Today: $\boxed{\text{Ken}}$

$\boxed{5.3}$ Mathematical Induction II

$\boxed{5.4}$ Strong Mathematical Induction

Last time:

$\boxed{5.2}$ Mathematical Induction I

$\boxed{5.3}$ Mathematical Induction II

#3 Stamps are sold in packages containing either 5 stamps or 8 stamps.

ⓐ Show that a person can obtain 5, 8, 10, 13, 15, 16, 20, 21, 24, or 25 stamps by buying a collection of 5-stamp packages and 8-stamp packages.

ⓑ Use mathematical induction to show that any quantity of at least 28 stamps can be obtained by buying a collection of 5-stamp packages and 8-stamp packages.

ⓐ Show that a person can obtain $5, 8, 10, 13, 15, 16, 20, 21, 24$, or 25 stamps by buying a collection of 5-stamp packages and 8-stamp packages.

$5 = 5(1)$
$8 = 8(1)$
$10 = 5(2)$
$13 = 5(1) + 8(1)$

$15 = 5(3)$
$16 = 8(2)$
$20 = 5(4)$
$21 = 5(1) + 8(2)$

$24 = 8(3)$
$25 = 5(5)$

ⓑ Let $P(n)$ be the statement

$n \geq 28$ stamps can be obtained using 5- or 8-stamp packages.

Suppose $P(n)$ for $n \geq 28$.

Goal: Show $P(n+1)$.

$\boxed{\geq 28}$

$3(5) + 0(8) = 15 < 28$
$2(8) = 16$

$3(5) + 1(8) < 28$
$2(8) = 16$

$3(5) + 2(8) = 31$
$2(8) = 16$

$3(8) = 24$ $\quad$ $4(8)$ and larger?

$5(5) = 25$

If there are no 8-stamp packages, one 8-stamp package, or two 8-stamp packages, then there are at least three 5-stamp packages and replace those three packages with two 8-stamp packages to obtain +1 more stamp.

If there are three 8-stamp packages, then we replace those three packages, with five 5-stamp packages to obtain exactly one more stamp.

Otherwise, if there are four or more 8 stamp packages, then via induction hypothesis, $n \geq 28$ stamps can be obtained.

So $P(n+1)$ is true.

Proposition 5.3.1

For every integer $n \geq 8$, n¢ can be obtained using 3¢ and 5¢ coins.

Let $P(n)$ be the statement n¢ can be obtained using 3¢ and 5¢ coins.

1. Base case: $P(8)$

   8¢ can be obtained with one 3¢ coin & one 5¢ coin.

2. Suppose $P(n)$ is true for $n \geq 8$.

   So we can obtain n¢ with 3¢ and 5¢ coins.

   Either we have a 5¢ coin or not.

   Suppose we have a 5¢ coin. How do we obtain exactly one more cent?

Answer: replace that 5¢ coin with two 3¢ coins to 6¢, exactly one more cent.

Suppose we have no 5¢ coins. Since we have $n \geq 8$ cents, we must have at least three 3¢ coins. How do we obtain exactly one more cent? Answer: Replace three 3¢ coins with two 5¢ coins.

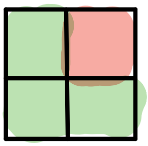
<u>Theorem 5.3.4</u>

Covering a Board with Trominoes

For any $n \in \mathbb{Z}^+$, if one square is removed from a $2^n \times 2^n$ checkerboard, the remaining squares can be completely covered by L-shaped trominoes.

Let $n \in \mathbb{Z}^+$, i.e. $n \geq 1$.

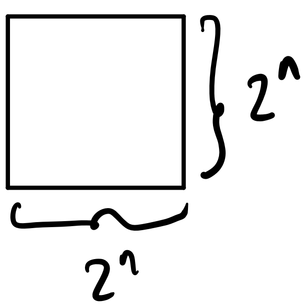
Let $P(n)$ be the statement: a $2^n \times 2^n$ checkerboard with one square removed can be covered by L-shaped trominoes.

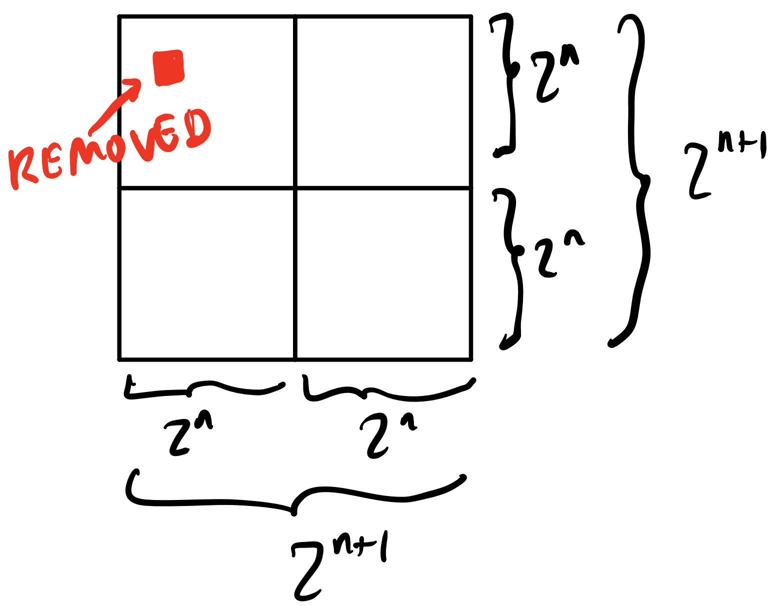


① Base case: $n=1$, 2×2 checkerboard with 1 square removed and L-shape tromino covering.

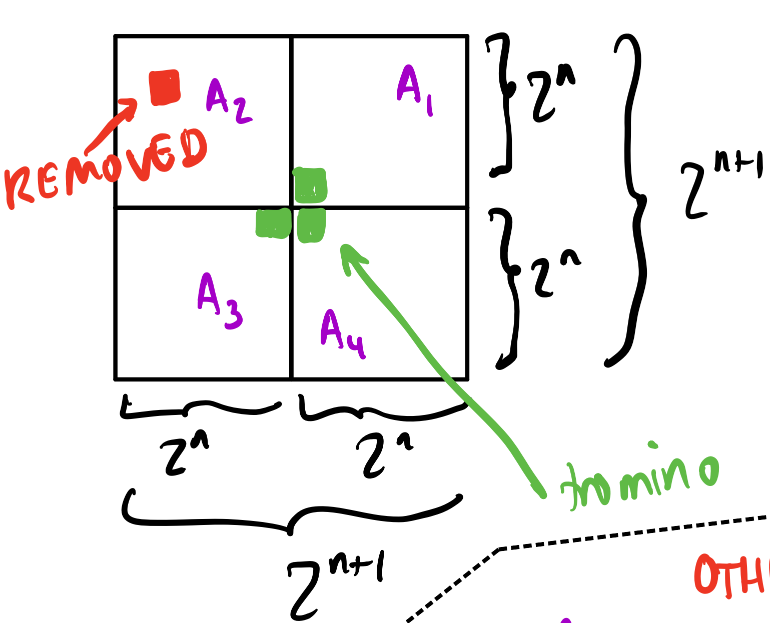
② Suppose $P(n)$ for $n \geq 1$.

Goal: Show $2^{n+1} \times 2^{n+1}$ checkerboard can be covered with the trominoes while 1 square is removed.





Consider the $2^{n+1} \times 2^{n+1}$ checkerboard such that the size of the $2^{n+1} \times 2^{n+1}$ board is quadruple the $2^n \times 2^n$ board, as sketched above. Remove exactly one square from the $2^{n+1} \times 2^{n+1}$ board.



Board A_2 is $2^n \times 2^n$ with one square removed, so covered by trominoes via induction hypothesis. The remaining boards A_1, A_3, and A_4 form a tromino where a square from each might OTHERWISE be removed, also A_1, A_3, A_4 covered by induction hypothesis.

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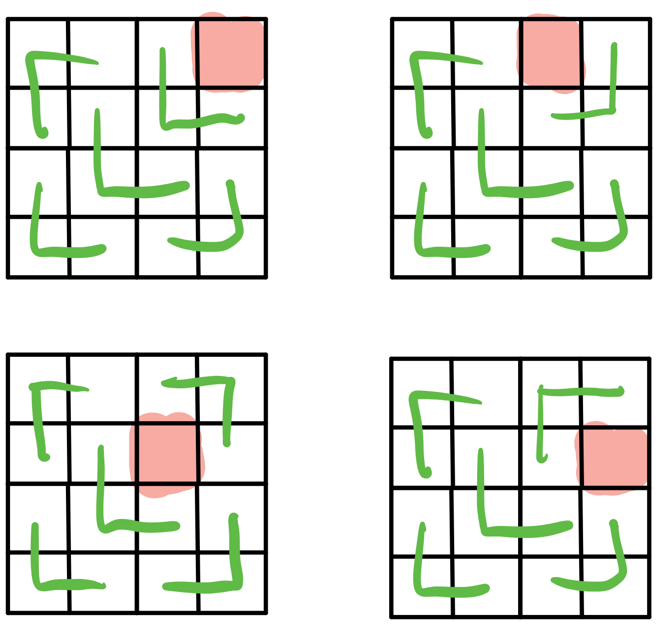
e.g.



$2^2 \times 2^2$

4×4

5.4 Strong Mathematical Induction and the Well-Ordering Principle for the Integers

Principle of Strong Mathematical Induction

Let $P(n)$ be a property that is defined for $n \in \mathbb{Z}$ and let $a, b \in \mathbb{Z}$ be fixed integers such that $a \leq b$. Suppose the two statements are true:

1. $P(a), P(a+1), \ldots, P(b)$ are all true

   (base case, basis step)

2. For every integer $k \geq b$, if $P(i)$ is true for all $i \in \{a, \ldots, k\}$ then $P(k+1)$ is true

   (induction hypothesis, inductive step)

Then the statement

for every integer $n \geq a$, $P(n)$

is true.

Another way to state the inductive hypothesis is to say that $P(a), P(a+1), \ldots, P(k)$ are all true.

Proving a Property of a Sequence with Strong Induction

#2

Suppose $\{b_n\}_{n=1}^{\infty}$ is a sequence such that $b_1 = 4$, $b_2 = 12$, and $b_k = b_{k-1} + b_{k-2}$ for all $k \geq 3$.

Prove that b_n is divisible by 4 for all $n \geq 1$.

$P(n)$ is property that $4 \mid b_n$ for $n \geq 1$.

① Base case

$b_1 = 4$, $\quad 4 \mid b_1$ because $4(1) = 4$

$b_2 = 12$, $\quad 4 \mid b_2$ because $4(3) = 12$

② to be continued next time