Today: $\boxed{\text{Ken}}$

$\boxed{3.1}$ Predicates & Quantifiers I

$\boxed{3.2}$ Predicates & Quantifiers II

Last time:

$\boxed{3.1}$ Predicates & Quantifiers I

Universal Conditional Statements

A universal conditional statement takes form

$$\forall x \, (P(x) \rightarrow Q(x)).$$

e.g. Writing Universal Conditional Statements Informally

ⓐ $\forall x \in \mathbb{R} \, (|x| = 0 \rightarrow x = 0)$

ⓑ $\forall z \in \mathbb{C} \, (z = -\bar{z} \rightarrow z \notin \mathbb{R})$ false if $z = 0$

true for $\mathbb{C} - \{0\}$

ⓐ For any real number x, if the absolute value of x is zero, then x equals zero.

ⓑ Let $z \in \mathbb{C} - \{0\}$. Suppose $z = -\bar{z}$. Since $z \in \mathbb{C}$, there exist $a, b \in \mathbb{R}$ such that $z = a + bi$. Also $-\bar{z} = -(a - bi)$

so $a + bi = -a + bi$

$$2a + 0 = 0$$
$$a = 0.$$

Thus $z = 0 + bi = bi \notin \mathbb{R}$.

(a sample proof)

e.g. Writing Universal Conditional Statements Formally

(a) All bicycles have two wheels.

(b) If a complex number z equals its conjugate, then z is a real number.

(a) $\forall x$, if x is a bicycle, then x has two wheels.

(b) $\forall z \in \mathbb{C} \, (z = \bar{z} \longrightarrow z \in \mathbb{R})$

2 z Z u n 4

Equivalent Forms of Universal & Existential Statements

If $P(x)$ and $Q(x)$ are predicates and B a set such that $A := \{x \in B \mid P(x)\}$, then $A \subset B$ and we can rewrite

$$\forall x \in B\,(P(x) \to Q(x))$$

as

$$\forall x \in A\,(Q(x)).$$

e.g. ⓐ All integers are rational numbers.

ⓑ All American bison are quadrupeds.

ⓐ $\forall x\,(x \in \mathbb{Z} \to x \in \mathbb{Q})$

$\forall x \in \mathbb{Z}\,(x \in \mathbb{Q})$ $\mathbb{Q}$ \mathbb{Q}

ⓑ $\forall x$ (x is an American bison $\longrightarrow$ x is a quadruped)

$\forall$ American bison x (x is a quadruped)

e.g. There is a rabbit that is a New England cottontail and enjoys grapes.

Predicates:

Cottontail(x) : x is a New England cottontail

Grapes(x) : x enjoys grapes

(a) $\exists$ rabbit x (Cottontail(x) $\wedge$ Grapes(x))

(b) $\exists$ a New England cottontail x such that Grapes(x)

<u>Notation:</u>

Let $P(x)$ and $Q(x)$ be predicates and suppose the common domain of x is D.

- $P(x) \Rightarrow Q(x)$ means $\forall x\,(P(x) \rightarrow Q(x))$
- $P(x) \Leftrightarrow Q(x)$ means $\forall x\,(P(x) \leftrightarrow Q(x))$

e.g. let

$P(x)$ be "x is prime"

$Q(x)$ be "x is rational"

$R(x)$ be "x is irrational"

$S(x,y)$ be "x is the square root of y"

Suppose the domain of all variables is $\mathbb{R}^+$.

Use $\Rightarrow$ and $\Leftrightarrow$ to indicate some true relationships.

e.g. (a) x is irrational if and only if x is not rational.

$$R(x) \Leftrightarrow \neg Q(x)$$

(b) if x is prime, then x is rational.

$$P(x) \Rightarrow Q(x)$$

(c) if x is prime and y is the square root of x, then y is irrational.

$$P(x) \land S(y,x) \Rightarrow R(y)$$

$\boxed{3.2}$ Predicates & Quantifiers II

Theorem 3.2.1

Given set A and predicate $P(x)$

$\neg \forall x \in A (P(x)) \equiv \exists x \in A (\neg P(x))$.

note: The negation of a universal statement ("all are") is logically equivalent to an existential statement ("some are not" or "there is at least one that is not").

Theorem 3.2.2

Given set A and predicate $P(x)$

$\neg \exists x \in A (P(x)) \equiv \forall x \in A (\neg P(x))$

note: the negation of an existential statement ("some are") is logically equivalent to a universal statement ("none are" or "all are not").

Negation of a Universal Conditional Statement

$\neg \forall x (P(x) \rightarrow Q(x)) \equiv \exists x (P(x) \wedge \neg Q(x))$

$$
\begin{aligned}
\neg \forall x (P(x) \rightarrow Q(x)) &\equiv \neg \forall x (\neg P(x) \vee Q(x)) \\
&\equiv \exists x (\neg (\neg P(x) \vee Q(x))) \\
&\equiv \exists x (\neg\neg P(x) \wedge \neg Q(x)) \\
&\equiv \exists x (P(x) \wedge \neg Q(x))
\end{aligned}
$$

e.g. Negate the following quantified statments.

① There exists a real number x such that $\sin(x) < -1$.

$\exists x \in \mathbb{R}\,(\sin(x) < -1)$ false

$\neg \exists x \in \mathbb{R}\,(\sin(x) < -1) \equiv \forall x \in \mathbb{R}\,(\sin(x) \geq -1)$ true

② All real numbers are transcendental. false

There exists a real number that is not transcendental.

e.g. $x^2 - 2 = 0$

③ All rabbits are New England cottontails. false

There exists a rabbit that is not a New England cottontail.

② $\forall x \in \mathbb{R}\,(x \text{ is transcendental})$

$\neg \forall x \in \mathbb{R}\,(x \text{ is transcendental}) \equiv \exists x \in \mathbb{R}\,(x \text{ is not transcendental})$

$\equiv \exists x \in \mathbb{R}\,(x \text{ is algebraic}).$

③ ⓘ There is a rabbit that is not a New England cottontail.
ⓘⓘ Some rabbit is not a New England cottontail.

The Relation among $\forall, \exists, \wedge, \vee$

If $P(x)$ is a predicate and the domain of x is $A = \{x_1, \ldots, x_n\}$ then

$$\forall x \in A\,(P(x)) \equiv P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n)$$

and

$$\exists x \in A\,(P(x)) \equiv P(x_1) \vee P(x_2) \vee \cdots \vee P(x_n).$$

e.g. $A = \{0, 1\}$

ⓐ Let $P(x)$ be $x \cdot x = x$.

$\forall x \in A\,(P(x)) \equiv P(0) \wedge P(1)$ true

$P(0)$: $0 \cdot 0 = 0$ true

$P(1)$: $1 \cdot 1 = 1$ true

ⓑ Let $P(x)$ be $x + x = x$.

$\exists x \in A\,(P(x)) \equiv P(0) \vee P(1)$ true

$P(0)$: $0 + 0 = 0$ true

$P(1)$: $1 + 1 = 1$ false

If $P(x)$ above is true for some x together with a binary operation, then we say x is *idempotent*.

Vacuous Truth of Universal Statements

Let $P(x)$, $Q(x)$ be predicates with domain A.
A statement of the form

$$\forall x \in A\, (P(x) \to Q(x))$$

is called vacuously true or true by default if and only if $P(x)$ is false for all x in A.

e.g. (a) (integers)

Let $P(n)$ be "$n+561<0$" and $Q(n)$ be "n is prime", both with domain $\mathbb{Z}^+$. For any positive integer n, if $n+561<0$ then n is prime.

$$\forall n \in \mathbb{Z}^+ \,(n+561<0 \to n \text{ is prime})$$

(b) (metric topology)

All points in the empty set $\emptyset$ are interior points.