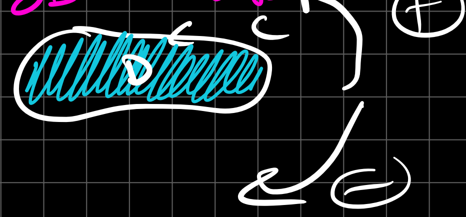
Green's Thm

$$\int_C \underbrace{P\,dx + Q\,dy}_{\vec{F} = P\hat{i} + Q\hat{j}} = \iint_D Q_x - P_y \, dA$$

∂D - boundary of D



if hole in D
need to take further action.

$$\underbrace{\int_C \vec{F} \cdot d\vec{r}}_{\text{Work}} = \iint_D Q_x - P_y \, dA$$

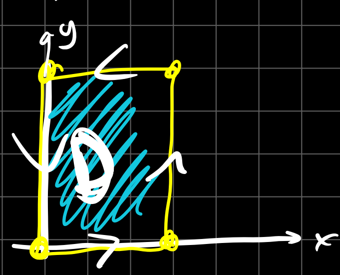
$\hat{k}$ component of curl $\vec{F}$

curl $\vec{F} \cdot \hat{k}$

16.4

(5) $\int_C ye^x\,dx + 2e^x\,dy$

C is rectangle with vertices (0,0) (3,0), (3,4) and (0,4).
positively oriented.



$P = ye^x$ $\quad Q = 2e^x$

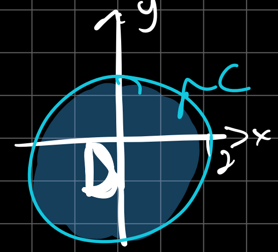
$P_y = e^x$ $\quad Q_x = 2e^x$

$$\int_C ye^x\,dx + 2e^x\,dy = \iint_D 2e^x - e^x\,dA = \iint_D e^x\,dA = \int_0^4\int_0^3 e^x\,dx\,dy$$

$$= 4(e^3 - 1)$$

$0 \le x \le 3$
$0 \le y \le 4$

⑨ $\int_C y^3\,dx - x^3\,dy$ C is $x^2+y^2=4$ positively oriented.



$P = y^3 \qquad Q = -x^3$

$P_y = 3y^2 \qquad Q_x = -3x^2$

$Q_x - P_y = -3x^2 - 3y^2$

$$\int_C y^3\,dx - x^3\,dy = \iint_D -3x^2-3y^2\,dA = -3\iint_D x^2+y^2\,dA = -3\int_0^{2\pi}\int_0^{2} r^2\,r\,dr\,d\theta$$

$0 \le r \le 2$
$0 \le \theta \le 2\pi$

$$= -3\int_0^{2\pi}\int_0^{2} r^3\,dr\,d\theta = (-3)(2\pi)\left(\frac{r^4}{4}\Big|_{r=0}^{r=2}\right) = -24\pi$$

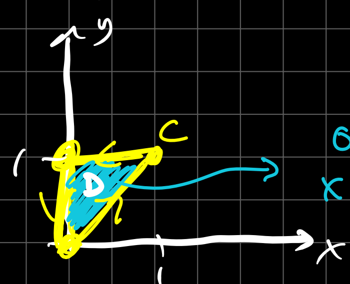
$\vec{F} = \langle P, Q\rangle$

tends to work against path C.

⑭ $\vec{F}(x,y) = \langle \sqrt{x^2+1}, \tan^{-1}x\rangle$

C is triangle from $(0,0)$ to $(1,1)$ to $(0,1)$ to $(0,0)$

Evaluate $\int_C \vec{F}\cdot d\vec{r}$



$0 \le x \le 1$
$x \le y \le 1$

or

$0 \le y \le 1$
$0 \le x \le y$

$P = \sqrt{x^2+1} \qquad Q = \tan^{-1}(x)$

$P_y = 0 \qquad Q_x = \frac{1}{x^2+1}$

$$\int_C \vec{F}\cdot d\vec{r} = \iint_D \frac{1}{x^2+1}\,dA = \int_0^1\underbrace{\int_0^y \frac{1}{x^2+1}\,dx}_{\arctan x\big|_{x=0}^{x=y}}\,dy$$

$$= \int_0^1 \arctan y \, dy$$ use IBP.

or switch order.

$u = \arctan y \quad dv = dy$

$du = \frac{1}{1+y^2} dy \quad v = y$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \int_x^1 \frac{1}{x^2+1} \, dy \, dx$$

$$= \int_0^1 \frac{1}{x^2+1} \int_x^1 dy \, dx$$

$$= \int_0^1 \frac{1}{x^2+1}(1-x) \, dx$$

$$= \int_0^1 \frac{1}{x^2+1} - \frac{x}{x^2+1} \, dx$$

$$= \int_0^1 \frac{1}{x^2+1} \, dx - \int_0^1 \frac{x}{x^2+1} \, dx$$

$u = x^2+1$

$du = 2x\,dx \rightarrow \frac{1}{2} du = x\,dx$

$$= \underbrace{\arctan(1)}_{\frac{\pi}{4}} - \frac{1}{2} \int_1^2 \frac{1}{u} \, du$$

$$= \frac{\pi}{4} - \frac{1}{2}(\ln(2) - \ln(1)) = \frac{\pi}{4} - \frac{1}{2}\ln 2$$

16.5 Curl & Divergence.

$$\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$$

$$\text{Curl } \vec{F} = \vec{\nabla} \times \vec{F} = (R_y - Q_z)\hat{i} + (P_z - R_x)\hat{j} + (Q_x - P_y)\hat{k}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$\left(\quad \vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right.$$

measures tendency of vector field to rotate

$$\neq \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$ not defined.

divergence. $\text{div} \vec{F} = \vec{\nabla} \cdot \vec{F} = P_x + Q_y + R_z$

defined for other dimensions

Interesting Results: $\vec{F}$ defined on all of $\mathbb{R}^3$ components have cont. p.d.
AND $\text{curl}\,\vec{F} = \vec{0}$ then $\vec{F}$ is conservative.

Moreover: $\text{curl}(\nabla f) = \vec{0}$, if f has continuous 2nd order p.d.

irrotational.

If $\vec{F} = \vec{\nabla} \times \vec{G}$ then $\text{div}\,\vec{F} = 0$

$\vec{F}$ is incompressible

(2) $\vec{F}(x,y,z) = x^3 y z^2 \hat{j} + y^4 z^3 \hat{k}$ $\quad P=0 \quad Q = x^3yz^2 \quad R = y^4z^3$

find curl & div.

$$\text{div}\,\vec{F} = \vec{\nabla} \cdot \vec{F} = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(x^3yz^2) + \frac{\partial}{\partial z}(y^4z^3)$$

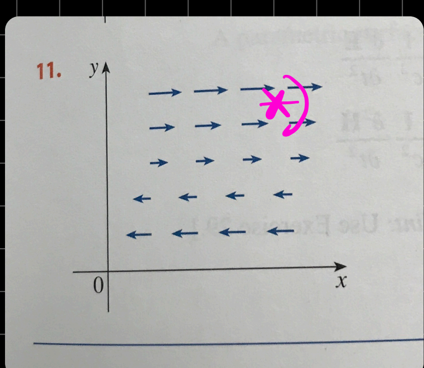
$$= x^3z^2 + 3y^4z^2$$

$$\text{curl}\,\vec{F} = \vec{\nabla} \times \vec{F} = (R_y - Q_z)\hat{i} + (P_z - R_x)\hat{j} + (Q_x - P_y)\hat{k}$$

$$= (4y^3z^3 - 2x^3yz)\hat{i} + (0-0)\hat{j} + (3x^2yz^2 - 0)\hat{k}$$

$$= (4y^3z^3 - 2x^3yz)\hat{i} + (3x^2yz^2)\hat{k}$$

in $\mathbb{R}^3$ but z component $=0$.

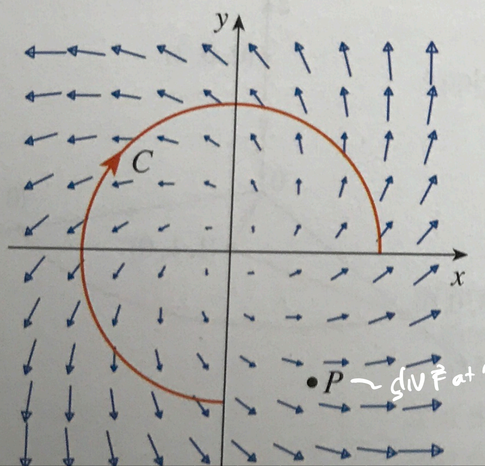
(11) 

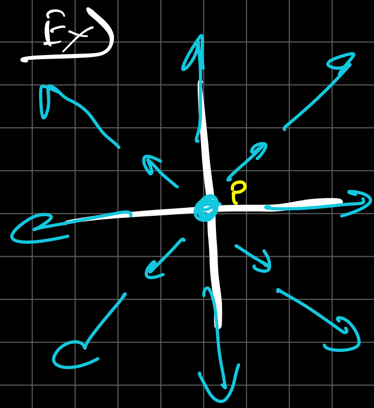
$\text{div}\,\vec{F} = 0$

$\text{curl}\,\vec{F} \neq \vec{0}$ clockwise rotation.

curl $\vec{F}$ points in negative z direction

vector field **F**, a curve C, and a point P are shown.
) Is $\int_C \mathbf{F} \cdot d\mathbf{r}$ positive, negative, or zero? Explain.
) Is div $\mathbf{F}(P)$ positive, negative, or zero? Explain.





div $\vec{F}$ at P > 0

⑫ Let f be a scalar field and **F** a vector field. State whether each expression is meaningful. If not, explain why. If so, state whether it is a scalar field or a vector field.

(a) curl f ✗
(b) grad f ✓
(c) div **F** ✓
(d) curl(grad f) ✓
(e) grad **F** ✗
(f) grad(div **F**) ✓
(g) div(grad f) ✓
(h) grad(div f) ✗
(i) curl(curl **F**) ✓
(j) div(div **F**) ✗
(k) (grad f) × (div **F**) ✗
(l) div(curl(grad f)) ✓

⑬ $\vec{F}(x,y,z) = \overset{P}{y^2z^3}\hat{i} + \overset{Q}{2xyz^3}\hat{j} + \overset{R}{3xy^2z^2}\hat{k}$

Is $\vec{F}$ conservative? Check if Curl $\vec{F} = \vec{0}$

$$\text{Curl } \vec{F} = \vec{\nabla} \times \vec{F} = (R_y - Q_z)\hat{i} + (P_z - R_x)\hat{j} + (Q_x - P_y)\hat{k}$$

$$= (6xyz^2 - 6xyz^2)\hat{i} + (3y^2z^2 - 3y^2z^2)\hat{j} + (2yz^3 - 2yz^3)\hat{k} = \vec{0}$$

$\vec{F}$ conservative we can find $f(x,y,z)$ so that $\vec{F} = \underbrace{\vec{\nabla} f}_{\text{potential}}$

(19) $\text{Curl}\,\vec{G} = \langle x\sin y, \cos y, z - xy \rangle$

Is this possible? does the given vector field have a vector potential?

Check if $\text{div Curl}\,\vec{G} = 0$

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{G}) = 0$$

$$\vec{\nabla} \cdot \langle x\sin y, \cos y, z - xy \rangle$$

$$= \frac{\partial}{\partial x}(x\sin y) + \frac{\partial}{\partial y}(\cos y) + \frac{\partial}{\partial z}(z - xy)$$

$$= \sin y - \sin y + 1 = 1 \neq 0$$

No potential.

No G s.t. $\text{Curl}\,G =$ vector field.

(22) $\vec{F}(x,y,z) = f(y,z)\hat{i} + g(x,z)\hat{j} + h(x,y)\hat{k}$

Show $\vec{F}$ is $\boxed{\text{incompressible.}}$ — show $\text{div}\,\vec{F} = 0$

$$\text{div}\,\vec{F} = \underbrace{\frac{\partial}{\partial x} f(y,z)}_{0} + \underbrace{\frac{\partial}{\partial y} g(x,z)}_{0} + \underbrace{\frac{\partial}{\partial z} h(x,y)}_{0} = 0$$

Try 32, 33, 34