Introduction to Robot Intelligence
[Spring 2023]

Robot Transformations (Part 2)

February 21, 2023

Lerrel Pinto

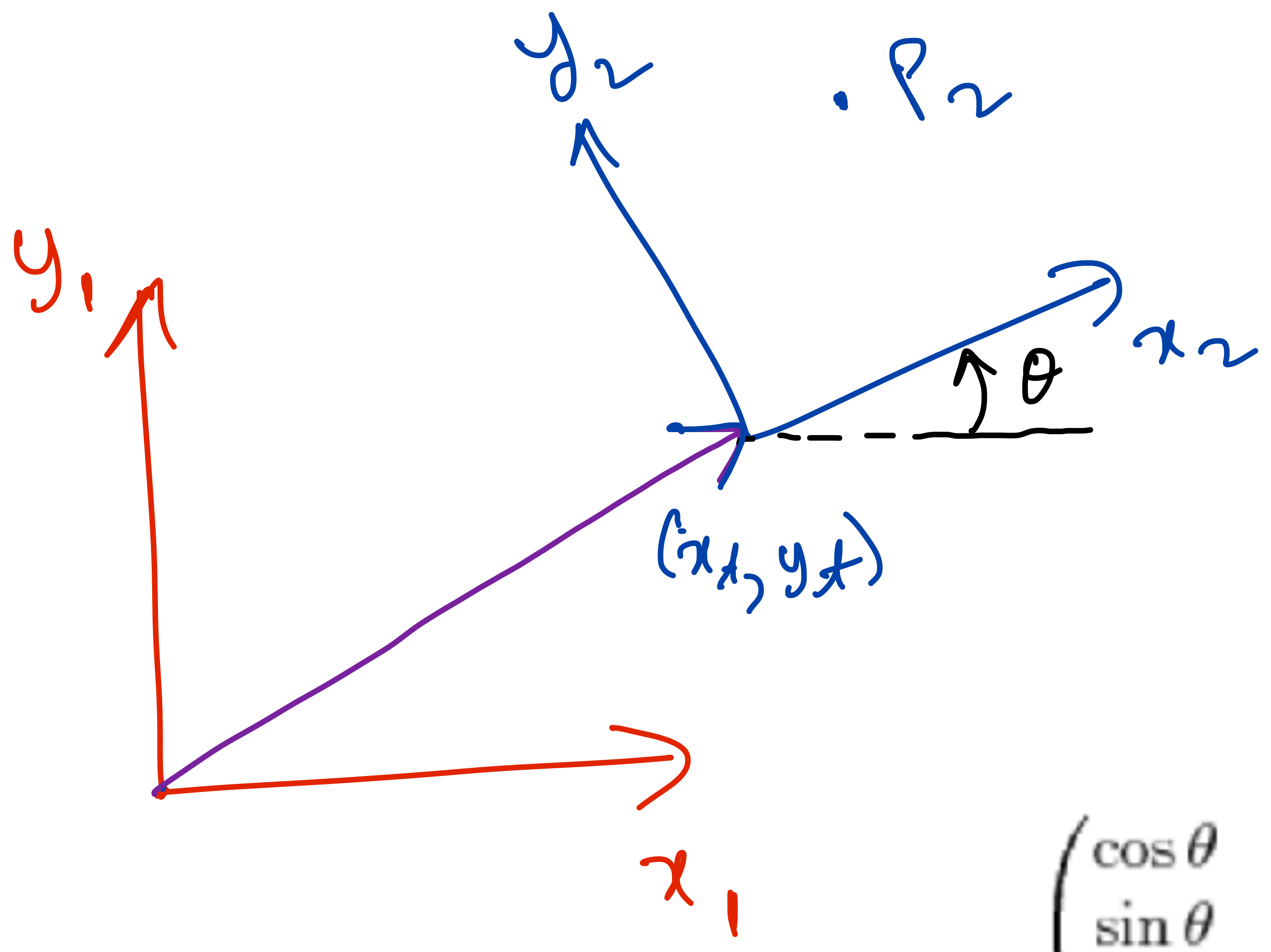
Review from last class

- Viewing a robot as a composition of rigid bodies.
- Fundamentals of rigid bodies

A mapping $g : \mathbb{R}^3 \to \mathbb{R}^3$ is a *rigid body transformation* if it satisfies the following properties:

1. Length is preserved: $\|g(p) - g(q)\| = \|p - q\|$ for all points $p, q \in \mathbb{R}^3$.
2. The cross product is preserved: $g_*(v \times w) = g_*(v) \times g_*(w)$ for all vectors $v, w \in \mathbb{R}^3$.

Summary of 2D Homogenous Transformation Matrix



$$\begin{pmatrix} x\cos\theta - y\sin\theta + x_t \\ x\sin\theta + y\cos\theta + y_t \end{pmatrix}.$$

$$\begin{pmatrix} \cos\theta & -\sin\theta & x_t \\ \sin\theta & \cos\theta & y_t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta + x_t \\ x\sin\theta + y\cos\theta + y_t \\ 1 \end{pmatrix}.$$

$$T = \begin{pmatrix} \cos\theta & -\sin\theta & x_t \\ \sin\theta & \cos\theta & y_t \\ 0 & 0 & 1 \end{pmatrix},$$

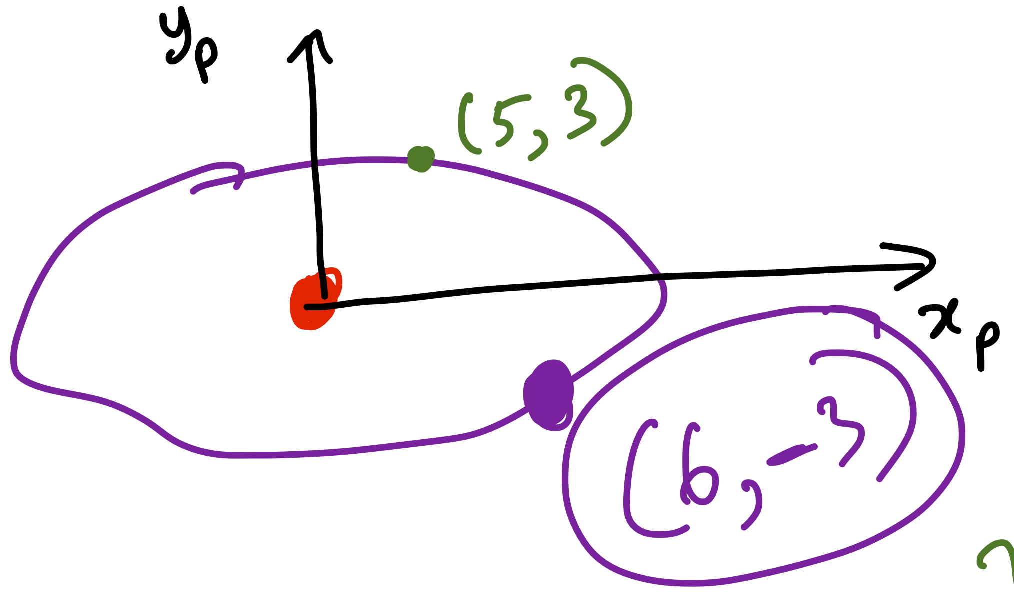
$$P_1 = \begin{bmatrix} \quad \end{bmatrix} P_2$$

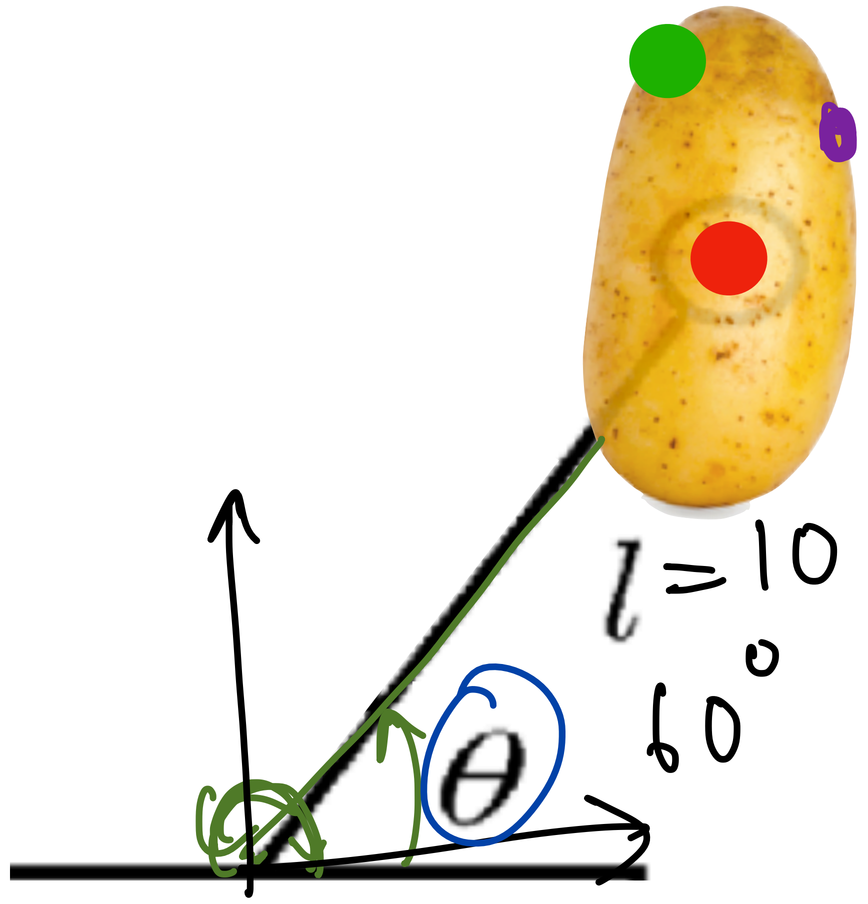
$$= \begin{bmatrix} \cos\theta & -\sin\theta & x_t \\ \sin\theta & \cos\theta & y_t \\ 0 & 0 & 1 \end{bmatrix}$$

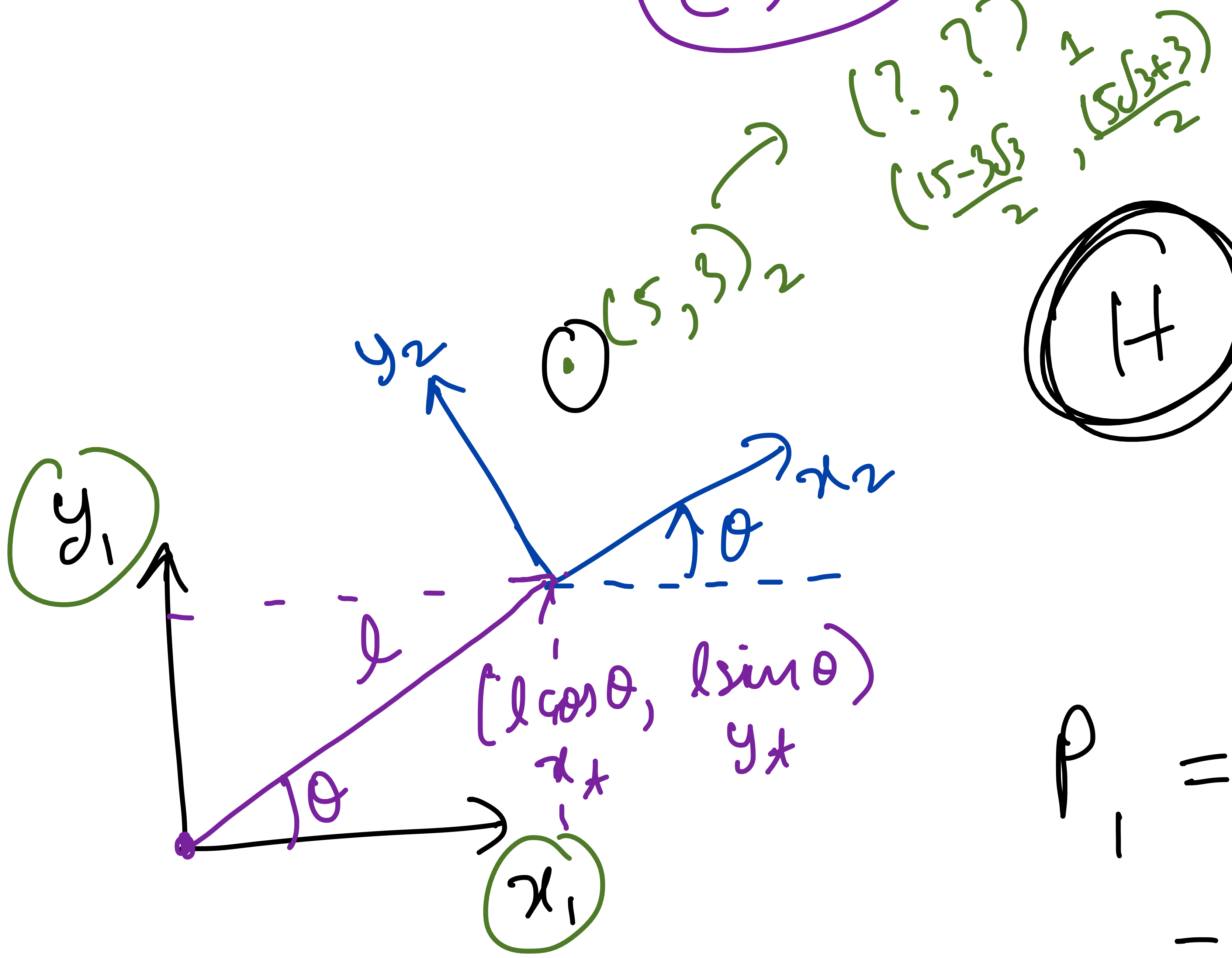
$$= \begin{bmatrix} R_{2\times 2} & t_{2\times 1} \\ 0_{1\times 2} & 1 \end{bmatrix}$$

http://planning.cs.uiuc.edu/node99.html

A localization exercise







$$\begin{array}{ccc} 1/2 & -\sqrt{3}/2 & 10\cdot\frac{1}{2}=5 \end{array}$$

$$H = \begin{bmatrix} \cos\theta & -\sin\theta & x_t \\ \sqrt{3}/2 & 1/2 & y_t \\ 0 & 0 & 1 \end{bmatrix} \quad 10\cdot\frac{\sqrt{3}}{2}$$

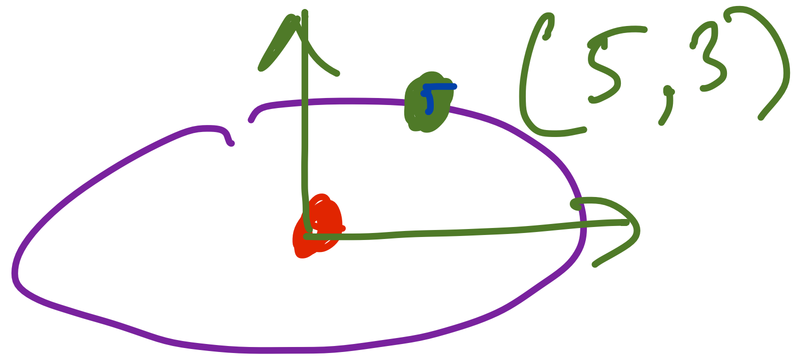
$$H = \begin{bmatrix} 1/2 & -\sqrt{3}/2 & 5 \\ \sqrt{3}/2 & 1/2 & 5\sqrt{3} \\ 0 & 0 & 1 \end{bmatrix}$$

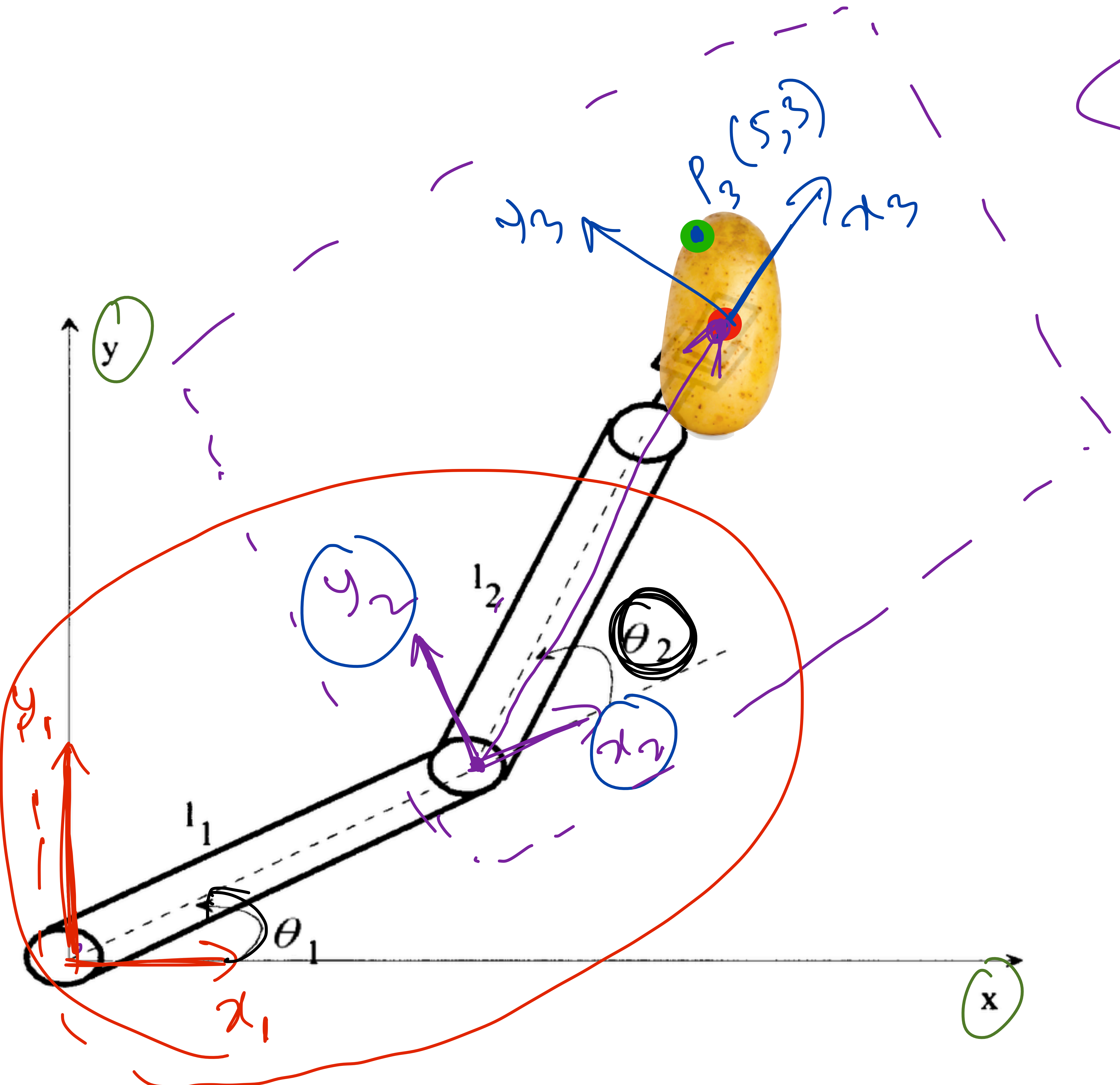
$$P_1 = H\,P_2$$

$$= H \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$$

$$P_1 = \begin{bmatrix} \frac{15-3\sqrt{3}}{2} \\ \frac{15\sqrt{3}+3}{2} \\ 1 \end{bmatrix}$$

A slightly more complicated localization exercise





$$H_{3\to2} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & l_2\cos\theta_2 \\ \sin\theta_2 & \cos\theta_2 & l_2\sin\theta_2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P_2 = H_{3\to2}\begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$$

$$H_{2\to1} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & l_1\cos\theta_1 \\ \sin\theta_1 & \cos\theta_1 & l_1\sin\theta_1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P_1 = H_{2\to1} P_2$$

$$P_1 = H_{2\to1} H_{3\to2}\begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$$

3D rotations

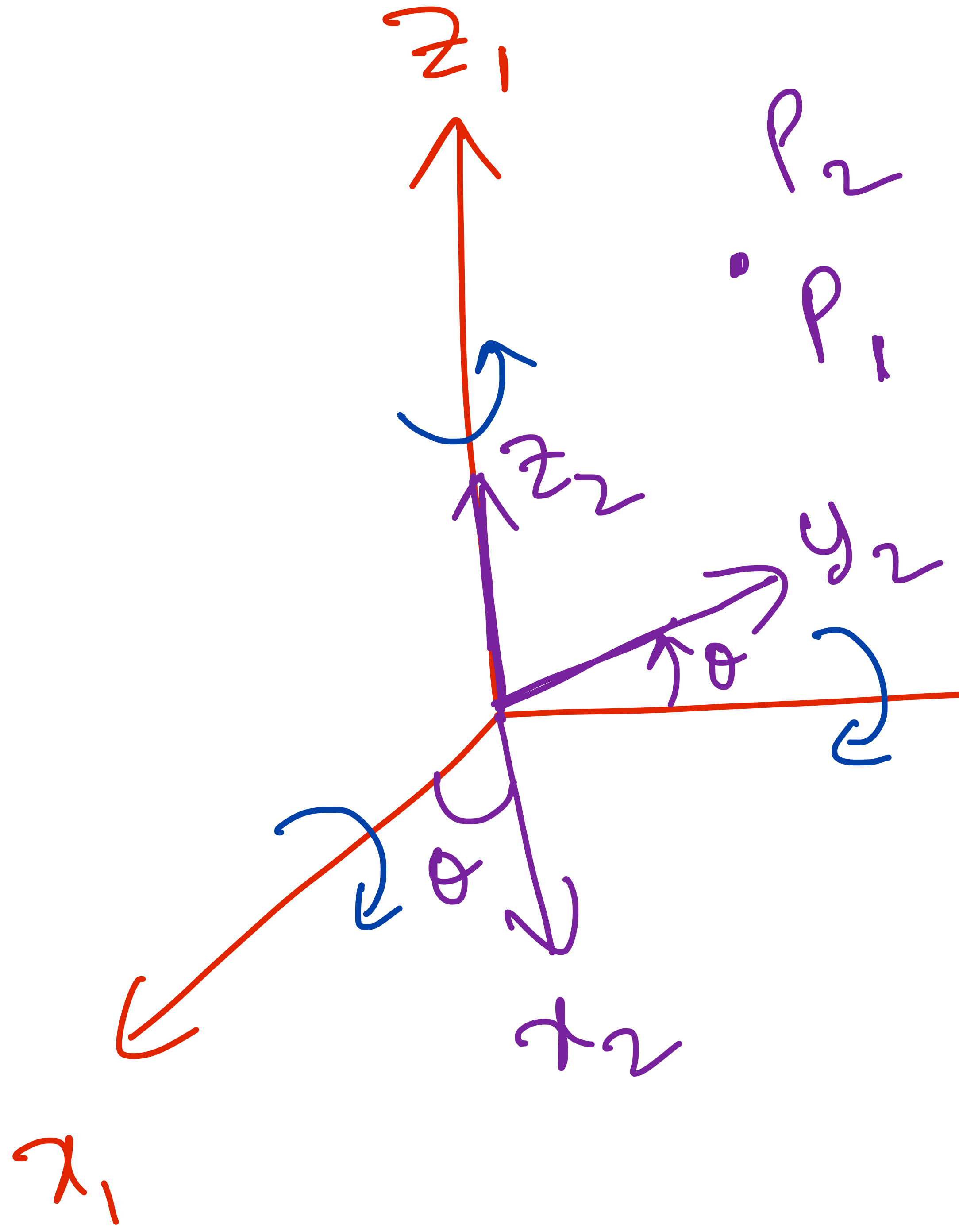
2D rotation

$P_2 \quad (a_2, b_2, c_2)$

$P_1 \quad (?, ?, ?)$

$c_1 = c_2$

c_1



$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}$$

rotations in z axis

$$\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$$

$a_1 = a_2$

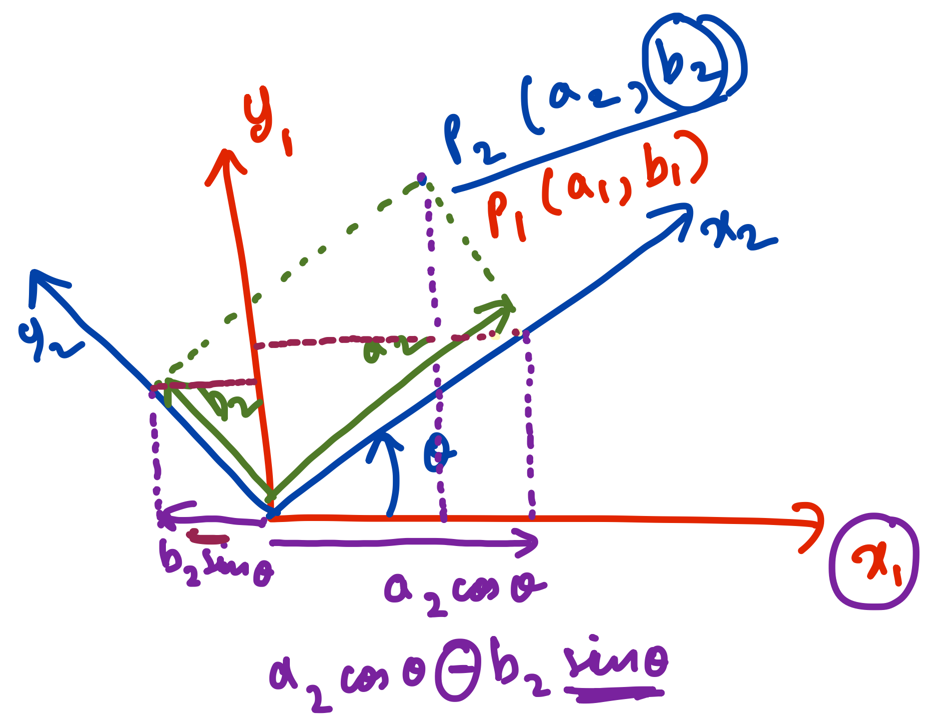
$$P_1 = R_z(\theta)\, P_2$$

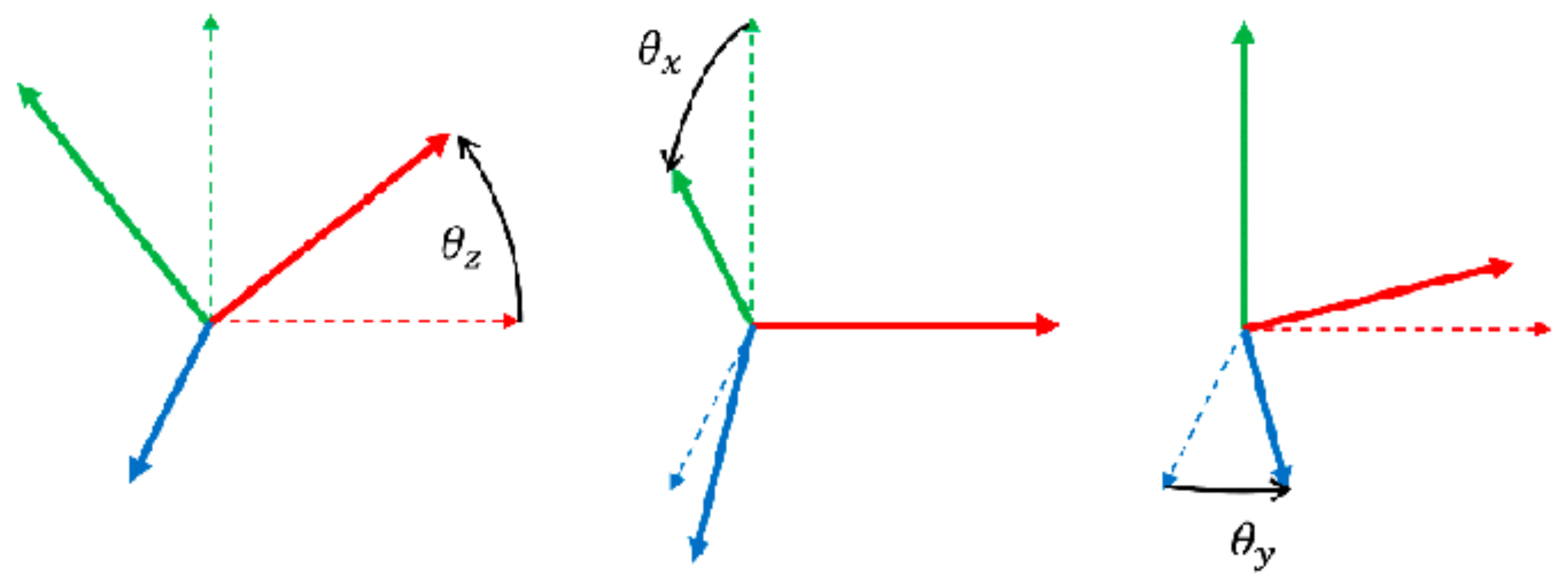
R_x

$$\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$$

$$R_y = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

Summary of 3D rotations





$$R = R_z(\theta_1) R_y(\theta_2) R_x(\theta_3) R_z(\theta_4)$$

First, the matrix for rotation about the Z axis contains a 2D rotation matrix in its upper corner:

$$R_Z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (5)$$

R

This can be interpreted by imagining the Z axis pointing out of the page, and the X and Y axes marking the axes of a standard graph on the page. The rotation about θ is a CCW rotation in the plane of the page. Notice that the Z coordinate of any point is preserved by this operation, a property maintained by the third row $(0, 0, 1)$, nor does it affect the X and Y coordinates, a property maintained by the first two 0 entries of the third column. In the (X, Y) plane, the upper 2×2 matrix is identical to a 2D rotation matrix.

The rotation about the X axis is similar, with a 2D rotation matrix appearing along the Y and Z axes:

$$R_X(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}. \quad (6)$$

R

Here the X coordinate is preserved while the Y and Z entries perform a 2D rotation.

Finally, the rotation about the Y axis is similar, but with a sign switch of the sin terms:

$$R_Y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}. \quad (7)$$

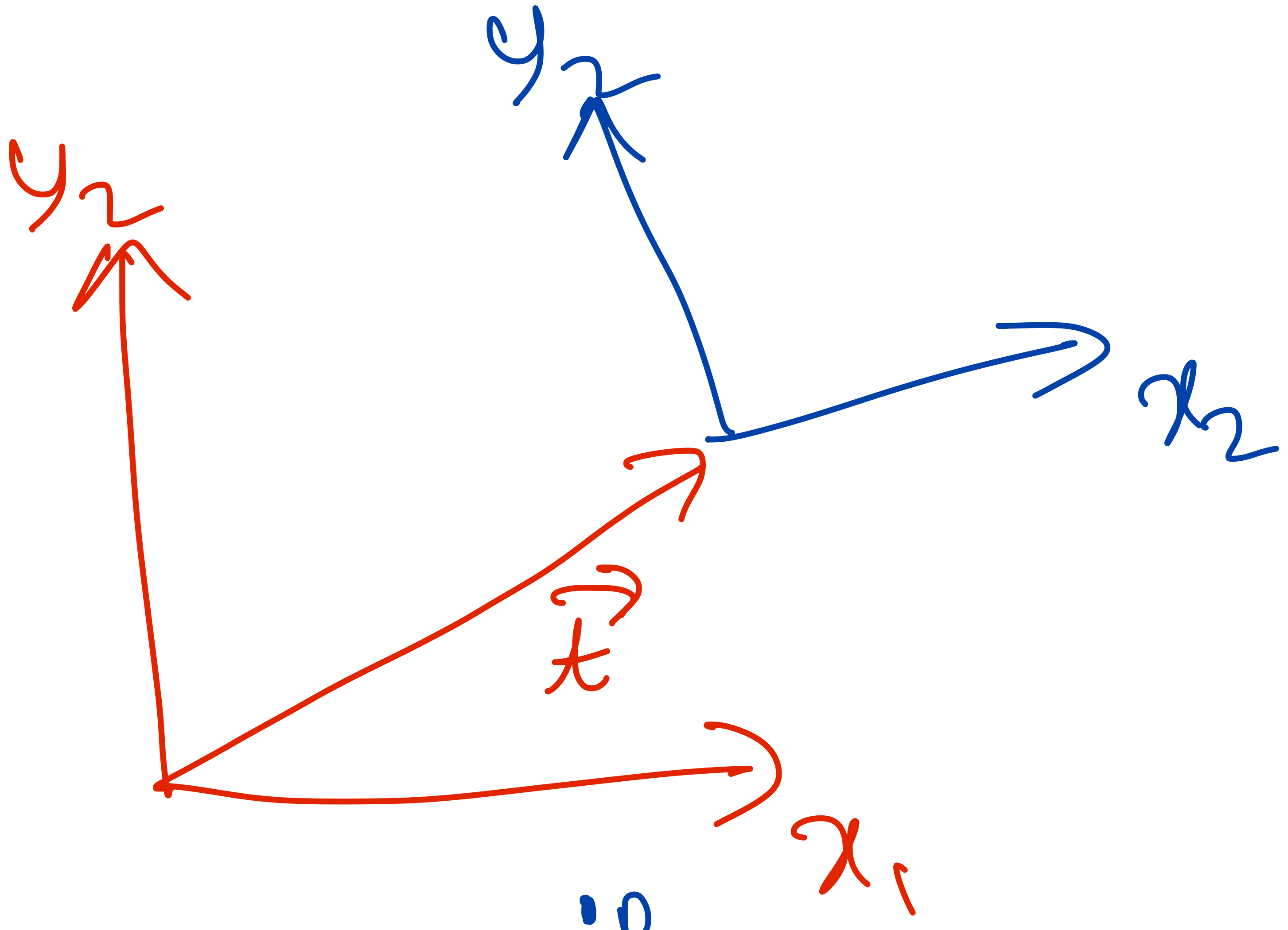
$-\sin\theta$

$\cos$ $\sin\theta$

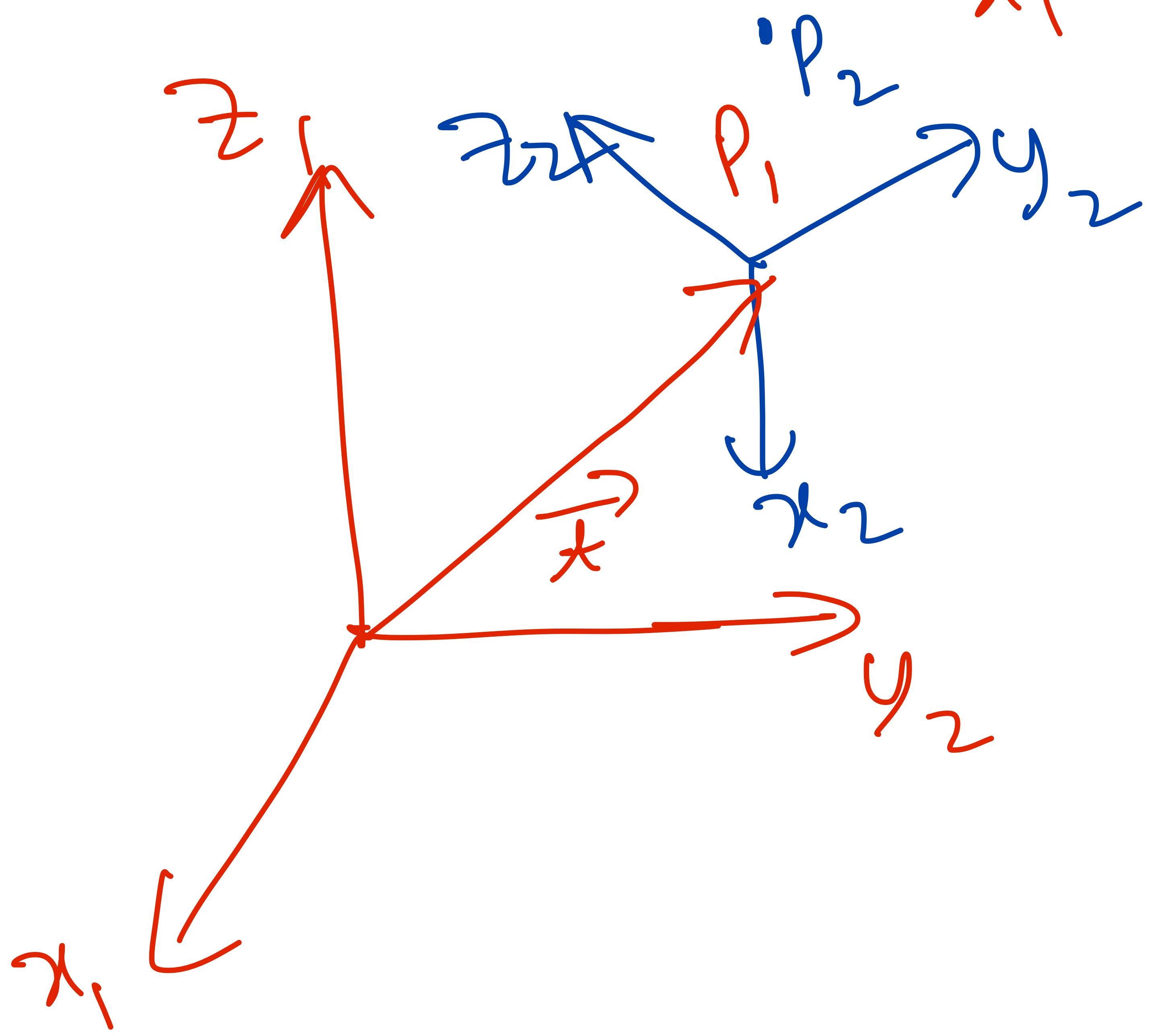
The reason why the $-\sin\theta$ term switches to below the diagonal is that if one were to orient a frame so the Y axis points out of the page, and align the X axis to point rightward, the Z axis would point downward instead of upward. Instead, the matrix can be derived by aligning the Z axis to the rightward direction and the X axis to the upward direction, so that the $-\sin\theta$ term arrives in the Z, X spot. A mnemonic to help remember the sign switch on rotations about Y is that the order of the two coordinate directions defining the orthogonal plane is derived from a *cyclic* ordering of the axes: Y follows X, Z follows Y, and X follows Z. So, the plane orthogonal to X is (Y, Z), the plane orthogonal to Y is (Z, X), and the plane orthogonal to Z is (X, Y).

1. http://scipp.ucsc.edu/~haber/ph216/rotation_12.pdf
2. http://motion.pratt.duke.edu/RoboticSystems/3DRotations.html

3D rotations + translations



$$H = \begin{bmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix}_{3\times3}$$



$$H = \begin{bmatrix} & R_{3\times3} & & \begin{matrix} t_x \\ t_y \\ t_z \end{matrix} \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4\times4}$$

$$P_1 = \underset{4\times4}{H}\, P_2$$

http://scipp.ucsc.edu/~haber/ph216/rotation_12.pdf

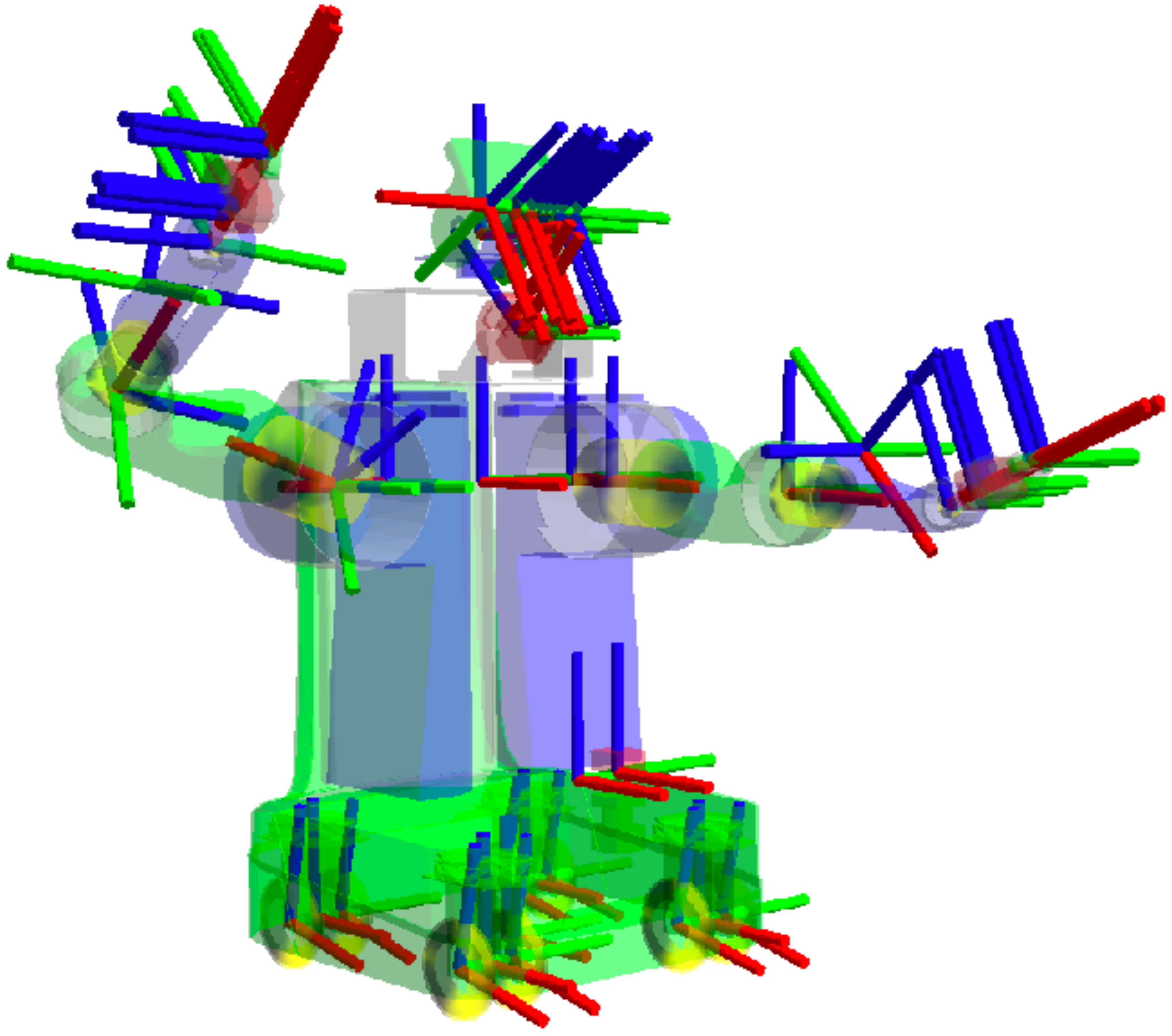
3D Homogenous Transformation Matrix

Properties of H

Inverse

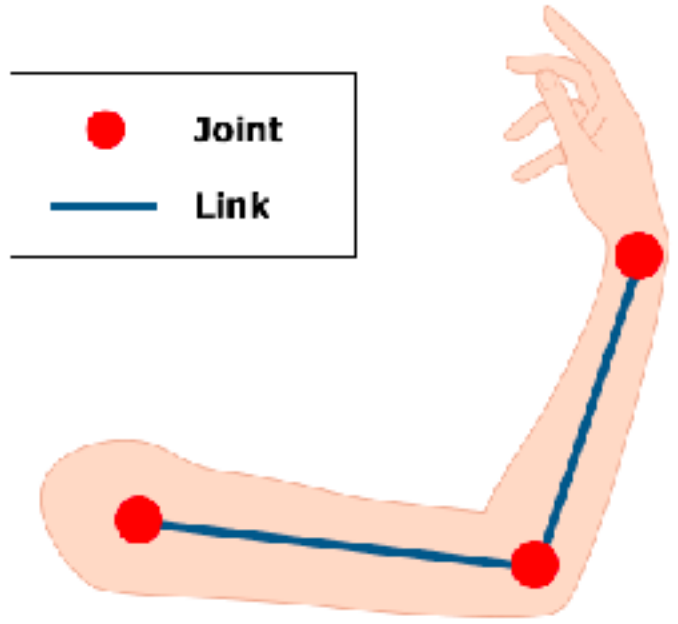
Composition

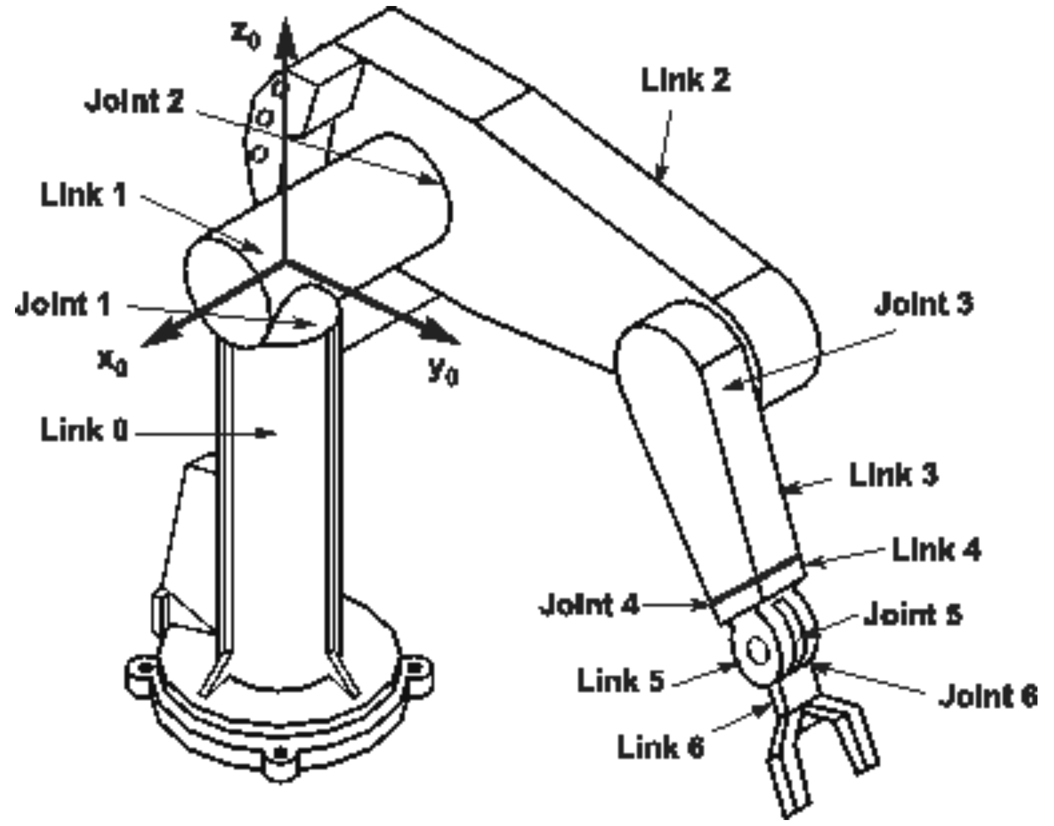
Robot as a collection of rigid bodies + transformations!



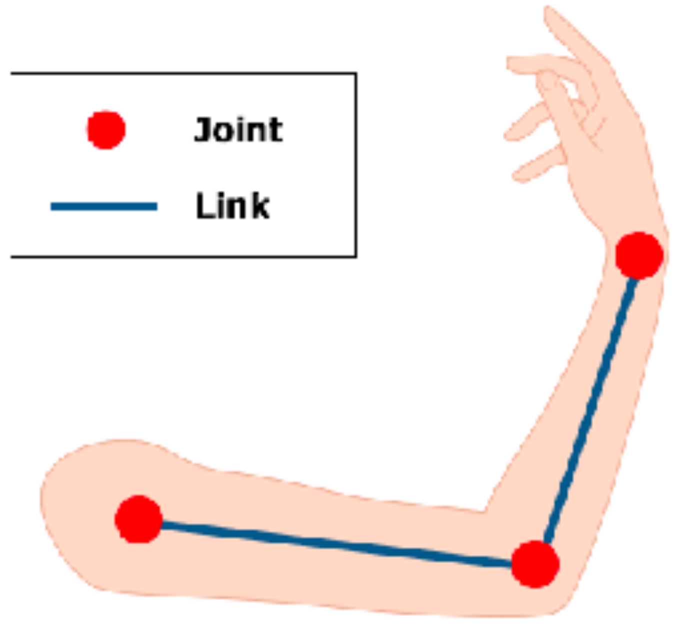
Modelling a robot

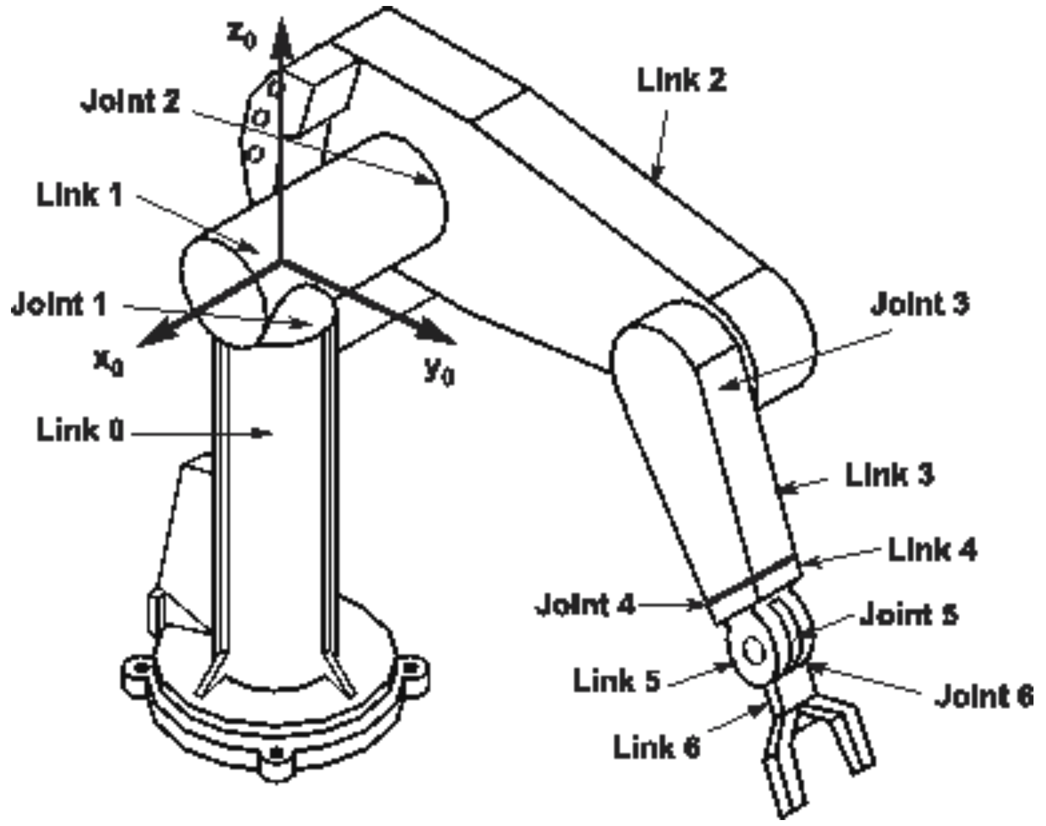
Modelling a robot



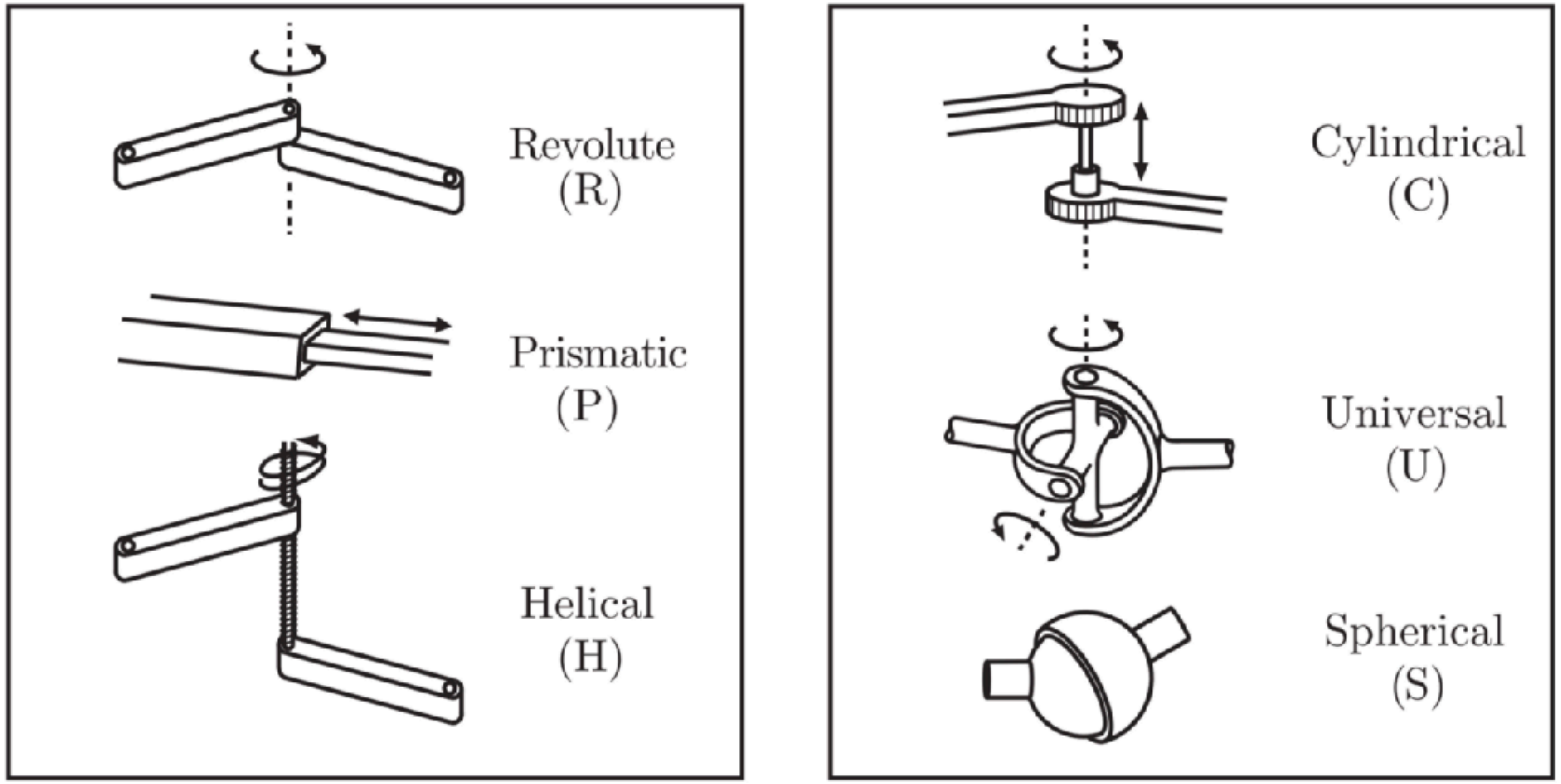


Modelling a robot





Types of joints



Modelling a robot