Today: $\boxed{\text{Ken}}$

$\boxed{6.1}$ Set Theory

Last time:

$\boxed{5.4}$ Strong Mathematical Induction

$\boxed{6.1}$ Set Theory: Definitions and the Element Method of Proof

$A \subset B \iff \forall x\,(x \in A \rightarrow x \in B)$ $\quad A \subseteq B$

$A \not\subset B \iff \exists x\,(x \in A \wedge x \notin B)$ $\quad A \not\subseteq B$

$A \subsetneq B \iff$ ① $A \subset B$ $\quad A \subset B$

\subsetneq

② $\exists x \in B\,(x \notin A)$

Element Argument:
Basic Method for Proving that One Set is a Subset of Another

Given sets A, B, to prove $A \subset B$

① Suppose $x \in A$ is some particular but arbitrarily chosen element.

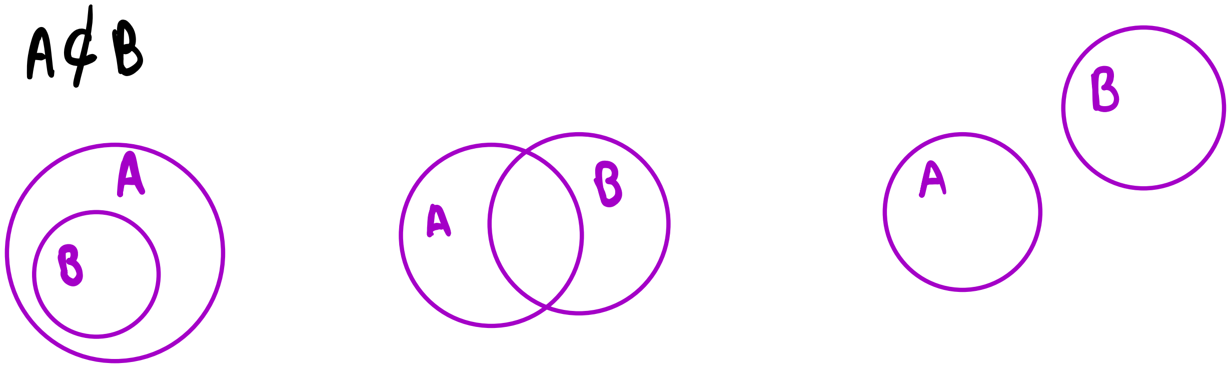
② Show that $x \in B$.

Definition Set Equality

Given sets A, B,

$$A = B \iff A \subset B \text{ and } B \subset A.$$

$A \not\subset B$



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singleton $\{a\}$

#1 f $\quad A = \{x \in \mathbb{R} \mid \cos(x) \in \mathbb{Z}\}$

$\quad = \{x \in \mathbb{R} \mid x = \frac{\pi k}{2}, k \in \mathbb{Z}\}$

$B = \{x \in \mathbb{R} \mid \sin(x) \in \mathbb{Z}\}$

$A \subset B, B \subset A, A \subsetneq B, B \subsetneq A$?

$A \subset B$? yes

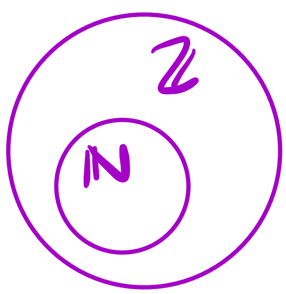
$B \subset A$? yes

$\Big\}\ A = B$

$A \subsetneq B$? No

$B \subsetneq A$? No

$\mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R} \subsetneq \mathbb{C}$



Definition

Let A, B be subsets of a universal set $\mathcal{U}$.

1. The union of A and B,
$$A \cup B := \{x \in \mathcal{U} \mid x \in A \text{ or } x \in B\}.$$

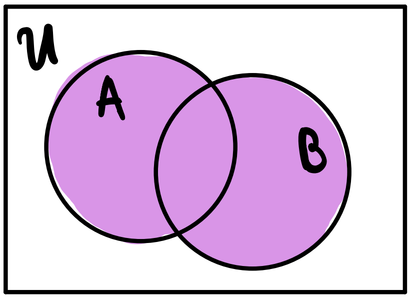
2. The intersection of A and B,
$$A \cap B := \{x \in \mathcal{U} \mid x \in A \text{ and } x \in B\}.$$

3. The (set) difference or relative complement of A minus B
$$A - B := \{x \in \mathcal{U} \mid x \in A \text{ and } x \notin B\}.$$

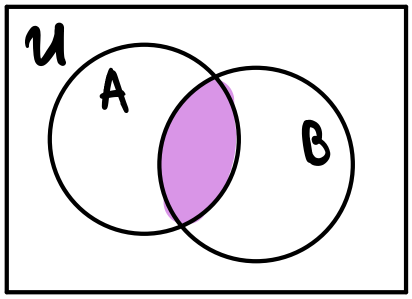
4. The complement of A,
$$A^c := \{x \in \mathcal{U} \mid x \notin A\}.$$

Venn Diagrams for $A \subset B$, $A \not\subset B$, $A = B$, $A \cup B$, $A \cap B$, $A - B$, A^c

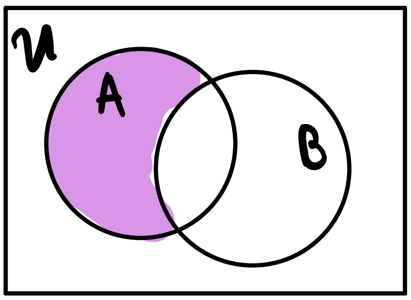
$A \cup B$



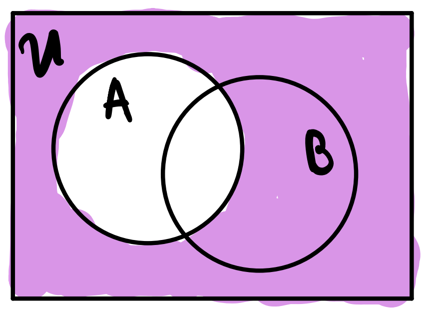
$A \cap B$



$A - B = A \setminus B \neq A/B$



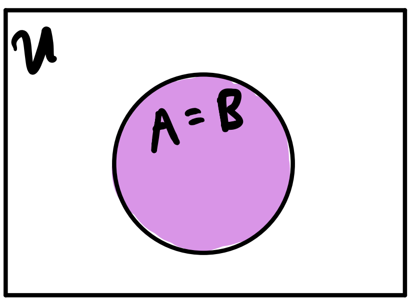
A^c



$\mathbb{Q} \subset \mathbb{R}$

$\mathbb{Q}^c = \mathbb{R} - \mathbb{Q}$ set of all irrational numbers

$A = B$



Basic Method for Proving that Sets are Equal

Let sets X and Y be given. To prove that $X = Y$:

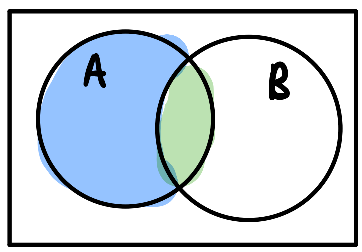
① Prove that $X \subset Y$.

② Prove that $Y \subset X$.

example

Let A, B be sets. Prove that

$A = (A-B) \cup (A \cap B)$



Proof: (element method)

① Let $x \in A$. Either $x \in B$ or $x \notin B$.

Suppose $x \in B$. Then $x \in A$ and $x \in B$.

So $x \in A \cap B$ by def of intersection.

By generalization $x \in A \cap B$ or $x \in A - B$.

So $x \in (A-B) \cup (A \cap B)$. Thus $A \subset (A-B) \cup (A \cap B)$.

Suppose $x \notin B$. Then $x \in A$ and $x \notin B$.
So $x \in A - B$ by def. of set difference.
$x \in A - B$ or $x \in A \cap B$ by generalization.
Also $x \in (A-B) \cup (A \cap B)$ by def. of union. So $A \subset (A-B) \cup (A \cap B)$.

② Let $x \in (A-B) \cup (A \cap B)$.
By def. of union $x \in A - B$ or $x \in A \cap B$.
Suppose $x \in A - B$. By def. of set difference, $x \in A$ and $x \notin B$. By specialization $x \in A$.
so $(A-B) \cup (A \cap B) \subset A$.

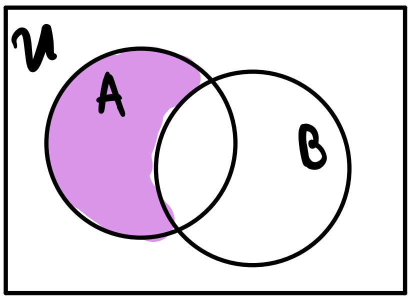
Suppose $x \in A \cap B$. Then by def. of intersection $x \in A$ and $x \in B$. By specialization $x \in A$. so $(A-B) \cup (A \cap B) \subset A$.

③ Since $A \subset (A-B) \cup (A \cap B)$ and $(A-B) \cup (A \cap B) \subset A$, $A = (A-B) \cup (A \cap B)$.

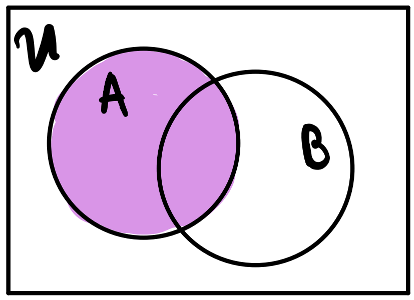
NOT element method:

$$
\begin{aligned}
(A-B)\cup(A\cap B) &= (A\cap B^c)\cup(A\cap B)\\
&= A\cap(B^c\cup B)\\
&= A\cap(B\cup B^c)\\
&= A\cap \mathcal{U}\\
&= A
\end{aligned}
$$

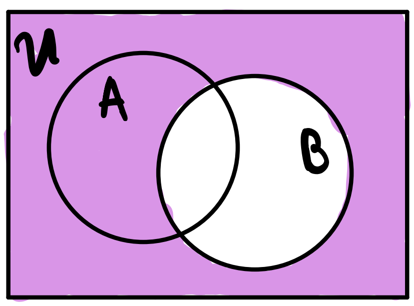
$A-B = A\setminus B \neq A/B$



A



B^c



#10 Let $A = \{1,3,5,7,9\}$, $B = \{3,6,9\}$, and $C = \{2,4,6,8\}$.

$A \cup B = \{1,3,5,6,7,9\}$

$A \cap B = \{3,9\}$

$A \cup C = \{1,2,3,4,5,6,7,8,9\}$

$A \cap C = \{\} = \varnothing$

$A - B = \{1,5,7\}$

$B - A = \{6\}$

$B \cup C = \{2,3,4,6,8,9\}$

$B \cap C = \{6\}$

don't: $A \cup B \cap C$ (ambiguous)

$\{a,a,b,b\}$

$\{(1,a),(2,a),(1,b),(2,b)\}$

$\{\{a\},\{\},a\}$

$A \cap B \subset X$

$(A \cap B) \subset X$

Interval Notation

Given $a, b \in \mathbb{R}$ such that $a \leq b$,

$(a,b) := \{x \in \mathbb{R} : a < x < b\}$ open

$[a,b) := \{x \in \mathbb{R} : a \leq x < b\}$

$(a,b] := \{x \in \mathbb{R} : a < x \leq b\}$

$[a,b] := \{x \in \mathbb{R} : a \leq x \leq b\}$ closed

$(a,\infty) := \{x \in \mathbb{R} : x > a\}$

$[a,\infty) := \{x \in \mathbb{R} : x \geq a\}$

$(-\infty,b) := \{x \in \mathbb{R} : x < b\}$

$(-\infty,b] := \{x \in \mathbb{R} : x \leq b\}$

$\mathbb{R} = (-\infty,\infty)$ open & closed, "clopen"

$\emptyset$ open & closed, "clopen"

$\mathbb{R}^c = \emptyset$; $\emptyset^c = \mathbb{R}$

Definition

Unions and Intersections of an Indexed Collection of Sets

Given $A_1, A_2, A_3, \ldots$ are subsets of a universal set $\mathcal{U}$ and given a positive integer n,

$$\bigcup_{i=1}^{n} A_i = \{x \in \mathcal{U} \mid \exists i \in \{1, \ldots, n\} (x \in A_i)\}$$

$$\bigcup_{i=1}^{\infty} A_i = \{x \in \mathcal{U} \mid \exists i \in \mathbb{Z}^+ (x \in A_i)\}$$

$$\bigcap_{i=1}^{n} A_i = \{x \in \mathcal{U} \mid \forall i \in \{1, \ldots, n\} (x \in A_i)\}$$

$$\bigcap_{i=1}^{\infty} A_i = \{x \in \mathcal{U} \mid \forall i \in \mathbb{Z}^+ (x \in A_i)\}$$

examples

What is the domain of $f(x) = \tan(x)$?

$$\{x \in \mathbb{R} : x \neq \pi/2 + \pi k,\ \forall k \in \mathbb{Z}\}$$

$$= \bigcup_{\forall k \in \mathbb{Z}} \left(\frac{\pi(2k-1)}{2}, \frac{\pi(2k+1)}{2} \right)$$

$$A_1 = \bigcup_{n=1}^{\infty} \left(1 - 1/n^2, 2 - 1/n\right] = (0,2)$$

Definition

The empty set or null set is the set with no elements denoted $\emptyset$ or $\{\}$.