Wednesday - Exam · 2 metro tech 878

or Zoom office hrs if you have ?

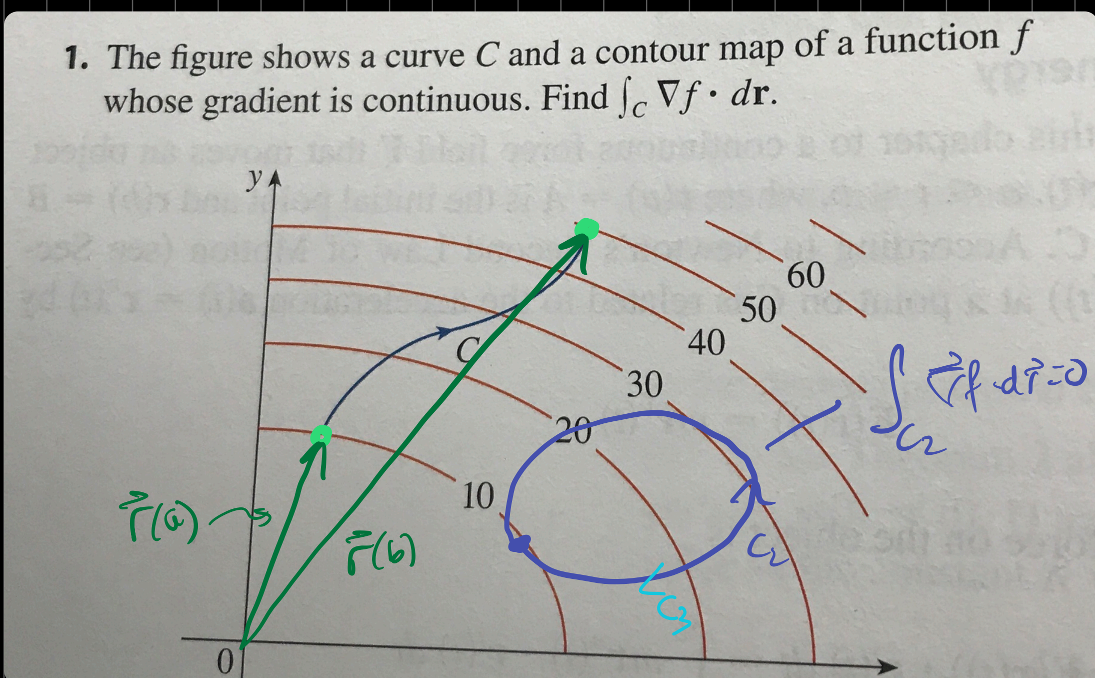
Fundamental Thm for Line Integrals.

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)) \quad \leftrightarrow \quad \int_a^b f'(t)\,dt = f(b) - f(a)$$

C = smooth

t=a   t=b

If $\vec{F} = \nabla f$ then $\vec{F}$ is conservative.



1. The figure shows a curve C and a contour map of a function f whose gradient is continuous. Find $\int_C \nabla f \cdot d\mathbf{r}$.

$$\int_C \vec{\nabla} f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

$$= 50 - 10 = 40$$

(3) $\vec{F}(x,y) = (xy + y^2)\,\hat{\imath} + (x^2 + 2xy)\,\hat{\jmath}$

is $\vec{F}$ conservative? $\iff$ can we find $f(x,y)$ s.t. $\vec{\nabla} f = \vec{F}$

$f_x = xy + y^2 \xrightarrow{\text{int. wrt. } x} \frac{x^2y}{2} + xy^2 + g(y) = f(x,y)$

$\downarrow \frac{\partial}{\partial y}$

$f_y = x^2 + 2xy \underset{\text{be equal}}{\overset{\text{must}}{=}} \frac{x^2}{2} + 2xy + g'(y)$

find $g'(y)$

$$x^2 + \cancel{2xy} = \frac{x^2}{2} + \cancel{2xy} + g'(y)$$

$$\frac{x^2}{2} = g'(y) \quad \text{Contradiction!!}$$

Vector field is not conservative!!

$\vec{F}$ does not have a scalar potential

(11) $\vec{F}(x,y) = \langle 2xy, x^2 \rangle$ is $\vec{F}$ conservative?

$f_x = 2xy \longrightarrow f(x,y) = x^2y + g(y)$

$\downarrow \frac{\partial}{\partial y}$

$f_y = x^2 \qquad f_y = x^2 + g'(y)$

should be same.

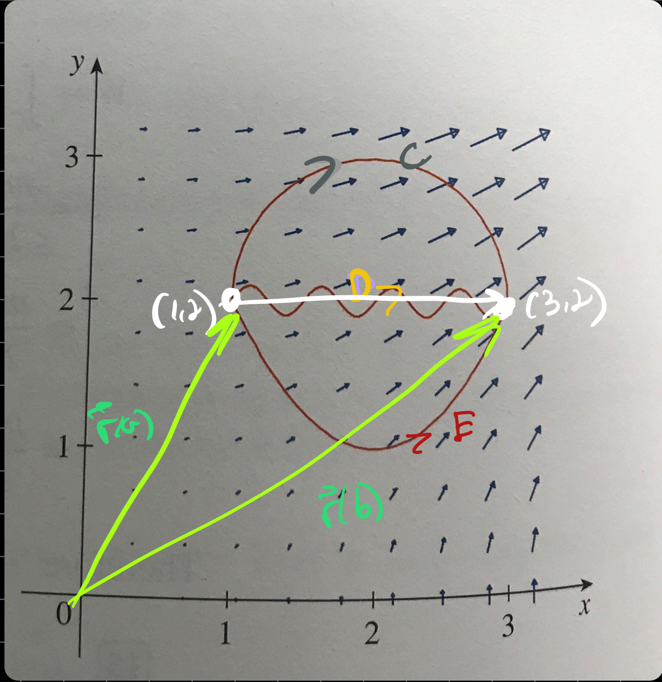
$$x^2 = x^2 + g'(y) \Rightarrow g'(y) = 0 \Rightarrow g(y) = C_1$$

Potential: $f(x,y) = x^2y + C_1$

3 curves that start at $(1,2)$ end at $(3,2)$

Evaluate $\int_C \vec{F} \cdot d\vec{r} = f(3,2) - f(1,2) = \int_D \vec{F} \cdot d\vec{r} = \int_E \vec{F} \cdot d\vec{r}$

$= ((3)^2(2) + C_1) - ((1)^2(2) + C_1) = \boxed{16}$



Path Independent.

$f(\vec{r}(a)) = f(1,2)$

$f(\vec{r}(b)) = f(3,2)$

(13) $\vec{F}(x,y) = x^2y^3\,\hat{\imath} + x^3y^2\,\hat{\jmath}$

$C: \vec{r}(t) = \langle t^3-2t,\ t^3+2t\rangle \quad 0 \le t \le 1$

evaluate $\int_C \vec{F}\cdot d\vec{r}$ from last class: $\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$

$$= \int_0^1 \left[(t^3-2t)^2(t^3+2t)^3\,\hat{\imath} + (t^3-2t)^3(t^3+2t)^2\,\hat{\jmath}\right] \cdot \left[(3t^2-2)\,\hat{\imath} + (3t^2+2)\,\hat{\jmath}\right] dt$$

looks complicated!!!

Does $\vec{F}$ have a potential?

use Thm: $\vec{F} = P\hat{\imath} + Q\hat{\jmath}$ defined on open / simply connected domain.

P, Q have continuous partial deriv.

and $P_y = Q_x$

then $\vec{F}$ is conservative.

$\vec{F}(x,y) = \underbrace{x^2y^3}_{P}\hat{i} + \underbrace{x^3y^2}_{Q}\hat{j}$

F, P_y, Q_x continuous on $\mathbb{R}^2$

$P_y = 3x^2y^2$ $\qquad Q_x = 3x^2y^2$

$P_y = Q_x \Rightarrow \vec{F}$ conservative.

now find potential.

$f_x = x^2y^3 \xrightarrow{\int f_x\,dx} \frac{x^3y^3}{3} + g(y) = f(x,y)$

$\downarrow \frac{\partial}{\partial y}$

$f_y = x^3y^2$ $\qquad x^3y^2 + g'(y)$

Set equal.

$x^3y^2 = x^3y^2 + g'(y)$

$g'(y) = 0$

$g(y) =$ constant.

Choose any constant

let's choose $0. = g(y)$

potential: $\frac{x^3y^3}{3} = f(x,y)$

now:

$\vec{F}(x,y) = x^2y^3\hat{i} + x^3y^2\hat{j}$

$C: \vec{r}(t) = \langle t^3 - 2t,\ t^3 + 2t\rangle \quad 0 \le t \le 1$

$\vec{r}(0) = \langle 0,0\rangle \qquad \vec{r}(1) = \langle -1, 3\rangle$

$\int_C \vec{F}\cdot d\vec{r} = f(-1,3) - f(0,0) = -9.$

(17) $\vec{F}(x,y,z) = yze^{xz}\hat{i} + e^{xz}\hat{j} + xye^{xz}\hat{k}$

$C: \vec{r}(t) = (t^2+1)\hat{i} + (t^2-1)\hat{j} + (t^2-2t)\hat{k} \qquad 0 \le t \le 2.$

Evaluate $\int_C \vec{F}\cdot d\vec{r}$

Then above can become:

$P_y = Q_x, \quad P_z = R_x, \quad Q_z = R_y$

Try to find f.

$f_x = yze^{xz} = yze^{xz} + \frac{\partial}{\partial x} g(x,z) \Rightarrow \frac{\partial}{\partial x} g(x,z) = 0$

$\Rightarrow$ only function of z. or constant

$f_y = e^{xz} \xrightarrow{\int f_y\, dy} ye^{xz} + g(x,z) = f(x,y,z)$

$f_z = xye^{xz}$

now let $f(x,y,z) = ye^{xz} + h(z)$

find $\frac{\partial}{\partial z}(ye^{xz} + h(z)) = xye^{xz} + h'(z)$

must equal xye^{xz}

$h'(z) = 0$

$\rightarrow h(z) = \text{constant}$

Choose 0.

potential: $f(x,y,z) = ye^{xz}$

now $\int_C \vec{F}\cdot d\vec{r} = f(\vec{r}(2)) - f(\vec{r}(0)) = 3 - (-1) = 4.$

$\vec{r}(t) = (t^2+1)\hat{i} + (t^2-1)\hat{j} + (t^2-2t)\hat{k}$

$\vec{r}(2) = 5\hat{i} + 3\hat{j} + 0\hat{k}$ $\quad f(\vec{r}(2)) = f(5,3,0) = 3e^{(5)(0)} = 3$

$\vec{r}(0) = \hat{i} - \hat{j} + 0\hat{k}$ $\quad f(\vec{r}(0)) = f(1,-1,0) = -1$

30 $\int_C y\,dx + x\,dy + xyz\,dz$ is this integral path independent?

$d\vec{r} = \langle dx, dy, dz\rangle$

Is $\vec{F}$ conservative - can we find f, s.t. $\vec{F} = \vec{\nabla} f$

$P=y \quad Q=x \quad R=xyz$

$\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$ then

$$\int_C y\,dx + x\,dy + xyz\,dz = \int_C \vec{F}\cdot d\vec{r}$$

$$\vec{F} = \underset{P}{y}\,\hat{i} + \underset{Q}{x}\,\hat{j} + \underset{R}{xyz}\,\hat{k}$$

use: $\underline{P_y = Q_x}$, $\underline{P_z = R_x}$, $\underline{Q_z = R_y}$

Check: $P_y = Q_x$ $\quad 1 = 1$ ✓

$P_z = R_x$ $\quad 0 = yz$ ✗

$Q_z = R_y$ $\quad 0 = xz$ ✗

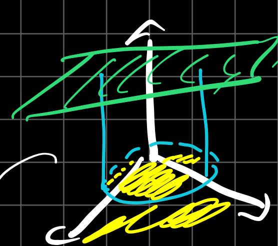
Not conservative!!

Exam - no curvature or a_T, a_N no arc length.

Double Integrals with general Regions.
(Triple Integrals).

Ex) Find Volume of Region bounded by $x^2+y^2=4$ $\quad z=0$ $\quad y=3-z$

plane.



in xy plane

$0 \le z \le 3-y = 3-r\sin\theta$

$$0 \le r \le 2$$
$$0 \le \theta \le 2\pi$$

$$V(E) = \int_0^{2\pi}\int_0^{2}\int_0^{3 - r\sin\theta} (1)\, r\, dz\, dr\, d\theta$$