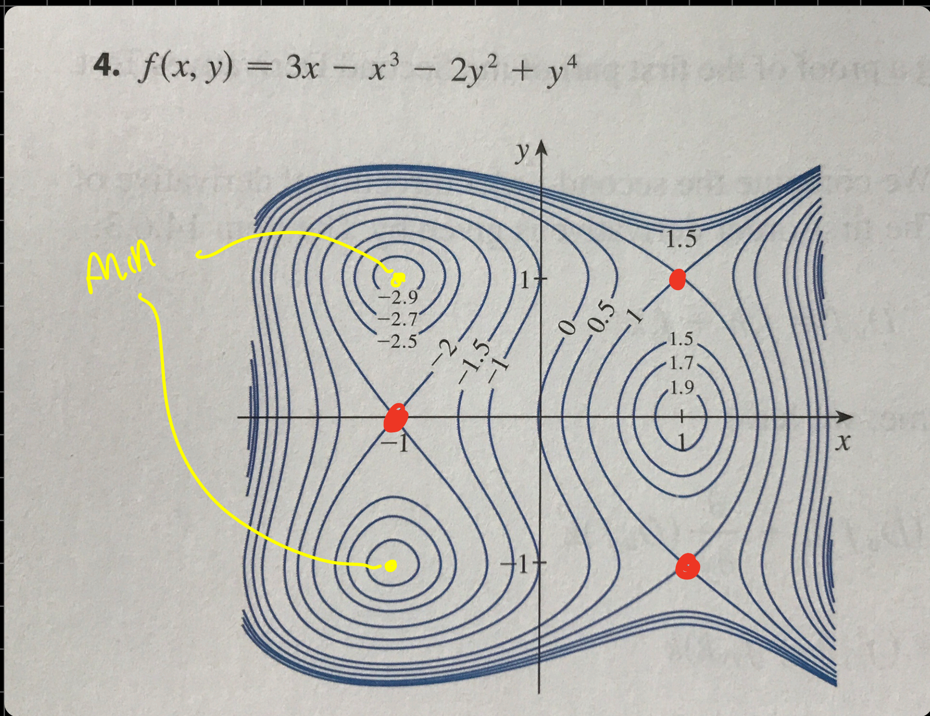
14.7 Optimization.



find c.p.

$\vec{\nabla} f = \vec{0}$

or undefined.

Graphically

Local max at $(1,0)$.

local min $(-1,\pm 1)$

using gradient

$$\vec{\nabla} f = \langle 3-3x^2, -4y+4y^3 \rangle = \langle 0,0 \rangle$$

$3-3x^2=0$ $\qquad$ $-4y+4y^3=0$

$x=\pm 1$ $\qquad$ $4y(-1+y^2)=0$

$y=0 \qquad y=\pm 1$

local max

c.p. : $\boxed{(1,0)}$ $\quad (1,1)$, $(1,-1)$

$(-1,0)$ $\quad \boxed{(-1,-1), (-1,1)}$ — local min.

use 2nd derivatives test

Hessian.

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

$$D = D(a,b) = f_{xx}(a,b)\, f_{yy}(a,b) - [f_{xy}(a,b)]^2$$

$D>0$ and $f_{xx}(a,b)>0$ $\quad$ local min

$D>0$ and $f_{xx}(a,b)<0$ $\quad$ local max

$D<0$ Saddle point

$D=0$ inconclusive.

$f(x, y) = 3x - x^3 - 2y^2 + y^4$

$f_{xx} = -6x$ $\qquad f_{yy} = -4+12y^2$ $\qquad f_{xy} = 0$

$D = (-6x)(-4+12y^2) - (0)^2$

$D(1,1) = (-6)(-4+12) = -48 < 0 \Rightarrow$ Saddle point.

$D(1,-1) = (-6)(-4+12) < 0 \Rightarrow$ Saddle point.

$D(-1,0) = (-6(-1))(-4) < 0 \Rightarrow$ saddle point.

(11) $f(x,y) = x^3 - 3x + 3xy^2$ $\qquad$ find any local max/min/saddle

$\vec{\nabla} f = \langle 3x^2 - 3 + 3y^2, 6xy \rangle$

now set $\vec{\nabla} f = \vec{0}$

$3x^2 - 3 + 3y^2 = 0$ $\qquad 6xy = 0$

$x^2 + y^2 = 1$ $\qquad x=0$ or $y=0$

if $x=0$ $\quad y = \pm 1$

if $y=0$ $\quad x = \pm 1$

C.P. $(0,1), (0,-1), (1,0), (-1,0)$

now classify: $\quad D = \underbrace{(6x)}_{f_{xx}}(6x) - (6y)^2 = 36x^2 - 36y^2$

$D(0,1) = -36 < 0$
$D(0,-1) = -36 < 0$ } Saddle points at $(0,\pm 1)$

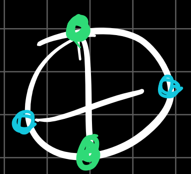
$D(1,0) = 36 > 0$
$D(-1,0) = 36 > 0$ } check $f_{xx}(1,0) > 0$
$f_{xx}(-1,0) < 0$

local Minimum at $(1,0)$
local Max at $(-1,0)$.

(19) $f(x,y) = y^2 - 2y\cos x \qquad -1 \le x \le 7.$ find max/min saddle.

grad $f = \vec{\nabla} f = \langle 2y\sin x,\ 2y - 2\cos x \rangle$

$y\sin x = 0 \qquad\qquad y - \cos x = 0$



$\boxed{y=0}$ | $\sin x = 0$ $\qquad y = \cos x.$

$\boxed{x = 0, \pi, 2\pi}$

{ if $y=0$ then $\cos x = 0 \Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$

$\left(\frac{\pi}{2}, 0\right), \left(\frac{3\pi}{2}, 0\right)$

{ if $x=0$, $y = \cos(0) = 1$ $\qquad (0,1)$
if $x = \pi$, $y = \cos\pi = -1$ $\qquad (\pi,-1)$
if $x = 2\pi$, $y = \cos(2\pi) = 1$ $\qquad (2\pi, 1)$

5 c.p. find D on own.

(42) find points on $x-2y+3z=6$ closest to $(0,1,1)$.

$$d=\sqrt{(x-0)^2+(y-1)^2+(z-1)^2}$$

3 var.

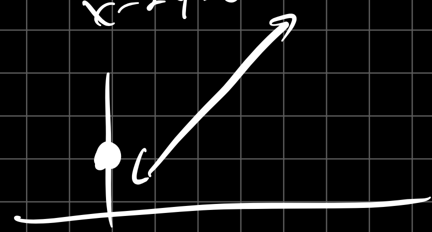
Sub to make 2 var.

$x=6+2y-3z$

$$d=f(y,z)=\sqrt{(6+2y-3z)^2+(y-1)^2+(z-1)^2}$$

Calc I

$x-2y=0$ $\quad y=\frac{x}{2}$



$$d=\sqrt{(x-0)^2+(y-1)^2}$$

2 var $\quad y=\frac{x}{2}$?

1 var.

$$=\sqrt{x^2+\left(\frac{x}{2}-1\right)^2}$$

now minimize.

$\nabla f=\langle f_y, f_z\rangle$

$$f_y=\frac{\partial}{\partial y}\sqrt{(6+2y-3z)^2+(y-1)^2+(z-1)^2}=\frac{2(6+2y-3z)(2)+2(y-1)}{2\sqrt{(6+2y-3z)^2+(y-1)^2+(z-1)^2}}$$

$$f_z=\frac{2(6+2y-3z)(-3)+2(z-1)}{2\sqrt{(6+2y-3z)^2+(y-1)^2+(z-1)^2}}$$

$\boxed{ax+by+cz=d}$

$f_y=0 \Rightarrow 2(6+2y-3z)(2)+2(y-1)=0$, $\quad 10y-12z=-22$

$f_z=0 \Rightarrow 2(6+2y-3z)(-3)+2(z-1)=0$, $\quad -12y+20z=38$

$10y-12z=-22$ $\qquad$ solve for y, z.

true for all x $\qquad -12y+20z=38$

Satisfying

$\boxed{x-2y+3z=6}$

$$6\,(5y-6z=-11) \rightarrow \quad 30y-36z=-66$$
$$5\,(-6y+10z=19) \qquad + \; -30y+50z=95$$

$$14z=29$$
$$z=\frac{29}{14}$$

then $y=$?

$$5y-6\left(\frac{29}{14}\right)=-11$$
$$5y=-11+\frac{6(29)}{14}$$
$$y=\left(-11+\frac{6(29)}{14}\right)\frac{1}{5}$$

now use y,z to find x using
$x-2y+3z=6$.

(45) find 3 positive #'s whose sum is 100.
and whose product is a max.

$$x+y+z=100 \qquad 0<x<100$$
$$0<y<100$$
$$z=100-x-y \qquad 0<z<100$$

$$f(x,y,z)=xyz$$

$$g(x,y)=xy(100-x-y)=100xy-x^2y-xy^2$$

$$\vec{\nabla} g=\langle 100y-2xy-y^2,\, 100x-x^2-2xy\rangle=\langle 0,0\rangle$$

$100y - 2xy - y^2 = 0 \rightarrow y(100 - 2x - y) = 0$

$100x - x^2 - 2xy = 0 \qquad x(100 - x - 2y) = 0$

$x, y \neq 0.$

we have $\quad 2x + y = 100$

$(-2)(x + 2y) = (100)(-2)$

$$\begin{array}{r} 2x + y = 100 \\ + \; -2x - 4y = -200 \\ \hline -3y = -100 \Rightarrow y = \frac{100}{3} \end{array}$$

$2x + \frac{100}{3} = 100 \rightarrow x = \frac{100}{3}$

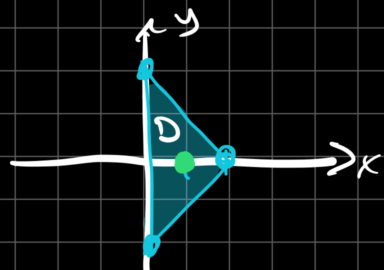
now we $\quad x + y + z = 100 \rightarrow z = \frac{100}{3}.$

Max product: $\left(\frac{100}{3}\right)^3$

(31) $f(x,y) = x^2 + y^2 - 2x \qquad$ D be closed triangle with vertices:

$(2,0), (0,2), (0,-2)$

find absolute max & min.



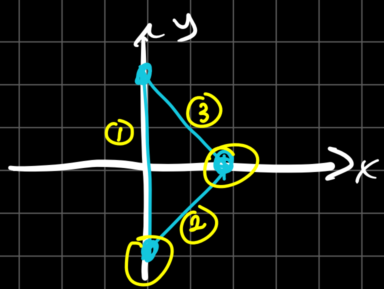
① find c.p. $\quad \vec{\nabla} f = \vec{0} \Rightarrow \langle 2x - 2, 2y \rangle = \langle 0, 0 \rangle$

$x=1 \quad y=0$

$f(1,0) = -1.$

② check boundary $\quad f(x,y) = x^2+y^2-2x$

$\left(\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2 - 3 = \frac{9}{4} + \frac{1}{4} - 3 = \frac{10}{4} - 3$



on ① $x=0 \quad -2 \le y \le 2$

$f(x,y)$ on ① $= y^2$

calc I optimiz. problem.

min $= 0$ $\quad$ max $= 4$

on ② $\quad y = x-2 \quad , 0 \le x \le 2$

$f(x,y)$ becomes $g(x) = x^2 + (x-2)^2 - 2x$

$= x^2 + x^2 - 4x + 4 - 2x$

$= 2x^2 - 6x + 4$

find max & min on ②

$g'(x) = 4x - 6 = 0 \Rightarrow x = \frac{3}{2}$

$y = -\frac{1}{2}$

$f\left(\frac{3}{2}, -\frac{1}{2}\right) = -\frac{1}{2}.$

check endpoints: $f(0,-2) = 4$

$f(2,0) = 0$

on ③ $\quad y = 2-x \quad , 0 \le x \le 2$

$f(x,y)$ becomes: $x^2 + (2-x)^2 - 2x$

$= x^2 + 4 - 4x + x^2 - 2x$

$= 2x^2 - 6x + 4$

$g(x) = 2x^2 - 6x + 4$, $g'(x) = 0$

$4x - 6 = 0$

$x = \frac{3}{2}$, $y = \frac{1}{2}$

$f(\frac{3}{2}, \frac{1}{2}) = -\frac{1}{2}$

list everything

$f(1,0) = -1$ — min $= -1$ at $(1,0)$.

$f(0,2) = 4$

$f(2,0) = 0$

$f(0,-2) = 4$ — max $= 4$ at these two points

$f(\frac{3}{2}, -\frac{1}{2}) = -\frac{1}{2}$

$f(\frac{3}{2}, \frac{1}{2}) = -\frac{1}{2}$