CSCI-UA 480 Robot Intelligence Homework 1

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Question 1 Two players take turns trying to kick a ball into a net while playing soccer. Player 1 succeeds at kicking the ball into the net with probability $\frac{1}{5}$, while Player 2 succeeds with the probability $\frac{1}{4}$. Whoever succeeds first, wins the game. Assuming that Player 1 takes the first shot, what is the probability that Player 1 wins the game? Show all steps of computation.

Solution.

$$\begin{aligned}
&\mathbb{P}(\text{Player 1 wins in the } n^{\text{th}} \text{ round}) \\
&= \mathbb{P}(\text{The game does not end after } (n-1) \text{ rounds}) \times \mathbb{P}(\text{Player 1 kicks the ball into the } n^{\text{th}} \text{ round}) \\
&= \left(\frac{4}{5} \cdot \frac{3}{4}\right)^{n-1} \cdot \frac{1}{5}
\end{aligned}$$

We can sum up the probability that Player 1 wins in each round to obtain the toal probability

$$\begin{aligned}
&\mathbb{P}(\text{Player 1 wins}) \\
&= \sum_{n=1}^{\infty} \left(\frac{4}{5} \cdot \frac{3}{4}\right)^{n-1} \cdot \frac{1}{5} \\
&= \frac{1}{5} \cdot \sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^{n} \\
&= \frac{1}{5} \cdot \frac{1}{1-\frac{3}{5}} \\
&= \frac{1}{2}
\end{aligned}$$

□

Question 2 You know that 1% of the population has COVID. You also know that 90% of the people who have COVID get a positive test result, while 10% of people who do not have COVID also test positive. What is the probability that you have COVID, given that you tested positive?

Solution.

Given that you tested positive, the probability that you have COVID is

$$\begin{aligned}
\frac{\text{true positive}}{\text{tested positive}} &= \frac{\text{true positive}}{\text{true positive} + \text{false positive}} \\
&= \frac{1\% \cdot 90\%}{1\% \cdot 90\% + 99\% \cdot 10\%} \\
&= 8.33\%
\end{aligned}$$

□

Question 3 Let the function f (x) be defined as below.

$$f(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{1+x} & \text{otherwise.} \end{cases}$$

Is $f(x)$ a valid probability-density function (PDF)? If yes, then prove that it is a PDF. If not, then prove that it is not a PDF.

Solution.

$$\int_{\mathbb{R}} f(x)dx = \int_{-\infty}^{0} 0dx + \int_{0}^{\infty} \frac{1}{1+x}dx = \int_{0}^{\infty} \frac{1}{1+x}dx = \ln(1+x)\Big|_{0}^{\infty} = \infty$$

Since the integration over $\mathbb{R}$ isn't equal to 1, $f(x)$ is not a valid probability-density function (PDF). □

Question 4 Assume that X and Y are two independent, identically distributed (i.i.d.) random variables with probability density function f(x), such that

$$f(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

What is the value of $\mathbb{P}(X + Y \leq 1)$?

Solution.

We can integrate the joint probability density function (pdf) of X and Y over the region where $X + Y \leq 1$.

$$\mathbb{P}(X + Y \leq 1) = \int_{\mathbb{R}} \int_{-\infty}^{1-y} (2x)(2y)dxdy = \int_{0}^{1} \int_{0}^{1-y} 4xydxdy = \int_{0}^{1} (2y^3 - 4y^2 + 2y)dy = \frac{1}{6}$$

□

Question 5 Let X be a uniformly distributed random variable over the open interval $(0, 1)$, $X \sim Unif(0, 1)$. Let $Y = e^X$, so that Y is the random variable that results from exponentiating X. Compute the expectation of Y. Show your work.

Solution.

The probability density function of X is

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

The expectation of Y is

$$\mathbb{E}[Y] = \int_{\mathbb{R}} yp(y)dy = \int_{0}^{1} e^x f(x)dx = \int_{0}^{1} e^x dx = e - 1$$

□

Question 6 The number of errors associated with a computer program follows a Poisson distribution with mean $\lambda = 5$. Suppose that we have 125 program submissions and let $X_i \in \{X_1, ..., X_{125}\}$ be an i.i.d. random variable denoting the number of errors associated with each submission i. Compute the probability that the average number of errors in this sample, $\bar{X} = \frac{1}{125}\sum_{i=1}^{125} X_i$, is strictly less than 5.5. Show your work.

Solution.

$$\mathbb{P}(X = x) = \frac{e^{-5}5^x}{x!}$$
$$E[X] = 5, Var[X] = 5$$

Since each $X_i \in \{X_1, ..., X_125\}$ is an i.i.d. random variable, by Central Limit Theorem (CLT),

$$Z_n = \sqrt{n} \cdot \frac{\bar{X} - E[X]}{\sqrt{Var[X]}} = \sqrt{125} \cdot \frac{\bar{X} - 5}{\sqrt{5}} = 5(\bar{X} - 5) \approx Z = N(1, 0)$$

$$\mathbb{P}(\bar{X} < 5.5) = \mathbb{P}(z < 5 \cdot (5.5 - 5)) = \mathbb{P}(z < 2.5) = 0.9938$$

□

Question 7 Let $X_n = f(W_n, X_{n-1})$ represent the value of a recurrence defined over $n \in \{1, ..., p\}$ for some unknown function $f(\cdot)$. Consider E given by $E = ||c - X_p||^2$, where c is an unknown constant.
Compute the value of gradient $\frac{\partial E}{\partial X_0}$. Write your answer in terms of f, c, and $W_i \in \{W_1, ..., W_p\}$.

Solution.

$$\frac{\partial X_n}{\partial X_0} = \frac{\partial f(W_n, X_{n-1})}{\partial X_0} = \frac{\partial f(W_n, X_{n-1})}{\partial X_{n-1}} \cdot \frac{\partial X_{n-1}}{\partial X_0} = \prod_{i=1}^{n} \frac{\partial f(W_i, X_{i-1})}{\partial X_{i-1}}$$

$$\begin{aligned}
\frac{\partial E}{\partial X_0} &= \frac{\partial ||c - X_p||^2}{\partial X_0} \\
&= 2(X_p - c) \cdot \frac{\partial X_p}{\partial X_0} \\
&= 2(X_p - c) \cdot \prod_{i=1}^{p} \frac{\partial f(W_i, X_{i-1})}{\partial X_{i-1}}
\end{aligned}$$

□

Question 8 Let A denote a matrix,

$$A = \begin{bmatrix} 2 & 6 & 7 \\ 3 & 1 & 2 \\ 5 & 3 & 4 \end{bmatrix}$$

and let x denote a column vector

$$x = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

Let A^T and x^T denote the transposes of A and x, respectively. Compute the matrix-vector products Ax, $A^T x$, and $x^T A$.

Solution.

$$Ax = \begin{bmatrix} 2 & 6 & 7 \\ 3 & 1 & 2 \\ 5 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 50 \\ 17 \\ 35 \end{bmatrix}$$

$$A^T x = \begin{bmatrix} 2 & 3 & 5 \\ 6 & 1 & 3 \\ 7 & 2 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 33 \\ 27 \\ 36 \end{bmatrix}$$

$$x^T A = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 6 & 7 \\ 3 & 1 & 2 \\ 5 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 33 & 27 & 36 \end{bmatrix}$$

□

Question 9 Determine whether the following matrices are invertible. If so, compute their inverses. Show all steps of computation.

(a)

$$\begin{bmatrix} 6 & 2 & 3 \\ 3 & 1 & 1 \\ 10 & 3 & 4 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 1 & 4 & 5 \end{bmatrix}$$

Solution.

(a)

$$\det\left(\begin{bmatrix} 6 & 2 & 3 \\ 3 & 1 & 1 \\ 10 & 3 & 4 \end{bmatrix}\right) = 6 \times 1 \times 4 + 2 \times 1 \times 10 + 3 \times 3 \times 3 - 3 \times 1 \times 10 - 2 \times 3 \times 4 - 6 \times 1 \times 3 = -1$$

Therefore it is invertible.

$$\left[\begin{array}{ccc|ccc} 6 & 2 & 3 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 & 1 & 0 \\ 10 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & \frac{1}{3} & \frac{1}{2} & \frac{1}{6} & 0 & 0 \\ 3 & 1 & 1 & 0 & 1 & 0 \\ 10 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & \frac{1}{3} & \frac{1}{2} & \frac{1}{6} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{1}{3} & -1 & -\frac{5}{3} & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & \frac{1}{3} & \frac{1}{2} & \frac{1}{6} & 0 & 0 \\ 0 & 1 & 3 & 5 & 0 & -3 \\ 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{2} & -\frac{3}{2} & 0 & 1 \\ 0 & 1 & 3 & 5 & 0 & -3 \\ 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 1 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{2} & -\frac{3}{2} & 0 & 1 \\ 0 & 1 & 3 & 5 & 0 & -3 \\ 0 & 0 & 1 & 1 & -2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -1 & 1 \\ 0 & 1 & 0 & 2 & 6 & -3 \\ 0 & 0 & 1 & 1 & -2 & 0 \end{array}\right]$$

Therefore the inverse is

$$\begin{bmatrix} -1 & -1 & 1 \\ 2 & 6 & -3 \\ 1 & -2 & 0 \end{bmatrix}.$$

(b)

$$\det\left(\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 1 & 4 & 5 \end{bmatrix}\right) = 1\times 2\times 5 + 2\times 2\times 1 + 3\times 0\times 4 - 3\times 2\times 1 - 2\times 0\times 5 - 1\times 2\times 4 = 0$$

Therefore it is not invertible.

□

Question 10 What is an eigenvalue of a matrix? What is an eigenvector of a matrix? Describe one method (any method) you could use to compute both of them. Use the method you described in order to compute the eigenvalues of the matrix below. Show all steps of computation.

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & 0 \\ -2 & 2 & 1 \end{bmatrix}$$

Solution.

An eigenvector of a matrix is a non-zero vector that, when multiplied by the matrix, results in a scalar multiple of the original vector. An eigenvalue tells the eigenvector is stretched or shrunken. They satisfy the equation $Av = \lambda v$, where A is the matrix, v the eigenvector, and λ is the eigenvalue.

We can compute them using the characteristic polynomial of the matrix. The characteristic polynomial is defined as $\det(A - \lambda I)$. Setting this polynomial equal to zero and solving for λ will give the eigenvalues. Then, by plugging each λ into the equation $Av = \lambda v$, we can solve for the corresponding eigenvectors.

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & -\lambda & 0 \\ -2 & 2 & 1-\lambda \end{bmatrix}\right) = -(\lambda+1)(\lambda-1)(\lambda-2) = 0$$

$$\lambda_1 = -1, \lambda_2 = 1, \lambda_3 = 2$$

Therefore the eigenvalues are $-1, 1$ and 2.

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & 0 \\ -2 & 2 & 1 \end{bmatrix} v_1 = -v_1 \to v_1 = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & 0 \\ -2 & 2 & 1 \end{bmatrix} v_2 = v_2 \to v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & 0 \\ -2 & 2 & 1 \end{bmatrix} v_3 = 2v_3 \to v_3 = \begin{bmatrix} -1 \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

Therefore the eigenvectors are $\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ -\frac{1}{2} \\ 1 \end{bmatrix}$.

□