Today: $\boxed{\text{Ken}}$

$\boxed{4.7}$ Indirect Argument: Contradiction & Contraposition

$\boxed{4.5}$ Direct Proof and Counterexample V

Last time:

$\boxed{4.5}$ Direct Proof and Counterexample V

$\boxed{4.4}$ Direct Proof and Counterexample IV

Theorem 4.5.2

The Parity Property

The parity of an integer refers to whether the integer is even or odd.

Any two consecutive integers have opposite parity.

Let $m, n \in \mathbb{Z}$ such that $n = m + 1$.

Suppose $m \in 2\mathbb{Z}$. Then there exists $k \in \mathbb{Z}$ such that $m = 2k$. So $n = 2k+1$ and $n \in \mathbb{Z} - 2\mathbb{Z}$. Hence m, n have opposite parity.

Suppose $m \in \mathbb{Z} - 2\mathbb{Z}$. Then there exists $k \in \mathbb{Z}$ such that $m = 2k+1$. So

$n = m+1 = 2k+1+1 = 2k+2 = 2(k+1)$,

$k+1 \in \mathbb{Z}$, and $n \in 2\mathbb{Z}$. Hence m, n have opposite parity.

Therefore m, n have opposite parity. $\square$

Theorem 4.5.3

The square of any odd integer has the form $8m+1$ for some integer m.

Proof:

Let $m \in \mathbb{Z} - 2\mathbb{Z}$. For any integer n, either $n = 4q$, $n = 4q+1$, $n = 4q+2$, or $n = 4q+3$, via quotient remainder theorem. Since m is odd, there exists $q \in \mathbb{Z}$ such that either $m = 4q+1$ or $m = 4q+3$. $p_1 \vee p_2$

Suppose $m = 4q+1$. Then

$$m^2 = (4q+1)^2 \quad \text{by substitution}$$
$$= 16q^2 + 8q + 1 \quad \text{by algebra}$$
$$= 8(2q^2 + q) + 1$$

and define $n_1 := 2q^2 + q \in \mathbb{Z}$ such that $m^2 = 8n_1 + 1$. $p_1 \to r$

Suppose $m = 4q+3$. Then

$$m^2 = (4q+3)^2$$
$$= 16q^2 + 24q + 9 = 16q^2 + 24q + 8 + 1$$
$$= 8(2q^2 + 3q + 1) + 1$$
and define $n_2 := 2q^2 + 3q + 1 \in \mathbb{Z}$ such that

$m^2 = 8n_2 + 1$. $P_2 \to r$

Therefore $m^2 = 8n+1$ for some $n \in \mathbb{Z}$. $\square$

$\therefore r$

Quotient Remainder theorem

Given $n, d \in \mathbb{Z}$, $d > 0$, $\exists! q, r \in \mathbb{Z}$

such that $n = dq + r$ where $0 \leq r < d$.

Definition

For any real number x,

$$|x| := \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases}$$

Lemma 4.5.4

$-|-2| \leq -2 \leq |-2|$

For any real number x, $-|x| \leq x \leq |x|$.

Let $x \in \mathbb{R}$.

Either $x > 0$, $x = 0$, or $x < 0$. $p_1 \vee p_2 \vee p_3$

Suppose $x < 0$. Then $|x| = -x$ and $-x > 0$. So $-|x| = x$ and, via generalization $-|x| \leq x$. Since $|x| = -x > 0$ and $0 > x$, $|x| > x$ and, again via generalization, $|x| \geq x$.

$a \leq b \equiv a < b \vee a = b$

Via conjunction, $-|x| \leq x \leq |x|$. $p_3 \to r$

Suppose $x = 0$. Then

$-|x| = -|0| = -0 = 0 = x$ so $-|x| \leq x$

via generalization. Likewise

$x = 0 = |0| = |x|$ so $x \leq |x|$ via generalization. Via conjunction, $-|x| \leq x \leq |x|$.

$p_2 \to r$

Suppose $x > 0$. By def. of abs. val. function, $|x| = x$ so $|x| \geq x$ via generalization. Also $|x| = x > 0$, so $-|x| = -x < 0$ and $0 < x$ by assumption. Thus $-|x| < x$ and, via generalization, $-|x| \leq x$. Finally, via conjunction, $-|x| \leq x \leq |x|$. $p_1 \to r$

In conclusion, $-|x| \leq x \leq |x|$. $\therefore r$ □

Lemma 4.5.5

For every real number x, $|-x| = |x|$.

Let $x \in \mathbb{R}$.

$$|-x| = \begin{cases} -(-x) & -x < 0 \\ 0 & -x = 0 \\ -x & -x > 0 \end{cases}$$

$$= \begin{cases} x & x > 0 \\ 0 & x = 0 \\ -x & x < 0 \end{cases}$$

↑ each inequality multiply both by -1 so the sign flips and $x = +0 = -0 = 0$.

$$= \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

$= |x|$. $\square$

Theorem 4.5.6 triangle inequality

For all real numbers x and y, $|x+y| \leq |x| + |y|$.

Let $x, y \in \mathbb{R}$. Via trichotomy, either $x+y \geq 0$ or $x+y < 0$. $P_1 \vee P_2$

Suppose $x+y \geq 0$. P_1

Via Lemma 4.5.4, $x \leq |x|$ and $y \leq |y|$.

Since $x+y \geq 0$, $|x+y| = x+y$ by

definition of abs. value, and

$|x+y| = x+y \leq |x|+|y|$ via the previous

statement. r

$p_1 \to r$

p_2

Suppose $x+y<0$. Then by def. of

abs. val., $|x+y| = -(x+y) = (-x)+(-y)$

by lemma 4.5.4 $-x \leq |-x|$ and

$-y \leq |-y|$. So by lemma 4.5.5

$|-y| = |y|$ and $|-x| = |x|$. Thus

r

$|x+y| = (-x)+(-y) \leq |-x|+|-y| = |x|+|y|$.

$p_2 \to r$

Therefore, $|x+y| \leq |x|+|y|$. $\therefore r$ $\square$

$q \vee \neg q$

4.7 Indirect Argument: Contradiction and Contraposition

Method of Proof by Contradiction

1. Suppose the statement to be false — i.e. suppose that its negation is true.
2. Show that this supposition leads logically to a contradiction.
3. Conclude that the statement to be proved is true.

Theorem 4.7.1

There is no greatest integer.

Theorem 4.7.2

There is no integer that is both even and odd.

$\forall x \in \mathbb{Z}\,(\neg(x \in 2\mathbb{Z} \wedge x \in \mathbb{Z} - 2\mathbb{Z}))$

Suppose there exists $n \in \mathbb{Z}$ such that n is even and n is odd.
By def of even and odd,

$n = 2k$ for some $k \in \mathbb{Z}$ and
$n = 2\ell + 1$ for some $\ell \in \mathbb{Z}$. So

$2k = n = 2\ell + 1$ and

$$2k = 2\ell + 1$$
$$2k - 2\ell = 1$$
$$2(k - \ell) = 1$$

where $t := k - \ell \in \mathbb{Z}$ so $2 \mid 1$,
a contradiction.
Therefore there is no integer that
is both even and odd. $\square$