Introduction to Robot Intelligence
[Spring 2023]

LQR and variants

March 28, 2023

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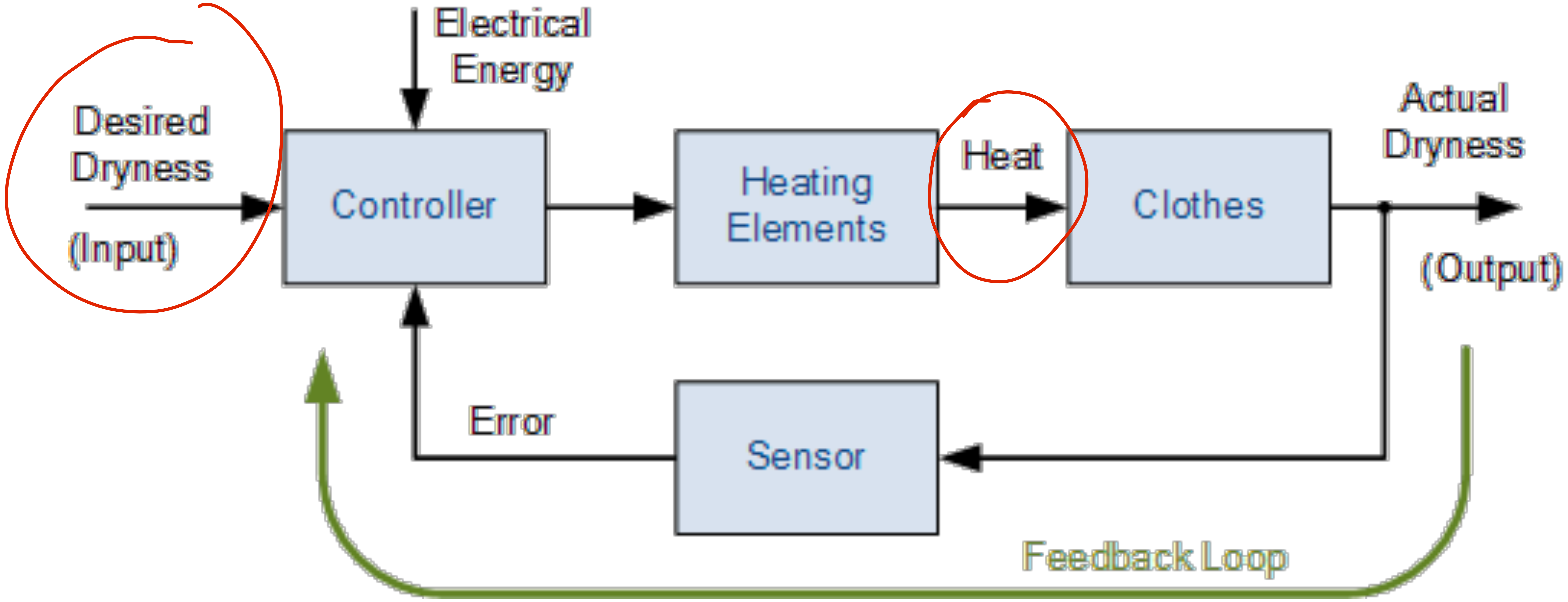
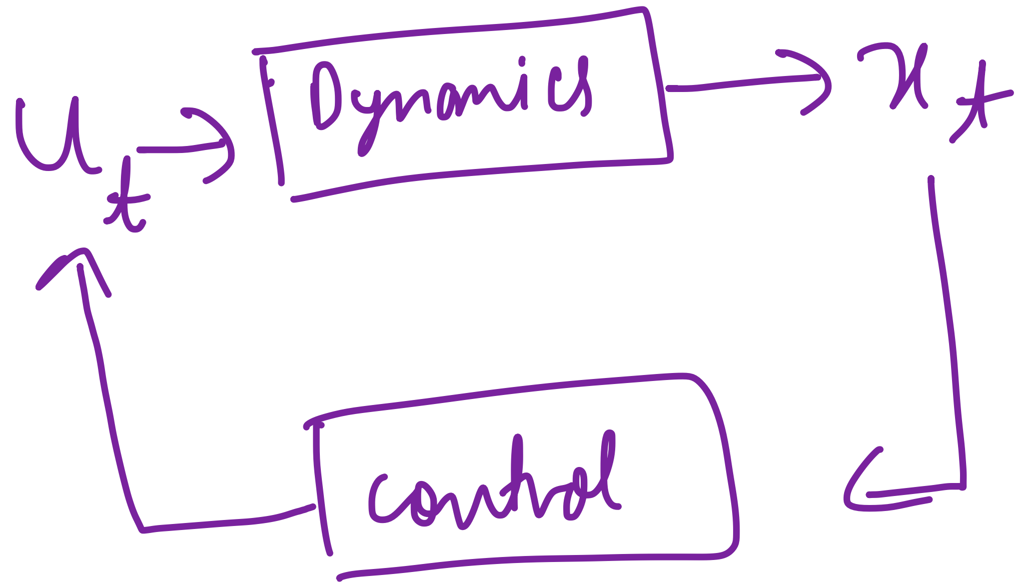
What is a controller?



LQR – Setup

Linear Quadratic Regulator

Dynamics.



$$x_{t+1} = f(x_t, u_t)$$

$$x_{t+1} = \underbrace{A}\, x_t + \underbrace{B}\, u_t$$

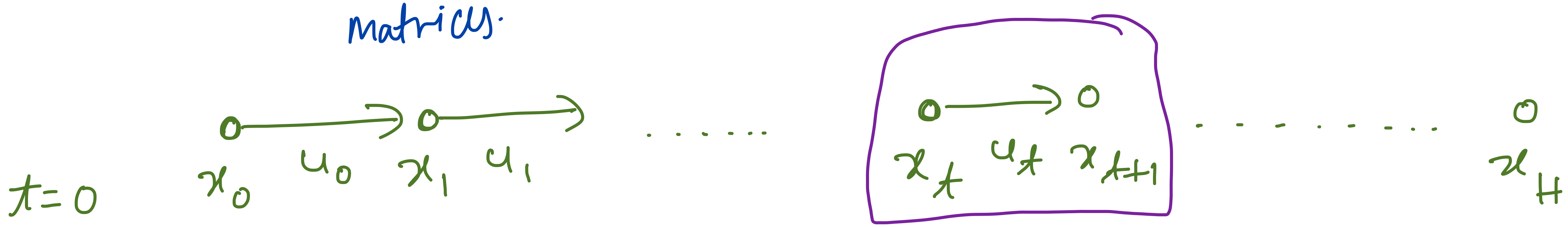
matrices.

Cost function

$$\min_{(u_0, u_1, \dots u_{H-1})} \sum_{t=0}^{H} c_t(x_t, u_t)$$

$$c(x_t, u_t) = x_t^T \underbrace{Q}\, x_t + u_t^T \underbrace{R}\, u_t$$

Matrices.



LQR solving through dynamic programming

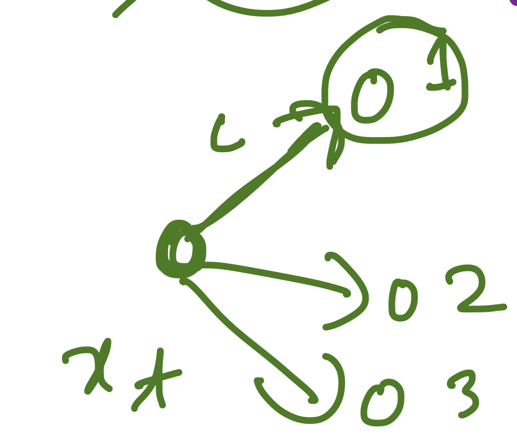
$t=0 \qquad x_0 \xrightarrow{u_0} x_1 \xrightarrow{u_1} \cdots\cdots \; x_t \xrightarrow{u_t} x_{t+1} \; \cdots\cdots \; x_{H-1} \xrightarrow{u_{H-1}} x_H$

$$V_t(x_t) = \sum_{t'=t}^{H} c(x_{t'}, u_{t'})$$

$$V_H(x_H) = \sum_{t'=H}^{H} c(x_{t'}, u_{t'}) = c(x_H, u_H) = x_H^T Q x_H = x_H^T P_H x_H \Big|_{P_H = Q}$$

$$V_t^*(x_t) = \min_{u_t} \left[c(x_t, u_t) + V_{t+1}^*(x_{t+1}) \right]$$

$$= \min_{u_t} \left[x_t^T Q x_t + u_t^T R u_t + V_{t+1}^*(Ax_t + Bu_t) \right]$$



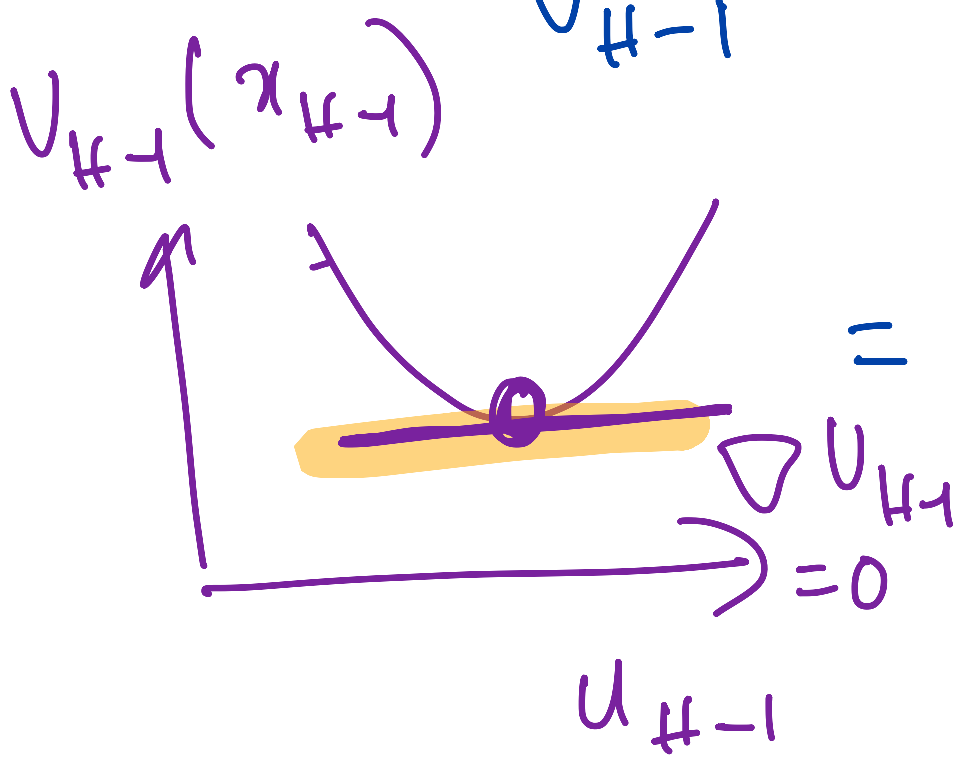
$$V^*_{x_t} = \min \begin{bmatrix} c(x_t \to 1) + V^*(1) \\ c(x_t \to 2) + V^*(2) \\ c(x_t \to 3) + V^*(3) \end{bmatrix}$$

LQR solving through dynamic programming

Initial condition $P_H = Q$

$$V_H = x_H^T P_H x_H$$

$$V^*_{H-1} = \min_{u_{H-1}} \Big[x_{H-1}^T Q x_{H-1} + u_{H-1}^T R u_{H-1} + V_H(\overbrace{A x_{H-1} + B u_{H-1}}^{x_H}) \Big]$$



$$= \min_{u_{H-1}} \Big[x_{H-1}^T Q x_{H-1} + u_{H-1}^T R u_{H-1} + (A x_{H-1} + B u_{H-1})^T P_H (A x_{H-1} + B u_{H-1}) \Big]$$

$$\nabla_{u_{H-1}} = 2 R u_{H-1} + 2 B^T P_H (A x_{H-1} + B u_{H-1}) = 0$$

$$u_{H-1} = -(R + B^T P_H B)^{-1} B^T P_H A x_{H-1}$$

$$= K_{H-1} x_{H-1}$$

$$V^*_{H-1} = x_{H-1}^T P_{H-1} x_{H-1}$$

$$\rightarrow P_{H-1} = Q + K_{H-1}^T R K_{H-1} + (A + B K_{H-1})^T P_H (A + B K_{H-1})$$

LQR solving through dynamic programming

$P_H = Q$ $\qquad V_H = x_H^T P_H x_H$

iterate from $t = H - 1$ to 0

$$K_t = -(R + B^T P_{t+1} B)^{-1} B^T P_{t+1} A$$

$$P_t = Q + K_t^T R K_t + (A + BK_t)^T P_{t+1} (A + BK_t)$$

$$\boxed{u_t^* = K_t x_t} \longrightarrow \text{Policy}$$

$$V_t(x_t) = x_t^T P_t x_t$$

Policy

$u_t = \pi(x_t)$

$\pi(x_t) = K_t x_t$

$\circ \xrightarrow[\pi]{} \circ$ $\qquad \circ \longleftarrow \circ$

Limitations of LQR

- Extension to Affine systems
- Stochastic dynamics
- Non-linear dynamics

Extension to affine systems

LQR

Linear Dynamics

$$x_{t+1} = A x_t + B u_t$$

Affine

$$x_{t+1} = A x_t + B u_t + c$$

??

$$z_t = \begin{bmatrix} 1 \\ x_t \\ 1 \end{bmatrix}$$

$$z_{t+1} := \begin{bmatrix} x_{t+1} \\ 1 \end{bmatrix} = \begin{bmatrix} A & c \\ 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} x_t \\ 1 \end{bmatrix}}_{z_t} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_t$$

$$z_{t+1} = A' z_t + B' u_t$$

$$A' = \begin{bmatrix} A & c \\ 0 & 1 \end{bmatrix} \qquad B' = \begin{bmatrix} B \\ 0 \end{bmatrix}$$

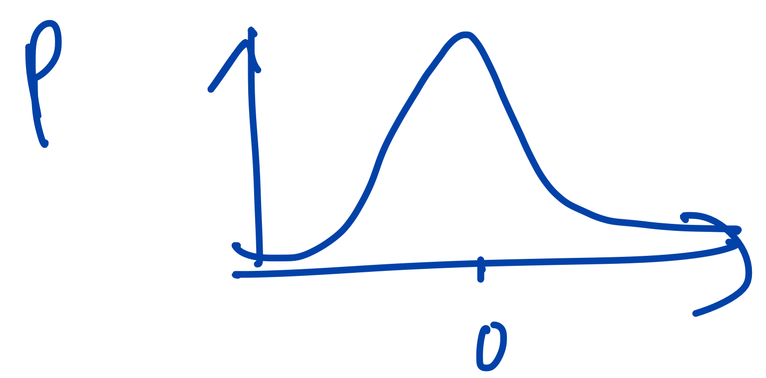
$$c = x^T Q x + u^T R u$$

$$z^T \begin{bmatrix} \ddots & 0 \\ 0 & 1 \end{bmatrix} z$$

Extension to stochastic systems

LQR

$$x_{t+1} = Ax_t + Bu_t + w_t \longrightarrow \mathcal{N}(0, 1I)$$



LQG

Solution is same as LQR

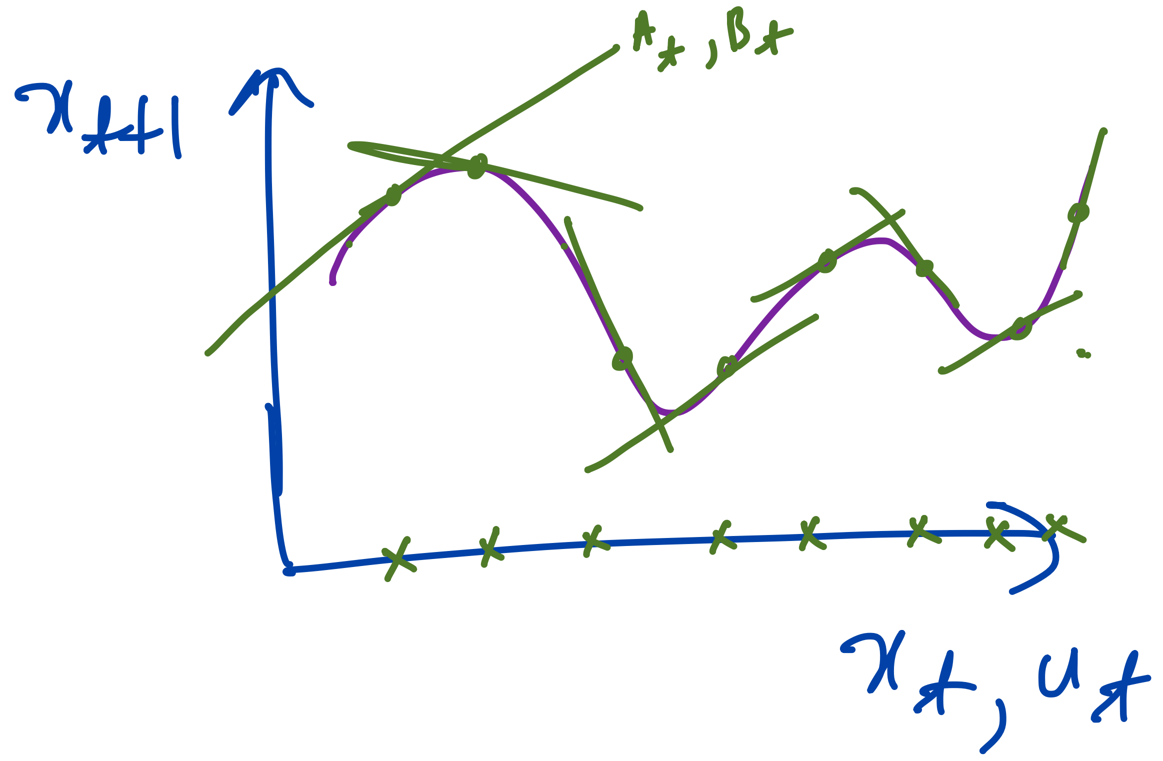
* $V_t(x)$ changes.

Extension to non-linear systems

$x_{t+1} = Ax_t + Bu_t$

$x_{t+1} = f(x_t, u_t)$

$$\min_{(u_0, u_1, \ldots u_{H-1})} \sum_{t=0}^{H} c(x_t, u_t) \rightarrow \text{Quadratic}$$



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iterative LQR (iLQR)

$f(x_t, u_t)$

$x_0, u_0, u_1, \ldots u_H$

step 1: (randomly) initialize a sequence of actions $:= (u_0, u_1, \ldots u_H)$

step 2: Compute sequence of states $\underbrace{(x_0, x_1, \ldots x_H)}$, given x_0 and seq. of us

step 3: Compute linearization about x_s.

$$A_t, B_t$$

step 4: Use LQR for obtaining $\underline{K_t}$ vectors

$$u_t = K_t x_t \quad \Leftarrow \text{From LQR}$$

step 5: compute new actions u_s and repeat

Extension to non-linear systems

Iterative Linear Quadratic Regulator Design for Nonlinear Biological Movement Systems

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