$f : A \to B$ $\boxed{K}$

$g : S \to T$

① $A = S$ $\quad (A \subset S) \wedge (S \subset A)$

② $B = T$ $\quad (B \subset T) \wedge (T \subset B)$

③ $\forall x \in A = S \, (f(x) = g(x))$

$\mathbb{Z}/5\mathbb{Z} \cong \mathbb{Z}_5 = \{0,1,2,3,4\} = A = B = S = T$

$f(x) = x^2 + 3x + 1 \mod 5$

$g(x) = 11x^2 - 2x + 6 \mod 5$

$f(0) = 0 + 0 + 1 \mod 5 = 1 = 6 \mod 5 = g(0)$

$f(1) = 1 + 3 + 1 \mod 5 = 5 \mod 5 = 0$

$g(1) = 11 - 2 + 6 \mod 5 = 15 \mod 5 = 0$

$f : \mathbb{Z} \to \mathbb{Z}$

$g : \mathbb{Z} \to \mathbb{Z}$

$f(x) = \cos(x)$

$g(x) = \cos(x + 2\pi)$

Let $x \in \mathbb{Z}$. $\quad \cos(x + 2\pi) = \cos(x)\cos(2\pi) - \sin(x)\sin(2\pi)$

$= \cos(x)(1) - \sin(x)(0)$

$= \cos(x)$

$\cos(x+p) = \cos(x)$ for any $p = 2\pi k$ $\forall k \in \mathbb{Z}$

$\{A_1, A_2, \dots, A_n\}$

$\forall i \in \{1, \dots, n\}$ $A_i \subset A$

① $A_i \cap A_j = \emptyset$ for any $i \neq j$

② $\bigcup_{i=1}^{n} A_i = A$

$\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$

$\{0, 2, 4\} = A_1$

$\{1, 3\} = A_2$

① $A_1 \cap A_2 = \emptyset$

② $A_1 \cup A_2 = \mathbb{Z}_5$

$\mathbb{Z}_3 \cong \mathbb{Z}/3\mathbb{Z}$

$\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$

$\mathbb{Z}_0 = \{0, \pm 3, \pm 6, \dots\} = \{n \in \mathbb{Z} : \forall k \in \mathbb{Z} (n = 3k)\}$

$\mathbb{Z}_1 = \{\dots, -5, -2, 1, 4, 7, \dots\}$

$= \{n \in \mathbb{Z} : \forall k \in \mathbb{Z} (n = 3k+1)\}$

$\mathbb{Z}_2 = \{n \in \mathbb{Z} : \forall k \in \mathbb{Z} (n = 3k+2)\}$

① $\mathbb{Z}_0 \cap \mathbb{Z}_1 = \emptyset$
$\mathbb{Z}_0 \cap \mathbb{Z}_2 = \emptyset$
$\mathbb{Z}_1 \cap \mathbb{Z}_2 = \emptyset$

② $\mathbb{Z}_0 \cup \mathbb{Z}_1 \cup \mathbb{Z}_2 = \mathbb{Z}$

$\{\mathbb{Z}_0, \mathbb{Z}_1, \mathbb{Z}_2\}$

"Quotient Remainder Theorem"
"Euclidean Algorithm"

$\mathbb{Z}_0 \in \{\mathbb{Z}_0, \mathbb{Z}_1, \mathbb{Z}_2\}$ Yes

$\mathbb{Z}_0 \subset \{\mathbb{Z}_0, \mathbb{Z}_1, \mathbb{Z}_2\}$ No $\quad 6 \in \mathbb{Z}_0 \not\rightarrow 6 \in \{\mathbb{Z}_0, \mathbb{Z}_1, \mathbb{Z}_2\}$

$\{\mathbb{Z}_0\} \subset \{\mathbb{Z}_0, \mathbb{Z}_1, \mathbb{Z}_2\}$ Yes $\quad \mathbb{Z}_0 \in \{\mathbb{Z}_0\} \rightarrow \mathbb{Z}_0 \in \{\mathbb{Z}_0, \mathbb{Z}_1, \mathbb{Z}_2\}$

$\emptyset \subset \{\mathbb{Z}_0, \mathbb{Z}_1, \mathbb{Z}_2\}$ Yes $\quad \underbrace{x \in \emptyset}_{F} \overset{T}{\Rightarrow} \underbrace{x \in \{\mathbb{Z}_0, \mathbb{Z}_1, \mathbb{Z}_2\}}_{F/T}$

$\emptyset \in \{\mathbb{Z}_0, \mathbb{Z}_1, \mathbb{Z}_2\}$ No

$A \subset B$

$\forall x \in U (x \in A \rightarrow x \in B)$

7.2 #40

Suppose $F: X \to Y$, F is 1-1, and $A \subset X$.
$F^{-1}(F(A)) = A$.

Need to show: $F^{-1}(F(A)) \subset A$ and $A \subset F^{-1}(F(A))$

Suppose $F: X \to Y$, F is 1-1, and $A \subset X$.

$F(A) = \{y \in Y : \exists a \in A (F(a) = y)\}$

For any $B \subset Y$, $F^{-1}(B) = \{x \in X : F(x) \in B\}$.

Since $F(A) \subset Y$ because $F: X \to Y$,

$F^{-1}(F(A)) = \{x \in X : F(x) \in F(A)\}$

Let $x \in F^{-1}(F(A))$. So $F(x) \in F(A)$ by the definition above. Since $F(x) \in F(A)$, there exists $a \in A$ such that $F(a) = y = F(x)$. Since F is one-to-one $F(a) = F(x)$ implies $a = x$ where $a \in A$. So $x \in A$. $F^{-1}(F(A)) \subset A$

Let $a \in A$. Then $F(a) \in F(A)$ since F is well-defined. $a \in F^{-1}(F(A))$ because $F(a) \in F(A)$.
$A \subset F^{-1}(F(A))$

$\log_b : (0,\infty) \to \mathbb{R}$

$b^{\cdot} : \mathbb{R} \to (0,\infty)$

Prove: $\forall x \in (0,\infty) \left(b^{\log_b(x)} = x\right)$

Let $\tilde{x} \in (0,\infty)$. Define $\tilde{y} := \log_b(\tilde{x})$.

Since the exponential function $b^{\cdot}$ has the inverse function $\log_b(\cdot)$, for any $x \in (0,\infty)$, $\log_b(x) = y$ if and only if $b^y = x$.

So $b^{\tilde{y}} = \tilde{z}$ if and only if $\log_b(\tilde{z}) = \tilde{y}$ for some $\tilde{z} \in (0,\infty)$. $\log_b(\tilde{z}) = \tilde{y} = \log_b(\tilde{x})$ by def of $\tilde{y}$. Since $\log_b$ is one-to-one, $\log_b(\tilde{z}) = \log_b(\tilde{x})$ implies $\tilde{z} = \tilde{x}$. So $b^{\log_b(\tilde{x})} = b^{\tilde{y}} = \tilde{z} = \tilde{x}$. Thus, for any $x \in (0,\infty)$, $b^{\log_b(x)} = x$.

§9.4 #10

$S = \{1, 2, 3, 4, 5, \ldots, 2n-1, 2n\}$

Let $X = \{x_1, x_2, x_3, \ldots, x_{n+1}\} \subset S$.

$S_0 := \{1, 3, 5, 7, \ldots, 2n-1\} = \{m \in \mathbb{Z} : \forall k \in \{1, 2, \ldots, n\} (m = 2k-1)\}$

$1, 2, 3, 4, \ldots, n-1, n$

$S_E := \{2, 4, 6, \ldots, 2n\} = \{m \in \mathbb{Z} : \forall k \in \{1, 2, \ldots, n\} (m = 2k)\}$

$1, 2, 3, \ldots, n-1, n$

$N(S_0) = n$ Note that $N(X) = n+1 > n = N(S_E)$.

$N(S_E) = n$ $X \subset S = S_0 \cup S_E$ because

① $S_0 \cap S_E = \emptyset$

② $S_0 \cup S_E = S$

Suppose $X \cap S_0 = \emptyset$. Let $x \in X$. Since $X \subset S = S_0 \cup S_E$, $x \in S_0 \cup S_E$. By definition of set union, $x \in S_0$ or $x \in S_E$. Suppose $x \in S_0$. Then $x \in X$ and $x \in S_0$, so $x \in X \cap S_0$ by definition of set intersection. However $X \cap S_0 = \{x\} \neq \emptyset$ contradicts $X \cap S_0 = \emptyset$. So $x \notin S_0$.

Then $x \in S_E$ by elimination. Thus $X \subset S_E$.

Since $X \subset S_E$, $n+1 = N(X) \leq N(S_E) = n$ so

$n+1 \leq n$ but $1 \leq 0$ is a contradiction.

Therefore $X \cap S_0 \neq \emptyset$. So there exists $c \in X \cap S_0$

such that $c \in X$ and $c \in S_0$. By specialization,

$c \in S_0$ so c is odd.

$f : X \to Y$

to show f is onto,

$\forall y \in Y \, \exists x \in X \, (f(x) = y)$

$f : \mathbb{R} \to \mathbb{R}$

$f(x) = x^3 + 1$

scrap

$y = x^3 + 1$

$\sqrt[3]{y-1} = x$

Let $y \in \mathbb{R}$. Choose $x := \sqrt[3]{y-1}$. Then

$f(x) = f(\sqrt[3]{y-1}) = (\sqrt[3]{y-1})^3 + 1 = y - 1 + 1 = y - 0 = y.$

So f is onto.

$A = B$

$A \subset B \wedge B \subset A$

① $\forall x \in A \, (x \in B) \equiv \forall x \in U \, (x \in A \to x \in B) \equiv (x \in A \Rightarrow x \in B)$

② $\forall x \in B \, (x \in A)$

$A \subset B$

$x \in A \Rightarrow x \in B$

$(A \cap C = B \cap C) \wedge (A \cup C = B \cup C) \Rightarrow A = B$

Let A, B, C be any sets (with universal set $\mathcal{U}$).

Suppose $A \cap C = B \cap C$ and $A \cup C = B \cup C$.

Let $x \in A$. ... show $x \in B$. Then show $B \subset A$ so $A = B$, provided this is true. If it's not true, find a counterexample via making choices of A, B, C so that $A \cap C = B \cap C$ and $A \cup C = B \cup C$ but $A \neq B$.

$(A \cap B)^c = A^c \cup B^c$ element method

Show $(A \cap B)^c \subset A^c \cup B^c$

Show $A^c \cup B^c \subset (A \cap B)^c$

Let f, g be functions such that

(i) $f: A \to B$ and $g: B \to A$

(ii) $f \circ g = id_B$ and $g \circ f = id_A$

(a) Prove that f is injective and surjective.

(b) Prove that $g = f^{-1}$.

Let f, g be functions such that

(i) $f: A \to B$ and $g: B \to A$

(ii) $f \circ g = id_B$ and $g \circ f = id_A$

(b) Since f is injective and surjective, f is bijective by definition (part (a)).
Since f is bijective, there exists $f^{-1}: B \to A$ such that

$$f^{-1}(x) = y \iff x = f(y)$$

via Theorem 7.2.2.

1. g & f^{-1} have the same domain B
2. g & f^{-1} have the same codomain A
3. need to show: $\forall b \in B\ (g(b) = f^{-1}(b))$

Let $b \in B$. Since f^{-1} is defined for any $y \in B$, let $f^{-1}(b) =: a \in A$. Recall the identity function $id_A : A \to A$, $id_A(c) = c$ for any $c \in A$.

$$f^{-1}(b) = a = id_A(a) = (g \circ f)(a) = g(f(a))$$

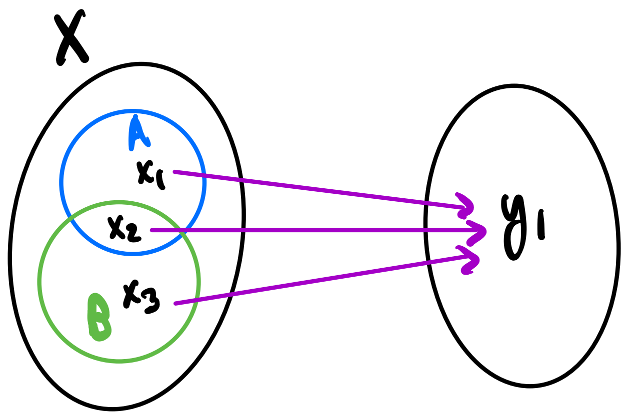
but recall $f^{-1}(b) = a \iff b = f(a)$ so

$$f^{-1}(b) = g(f(a)) = g(b).$$

$f : X \to Y$

$A, B \subset X$

$f(A) \subset f(B) \Rightarrow A \subset B$



$A = \{x_1, x_2\}$

$B = \{x_2, x_3\}$

$f(A) = \{y_1\} = f(B)$

$f(A) \subset f(B)$

but

$A \not\subset B$

because $x_1 \notin B$

but $x_1 \in A$

for this to be true

f must be injective

$f = \{(x_1, y_1), (x_2, y_1), (x_3, y_1)\}$

$f : X \to Y$

$A \subset B \Longrightarrow f(A) \subset f(B)$

$f(A) \not\subset f(B) \Rightarrow A \not\subset B$ (contrapositive proof)

Suppose $A \subset B$ and $f(A) \not\subset f(B)$. (contradiction proof)

(would have to prove all this, on this page)

$f: X \hookrightarrow\!\!\!\!\rightarrow Y$

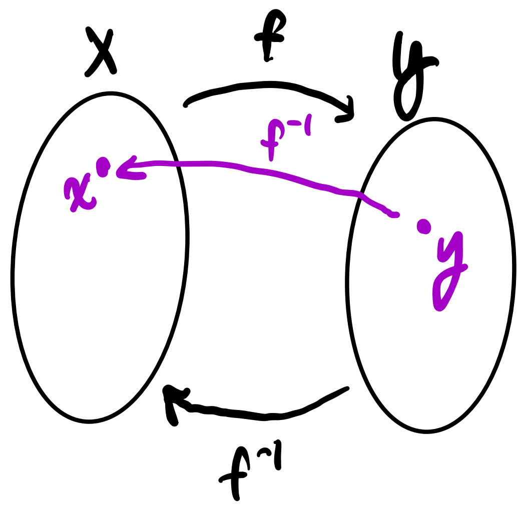
$f^{-1}: Y \to X$

Let $x \in X$. Then $f(x) = y$ for some $y \in Y$ since f is a function. By Theorem 7.2.2,

$f(x) = y \iff f^{-1}(y) = x$. So, for any $a \in X$, there exists $b \in Y$ such that $f^{-1}(b) = a$.

So f^{-1} is onto.

$\forall a \in X \, \exists b \in Y \, (f^{-1}(b) = a)$



Definition

A function f from a set A to a set B is a set such that

① $\forall x \in A\ \exists y \in B\ ((x,y) \in f)$ (not the same as ONTO which is $\forall y \in B\ \exists x \in A\ (f(x) = y)$) flipped

where $f \subset A \times B$

② $\forall x \in A\ \forall y, z \in B\ ((x,y) \in f \wedge (x,z) \in f \Rightarrow y = z)$

$A = \{a\}$

$\mathcal{P}(A) = \{\emptyset, \{a\}\}$

$(\mathcal{P}(A), \subset)$	$(\{0,1\}, \leq)$
$\cancel{\emptyset \subset \emptyset}$	$0 \leq 0$
$\emptyset \subset \{a\}$	$0 \leq 1$
$\{a\} \subset \{a\}$	$1 \leq 1$

well-ordering principle

finite, nonempty

Let $S \subset \mathbb{Z}$ where $N(S) = n \in \mathbb{Z}^+$. Then there exists $\alpha \in S$ such that, for any $x \in S$, $\alpha \leq x$.

$\{-329, -5, 10, 11201, 23537\}$

Principle of Strong Mathematical Induction

Let $P(n)$ be a property that is defined for $n \in \mathbb{Z}$ and let $a, b \in \mathbb{Z}$ be fixed integers such that $a \le b$. Suppose the two statements are true:

① $P(a), P(a+1), \dots, P(b)$ are all true

(base case, basis step) $P(1), P(2), P(3)$ $a=1$ $b=3$

② For every integer $k \geq b$, if $P(i)$ is true for all $i \in \{a, \dots, k\}$ then $P(k+1)$ is true

Suppose for every integer $k \geq 3$, $P(i)$ is true for all $i \in \{1, 2, \dots, k\}$.

(induction hypothesis, inductive step)

Then the statement

for every integer $n \geq a$, $P(n)$

is true.

Another way to state the inductive hypothesis is to say that $P(a), P(a+1), \dots, P(k)$ are all true.

Method of Proof by Induction $\forall n \in \mathbb{Z}\,(n \geq a \rightarrow P(n))$

Consider a statement of the form,

"For every integer $n \geq a$, a property $P(n)$ is true."

To prove such a statement, perform the following two steps.

Step 1 (base case): Show that $P(a)$ is true.

Step 2 (inductive step): Show that for every integer $k \geq a$, if $P(k)$ is true, then $P(k+1)$ is true. To perform this step,

Suppose that $P(k)$ is true, where k is any particular but arbitrarily chosen integer with $k \geq a$.

(This assumption is called the inductive hypothesis.)

Then show that $P(k+1)$ is true.

step 1: Show $P(1)$ is true.

Step 2: Suppose ...

$a_0 = 2$

$a_1 = 3$

$a_2 = 5$

$a_n = \ldots$