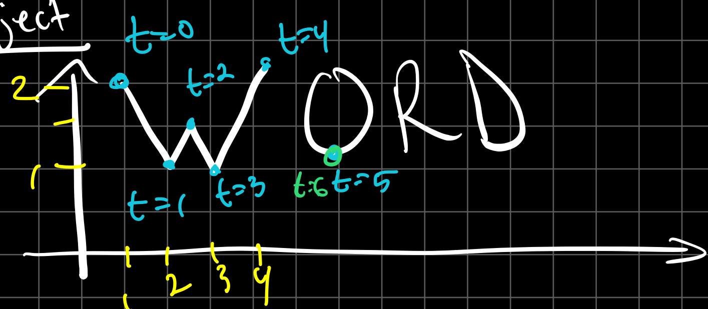
2^{nd} worksheet out.

$$\frac{d}{dx}\int_0^{x} \cos\left(\tfrac{1}{2}\theta^2\right)d\theta \qquad 2^{nd}\ \text{F.T.C.}$$

$$\cos\left(\tfrac{1}{2}x^2\right)$$

3 questions.

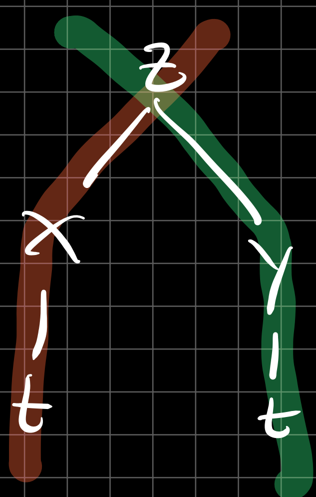
Project



Submit 1^{st} part Friday.

Chain Rule $\quad z=f(x,y) \qquad x(t),\ y(t)$

$$z=f(\underbrace{x(t),y(t)}_{t})$$



find $\dfrac{dz}{dt}$

$$=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}.$$

② $\quad z=\dfrac{x-y}{x+2y} \qquad x=e^{\pi t} \qquad y=e^{-\pi t}$

find $\dfrac{dz}{dt}$ $\quad$ option ① $\quad z=\dfrac{e^{\pi t}-e^{-\pi t}}{e^{\pi t}+2e^{-\pi t}}$

<u>option 2</u> $\qquad \dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt}$

$$z_x = \frac{\partial}{\partial x}\left(\frac{x-y}{x+2y}\right) = \frac{(x+2y)-(x-y)}{(x+2y)^2} = \frac{3y}{(x+2y)^2}$$

$$z_y = \frac{\partial}{\partial y}\left(\frac{x-y}{x+2y}\right) = \frac{(x+2y)(-1)-(x-y)(2)}{(x+2y)^2} = \frac{-3x}{(x+2y)^2}$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= \frac{3y}{(x+2y)^2}\,\pi \underbrace{e^{\pi t}}_{x} + \frac{-3x}{(x+2y)^2}(-\pi)\underbrace{e^{-\pi t}}_{y}$$

$$= \frac{3\pi xy}{(x+2y)^2} + \frac{3\pi xy}{(x+2y)^2} = \boxed{\frac{6\pi xy}{(x+2y)^2}}$$

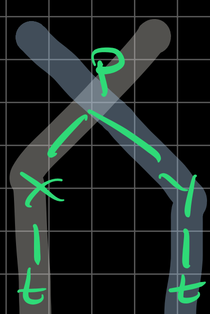
(13) $p(t) = f(x,y)$

$x = g(t)$ $\qquad g(2)=4 \qquad h(2)=5$

$y = h(t)$ $\qquad g'(2)=-3 \qquad h'(2)=6$

$\qquad f_x(4,5)=2 \qquad f_y(4,5)=8$

find $p'(2)$.



$$p'(t) = \frac{\partial p}{\partial x}\frac{dx}{dt} + \frac{\partial p}{\partial y}\frac{dy}{dt}$$

when $t=2$, $\left.\frac{dx}{dt}\right|_{t=2} = g'(2) = -3$ $\quad \left.\frac{dy}{dt}\right|_{t=2} = h'(2) = 6$

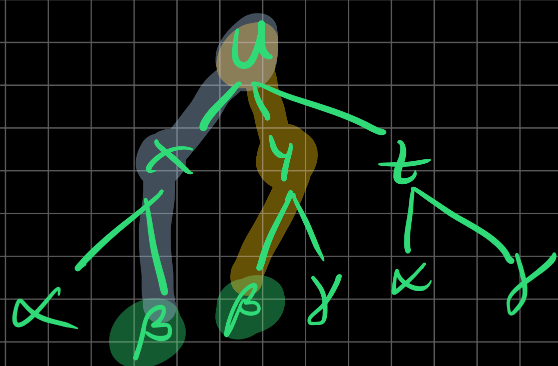
when $t=2$ $\quad x = g(2) = 4$ $\quad y = h(2) = 5$

$P_x(4,5) = 2$ $\quad P_y(4,5) = 8$

$$p'(2) = (2)(-3) + (8)(6) = 48 - 6 = 42$$

(26) $u = xe^{ty}$ $\quad x = \alpha^2\beta$ $\quad y = \beta^2\gamma$ $\quad t = \gamma^2\alpha$

find $\frac{\partial u}{\partial \alpha}$, $\boxed{\frac{\partial u}{\partial \beta}}$



$$\frac{\partial u}{\partial \alpha} = u_x x_\alpha + u_t t_\alpha$$

$$\boxed{= e^{ty}(2\alpha\beta) + xye^{ty}\gamma^2}$$

$$\frac{\partial u}{\partial \beta} = u_x x_\beta + u_y y_\beta = \boxed{e^{ty}(\alpha^2) + xte^{ty}(2\beta\gamma)}$$

(29) $\tan^{-1}(x^2 y) = x + xy^2$ $\quad$ find $\frac{dy}{dx}$

$$\tan^{-1}(x^2y) - x - xy^2 = 0$$

$$\underbrace{F(x,y)}\ = 0$$ use chain Rule. on $F(x,y)$ solve for $\frac{dy}{dx}$.

$$\boxed{\frac{dy}{dx} = -\frac{F_x}{F_y}}$$

$F(x,y) = \tan^{-1}(x^2y) - x - xy^2$

$$F_x = \frac{1}{1+(x^2y)^2}(2xy) - 1 - y^2$$

$$F_y = \frac{1}{1+(x^2y)^2}(x^2) - 2xy$$

$$\frac{dy}{dx} = -\frac{\frac{2xy}{1+(x^2y)^2} - 1 - y^2}{\frac{x^2}{1+(x^2y)^2} - 2xy}$$

33 $e^z = xyz$

find $\frac{\partial z}{\partial x}$

Also

$F(x,y,z) = 0$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

$$e^z - xyz = 0$$

$F_x = -yz \qquad F_z = e^z - xy$

$$\frac{\partial z}{\partial x} = -\frac{-yz}{e^z - xy} = \frac{yz}{}$$

$$e^{z}-xy$$

(*) when is $\frac{\partial z}{\partial x}=0$? $y=0$ or $z=0$

xz plane, xy plane.

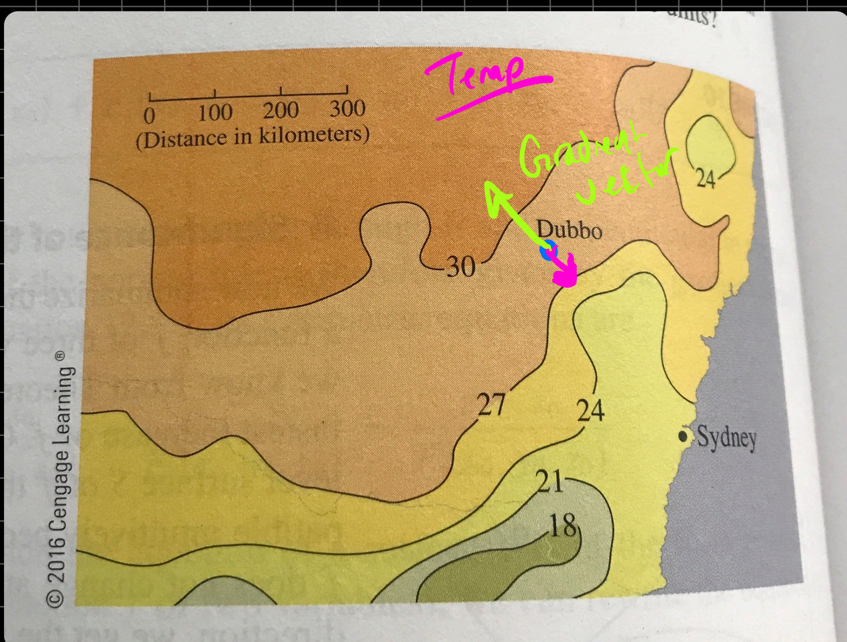
$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{-xz}{e^{z}-xy} = \frac{xz}{e^{z}-xy}$$

(*) when is $\frac{\partial z}{\partial y}=0$? $x=0$, $z=0$

yz plane xy plane.

14.6 Gradient & directional derivative.

(2)



value of directional deriv from Dubbo in direction of sydney

$$\frac{\Delta \text{Temp}}{\text{distance}} = \frac{-1.5\,°C}{50\,km}$$

directional deriv

$$\approx -\frac{1.5}{50}\ °C/km$$

Ex

$$\boxed{f(x,y)=x^2+xy,\quad \nabla f = \langle \underbrace{2x+y}_{f_x}, \underbrace{x}_{f_y}\rangle}$$

(15) $f(x,y,z)=x^2y+y^2z$ direction is $\vec{v}=\langle 2,-1,2\rangle$

$$D_{\vec{u}} f = \nabla f\cdot\vec{u} \qquad \|\vec{u}\|=1$$

$$\vec{\nabla} f = \langle f_x, f_y, f_z \rangle$$

$$\vec{\nabla} f = \langle \underbrace{2xy}_{f_x}, \underbrace{x^2+2yz}_{f_y}, \underbrace{y^2}_{f_z} \rangle$$

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle 2,-1,2\rangle}{\sqrt{2^2+1^2+2^2}} = \langle \tfrac{2}{3}, -\tfrac{1}{3}, \tfrac{2}{3} \rangle$$

$$D_{\vec{u}} f = \vec{\nabla} f \cdot \vec{u} = \langle 2xy, x^2+2yz, y^2\rangle \cdot \langle \tfrac{2}{3}, -\tfrac{1}{3}, \tfrac{2}{3}\rangle$$

$$= \frac{4}{3}xy - \frac{x^2}{3} - \frac{2yz}{3} + \frac{2y^2}{3}$$

Calc $D_u f(1,2,3)$

$$= \frac{4}{3}(2) - \frac{1}{3} - \frac{2(2)(3)}{3} + \frac{2(2)^2}{3}$$

$$= \frac{8}{3} - \frac{1}{3} - 4 + \frac{8}{3} = 5 - 4 = 1.$$

(22) $f(s,t) = te^{st}$ find max rate of change at $(0,2)$ and direction.

$s=0$
$t=2$

$$f_s = t^2 e^{st}$$

$$f_t = \frac{\partial}{\partial t}\left(te^{st}\right) = e^{st} + ste^{st}$$

$$\vec{\nabla} f = \text{grad} f = \langle t^2 e^{st}, e^{st} + ste^{st} \rangle$$

$$\text{grad} f(0,2) = \langle 4, 1 \rangle$$

Max rate $= \|\text{grad} f(0,2)\| = \|\langle 4,1\rangle\| = \sqrt{4^2+1^2} = \sqrt{17}$

(29) $f(x,y) = x^2 + y^2 - 2x - 4y$

find all points where fastest rate of change is $\hat{i} + \hat{j}$.

$$\vec{\nabla} f = \langle 2x-2, 2y-4 \rangle = \langle 1, 1 \rangle$$

$2x - 2 = 1$ and $2y - 4 = 1$

$x = \frac{3}{2}$ $\quad$ $y = \frac{5}{2}$

at $(\frac{3}{2}, \frac{5}{2})$ fastest rate of change is in direction of $\hat{i} + \hat{j}$.

(37) u, v diff'able $u(x,y)$, $v(x,y)$

a, b constants.

find $\nabla(au + bv)$

$f(x,y) = au + bv$

$f_x = \frac{\partial}{\partial x}(au + bv) = \frac{\partial}{\partial x}(au) + \frac{\partial}{\partial x}(bv)$

$= au_x + bv_x$

$f_y = au_y + bv_y$

$$\nabla(\ ) = \langle au_x + bv_x, au_y + bv_y \rangle$$

$$= \langle au_x, au_y \rangle + \langle bv_x, bv_y \rangle$$

$$= a\langle u_x, u_y \rangle + b\langle v_x, v_y \rangle$$

$$= a\nabla u + b\nabla v$$

$$\nabla(uv) = (\nabla u)v + u\nabla v$$

$$\nabla\left(\frac{u}{v}\right) = \frac{v\nabla u - u\nabla v}{v^2}$$

$$\nabla(u^n) = n u^{n-1} \nabla u$$

(39) $D_{\vec{u}} f = \nabla f \cdot \vec{u}$, $\|\vec{u}\| = 1$

$$D^2_{\vec{u}} f(x,y) = D_{\vec{u}}\left[D_{\vec{u}} f(x,y)\right]$$

$f(x,y) = x^3 + 5x^2y + y^3 \qquad \vec{u} = \langle \frac{3}{5}, \frac{4}{5} \rangle$

find $D_{\vec{u}}^2 f(2,1)$.

make sure $\|\vec{u}\| = 1$, $\|\vec{u}\| = \sqrt{(\frac{3}{5})^2 + (\frac{4}{5})^2} = 1$

need $\nabla f = \langle 3x^2 + 10xy, 5x^2 + 3y^2 \rangle$

$$D_{\vec{u}} f = \nabla f \cdot \vec{u} = \langle 3x^2 + 10xy, 5x^2 + 3y^2 \rangle \cdot \langle \frac{3}{5}, \frac{4}{5} \rangle$$

$$= \underbrace{\frac{9}{5}x^2 + 6xy + 4x^2 + \frac{12}{5}y^2}_{\text{Scalar field.}}$$

call it $g(x,y)$

$$D_u^2 f = D_u g(x,y) = \vec{\nabla} g \cdot \vec{u}$$

$$\vec{\nabla} g = \langle \frac{18}{5}x + 6y + 8x, 6x + \frac{24}{5}y \rangle$$

$$D_u^2 f = \langle \frac{58}{5}x + 6y, 6x + \frac{24}{5}y \rangle \cdot \langle \frac{3}{5}, \frac{4}{5} \rangle$$

$$= \frac{58}{5}\left(\frac{3}{5}\right)x + \frac{18}{5}y + \frac{24}{5}x + \frac{24(4)}{25}y$$

$$D_u^2 f(2,1) = \frac{58}{5}\left(\frac{3}{5}\right)(2) + \frac{18}{5} + \frac{24}{5}(2) + \frac{24(4)}{25}$$, Simplify

in general $\vec{u} = \langle a, b \rangle$ $\|\vec{u}\| = 1$

$$D_{\vec{u}}^2 f = f_{xx} a^2 + 2 f_{xy} ab + f_{yy} b^2$$

<u>Next</u>: find max/min of functions.