Today: $\boxed{\text{Ken}}$

$\boxed{3.2}$ Predicates & Quantifiers II

$\boxed{3.3}$ Statements with Multiple Quantifiers

Last time:

$\boxed{3.1}$ Predicates & Quantifiers I

$\boxed{3.2}$ Predicates & Quantifiers II

$A \subset B$ $\qquad$ $A = \{x \in B : p(x)\} = \{\} = \emptyset$

$p(x)$

$q(x)$

$\forall x \in B\, (p(x) \to q(x))$

$\forall x \in A\, (q(x))$

Variants of Universal Conditional Statements

Given predicates $P(x)$, $Q(x)$ both with domain A, consider the universal conditional statement

$$\forall x \in A\, (P(x) \longrightarrow Q(x))$$

① Its contrapositive is $\forall x \in A\, (\neg Q(x) \to \neg P(x))$.

$$\forall x \in A\, (P(x) \longrightarrow Q(x)) \equiv \forall x \in A\, (\neg Q(x) \to \neg P(x))$$

② Its converse is $\forall x \in A\, (Q(x) \longrightarrow P(x))$.

$$\forall x \in A\, (P(x) \longrightarrow Q(x)) \not\equiv \forall x \in A\, (Q(x) \longrightarrow P(x))$$

e.g. All American bison are quadrupeds.

converse? All quadrupeds are American bison. F

counterexample?

dogs are quadrupeds.

③ Its inverse is $\forall x \in A(\neg P(x) \to \neg Q(x))$.

$$\forall x \in A(P(x) \to Q(x)) \not\equiv \forall x \in A(\neg P(x) \to \neg Q(x))$$

e.g. $\forall x \in \mathbb{R}\,(x \in [0,\pi] \to \sin(x) \geq 0)$

inverse? $\forall x \in \mathbb{R}\,(x \notin [0,\pi] \to \sin(x) < 0)$

counterexample? $x = 2\pi$ & $\sin(2\pi) = 0$

$$\forall x \in \mathbb{R}\,(x \in (-\infty,0) \cup (\pi,\infty) \to \sin(x) < 0)$$

$$\neg \forall x \in \mathbb{R}\,(x \in (-\infty,0) \cup (\pi,\infty) \to \sin(x) < 0)$$

$$\equiv \neg \forall x \in \mathbb{R}\,(x \in [0,\pi] \vee \sin(x) < 0)$$

$$\equiv \exists x \in \mathbb{R}\,(\neg(x \in [0,\pi] \vee \sin(x) < 0))$$

$$\equiv \exists x \in \mathbb{R}\,(x \in (-\infty,0) \cup (\pi,\infty) \wedge \sin(x) \geq 0)$$

Definition

① $\forall x\, P(x)$ is a sufficient condition for $Q(x)$ if

$\forall x\,(P(x) \to Q(x))$

e.g. All whales are mammals.

Being a whale is a sufficient condition for being a mammal.

② $\forall x(Q(x))$ is a necessary condition for $P(x)$ if $\forall x\,(P(x) \rightarrow Q(x))$

e.g. $\forall n \in \mathbb{Z}\ \left(n \geq 1 \longrightarrow \frac{1}{2n+1} \leq 3\right)$

③ $\forall x\ P(x)$ only if $Q(x)$ means $\forall x\,(\neg Q(x) \longrightarrow \neg P(x))$ or equivalently $\forall x\,(P(x) \rightarrow Q(x))$

$\boxed{3.3}$ Statements with Multiple Quantifiers

The ordering of quantifiers matter.

e.g. Are the following statements true or false?

① For any real number x, there exists a real number y such that $x<y$.

$\forall x \in \mathbb{R}\ \exists y \in \mathbb{R}\,(x<y)$

true

② $\exists y \in \mathbb{R}\,(\forall x \in \mathbb{R}\,(x<y))$ false

There exists a real number y such that for any real number x, $x<y$.

In a statement containing both $\forall$ and $\exists$, changing the order of the quantifiers can significantly change the meaning of the statement.

Interpreting Statements with Two Different Quantifiers

Given domains A, B and predicate $P(x,y)$,

① if we want to prove true the statement

$$\forall x \in A (\exists y \in B (P(x,y)))$$

then, we let someone pick an arbitrary $x \in A$ and we must find some $y \in B$ such that $P(x,y)$ is true.

② if we want to prove true the statement

$$\exists x \in A (\forall y \in B (P(x,y)))$$

then, we must provide an $x \in A$ such that, no matter what $y \in B$ someone picks, $P(x,y)$ is still true.

e.g. Can we prove that for any real number x, there exists a real number y such that $x<y$?

Let $x \in \mathbb{R}$. Choose $y := x+1 \in \mathbb{R}$.

So $y = x+1 > x$.

e.g. Can we prove there exists $K \in \mathbb{Z}$ such that $K<n$ for all $n \in \mathbb{Z}^+$?

Consider $-1 \in \mathbb{Z}$. Let $n \in \mathbb{Z}^+$. So $-1 < n$.

e.g. For any $n \in \mathbb{Z}^+$, i.e. n is any positive integer, a sequence is an ordered collection of real numbers

$$a_1, a_2, a_3, \dots$$

and the limit of a convergent sequence is denoted

$$\lim_{n \to \infty} a_n = L$$

where $L \in \mathbb{R}$, i.e. L is a unique real number.

e.g. If $a_n = 1/n$ then

$1, 1/2, 1/3, 1/4, \ldots$

$$\lim_{n\to\infty} 1/n = 0.$$

What's the precise definition of the limit of a convergent sequence?

For any $\varepsilon > 0$, there exists a positive integer n_0 such that for any integer n, $n \geq n_0$ implies $|a_n - L| < \varepsilon$.

$$\forall \varepsilon > 0\ \exists n_0 \in \mathbb{Z}^+\ \forall n \in \mathbb{Z}\ (n \geq n_0 \to |a_n - L| < \varepsilon)$$

What's the negation?

$$\neg \forall \varepsilon > 0\ \exists n_0 \in \mathbb{Z}^+\ \forall n \in \mathbb{Z}\ (n \geq n_0 \rightarrow |a_n - L| < \varepsilon)$$

$$\equiv \neg \forall \varepsilon > 0\ \exists n_0 \in \mathbb{Z}^+\ \forall n \in \mathbb{Z}\ (n < n_0 \vee |a_n - L| < \varepsilon)$$

$$\equiv \exists \varepsilon > 0\ \forall n_0 \in \mathbb{Z}^+\ \exists n \in \mathbb{Z}\ (\neg(n < n_0 \vee |a_n - L| < \varepsilon))$$

$$\equiv \exists \varepsilon > 0\ \forall n_0 \in \mathbb{Z}^+\ \exists n \in \mathbb{Z}\ (n \geq n_0 \wedge |a_n - L| \geq \varepsilon)$$

example 3.3.3

A college cafeteria line has four stations: salad, main courses, desserts, and beverages. The salad station offers a choice of green salad or fruit salad; the main course station offers spaghetti or fish; the dessert station offers pie or cake; and the beverage station offers milk, soda, or coffee. Three students, Uta, Tim, and Yuen, go through the line and make the following choices:

Uta: green salad, spaghetti, pie, milk

Tim: fruit salad, fish, pie, cake, milk, coffee

Yuen: spaghetti, fish, pie, soda

Write each of the statements informally and find its truth value.

ⓐ $\exists$ an item I such that $\forall$ student S, S chose I. true

ⓑ $\exists$ a student S such that $\forall$ item I, S chose I. false

ⓒ $\exists$ a student S such that $\forall$ station Z, $\exists$ an item in I in Z such that S chose I true, Uta, Tim

ⓓ $\forall$ student S and $\forall$ station Z, $\exists$ an item I in Z such that S chose I False, Yuen has no salad

Negations of Statements with Two Different Quantifiers

Given domains A, B and predicate $P(x,y)$,

① $\neg \forall x \in A\,(\exists y \in B\,(P(x,y))) \equiv \exists x \in A\,(\neg \exists y \in B\,(P(x,y)))$

$\equiv \exists x \in A\,(\forall y \in B\,(\neg P(x,y)))$

② $\neg \exists x \in A\,(\forall y \in B\,(P(x,y))) \equiv \forall x \in A\,(\neg \forall y \in B\,(P(x,y)))$

$\equiv \forall x \in A\,(\exists y \in B\,(\neg P(x,y)))$