Today: $\boxed{\text{Ken}}$

$\boxed{5.1}$ Sequences

$\boxed{4.7}$ Indirect Argument: Contradiction & Contraposition

Last time:

$\boxed{4.7}$ Indirect Argument: Contradiction & Contraposition

$\boxed{4.5}$ Direct Proof and Counterexample V

Theorem 4.7.3

The sum of any rational number and any irrational number is irrational.

Let $r \in \mathbb{Q}$ and $q \in \mathbb{R} - \mathbb{Q}$. Suppose $r + q \in \mathbb{Q}$. Since $r \in \mathbb{Q}$ and $r+q \in \mathbb{Q}$ there exist $m_1, m_2, n_1, n_2 \in \mathbb{Z}$ such that $n_1 \neq 0$, $n_2 \neq 0$, and $r = m_1/n_1$ and $r+q = m_2/n_2$.

$$\frac{m_2}{n_2} = r + q = \frac{m_1}{n_1} + q$$

so

$$q = \frac{m_2}{n_2} - \frac{m_1}{n_1} = \frac{n_1}{n_1}\frac{m_2}{n_2} - \frac{m_1}{n_1}\frac{n_2}{n_2}$$

$$= \frac{n_1 m_2 - m_1 n_2}{n_1 n_2}.$$

Then $n_1 m_2 - m_1 n_2, n_1 n_2 \in \mathbb{Z}$ because $\mathbb{Z}$ is closed under products and sums. $n_1 n_2 \neq 0$ because $n_1 \neq 0$ and $n_2 \neq 0$

via zero product property. So

$q \in \mathbb{Q}$ but recall $q \in \mathbb{R} - \mathbb{Q}$.

Therefore $r+q \notin \mathbb{Q}$, i.e. $r+q \in \mathbb{R} - \mathbb{Q}$,

the sum $r+q$ is irrational.

Argument by Contraposition

Method of Proof by Contraposition

① Express the statement to be proved in the form

$$\forall x \in D\,(P(x) \rightarrow Q(x)).$$

② Rewrite the statement in the contrapositive form

$$\forall x \in D\,(\text{if } Q(x) \text{ is false then } P(x) \text{ is false})$$

i.e.

$$\forall x \in D\,(\neg Q(x) \rightarrow \neg P(x)).$$

Note: Steps ① & ② may be done mentally.

③ Prove the contrapositive by a direct proof.

ⓐ Suppose x is a (particular but arbitrarily chosen) element of D such that $Q(x)$ is false.

ⓑ Show that $P(x)$ is false.

<u>Proposition 4.7.4</u>

For every integer n, if n^2 is even then n is even.

$\forall n \in \mathbb{Z}\,(n^2 \in 2\mathbb{Z} \to n \in 2\mathbb{Z})$

$\forall n \in \mathbb{Z}\,(n \notin 2\mathbb{Z} \longrightarrow n^2 \notin 2\mathbb{Z})$

$\equiv \forall n \in \mathbb{Z}\,(n \in \mathbb{Z} - 2\mathbb{Z} \to n^2 \in \mathbb{Z} - 2\mathbb{Z})$

Let $n \in \mathbb{Z}$. Suppose $n \in \mathbb{Z} - 2\mathbb{Z}$, i.e. n is odd. So, via definition of odd, $n = 2k+1$ for some $k \in \mathbb{Z}$. Then

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1$$
$$= 2(2k^2 + 2k) + 1$$

where $\ell := 2k^2 + 2k \in \mathbb{Z}$ by closure of $\mathbb{Z}$ under products and sums, so $n^2 = 2\ell + 1 \in \mathbb{Z} - 2\mathbb{Z}$. Therefore if n is odd then n^2 is odd and, via logical equivalence, n^2 is even implies n is even. (Proof 1)

(Proof 2) $\forall n \in \mathbb{Z} \, (n^2 \in 2\mathbb{Z} \rightarrow n \in 2\mathbb{Z})$

$\boxed{5.1}$ Sequences

Definition

A sequence is a function whose domain is either all the integers between two given integers or all the integers greater than or equal to a given integer.

e.g.

$$a_m, a_{m+1}, a_{m+2}, \dots, a_n$$

where each a_k is called a term, the k in a_k is called a subscript or index, and m is the subscript of the initial term while n is the subscript of the final term.

$$a_m, a_{m+1}, a_{m+2}, \dots$$

denotes an infinite sequence. An explicit formula or general formula for a sequence is a rule that shows how the values of a_k depend on k.

#4 write the first five terms of the sequence defined by the formula

$a_n = 1 + (1/2)^n$ for all $n \geq 0$

$a_0 = 2,\ a_1 = 3/2,\ a_2 = 5/4,\ a_3 = 9/8,\ a_4 = 17/16, \ldots$

examples

(a) $\{(-1)^n\}_{n=0}^{\infty}$

$1, -1, 1, -1, 1, -1, \ldots$

(b) $\{1/n\}_{n=1}^{\infty}$

$1, 1/2, 1/3, 1/4, \ldots$

examples

Find the general term of the sequences

(a) $\pi, 0, \pi, 0, \pi, 0, \ldots$

$\{1 + (-1)^n\}_{n=0}^{\infty}$

$2, 0, 2, 0, 2, 0, \ldots$

$\{\pi/2\,(1 + (-1)^n)\}_{n=0}^{\infty}$

(b) $1/3, -1/5, 1/9, -1/17, 1/33, \ldots$

$$\left\{ \frac{(-1)^{n+1}}{2^n+1} \right\}_{n=1}^{\infty}$$

Summation Notation

Definition

Given a sequence $a_m, a_{m+1}, a_{m+2}, \ldots, a_n$, the notation $\sum_{k=m}^{n} a_k$ is the summation of all a_k from $k=m$ to $k=n$ and

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + a_{m+2} + \cdots + a_n$$

where the expression on the right hand side is the expanded form of the sum. We call k the index of the summation, m the lower limit of the summation, and n the upper limit of the summation.

Product Notation

Definition

If m and n are integers and $m \le n$, the symbol $\prod_{k=m}^{n} a_k$, read the product of all a_k from $k=m$ to $k=n$, is the product of the terms

$$\prod_{k=m}^{n} a_k = a_m a_{m+1} a_{m+2} \cdots a_{n-1} a_n.$$

Theorem 5.1.1

Properties of Sums and Products

If $\{a_k\}$ and $\{b_k\}$ are sequences of real numbers and c is any real number, then

1. $$\sum_{k=m}^{n} (a_k + b_k) = \left(\sum_{k=m}^{n} a_k\right) + \left(\sum_{k=m}^{n} b_k\right)$$
2. $$\sum_{k=m}^{n} c a_k = c \sum_{k=m}^{n} a_k$$
3. $$\prod_{k=m}^{n} (a_k b_k) = \left(\prod_{k=m}^{n} a_k\right)\left(\prod_{k=m}^{n} b_k\right)$$

④ $$\prod_{k=m}^{n}(ca_k) = c^{n-m+1}\left(\prod_{k=m}^{n} a_k\right), \quad n>m>0$$

e.g. Suppose $a_0=0, a_1=1, a_2=1, a_3=2, a_4=3, a_5=5$.
Calculate the sums and products.

ⓐ $$\sum_{k=0}^{5} a_k = 0+1+1+2+3+5 = 12$$

ⓑ $$\prod_{k=1}^{5} a_k = 1(1)(2)(3)(5) = 30$$

ⓒ $$\sum_{k=4}^{4} a_k = a_4 = 3$$

ⓓ $$\sum_{k=0}^{2} a_{2k} = a_0 + a_2 + a_4 = 0+1+3=4$$

ⓔ $$\sum_{k=0}^{3} ka_k = 0a_0 + 1a_1 + 2a_2 + 3a_3$$
$$= 0\cdot0 + 1\cdot1 + 2\cdot1 + 3\cdot2$$
$$= 0+1+2+6 = 9$$

ⓕ $$\sum_{k=3}^{6} k^2$$

$$(10)(8)(6)(4)(2) = \prod_{k=1}^{5}(2k) \neq (2k)!$$

e.g. Write sum or product in expanded form.

#30 (a) $\sum_{k=1}^{n} k(k+1) = 1(2)+2(3)+3(4)+\cdots+n(n+1)$

(b) $\sum_{k=1}^{n} \ln\left(\frac{k}{k+1}\right) = \ln\left(\frac{1}{2}\right)+\ln\left(\frac{2}{3}\right)+\cdots+\ln\left(\frac{n}{n+1}\right)$

$= \ln(1)-\ln(2)+\ln(2)-\ln(3)+\cdots+\ln(n)-\ln(n+1)$

$= -\ln(n+1)$

(c) $\prod_{k=1}^{n} k = n!$

(d) $\sum_{i=1}^{n} \prod_{j=1}^{i} 1/j$

$= \sum_{i=1}^{n}\left(\frac{1}{1}\left(\frac{1}{2}\right)\cdots\left(\frac{1}{i}\right)\right)$

$= 1+\frac{1}{1}\left(\frac{1}{2}\right)+\frac{1}{1}\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)+\cdots+\left(\frac{1}{1}\right)\left(\frac{1}{2}\right)\cdots\left(\frac{1}{n}\right)$

(e) $\left(\sum_{i=1}^{n} a_i\right)^2 = (a_1+a_2+\cdots+a_n)^2$

$= a_1^2+a_2^2+\cdots+a_n^2+2a_1a_2+2a_1a_3+\cdots+2a_{n-1}a_n$

$$= \left(\sum_{k=1}^{n} a_k^2\right) + 2\left(\sum_{\substack{i,j=1 \\ i<j}}^{n} a_i a_j\right)$$

$$(x+y+z)^2 = x^2 + y^2 + z^2 + 2xy + 2xz + 2yz$$

e.g. Write the sum or product in a compact notation.

(a) $\frac{n}{n+1} + \frac{n+1}{n+2} + \frac{n+2}{n+3} + \cdots + \frac{2n}{2n+1} = \sum_{k=n}^{2n} \frac{k}{k+1}$

(b) $1(3)(5)\cdots(25)(27)(29) = \prod_{k=1}^{14} (2k+1)$

(c) $2\left(\frac{4}{3}\right)\left(\frac{6}{5}\right)\cdots\left(\frac{24}{23}\right) = \prod_{k=1}^{12} \frac{2k}{2k-1}$

e.g. Write each expression as a single summation or product.

#59 (a) $3\sum_{k=1}^{n}(2k-3) + \sum_{k=1}^{n}(4-5k)$

apply Theorem 5.1.1

#61 (b) $\left(\prod_{k=1}^{n} \frac{k}{k+1}\right)\left(\prod_{k=1}^{n} \frac{k+1}{k+2}\right)$

e.g. Conduct a change of variable:

$$\sum_{k=0}^{n} \frac{(-1)^k (k+1)}{k+2} = \sum_{k=1}^{n+1} \frac{(-1)^{k-1}(k)}{k+1}$$

e.g. Assume the values of the variables are restricted so that the expressions are defined.

gamma function

ⓐ $\frac{7!}{4!} = 7(6)(5) = 210$

ⓑ $\frac{5!}{0!} = 5! = 120$

#66 ⓒ $\frac{(n-1)!}{(n+1)!} = \frac{1}{(n+1)n}$

#69 ⓓ $\frac{n!}{(n-k)!} = n(n-1)(n-2)\cdots(n-k+1)$

e.g. ⓐ Calculate the following values

$\binom{0}{0}, \binom{1}{0}, \binom{1}{1}, \binom{2}{0}, \binom{2}{1}, \binom{2}{2}, \binom{3}{0}, \binom{3}{1}, \binom{3}{2}, \binom{3}{3},$

$\binom{4}{0}, \binom{4}{1}, \binom{4}{2}, \binom{4}{3}, \binom{4}{4}$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

"n choose k"

⑥ If $n \in \mathbb{Z}^+ \cup \{0\}$, then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

$x=y=1$?

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$$