(\hat{\theta}_t, \Sigma_t)$

Answer:

at time t, before observe Y_t

$$\begin{pmatrix} \theta_t \\ Y_t \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \right)$$

$$\mu_1 = G_t \hat{\theta}_{t-1}$$

$$\mu_2 = F_t G_t \hat{\theta}_{t-1}$$

$$\Sigma_{11} = Cov(\theta_t, \theta_t) = W_t + G_t \Sigma_{t-1} G_t' \triangleq R_t$$

$$\Sigma_{12} = Cov(\theta_t, Y_t) = Cov(\theta_t, \theta_t) F_t' = R_t F_t'$$

$$\Sigma_{21} = \Sigma_{12}'$$

$$\Sigma_{22} = Cov(Y_t, Y_t) = V_t + F_t R_t F_t'$$

after observe Y_t

$$\theta_t | Y_t \sim N(\hat{\theta}_t, \Sigma_t)$$

$$\hat{\theta}_t = \mu_1 + \Sigma_{12} \Sigma_{22}^{-1} (Y_t - \mu_2)$$

$$\Sigma_t = \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21}$$

Let $e_t = Y_t - \mu_2 = Y_t - F_t G_t \hat{\theta}_{t-1}$

$$\begin{cases} \hat{\theta}_t = G_t \hat{\theta}_{t-1} + R_t F_t' (V_t + F_t R_t F_t')^{-1} e_t \\ \Sigma_t = R_t - R_t F_t' (V_t + F_t R_t F_t')^{-1} F_t R_t \end{cases}$$