Today:

$\boxed{6.1}$ Set Theory

$\boxed{6.2}$ Properties of Sets

Last time:

$\boxed{6.1}$ Set Theory

Definition

Unions and Intersections of an Indexed Collection of Sets

Given $A_1, A_2, A_3, \dots$ are subsets of a universal set $\mathcal{U}$ and given a positive integer n,

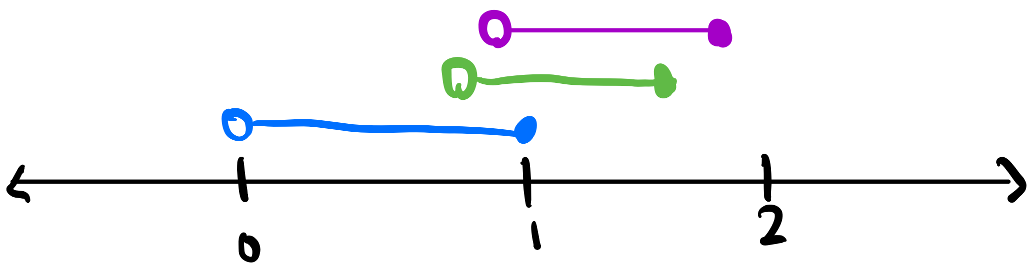
$$\bigcup_{i=1}^{n} A_i = \{x \in \mathcal{U} \mid \exists i \in \{1, \dots, n\} (x \in A_i)\}$$

$$\bigcup_{i=1}^{\infty} A_i = \{x \in \mathcal{U} \mid \exists i \in \mathbb{Z}^+ (x \in A_i)\}$$

$$\bigcap_{i=1}^{n} A_i = \{x \in \mathcal{U} \mid \forall i \in \{1, \dots, n\} (x \in A_i)\}$$

$$\bigcap_{i=1}^{\infty} A_i = \{x \in \mathcal{U} \mid \forall i \in \mathbb{Z}^+ (x \in A_i)\}$$

$$A_1 = \bigcup_{n=1}^{\infty} \left(1 - \tfrac{1}{n^2}, 2 - \tfrac{1}{n}\right] = (0,2)$$



$(1-1, 2-1] \cup (1-\frac{1}{4}, 2-\frac{1}{2}] \cup (1-\frac{1}{9}, 2-\frac{1}{3}] \cup \cdots$

$(0,1] \cup (\frac{3}{4}, \frac{3}{2}] \cup (\frac{8}{9}, \frac{5}{3}] \cup \cdots$

Definition

Bourbaki

The empty set or null set is the set with no elements denoted $\varnothing$ or $\{\}$.

\varnothing

example

Let S be any set.

e.g.

① Prove that $\varnothing \subset S$.

$\{1\} \subset \{1,2\}$

$\forall x (x \in \varnothing \rightarrow x \in S)$

$\{1\} \not\subset \{1,2\}$

vacuously true

$\{1\} \in \{\{1\}, 2\}$

② Is it always true that $\varnothing \in S$?

False in general, e.g. $S = \{2\}$, $\varnothing \notin S$

③ Is it always true that $\emptyset \notin S$?

False, e.g. $S = \{\emptyset\}$, $\emptyset \in S$

④ Prove that if $S \subset \emptyset$ then $S = \emptyset$.

Suppose $S \subset \emptyset$. By ①, $\emptyset \subset S$, so $S \subset \emptyset$ and $\emptyset \subset S$. Thus $S = \emptyset$.

e.g. $A = \{1, 2\}$

$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$

$N(A) = 2 = |A| = \#(A) = \mathrm{card}(A)$

$N(\mathcal{P}(A)) = 2^{N(A)}$

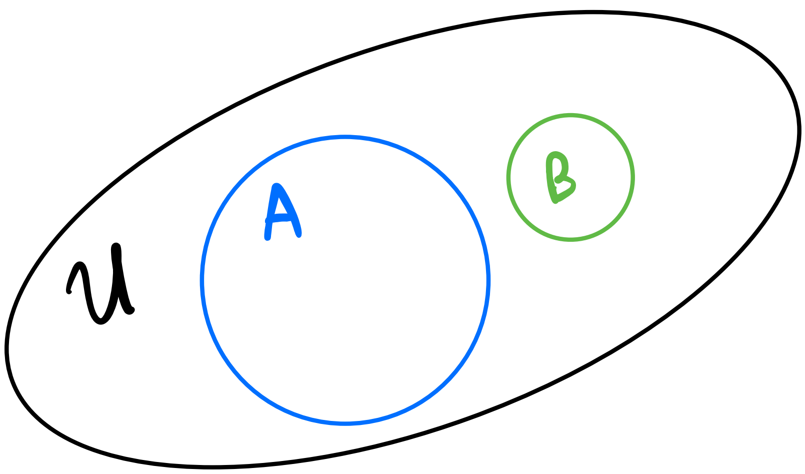
$B = \{\{1\}\} = \{B'\}$ let $B' = \{1\}$

$\mathcal{P}(B) = \{\emptyset, \{\{1\}\}\}$

Definition

Two sets are disjoint if and only if the two sets have no elements in common.

A and B are disjoint $\iff A \cap B = \varnothing$

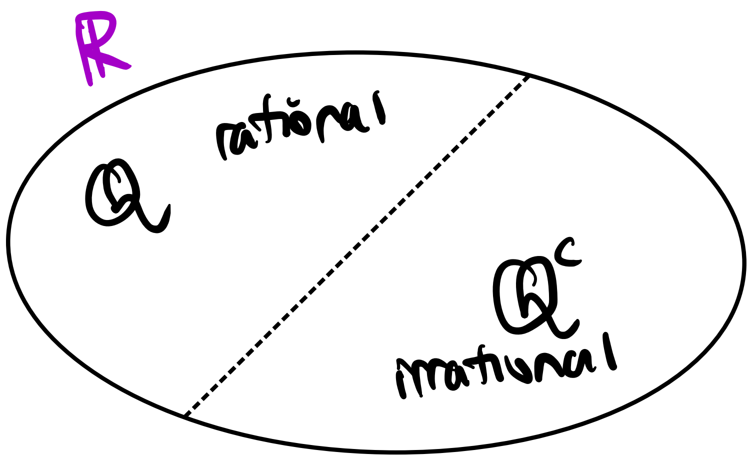


$A \cap B = \varnothing$

Definition

Sets $A_1, A_2, A_3, \ldots$ are **mutually disjoint**, **pairwise disjoint**, or **nonoverlapping** if and only if any two sets A_i and A_j such that $i \neq j$ have no elements in common.

$$i \neq j \Longrightarrow A_i \cap A_j = \emptyset$$



example

Let $A_i = (i-1, i]$ for any $i \in \mathbb{Z}^+$.

If $i, j \in \mathbb{Z}$ such that $i \neq j$, what is $A_i \cap A_j$?

$i=2 \quad A_2 = (2-1, 2] = (1,2]$

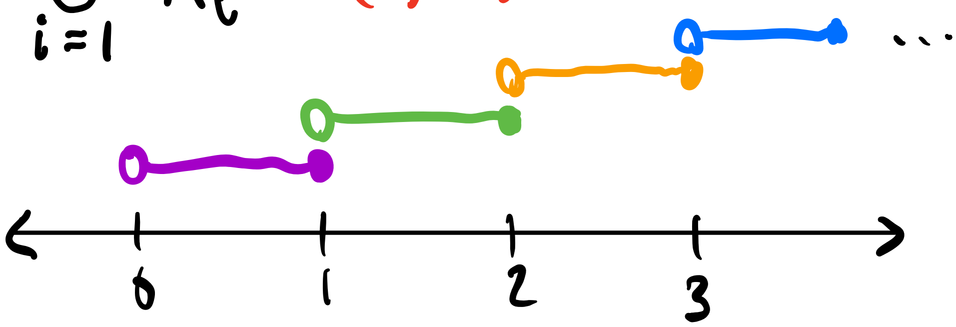
$j=12 \quad A_{12} = (11, 12]$

$(0,1] \cup (1,2] \cup (2,3] \cup \cdots$

$i \neq j \Rightarrow A_i \cap A_j = \emptyset$

How about $\bigcup_{i=1}^{\infty} A_i$?

$\bigcup_{i=1}^{\infty} A_i = (0, \infty)$



Definition

A finite or infinite collection of nonempty sets $\{A_1, A_2, A_3, \ldots\}$ is a partition of a set A if and only if

1. A is the union of all A_i
2. the sets $A_1, A_2, A_3, \ldots$ are mutually disjoint.

example 6.1.11

Let $n \in \mathbb{Z}^+$ such that, for any $i \in \{0, 1, \ldots, n-1\}$,

$\mathbb{Z}_i := \{m \in \mathbb{Z} : \forall k \in \mathbb{Z} \, (m = i + nk)\}$.

$n = 4$, $i \in \{0, 1, 2, 3\}$

$$\mathbb{Z}_0 = \{m \in \mathbb{Z} : \forall k \in \mathbb{Z} \, (m = 0 + 4k)\}$$
$$= \{0, \pm 4, \pm 8, \pm 12, \ldots\}$$

$Z_1 = \{m \in \mathbb{Z} : \forall k \in \mathbb{Z}\ (m = 1 + 4k)\}$

$= \{\dots, -7, -3, 1, 5, 9, \dots\}$

$Z_2 = \{\dots, -6, -2, 2, 6, 10, \dots\}$

$Z_3 = \{\dots, -5, -1, 3, 7, 11, \dots\}$

Then $\mathbb{Z} = \bigcup_{i=0}^{n-1} Z_i$ and $\{Z_i\}$ is a partition of $\mathbb{Z}$.

Definition

Given a set A, the power set of A, denoted $\mathcal{P}(A)$, is the set of all subsets of A.

Let $A_1 = \{1\}$, $A_2 = \{1, 2\}$, and $A_3 = \{1, 2, 3\}$.

$\mathcal{P}(A_1) = \{\emptyset, \{1\}\}$ $\quad 2^1 = 2$

$\mathcal{P}(A_2)$ see above $\quad 2^2 = 4 \quad 2^3 = 8$

$\mathcal{P}(A_3) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

$N(\mathcal{P}(A_1)) = |\mathcal{P}(A_1)| = 2$

$\boxed{6.2}$ Properties of Sets

Theorem 6.2.1

Some Subset Relations

1. Inclusion of Intersection

   For all sets A and B,

   (a) $A \cap B \subset A$

   (b) $A \cap B \subset B$

2. Inclusion in Union

   For all sets A and B,

   (a) $A \subset A \cup B$ but $A \cup B \subset A$ is false

   (b) $B \subset A \cup B$

3. Transitive Property of Subsets

   For all sets A, B, and C,

   if $A \subset B$ and $B \subset C$, then $A \subset C$.

e.g. $\mathbb{Z}^+ \subset \mathbb{Z}^+ \cup \{0, -1\}$

$\mathbb{Z}^+ \cup \{0, -1\} \not\subset \mathbb{Z}^+$

For any set A, $A \subset A$.

$$\forall x \in \mathcal{U}\,(x \in A \longrightarrow x \in A) \equiv \forall x \in A\,(x \in A)$$

Procedural Versions of Set Definitions

Let X and Y be subsets of a universal set $\mathcal{U}$ and suppose $x, y \in \mathcal{U}$.

1. $x \in X \cup Y \iff x \in X$ or $x \in Y$
2. $x \in X \cap Y \iff x \in X$ and $x \in Y$
3. $x \in X - Y \iff x \in X$ and $x \notin Y$
4. $x \in X^c \iff x \notin X$
5. $(x, y) \in X \times Y \iff x \in X$ and $y \in Y$

Theorem 6.2.2

Set Identities

Let all sets below be subsets of a universal set $\mathcal{U}$.

① Commutative Laws For all sets A and B,

ⓐ $A \cup B = B \cup A$

ⓑ $A \cap B = B \cap A$

② Associative Laws For all sets A, B, and C,

ⓐ $(A \cup B) \cup C = A \cup (B \cup C)$

ⓑ $(A \cap B) \cap C = A \cap (B \cap C)$

③ Distributive Laws For all sets A, B, and C,

ⓐ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

ⓑ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

④ Identity Laws For any set A,

ⓐ $A \cup \emptyset = A$

ⓑ $A \cap \mathcal{U} = A$

⑤ Complement Laws For any set A,

ⓐ $A \cup A^c = \mathcal{U}$

ⓑ $A \cap A^c = \emptyset$

⑥ Double Complement Law For any set A,

$(A^c)^c = A$.

Theorem 6.2.2 (continued)

Let all sets below be subsets of a universal set $\mathcal{U}$.

⑦ **Idempotent Laws** For any set A,

ⓐ $A \cup A = A$

ⓑ $A \cap A = A$

⑧ **Universal Bound Laws** For any set A,

ⓐ $A \cup \mathcal{U} = \mathcal{U}$

ⓑ $A \cap \emptyset = \emptyset$

⑨ **DeMorgan's Laws** For any sets A and B,

ⓐ $(A \cup B)^c = A^c \cap B^c$

ⓑ $(A \cap B)^c = A^c \cup B^c$

⑩ **Absorptions Laws** For any sets A and B,

ⓐ $A \cup (A \cap B) = A$

ⓑ $A \cap (A \cup B) = A$

⑪ **Complements of $\mathcal{U}$ and $\emptyset$**

ⓐ $\mathcal{U}^c = \emptyset$

ⓑ $\emptyset^c = \mathcal{U}$

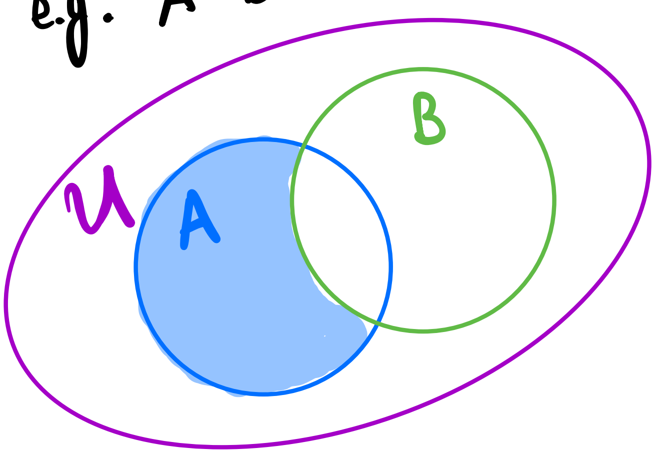
⑫ **Set Difference Law** For all sets A and B,

$A - B = A \cap B^c$

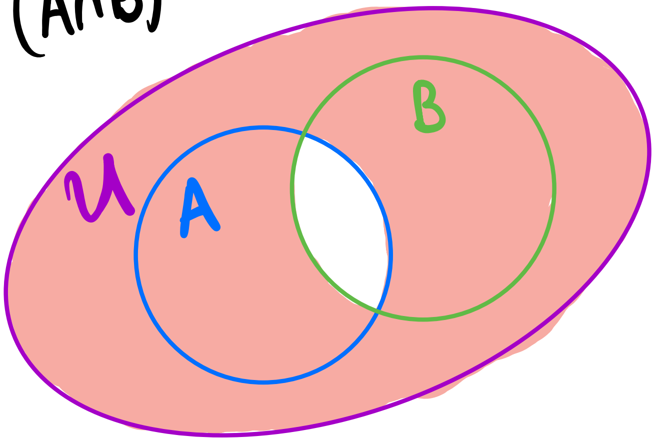
#38 For all sets A and B,

$$
\begin{aligned}
(A \cap B)^c \cap A &= (A^c \cup B^c) \cap A && \text{DeMorgan's Laws} \\
&= A \cap (A^c \cup B^c) && \text{commutativity} \\
&= (A \cap A^c) \cup (A \cap B^c) && \text{distributivity} \\
&= \emptyset \cup (A \cap B^c) && \text{complement law} \\
&= (A \cap B^c) \cup \emptyset && \text{commutativity} \\
&= A \cap B^c && \text{identity law} \\
&= A - B && \text{set difference law}
\end{aligned}
$$

e.g. $A-B$



$(A \cap B)^c$



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