Today: $\boxed{\text{Ken}}$

$\boxed{6.2}$ Properties of Sets

$\boxed{6.3}$ Disproofs & Algebraic Proofs

Last time:

$\boxed{6.1}$ Set Theory

$\boxed{6.2}$ Properties of Sets

Theorem 6.2.2
Set Identities
Let all sets below be subsets of a universal set $\mathcal{U}$.

① Commutative Laws For all sets A and B,
ⓐ $A \cup B = B \cup A$
ⓑ $A \cap B = B \cap A$

② Associative Laws For all sets A, B, and C,
ⓐ $(A \cup B) \cup C = A \cup (B \cup C)$
ⓑ $(A \cap B) \cap C = A \cap (B \cap C)$

③ Distributive Laws For all sets A, B, and C,
ⓐ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
ⓑ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

④ Identity Laws For any set A,
ⓐ $A \cup \emptyset = A$
ⓑ $A \cap \mathcal{U} = A$

⑤ Complement Laws For any set A,
ⓐ $A \cup A^c = \mathcal{U}$
ⓑ $A \cap A^c = \emptyset$

⑥ Double Complement Law For any set A,
$(A^c)^c = A$.

Theorem 6.2.2 (continued)

Let all sets below be subsets of a universal set $\mathcal{U}$.

(7) Idempotent Laws For any set A,

(a) $A \cup A = A$

(b) $A \cap A = A$

(8) Universal Bound Laws For any set A,

(a) $A \cup \mathcal{U} = \mathcal{U}$

(b) $A \cap \emptyset = \emptyset$

(9) DeMorgan's Laws For any sets A and B,

(a) $(A \cup B)^c = A^c \cap B^c$

(b) $(A \cap B)^c = A^c \cup B^c$

(10) Absorptions Laws For any sets A and B,

(a) $A \cup (A \cap B) = A$

(b) $A \cap (A \cup B) = A$

(11) Complements of $\mathcal{U}$ and $\emptyset$

(a) $\mathcal{U}^c = \emptyset$

(b) $\emptyset^c = \mathcal{U}$

(12) Set Difference Law For all sets A and B,

$A - B = A \cap B^c$

Theorem 6.2.2 (9)(b) Proof?

For all sets A and B, $(A\cap B)^c = A^c \cup B^c$.

Let A & B be sets.

(Goal: Show $(A\cap B)^c \subset A^c \cup B^c$ and $A^c \cup B^c \subset (A\cap B)^c$.)

Suppose $x \in (A\cap B)^c$. So $x \notin A\cap B$ (by def. of set complement). Either $x\in A$ or $x \notin A$.

Suppose $x \in A$. Suppose $x \in B$. Then $x \in A\cap B$. That's a contradiction since $x \in A\cap B$ and $x \notin A\cap B$. So $x\notin B$. Then $x \in B^c$.

By generalization, $x \in B^c$ or $x \in A^c$. So $x \in A^c \cup B^c$. Suppose $x \notin A$. So $x \in A^c$. Then, again by generalization, $x\in A^c$ or $x \in B^c$, hence $x \in A^c \cup B^c$. Thus

$(A\cap B)^c \subset A^c \cup B^c$.

Suppose $x \in A^c \cup B^c$. Suppose $x \in A^c$. So $x \notin A$. Suppose $x \in A \cap B$.

Then $x\in A$ and $x \in B$. By specialization, $x \in A$, so $x\in A$ and $x \notin A$ (contradiction).

Then $x \notin A \cap B$ so $x \in (A \cap B)^c$.

Suppose $x \in B^c$. So $x \notin B$. Suppose $x \in A \cap B$. Then $x \in A$ and $x \in B$.

By specialization, $x \in B$. Then $x \in B$ and $x \notin B$ (contradiction).

Thus $x \notin A \cap B$ so $x \in (A \cap B)^c$.

So $A^c \cup B^c \subset (A \cap B)^c$.

Since $(A \cap B)^c \subset A^c \cup B^c$, $A^c \cup B^c \subset (A \cap B)^c$, $(A \cap B)^c = A^c \cup B^c$.

Example 6.2.2 and Theorem 6.2.2 3 a

Proof of a Distributive Law for Sets

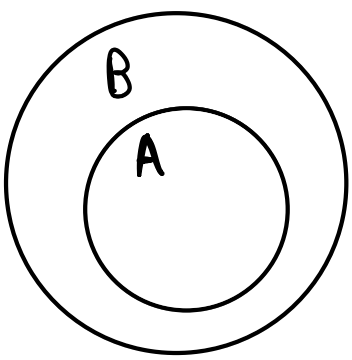
For all sets A, B, and C, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Theorem 6.2.2 3 b Proof?

For all sets A, B, and C, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Example 6.2.3 and Theorem 6.2.2 (a)(a)
Proof a De Morgan's Law for Sets
For all sets A and B, $(A \cup B)^c = A^c \cap B^c$.

Theorem 6.2.3
Intersection and Union with a Subset
For any sets A and B, if $A \subset B$, then
(a) $A \cap B = A$
(b) $A \cup B = B$



Theorem 6.2.4

A Set with No Elements is a Subset of Any Set
If E is a set with no elements and A is any set, then $E \subset A$.

$\forall x \in U (x \in E \rightarrow x \in A)$ (vacuously true)

Corollary 6.2.5 Uniqueness of the Empty Set
There is only one set with no elements.

recall we showed $\emptyset \subset S$ for any set S.

Proposition 6.2.6
For all sets $A, B,$ and C, if $A \subset B$ and $B \subset C^c$, then $A \cap C = \emptyset$.

Proof: Let A, B, C be sets. Suppose $A \subset B$ and $B \subset C^c$. Suppose $A \cap C \neq \emptyset$.
So there exists $x \in A \cap C$. By def. of intersection $x \in A$ and $x \in C$. *
Since $A \subset B$, $z \in A \Longrightarrow z \in B$. ☆
But since, $B \subset C^c$ $z \in B \Longrightarrow z \in C^c$. △

Since $x \in A$ by specialization, $x \in B$ by $\star$. Since $x \in B$, $x \in C^c$ by $\triangle$. Since $x \in C^c$, $x \notin C$. But, by $\ast$, $x \in C$, so $x \in C$ and $x \notin C$ (contradiction). Therefore $A \cap C = \emptyset$. $\square$

Element Method for Proving a Set Equals the Empty Set

To prove that a set X is equal to the empty set $\emptyset$, prove that X has no elements.
To accomplish this, suppose X has an element and derive a contradiction.

#22 For all sets A, B, and C,

$$A \times (B \cap C) = (A \times B) \cap (A \times C).$$

Definition (Cartesian product)
Let A, B be sets.
Define $A \times B = \{(a,b) : a \in A \text{ and } b \in B\}$

$\mathbb{R} \times \mathbb{R} = \{(x,y) : x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}$

$\boxed{6.3}$ Disproofs and Algebraic Proofs

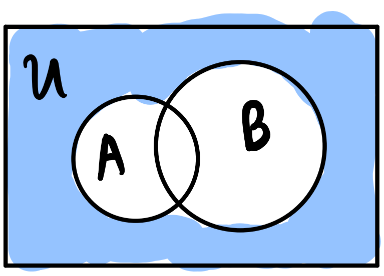
Disproving an Alleged Set Property

example

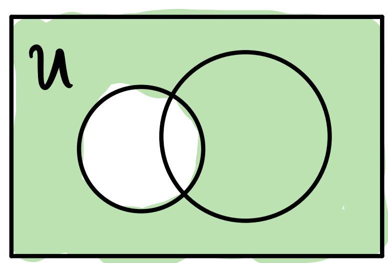
Finding a Counterexample for a Set Identity

Assume all sets are subsets of a universal set $\mathcal{U}$.

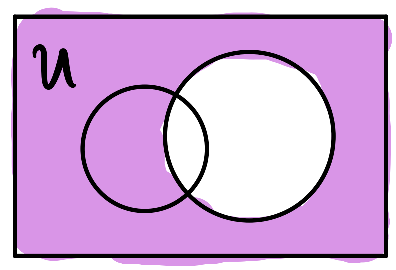
#2 For all sets A and B, $(A \cup B)^c = A^c \cup B^c$.



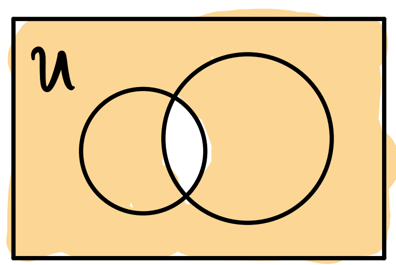
$(A \cup B)^c$



A^c



B^c



$A^c \cup B^c$

$\mathcal{U} = \{1, 2, 3, 4\}$

$A = \{1, 2\}$

$B = \{2, 3\}$

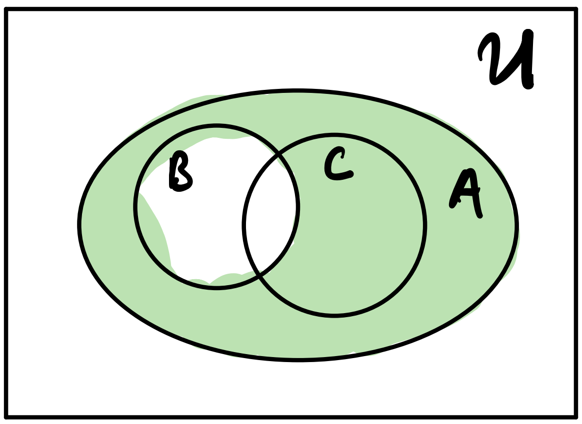
$(A \cup B)^c = \{4\}$

$A^c = \{3, 4\}$

$B^c = \{1, 4\}$

So $(A \cup B)^c \neq A^c \cup B^c$ $\quad A^c \cup B^c = \{1,3,4\}$

#4 For all sets A, B, and C, if $B \cup C \subset A$ then $(A-B) \cap (A-C) = \emptyset$.



A − B

Let A, B, C be sets.

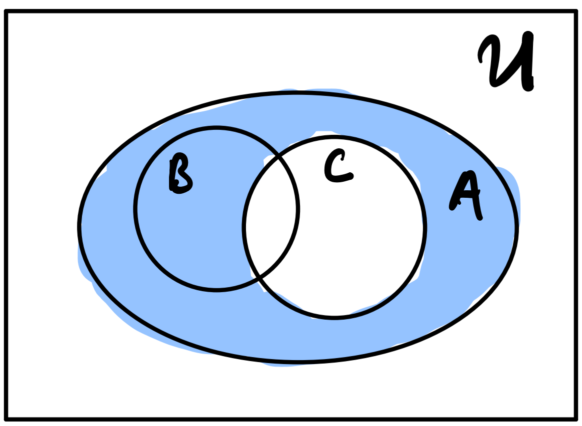
Suppose $B \cup C \subset A$.

$\mathcal{U} = \{1,2,3,4\}$

$A = \{1,2,3,4\}$

$B = \{1,2\}$

$C = \{2,3\}$



A − C

$B \cup C = \{1,2,3\}$

$A - B = \{3,4\}$

$A - C = \{1,4\}$

$(A-B) \cap (A-C) = \{4\} \neq \emptyset$

Theorem 6.3.1

For every integer $n \geq 0$, if a set X has n elements, then $\mathscr{P}(X)$ has 2^n elements.

#43 Simplify the expression

$((A \cap (B \cup C)) \cap (A - B)) \cap (B \cup C^c)$

$= ((A \cap (B \cup C)) \cap (A \cap B^c)) \cap (B \cup C^c)$ set difference law

$= (A \cap (B \cup C) \cap A \cap B^c) \cap (B \cup C^c)$ associative law

$= ((B \cup C) \cap A \cap A \cap B^c) \cap (B \cup C^c)$ commutative law

$= ((B \cup C) \cap A \cap B^c) \cap (B \cup C^c)$ idempotent law

$= ((B \cup C) \cap B^c \cap A) \cap (B \cup C^c)$ commutative law

$= (B^c \cap (B \cup C) \cap A) \cap (B \cup C^c)$ commutative law

$= ((B^c \cap B) \cup (B^c \cap C)) \cap A \cap (B \cup C^c)$ distributivity

$= ((B \cap B^c) \cup (B^c \cap C)) \cap A \cap (B \cup C^c)$ commutative law

$= (\emptyset \cup (B^c \cap C)) \cap A \cap (B \cup C^c)$ complement law

$= (B^c \cap C) \cap A \cap (B \cup C^c)$ identity law

$= B^c \cap C \cap A \cap (B \cup C^c)$ associative law

$= B^c \cap A \cap C \cap (B \cup C^c)$ commutative law

$= B^c \cap A \cap ((C \cap B) \cup (C \cap C^c))$ distributive law

$= B^c \cap A \cap ((C \cap B) \cup \emptyset)$ complement law

$= B^c \cap A \cap (C \cap B)$ identity law

$= B^c \cap A \cap C \cap B$ associative law

$= A \cap B^c \cap C \cap B$ commutative law

$= A \cap C \cap B^c \cap B$ commutative law

$= A \cap C \cap B \cap B^c$ commutative law

$= A \cap C \cap \emptyset$ complement law

$= A \cap \emptyset$ universal bound law

$= \emptyset$ universal bound law