①

CFD Spring 2012 A. Donev

Advection – Diffusion Equations

based on book by Hundsdorfer / Verwer

1D:

$$\frac{\partial}{\partial t} u(x,t) + \frac{\partial}{\partial x}\Big[\underbrace{a(x,t)\,u(x,t)}_{\text{advective flux}}\Big] = \frac{\partial}{\partial x}\Big[\underbrace{d(x,t)\,u_x(x,t)}_{\text{diffusive flux}}\Big]$$

conservation law

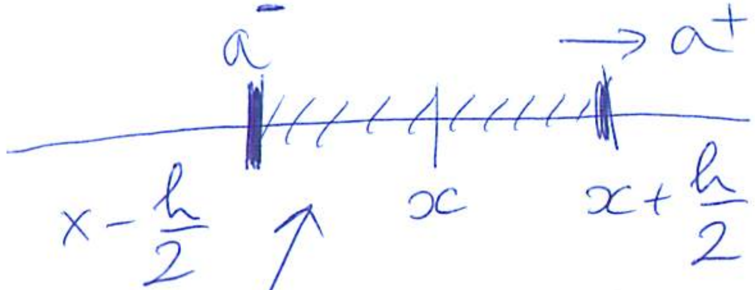
In this equation, $a(x,t)$ is a <u>given</u> advection velocity, and $d(x,t)$ is a <u>given</u> diffusion coefficient

$$\boxed{u_t + (au)_x = (d u_x)_x + f(u)} \quad \dots (*)$$

concentration of ions, pollutant, etc. (transported quantity)

forcing (reactions)

(2) The adv-diff equation is an expression of <u>conservation of mass</u>



finite volume cell

$$\overline{u}^x = \frac{1}{h}\int_{x-h/2}^{x+h/2} u(x,t)\,dx$$

$$h\frac{\partial}{\partial t}\overline{u}^x = -\left[a\left(x+\tfrac{h}{2}\right)u\left(x+\tfrac{h}{2}\right) - a\left(x-\tfrac{h}{2}\right)u\left(x-\tfrac{h}{2}\right)\right]$$

$$+\left[d\left(x+\tfrac{h}{2}\right)u_x\left(x+\tfrac{h}{2}\right) - d\left(x-\tfrac{h}{2}\right)u_x\left(x-\tfrac{h}{2}\right)\right]$$

<u>exact!</u>
(not a discretization)

lim $h \to 0$ gives $(*)$

Fick's law (constitutive relation) from <u>microscopic</u> physics

(3) For dimensions higher than one (typically dim $= 2$ or dim $= 3$)

nabla notation
$$\begin{cases} \nabla = (\partial_x, \partial_y, \partial_z)^T = \text{grad} \\ \Delta = \nabla \cdot \nabla = \text{div grad} = \partial_{xx} + \partial_{yy} + \partial_{zz} \end{cases}$$

I prefer ∇^2 instead of Δ

$$u_t + \nabla \cdot (\underline{a}\, u) = \nabla \cdot (D \nabla u)$$

$$\underline{a} \in \mathbb{R}^{dim}, \quad D \in \mathbb{R}^{dim \times dim} \succcurlyeq 0 \text{ (SPD)}$$

$$\nabla \cdot (\underline{a} u) = \partial_\alpha (a_\alpha u) = \quad (\alpha = 1, 2, .., dim)$$

$$(\partial_\alpha a_\alpha) u + a_\alpha \partial_\alpha u =$$

$$(\nabla \cdot \underline{a}) u + a \cdot \nabla u = a \cdot \nabla u$$

if $\boxed{\nabla \cdot \underline{a} = 0}$ solenoidal or incompressible

(4)

$$u_t + \underline{a} \cdot \nabla u = \nabla \cdot (D \nabla u)$$

advective form (non-conservative in general)

Another way in which advective form can arise:

$$\begin{cases} u_t + \nabla \cdot (\underline{a} u) = \nabla \cdot (D \nabla u) & \leftarrow \text{density of pollutant} \\ \rho_t + \nabla \cdot (\underline{a} \rho) = 0 & \leftarrow \text{true } \underline{\text{mass density}} \text{ (does } \underline{\text{NOT}} \text{ diffuse)} \end{cases}$$

Define $c = \dfrac{u}{\rho}$ as <u>concentration</u>

$$u = \rho c \Rightarrow u_t = \rho c_t + c \rho_t \Rightarrow$$

$$\rho c_t - c \nabla \cdot (\underline{a} \rho) + \nabla \cdot (\underline{a} u) = \rho (c_t + \underline{a} \cdot \nabla c)$$

$$\left\{ \begin{array}{l} \text{Note } \nabla \cdot (\underline{a} \rho c) = \\ = c \nabla \cdot (\underline{a} \rho) + \rho \underline{a} \cdot \nabla c \end{array} \right.$$

(5)

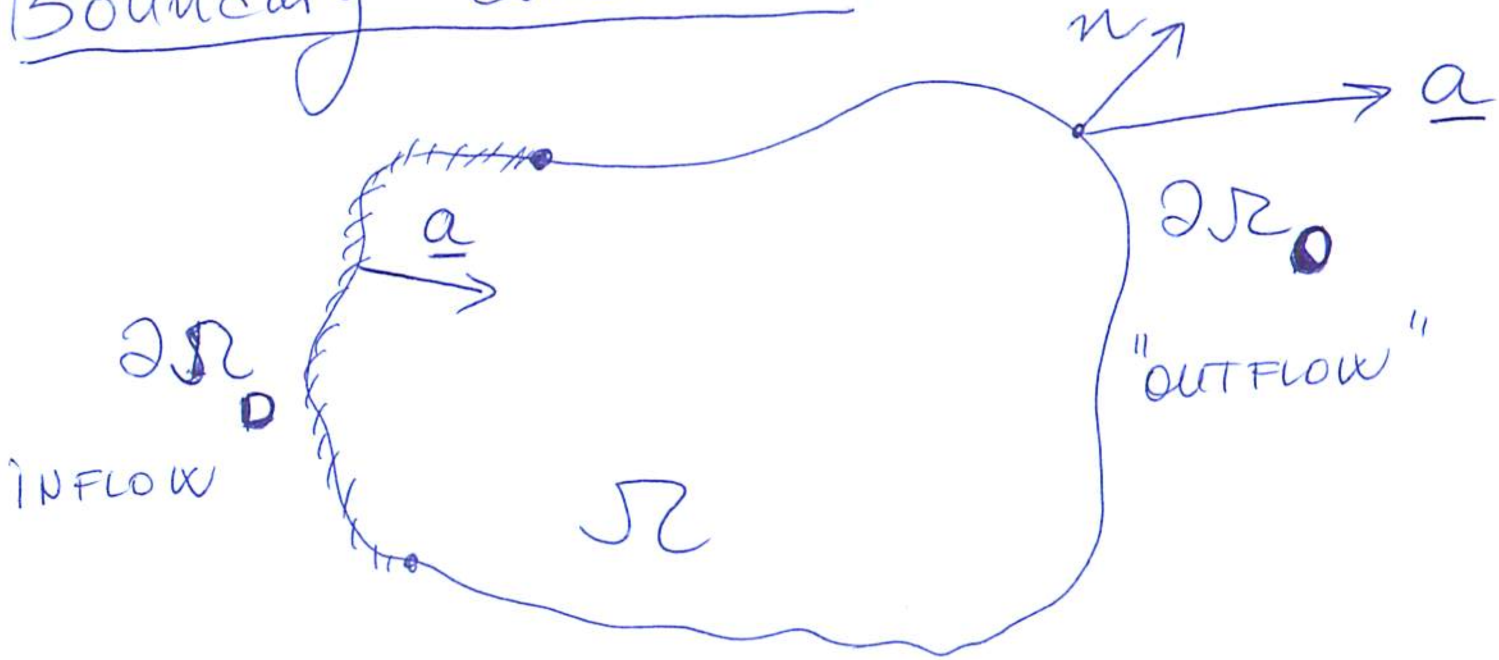
$$\rho(c_t + \underline{a}\cdot\nabla c) = \underbrace{\nabla\cdot(D\nabla u)}_{\text{Correct physics usually } \nabla\cdot(\rho D\nabla c)}$$

$$\boxed{D_t c = c_t + \underline{a}\cdot\nabla c} = \rho^{-1}\,\nabla\cdot(\rho D\nabla c)$$

advective or <u>material derivative</u>

Notice <u>only</u> conservative if $\rho = \text{const}$.

<u>Boundary conditions:</u>



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$$\partial\Omega = \partial\Omega_D \cup \partial\Omega_O$$

$\underline{n} \cdot \underline{a} < 0$ on $\partial\Omega_D$

normal vector (outward)

$\underline{n} \cdot \underline{a} \geq 0$ on $\partial\Omega_O$

a) For <u>pure advection</u>,

<u>Dirichlet</u> BC $\boxed{u = g_D}$ on $\partial\Omega_D$

is <u>sufficient</u>

b) For diffusion or adv-diff, one needs BCs on $\partial\Omega_O$ as well, e.g.

<u>Neumann</u> BC : $\boxed{\underline{n} \cdot \nabla u = g_N}$ on $\partial\Omega_N$

(in) homogeneous

<u>Robin</u> or <u>zero flux</u> BC : $\boxed{\underline{n} \cdot (\underline{a} u - D \nabla u) = g_M}$ on $\partial\Omega_M$

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We see that adding diffusion dramatically changes the character of the problem!

$$u_t + \nabla \cdot (\underline{a}\, u) = 0$$ is a pure <u>hyperbolic</u> eq.

$$u_t = \nabla \cdot (D \nabla u)$$ is a pure <u>parabolic</u> eq.

steady state $u_t = 0 = \nabla \cdot (D \nabla u)$ is a pure <u>elliptic</u> eq

In reality most equations have mixed character, but often problems are <u>advection-dominated</u> or <u>diffusion-dominated</u> and behave alike the limiting cases.

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DISCRETIZATIONS OF ADV-DIFF EQ.

$h = 1/m$ (grid spacing)

$$u_t(x_j, t) = \sum_k c_k \, u(x_{j+k}, t) + O(h^q)$$

stencil coefficients

local error
order of accuracy

This is a finite-difference scheme but for simple uniform discretizations it does not matter, and the distinction regards local conservation and variational structure

FINITE VOLUME FINITE ELEMENT

(2)

$$w_j(t) \simeq u(x_j, t) \quad \left(\begin{array}{c}\text{Finite}\\ \text{difference}\end{array}\right)$$

But could be

$$w_j(t) \simeq \frac{1}{h}\int_{x_j-h/2}^{x_j+h/2} u(x,t)\,dx \quad \left(\begin{array}{c}\text{Finite}\\ \text{Volume}\end{array}\right)$$

Or the relation could be expressed the other way, e.g., interpolation

$$u(x,t) = F(\underline{w}, x) \quad \left(\begin{array}{c}\text{Finite}\\ \text{Element,}\\ \text{SPECTRAL}\end{array}\right)$$

From here, we convert the PDE into a system of ODEs for w

$$\frac{dw}{dt} = w'(t) = \underset{\uparrow\ \text{stencil}}{A}\, w(t)$$

<u>Method of</u>
<u>Lines</u> approach
(SPLIT space and time)

(3) In finite difference methods we directly work with the PDE and replace derivatives with finite-difference approximations. In finite-volume methods we evaluate fluxes at the cell interfaces to maintain conservation. In finite element methods we use a weak (variational) form of the PDE, and in spectral methods we transform the PDE into another set of basis (Fourier) functions

For low-order (second-order) typically in the end all discretizations look the same and the main task is to analyze them!

⑦ <u>Modified Equations</u>

$$\frac{1}{h}\left[u(x-h) - u(x)\right] = -u_x(x) + \frac{1}{2} h\, u_{xx}(x) + O(h^2)$$

So the upwind scheme is actually <u>adding</u> diffusion or <u>artificial dissipation</u>

$$\begin{cases} \tilde{u}_t + a\tilde{u}_x = \frac{1}{2} a h\, \tilde{u}_{xx} & \leftarrow \text{modified equation} \\ d \equiv \frac{ah}{2} & \text{artificial dissipation} \end{cases}$$

The upwind scheme is a second-order approximation to the modified and <u>not</u> the original equation.

d/a

$$\frac{1}{h}\left(w_{j-1} - w_j\right) = \frac{1}{2h}\left(w_{j-1} - w_{j+1}\right) + \frac{h/2}{h^2}\left(w_{j-1} - 2w_j + w_{j+1}\right)$$

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$$\frac{1}{2h}\left[u(x-h)-u(x+h)\right] = -u_x(x) - \frac{h^2}{6}u_{xxx}(x) + O(h^4)$$

modified equation for centered scheme

$$\tilde{u}_t + a\tilde{u}_x = -\frac{ah^2}{6}\tilde{u}_{xxx}$$

artificial dispersion

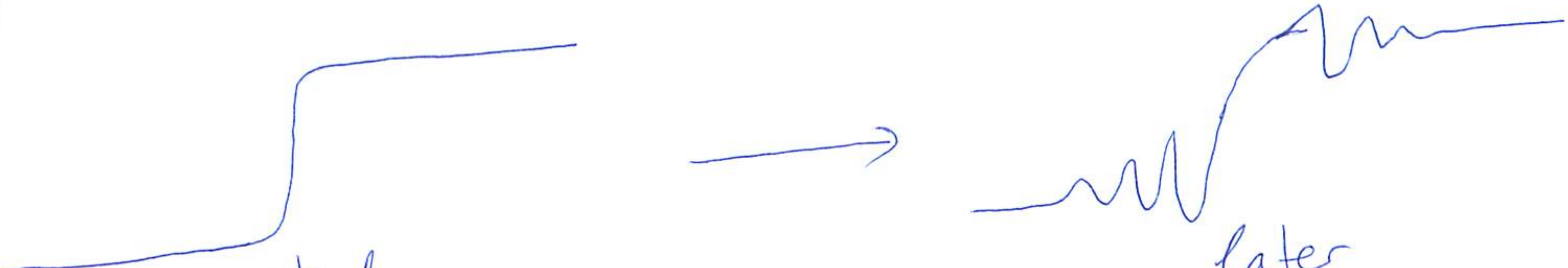
$$\begin{cases} \tilde{u}(x,0) = e^{2\pi i k x} \Rightarrow \\ \tilde{u}(x,t) = e^{2\pi i k (x - a_k t)} \end{cases}$$

numerical phase velocity

but $a_k = a\left(1 - \frac{2}{3}\pi^2 k^2 h^2\right)$

dispersion relation

So if the solution is not smooth (has high-frequency components), it will be distorted (Gibbs phenomenon)



initial later

Note $k_{max} = \frac{m}{2}$ so $\frac{2}{3}\pi^2 k_{max}^2 h^2 = \frac{2}{3}\pi^2 \cdot \frac{1}{4}$

independent of h. So the grid must be <u>much</u> finer than the width of regions with <u>sharp gradients</u>

<u>Dilemma</u> (Central issue in adv-diff)

$\left\{\right.$ Accept low accuracy (artificial diss.)
accept low robustness (non-monotonicity)
or find a way to trade off

(14) E.g. third-order upwind-biased method for $a>0$

$$w_j' = \frac{a}{h}\left[-\frac{1}{6}w_{j-2} + w_{j-1} - \frac{1}{2}w_j - \frac{1}{3}w_{j+1}\right]$$

Modified eq: $\tilde{\hat{u}}_t + a\tilde{\hat{u}}_x = -\frac{|a|}{12}h^3\tilde{\hat{u}}_{xxxx}$

So now the dissipative term is $O(h^3)$ and also fourth-order, not second-order diffusive

Note: $u_t = -u_{xxxx}$, $\hat{u}_t = -k^4\hat{u}$

does not satisfy a maximum principle and over/under shoots can appear

(29) Take $h \to 0$ to get a semi-discretization in time

$$u' = A u$$

where A is some <u>linear</u> (differential) operator.

Consider the θ-method:

$$\boxed{u^{n+1} = u^n + (1-\theta)\tau A u^n + \theta \tau A u^{n+1}}$$

$\theta = 0$: forward Euler

$\theta = 1$: backward Euler

$\theta = 1/2$: Crank-Nicolson.

<u>Temporal truncation error</u>:

$$\delta_n = \frac{1}{\tau}\left[u(t_{n+1}) - u(t_n)\right] - \left(\frac{u^{n+1} - u^n}{\tau}\right)$$

(30) By Taylor series of $u' = Au$

$$\rho_n = \left(\frac{1}{2} - \theta\right)\tau \underbrace{\mathbf{u}''(t_n)}_{A^2} + \left(\frac{1}{6} - \frac{1}{2}\theta\right)\tau^2 \underbrace{\mathbf{u}'''(t_n)}_{A^3}$$

This means that our scheme is closer to solving the <u>modified equation</u>

$$\tilde{u}' = \tilde{A}\tilde{u}$$

$$\tilde{A} = A + \left(\theta - \frac{1}{2}\right)\tau A^2 + \left(\frac{\theta}{2} - \frac{1}{6}\right)\tau^2 A^3$$

↑ modified operator

$$\begin{cases} \tilde{A} \approx A \text{ [illegible] } \dfrac{\tau^2}{12} A^3 & \text{for } \theta = 1/2 \text{ (CN)} \\ \tilde{A} \approx A + \dfrac{\tau}{2} A^2 & \text{for } \theta = \underline{1} \text{ (BE)} \end{cases}$$

31 For pure advection equation,

$$A \equiv -a\partial_x \Rightarrow A^2 = a^2\partial_{xx}$$

$$\Rightarrow A^3 = -a^3\partial_{xxx}$$

So the modified equation is

BE $\left[\ \tilde{u}_t + a\tilde{u}_x = \frac{\tau a^2}{2}\tilde{u}_{xx} \quad \text{for } \theta = 1\right.$

Artificial diffusion!

CN $\left[\ \tilde{u}_t + a\tilde{u}_x = -\frac{1}{12}\tau^2 a^3 \tilde{u}_{xxx} \quad \text{for } \theta = \frac{1}{2}\right.$

artificial dispersion

See HW 3!

One can combine together the spatial and temporal artificial dissipation & dispersion into one modified equation. This works for space-time schemes also. (1)

Take upwind scheme & see Section 10.9 in Le Veque

$$u_j^{n+1} = u_j^n - \frac{a\tau}{h}\left(u_j^n - u_{j-1}^n\right)$$

This can be seen as an MOL scheme:
Upwind + Forward Euler

Upwind modified equation

$$\tilde{u}_t + a\tilde{u}_x = \frac{1}{2} a h \tilde{u}_{xx}$$

Forward Euler modified equation

$$\tilde{u}_t + a\tilde{u}_x = -\frac{a^2 \tau}{2} \tilde{u}_{xx}$$

Put them together we get

$$\tilde{u}_t + a\tilde{u}_x = \frac{1}{2} a h \left(1 - \frac{a\tau}{h}\right) \tilde{u}_{xx}$$

$$= \frac{1}{2} a h (1-\nu) \tilde{u}_{xx}$$

Modified Eq. Upwind

We could have gotten this directly: (3)

$$v(x, t+\tau) = v(x, t) - \frac{a\tau}{h}\left[v(x,t) - v(x-h,t)\right]$$

$\Downarrow$ Taylor series $+O(h^2) \quad +O(\tau^2)$

$$v_t + a v_x = \frac{1}{2}\left(a h v_{xx} - \tau v_{tt}\right)$$

But $\quad v_{tt} \approx a^2 v_{xx}$ from PDE

$$v_t + a v_x = \frac{1}{2}\left(a h v_{xx} - a^2 \tau v_{xx}\right)$$
$$+ O(\tau^2, h^2)$$

Effective <u>artificial diffusion coefficient</u>

$$d = (ah - a^2\tau)/2 = \boxed{ah(1-\nu)/2 \geq 0}$$

Following the same procedure for Lax-Wendroff we have to keep terms of $O(\tau^2)$ to get modified equation

$$\tilde{u}_t + a\,\tilde{u}_x + \underbrace{\frac{a\tau^2}{6}(1-\nu^2)\,\tilde{u}_{xxx}}_{} = 0$$

Artifically Dispersive term

Oscillations will develop behind sharp peaks not resolved by grid.

SHOW FIG. 10.4 in Le Veque

If we kept next term we would see <u>some artificial diffusion</u> also (necessary to stabilize scheme)

$$\boxed{\tilde{u}_t + a\tilde{u}_x + \frac{a\tau^2}{6}(1-\nu^2)\tilde{u}_{xxx} = -\varepsilon\,\tilde{u}_{xxxx}}$$

Lax-Wendroff

where $\boxed{\varepsilon = O(\tau^3 + h^3) \geq 0}$

<u>Beam-warming scheme</u> is similar

$$\tilde{u}_t + a\tilde{u}_x = \frac{a\tau^2}{6}(2-3\nu+\nu^2)\,\tilde{u}_{xxx} = 0$$

Oscillations now <u>precede</u> sharp peaks

(10)

Variable Coefficients

$$u_t + [a(x)\, u]_x = [d(x)\, u_x]_x$$

conservation law

advective flux

diffusive flux

Recall $\quad \overline{u}_j(t) = w_j = \dfrac{1}{h} \displaystyle\int_{x_j - h/2}^{x_j + h/2} u(x,t)\, dx$

Tayler series $= u(x_j, t) + \dfrac{h^2}{24} u_{xx}(x_j) + \ldots$

We can choose finite-difference or finite-volume

$$\begin{cases} w_j \approx u(x_j, t) \\ w_j \approx \overline{u}_j(t) \end{cases}$$

$\rightarrow$ Makes <u>no difference</u> up to second order

(11) But for discretization, we use <u>finite-volume</u>, which means we use flux form or conservation form:

$$w_j'(t) = \frac{1}{h}\left[F_{j-1/2} - F_{j+1/2}\right]$$

fluxes through cell boundaries

Let's take the natural

$$F_{j+1/2} = \underbrace{a(x_{j+1/2})\,w_{j+1/2}}_{\text{advective flux}} - \underbrace{d(x_{j+1/2})(w_j - w_{j+1})}_{\text{diffusive flux}}$$

What is $w_{j+1/2}$?

How to go from <u>cell centers</u> to <u>cell faces</u>?

(12) If $a(x) > 0$ for $\forall x$

$w_{j+1/2} = w_j$ is upwind (first-order)

More generally, one writes

$$\begin{cases} a(x_{j+1/2})\, w_{j+1/2} = a^+(x_{j+1/2})\, w_j(t) - a^-(x_{j+1/2})\, w_{j+1} \\ \text{where } a^+ = \max(a,0),\ a^- = \min(a,0) \end{cases}$$

for first-order upwind scheme

or

$$\begin{cases} w_{j+1/2} = \frac{1}{2}(w_j + w_{j+1}) \\ \text{for second-order centered scheme} \end{cases}$$

DIY: Convince yourself that for
Do-It-Yourself
constant coeff. these are the same as before

(14) For <u>third-order upwind-biased</u>:

$$w_{j+1/2} = \begin{cases} \frac{1}{6}\left[-w_{j-1} + 5 w_j + 2 w_{j+1}\right] \\ \text{if } a(x_{j+1/2}) \geq 0 \text{, and similarly} \\ \text{flip direction for } a_{j+1/2} \leq 0 \end{cases}$$

Analytical <u>excercise</u>:
Show that the order of consistency of this scheme is $q = 2$ for finite-difference interpretation, but $q = 3$ for finite-volume

No stability result exists in general - we use <u>heuristics</u> and <u>frozen coefficients</u> arguments if a is smooth

Space - Time methods

for Advection - Diffusion

CFD Fall 2014

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(some notes & comments)

Recall that Lax-Wendroff is <u>NOT</u> an MOL (method-of-lines) scheme, it is a <u>space-time</u> scheme.

It's local truncation error is:

$$\tau = -\frac{1}{6} a \Delta x^2 (1 - \nu^2) u_{xxx} + O(\Delta t^3)$$

where $\nu = \frac{a \Delta t}{\Delta x}$ is CFL number

$$\rho = -\frac{1}{6} u_{xxx} \left(\Delta x^2 - a^2 \Delta t^2 \right)$$

which is a sum of a spatial and a temporal error just like MOL schemes. But this is <u>NOT</u> always the case.

Consider the **Lax-Friedrichs** method

$$u_i^{n+1} = \underbrace{\frac{1}{2}\left(u_{i-1}^n + u_{i+1}^n \right)}_{\text{stabilization (not MOL)}} - \underbrace{(\Delta t) a \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x}}_{\text{centered difference + Euler (UNSTABLE)}}$$

Stable if $CFL \leq 1$

Doing Taylor series on (3)

$$\frac{1}{2}\left(u_{i-1}^{n} + u_{i+1}^{n}\right) = u_i^n + \frac{1}{2}\Delta x^2 u_{xx}$$

we see that the modified equation is

$$u_t = -a u_x + \underbrace{\frac{1}{2}\frac{\Delta x^2}{\Delta t} u_{xx}}_{\text{Numerical (artificial) dissipation}}$$

The Lax-Friedrichs method is <u>not</u> even <u>consistent</u> if $\Delta t \to 0$ keeping Δx fixed: One must refine both space <u>and</u> time together for space-time schemes

The error for Lax-Friedrichs is (4)

$$\varepsilon = \frac{1}{2} a \Delta x \left(\nu^{-1} - \nu \right) u_{xx} + O(\Delta t^2)$$

so it <u>must</u> be run with $\nu \simeq 1$

When talking about non-MOL schemes we are looking at space-time error

$$\boxed{\nu = \text{const} \Rightarrow \Delta t = O(\Delta x) \quad \text{(for advection problems)}}$$

Note that for <u>explicit</u> (conditionally stable) methods we can never refine in space <u>only</u> due to $\nu \leq 1$ limit

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$$u_t + a u_x = 0$$

$$\Rightarrow u(x_j, t_{n+1}) = u(x_j - \tau a, t_n)$$

Denote $\boxed{\nu = \frac{\tau a}{h}}$ → advective Courant or CFL number

$$\Rightarrow \tau a = \nu h$$

Lax-Wendroff can be obtained by using quadratic interpolation to get $u(x_j - \tau a, t_n)$:

Assume $\boxed{-1 \leq \nu \leq 1}$ CFL condition

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$$\text{interpolate}\left\{ \begin{aligned} u(x_j - \nu h) \approx\ & \tfrac{1}{2}\nu(\nu+1)\, u(x_{j-1}) \\ & + (1-\nu^2)\, u(x_j) \\ & + \tfrac{1}{2}\nu(\nu-1)\, u(x_{j+1}) \end{aligned} \right.$$

$a \geq 0$ <u>or</u> $a \leq 0$

This leads to the scheme: Lax-Wendroff

$$\boxed{w_j^{n+1} = w_j^n + \underbrace{\frac{a\tau}{2h}\left(w_{j-1}^n - w_{j+1}^n\right)}_{\text{forward Euler for centered advection}} + \underbrace{\frac{1}{2}\left(\frac{a\tau}{h}\right)^2 \left[w_{j-1}^n - 2w_j^n + w_{j+1}^n\right]}_{\text{second-order diffusive correction}}}$$

4 Godunov methods

Consider a general conservation law

$$q_t + \left(f(q) \right)_x = 0$$

$f(q)$: flux function

fundamental hyperbolic equation

$$f(q) = a q \quad \text{for advection}$$

Define cell-average:

$$q_i^n = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} q(t^n, x)\, dx$$

Take time derivative and commute

(5)

$$\frac{dq_i^n}{dt} = -\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \frac{d f(q(x))}{dx}\, dx$$

$$= -\frac{1}{\Delta x}\left(f(q(t, x_{i+1/2})) - f(q(t, x_{i-1/2}))\right)$$

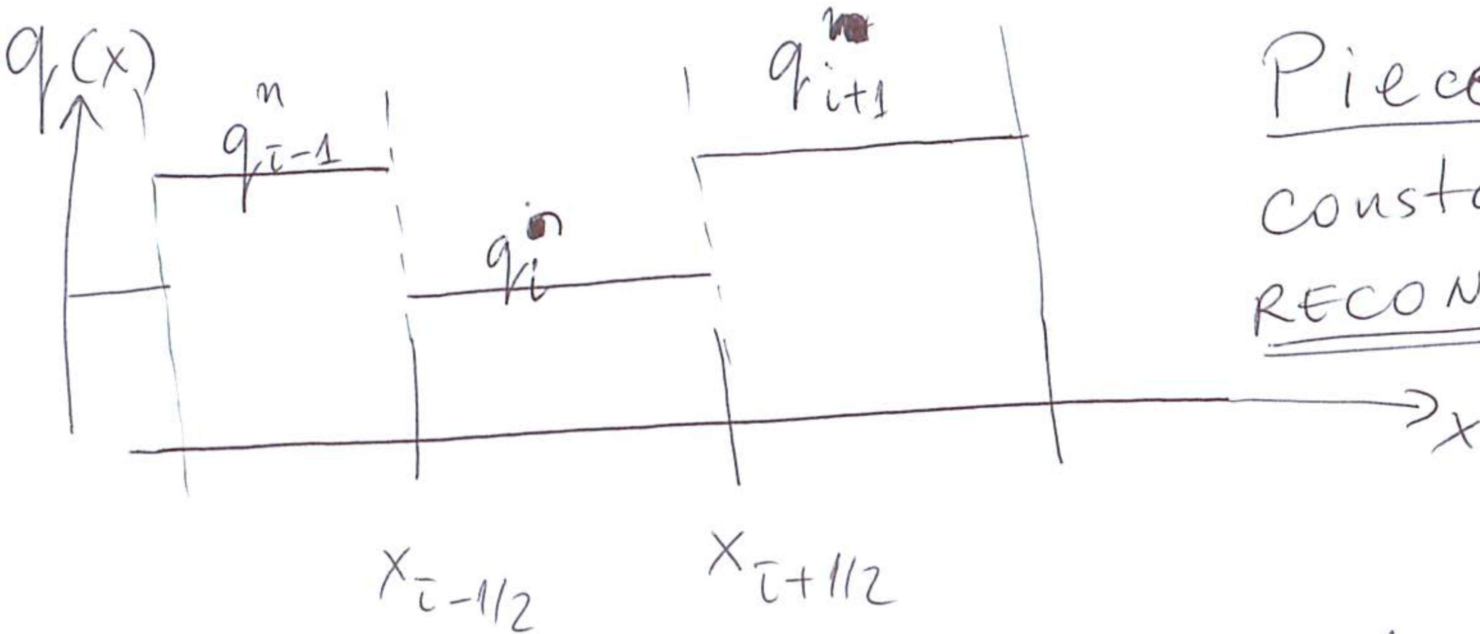
fluxes
through faces of
finite-volume grid

Now integrate in time

$$q_i^{n+1} = q_i^n - \frac{1}{\Delta x}\int_{t^n}^{t^{n+1}} \left(f(q(t, x_{i+1/2})) - f(q(t, x_{i-1/2}))\right) dt$$

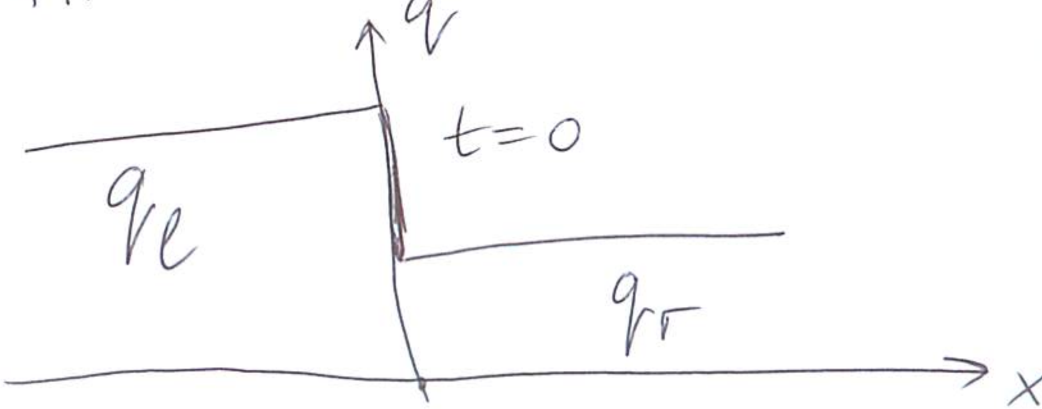
This is so far exact, now we need an approximation

⑥ Riemann problem



Piecewise constant RECONSTRUCTION

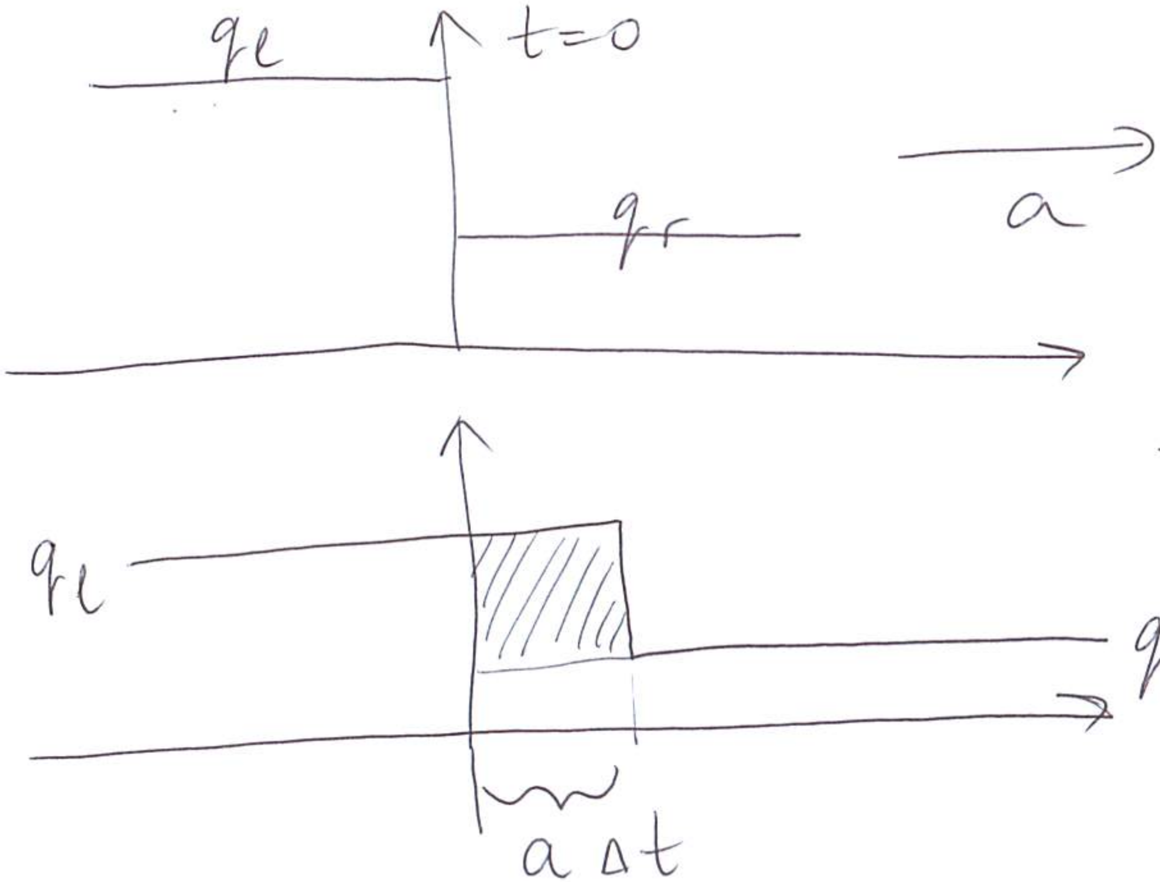
Assume that we can solve a piecewise constant problem exactly over a finite time interval



Riemann problem to compute $f^{\downarrow}(q_l, q_r)$

⑦ $f^{\downarrow}(q_l, q_r)$ — approximation of <u>average</u> flux in Riemann problem

For linear advection:



$$\boxed{\begin{array}{l}\text{If } f'(q) > 0 \\ f^{\downarrow} \simeq f(q_l)\end{array}}$$

$$f^{\downarrow} = a q_l$$

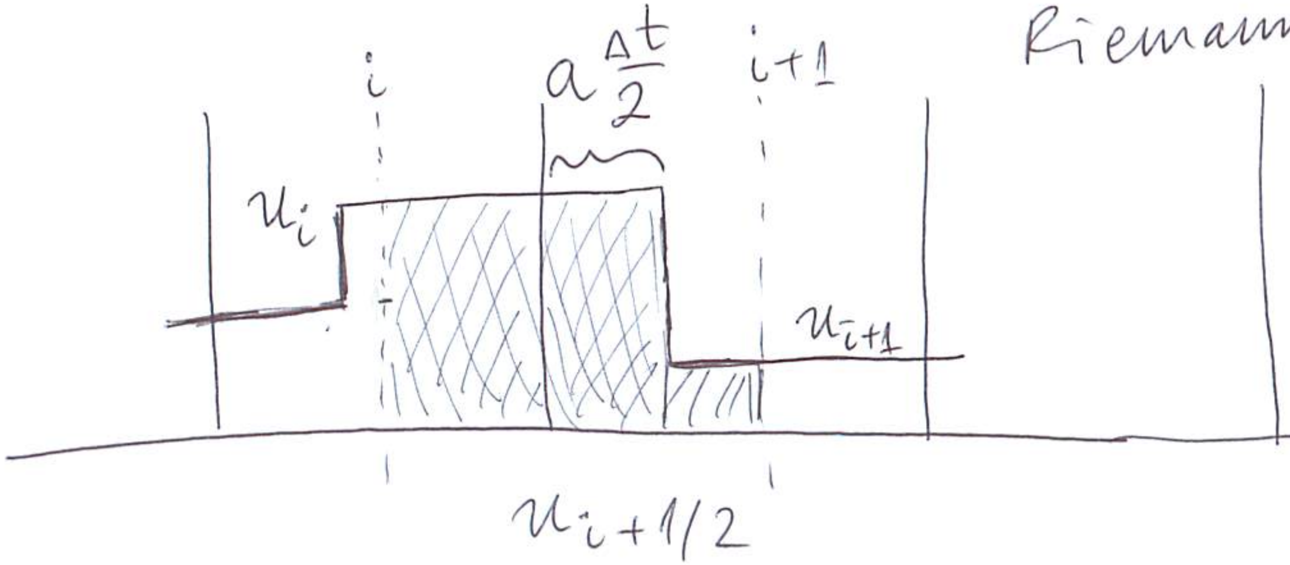
Gives simple <u>upwinding</u>

(8) How can we make this higher order?

Let's <u>center</u> the flux in <u>space</u> <u>and</u> <u>time</u>:

$$f_{i+1/2} = \frac{1}{\Delta x} \int_{-1/2\Delta x}^{1/2\Delta x} f\left[u_{i+1/2}\left(x, \frac{\Delta t}{2}\right)\right] dx$$

Solution of Riemann problem



<u>Assume</u>

CFL < 1

$$a\frac{\Delta t}{2} < \frac{\Delta x}{2}$$

(9) For linear advection

$$f_{i+1/2} = a\left[\frac{1}{2} u_i + \frac{c}{2} u_i + \left(\frac{1}{2} - \frac{c}{2}\right) u_{i+1}\right]$$

$$= \frac{a}{2}\left[(1+c)\, u_i + (1-c)\, u_{i+1}\right]$$

$$f_{i-1/2} = \frac{a}{2}\left[(1+c)\, u_{i-1} + (1-c)\, u_i\right]$$

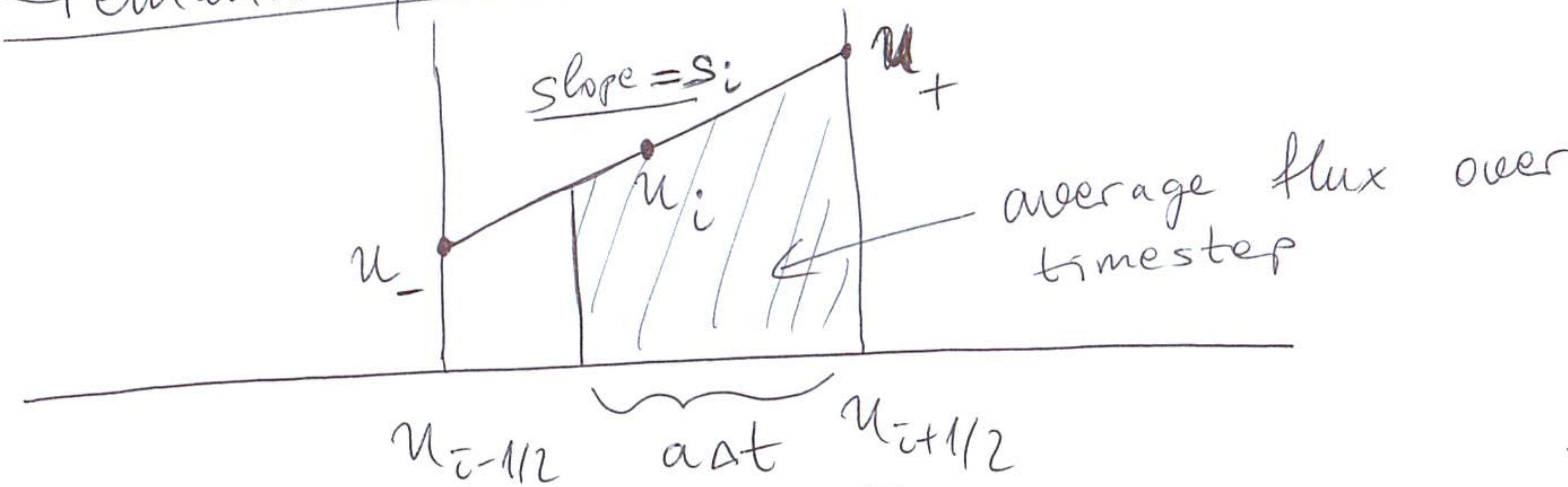
$$\frac{du_i}{dt} = -\frac{f_{i+1/2} - f_{i-1/2}}{\Delta x} = -\frac{a}{2\Delta x}\left[(1-c) u_{i+1} + 2c u_i - (1+c) u_{i-1}\right]$$

$$u_i^{n+1} = u_i^n \; c\left[\left(\frac{1-c}{2}\right) u_{i+1} + c u_i - \left(\frac{1+c}{2}\right) u_{i-1}\right]$$

$\equiv$ LAx-Wendroff !

(10) Another approach to getting higher order is to use **higher-order reconstruction**, such as linear reconstruction

Riemann problem $\longrightarrow$ a



$$f^{\downarrow}_{i+1/2} = \frac{\text{area} \cdot a}{\Delta x} = \frac{a}{2}\left[\underbrace{(u_- + u_+)}_{= 2u_i} + (1-c)(u_+ - u_-)\right]$$

conservation in reconstruction

11

$$f^{\downarrow}_{i+1/2} = a\,u_i + \frac{a\,\Delta x}{2}(1-c)\,\underset{\uparrow \text{slope}}{S_i}$$

Now we need to choose how to compute the slope of the local reconstruction

Set $\quad S_i = \dfrac{u_{i+1} - u_i}{\Delta x} \Longrightarrow$ Lax-Wendroff

$$f^{\downarrow}_{i+1/2} = a\,u_i + \frac{a}{2}(1-c)\,(u_{i+1} - u_i)$$

$$= \frac{a}{2}\left[(1+c)\,u_i + (1-c)\,u_{i+1}\right]$$

just as before

But there are other choices

(12) For example, choose <u>centered</u> slopes

$$S_i = \frac{u_{i+1} - u_i}{\Delta x}$$

$$f_{i+1/2} = a u_i + \frac{a}{4}(1-c)\left(u_{i+1} - u_{i-1}\right)$$

$$u_i^{n+1} = u_i^n - c\left(u_i - u_{i-1}\right) \leftarrow \text{upwind piece}$$

$$- \frac{c(1-c)}{4} \underbrace{\left(u_{i+1} - u_i - u_{i-1} + u_{i-2}\right)}_{\simeq u_{xx} \cdot \Delta x^2}$$

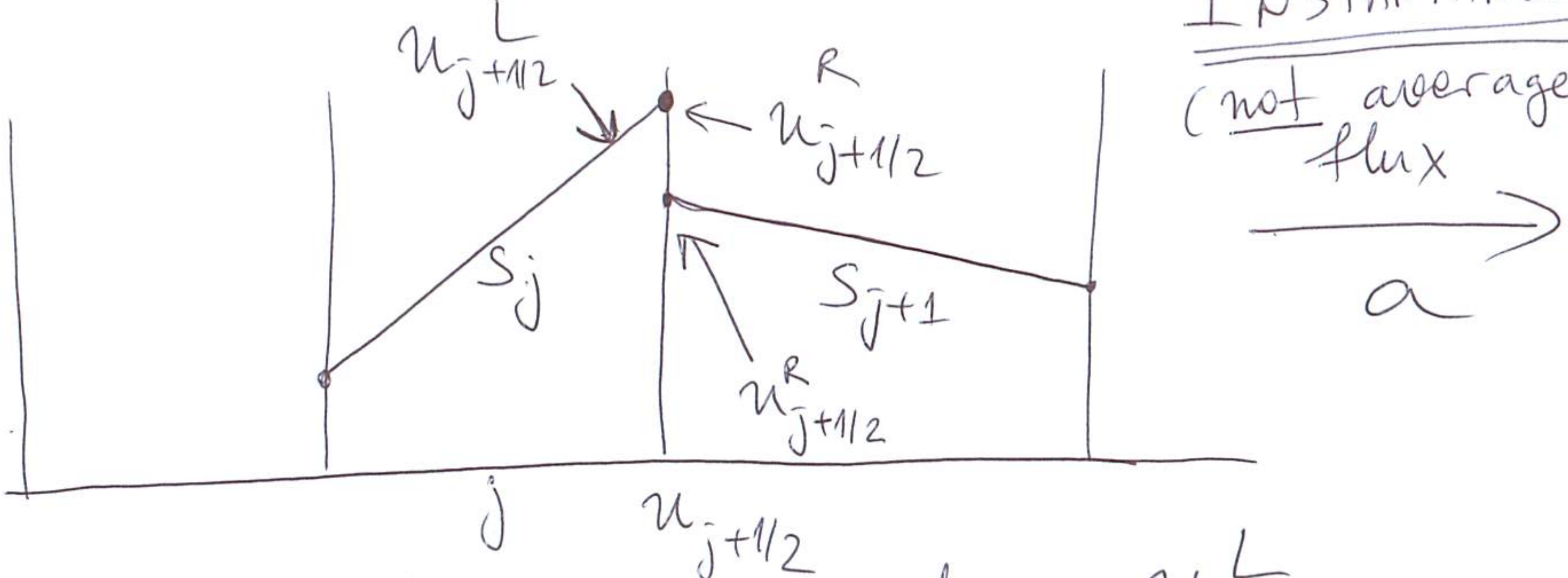
← second order correction

this is known as <u>FROMM'S METHOD</u>

(15) The higher-order reconstruction can also be used with a MOL (method of lines) approach:

$$f_{j+1/2} = f^{\text{instant}}_{\text{Riemann}}\left(u^L_{j+1/2}, u^R_{j+1/2}\right) \rightarrow$$

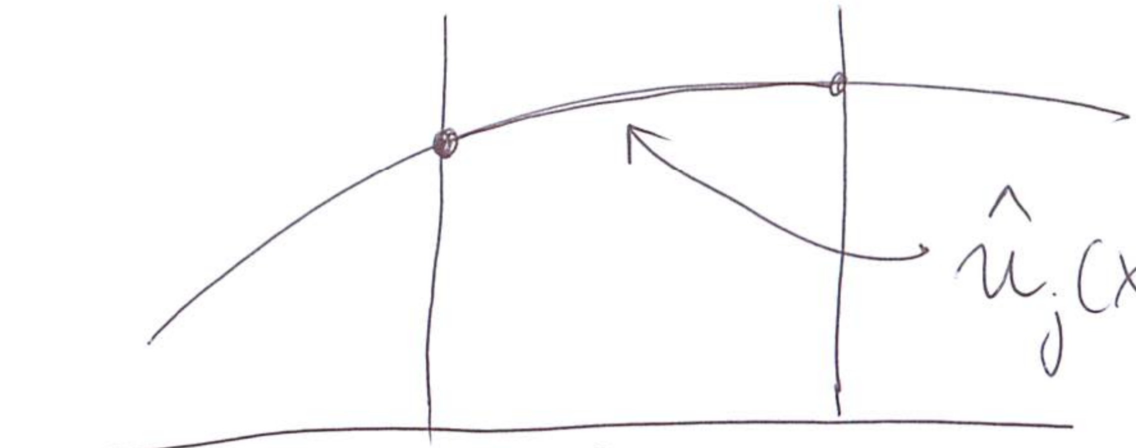
INSTANTANEOUS (not average) flux



For advection this picks $u^L_{j+1/2}$

(16) $$\frac{du_j}{dt} = -\left(\frac{f_{j+1/2} - f_{j-1/2}}{\Delta x}\right)$$

Let us consider a quadratic reconstruction



$$\hat{u}_j(x) = c_1 + c_2 x + c_2 x^2$$

cell j

Conditions for finite-volume averages:

$$\left\{ \begin{array}{l} \frac{1}{\Delta x} \int_{\text{cell } j} \hat{u}_j(x) = u_j \\ \text{and same for average over cells } j-1 \text{ and } j+1 \end{array} \right.$$

(17) Result:

$$\hat{u}_j(x) = u_j + \left(\frac{u_{j+1} - u_{j-1}}{2\Delta x}\right)(x - x_j)$$

$$+ \underbrace{\left(\frac{u_{j+1} - 2u_j + u_{j-1}}{2\Delta x^2}\right)\left((x-x_j)^2 - \frac{\Delta x^2}{12}\right)}_{\text{Taylor series}}$$

finite-volume piece

Choose upwinding for flux!

$$f_{j+1/2} = a\,\hat{u}_j\left(x = (j + 1/2)\Delta x\right)$$

$$= a\,u^L_{j+1/2}$$

(18) Putting it all together gives

$$\frac{du_j}{dt} = - \frac{f_{j+1/2} - f_{j-1/2}}{\Delta x}$$

$$= - \frac{a}{6\Delta x}\left(2u_{j+1} + 3u_j - 6u_{j-1} + u_{j-2}\right)$$

which is our familiar third-order upwind biased spatial discretization.

The second-order version (linear reconstruction) is

$$\frac{du_j}{dt} = - \frac{a}{4\Delta x}\left(u_{j+1} + 3u_j - 5u_{j-1} + u_{j-2}\right)$$

(NOT WORTH IT)