Partial derivatives.

given $f(x,y)$: $f_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$

leave y -constant
as x changes.

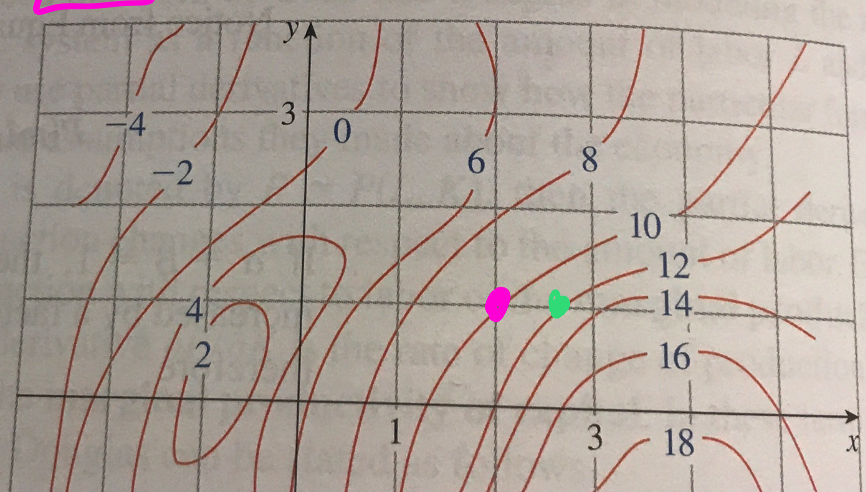
notation: $f_x(x,y) = \frac{\partial}{\partial x} \underbrace{f(x,y)}_{z} = \frac{\partial z}{\partial x} = z_x = f_x$

$\partial \neq d$ $\qquad$ $\cancel{\frac{dz}{dx}}$

LATEX: \partial

⑩

10. A contour map is given for a function f. Use it to estimate $\boxed{f_x(2, 1)}$ and $f_y(2, 1)$.



$f_x(2,1)$

$\approx \frac{\Delta z}{\Delta x}, y=1$

$\Delta z \approx 12-10$
$= 2$

$\Delta x \approx \frac{1}{2}$

$f_x(2,1) \approx \frac{2}{\frac{1}{2}} = 4.$

(15) $f(x,y) = x^4 + 5xy^3$

find $f_x(x,y)$, $f_y(x,y)$.

$$\frac{\partial}{\partial x}\left(x^4 + 5xy^3\right) = \underbrace{\frac{\partial}{\partial x} x^4}_{\underbrace{\frac{d}{dx}x^4}_{=\ 4x^3}} + \underbrace{\frac{\partial}{\partial x}(5xy^3)}_{\underbrace{5y^3 \underbrace{\frac{d}{dx}(x)}_{1}}_{5y^3}}$$

$$= 4x^3 + 5y^3$$

$$f_y(x,y) = \frac{\partial}{\partial y}\left(x^4 + 5xy^3\right)$$

$$= 5x(3y^2) = 15xy^2$$

(20) $z = x\sin(xy)$

find z_x z_y

$$z_x = \frac{\partial}{\partial x}\left(x\sin(xy)\right) = \underbrace{\left(\frac{\partial}{\partial x}x\right)}_{1}\sin xy + x\underbrace{\frac{\partial}{\partial x}(\sin xy)}_{y\cos(xy)}$$

$$= \sin(xy) + xy\cos(xy)$$

$$z_y = \frac{\partial}{\partial y}(x\sin(xy)) = x\frac{\partial}{\partial y}\sin(xy)$$

$$= x^2\cos(xy)$$

32 $f(x,y,z) = xy^2e^{-xz}$

find f_x & f_z

$$f_x(x,y,z) = \frac{\partial}{\partial x}(xy^2e^{-xz})$$

$$= y^2\left[\underbrace{\left(\frac{\partial}{\partial x}x\right)}_{1}e^{-xz} + x\underbrace{\left(\frac{\partial}{\partial x}e^{-xz}\right)}_{-ze^{-xz}}\right]$$

$$= y^2e^{-xz} - xy^2ze^{-xz}$$

$$f_z(x,y,z) = \frac{\partial}{\partial z}(xy^2e^{-xz}) = xy^2\underbrace{\frac{\partial}{\partial z}e^{-xz}}$$

$$= xy^2(-x)e^{-xz}$$

$$= -x^2y^2e^{-xz}$$

$$\overbrace{}^{f(x_1,x_2,\ldots,x_n)}$$

(40) $u = \sin(x_1 + 2x_2 + 3x_3 + \cdots + nx_n)$

find all 1^{st} partial derivatives

find u_{x_i} $\quad i = 1, \ldots, n$

and see that $\quad u = \sin\left(\sum_{j=1}^{n} jx_j\right)$

$$u_{x_i} = \frac{\partial}{\partial x_i} \sin\left(\sum_{j=1}^{n} jx_j\right)$$

$$= \cos\left(\sum_{j=1}^{n} jx_j\right) \frac{\partial}{\partial x_i} \sum_{j=1}^{n} jx_j \quad = i$$

$$= i\cos(x_1 + 2x_2 + \cdots + nx_n)$$

$i = 1, \ldots, n$

$$\frac{\partial}{\partial x_1} x_2 = 0$$

So, $u_{x_3} = 3\cos(x_1 + \cdots + nx_n)$

$$\frac{\partial}{\partial x_1} x_3 = 0$$

$u_{x_{1000}} = 1000\cos(\quad)$

$i \neq j \qquad \frac{\partial}{\partial x_i} x_j = 0$

$j = i \qquad \frac{\partial}{\partial x_i} x_j = 1$

$$\frac{\partial}{\partial x_i} \sum_{j=1}^{n} jx_j = \sum_{j=1}^{n} j \underbrace{\frac{\partial}{\partial x_i} x_j}_{=1 \text{ when } i=j} = i$$

$$= \frac{\partial}{\partial x_i}\left(\cancel{x_1 + 2x_2} + \cdots + i x_i + \cancel{(i+1)x_{i+1} + \cdots + n x_n}\right)$$

$$= \frac{\partial}{\partial x_i}(i x_i) = i$$

51 & 52

(a) $z = f(x) + g(y)$

(b) $z = f(x) g(y)$

(c) $z = f(xy)$

for each, find z_x

(a) $z_x = \frac{\partial}{\partial x}\left(f(x) + g(y)\right) = f'(x)$.

(b) $z_x = \frac{\partial}{\partial x}\left(f(x) g(y)\right) = g(y) \underbrace{\frac{\partial}{\partial x} f(x)}_{f'(x)}$

$= f'(x) g(y)$.

(c) $z_x = \frac{\partial}{\partial x} f(xy)$

$= f'(xy) \underbrace{\frac{\partial}{\partial x}(xy)}_{y}$

$= y f'(xy)$.

$\cancel{f'(xy)}$

$f'(u) \rightsquigarrow \frac{d}{du}$ $\quad f'(x) \sim \frac{d}{dx}$

$f'(☺) \sim \frac{d}{d☺}$

(53) $f(x,y) = x^4y - 2x^3y^2$

find all 2^{nd} partial derivatives.

$f_{xx} \quad f_{yy} \quad f_{xy} \quad f_{yx}$

$f_x = 4x^3y - 6x^2y^2$

$f_y = x^4 - 4x^3y$

$f_{xx} = \frac{\partial}{\partial x}(4x^3y - 6x^2y^2) = 12x^2y - 12xy^2$

$f_{yy} = \frac{\partial}{\partial y}(x^4 - 4x^3y) = -4x^3$

same b/c f is "nice enough"

$$\left\{\begin{aligned} f_{xy} &= \frac{\partial}{\partial y}(4x^3y - 6x^2y^2) = 4x^3 - 12x^2y \\ f_{yx} &= \frac{\partial}{\partial x}(x^4 - 4x^3y) = 4x^3 - 12x^2y \end{aligned}\right.$$

if f is smooth and f_x, f_y, f_z are continuous.

$f_{xyz} = f_{xzy} = f_{yzx} = f_{zxy} = \dots$

(54) $f(x,y) = \ln(ax + by)$

find all 2^{nd} partial derivatives

$$f_x = \frac{1}{ax+by}\frac{\partial}{\partial x}(ax+by) \qquad f_y = \frac{b}{ax+by}$$

$$= \frac{a}{ax+by}$$

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{a}{ax+by}\right) = \frac{\partial}{\partial x}\left[a(ax+by)^{-1}\right]$$

$$= a\frac{\partial}{\partial x}\left[(ax+by)^{-1}\right]$$

$$= a(-1)(ax+by)^{-2}(a)$$

$$= \frac{-a^2}{(ax+by)^2}$$

$$f_{yy} = \frac{-b^2}{(ax+by)^2}$$

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{a}{ax+by}\right) = \frac{-ab}{(ax+by)^2}$$

68 $V = \ln(r+s^2+t^3)$ find $\frac{\partial^3 V}{\partial r\,\partial s\,\partial t} = V_{tsr}$

$\frac{\partial^5 V}{\partial r^2 \partial s \partial t^2}$

$= V_{ttsrr}$

$$V_t = \frac{3t^2}{r+s^2+t^3}$$

$$(V_t)_s = \frac{\partial}{\partial s}\left(\frac{3t^2}{r+s^2+t^3}\right) = 3t^2 \frac{\partial}{\partial s}(r+s^2+t^3)^{-1}$$

$$= 3t^2(-1)(r+s^2+t^3)^{-2}(2s)$$

$$= -6st^2(r+s^2+t^3)^{-2}$$

$$(V_{ts})_r = \frac{\partial}{\partial r}\left(-6st^2(r+s^2+t^3)^{-2}\right)$$

$$= -6st^2 \frac{\partial}{\partial r}(r+s^2+t^3)^{-2}$$

$$= -6st^2(-2)(r+s^2+t^3)^{-3}(1)$$

$$= \boxed{\frac{12st^2}{(r+s^2+t^3)^3}}$$

76 $\quad U_{xx} + U_{yy} = 0 \quad$ Laplace's Equation.

$\Delta u = 0$

$u'' = 0$

$\Rightarrow u =$ line.

$mx+b.$

Determine if the following are solutions.

$u = x^2 + y^2$, $u = x^2 - y^2$, $u = \sin x \cosh y + \cos x \sinh y$.

$$\begin{aligned} u_{xx} &= 2 \\ + \; u_{yy} &= 2 \\ \hline & 4 \\ & \neq 0 \end{aligned}$$

$$\begin{aligned} u_{xx} &= 2 \\ + \; u_{yy} &= -2 \\ \hline & 0 \end{aligned}$$

$x = \cosh t \quad y = \sinh t$

solve $x^2 - y^2 = 1$

$$\cosh t = \frac{e^t + e^{-t}}{2}$$

$$\sinh t = \frac{e^t - e^{-t}}{2}$$

$$(\sinh t)' = \cosh t$$

$$(\cosh t)' = \sinh t$$

$u = \sin x \cosh y + \cos x \sinh y$.

$$u_x = \cos x \cosh y - \sin x \sinh y$$

$$u_{xx} = -\sin x \cosh y - \cos x \sinh y$$

$$u_y = \sin x \sinh y + \cos x \cosh y$$

$$u_{yy} = \sin x \cosh y + \cos x \sinh y$$

$$u_{xx} + u_{yy} = 0.$$

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also verify

$$u = e^{-\alpha^2 k^2 t} \sin kx$$

Solves $u_t = \alpha^2 u_{xx}$ heat conduction equation.

Wave equation

$$u_{tt} = \alpha^2 u_{xx}$$

(80)

$$u = e^{a_1x_1 + a_2x_2 + \cdots + a_nx_n}, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = 1$$

show: $u_{x_1x_1} + u_{x_2x_2} + \cdots + u_{x_nx_n} = u.$

$$u_{x_i} = \frac{\partial}{\partial x_i} e^{a_1x_1 + \cdots + a_ix_i + \cdots + a_nx_n}$$

$$= a_i e^{a_1x_1 + \cdots + a_nx_n} = a_i u$$

$$u_{x_ix_i} = a_i u_{x_i} = a_i^2 u \qquad i = 1, \ldots, n$$

$$\sum_{i=1}^{n} u_{x_ix_i} = a_1^2 u + a_2^2 u + \cdots + a_n^2 u = u(\underbrace{a_1^2 + \cdots + a_n^2}_{1})$$

$$= u.$$