CSCI-UA 480 Robot Intelligence Homework 4

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Question 1: Computing Jacobians

Derive the Jacobians of the functions below. Your derivations should be done by hand.

1. $g(x_1, x_2) = [x_1 x_2, \sin(x_1)]$

   Solution.

$$
\begin{aligned}
J_g(\mathbf{x}) &= \begin{bmatrix} \dfrac{\partial g_1(x_1, x_2)}{\partial x_1} & \dfrac{\partial g_1(x_1, x_2)}{\partial x_2} \\ \dfrac{\partial g_2(x_1, x_2)}{\partial x_1} & \dfrac{\partial g_2(x_1, x_2)}{\partial x_2} \end{bmatrix} \\
&= \begin{bmatrix} \dfrac{\partial x_1 x_2}{\partial x_1} & \dfrac{\partial x_1 x_2}{\partial x_2} \\ \dfrac{\partial \sin(x_1)}{\partial x_1} & \dfrac{\partial \sin(x_1)}{\partial x_2} \end{bmatrix} \\
&= \begin{bmatrix} x_2 & x_1 \\ \cos(x_1) & 0 \end{bmatrix}
\end{aligned}
$$

□

2. $f(x_1, x_2, x_3) = [\frac{x_1}{x_2}, \cos^2(x_1 x_3)]$

   Solution.

$$
\begin{aligned}
J_f(\mathbf{x}) &= \begin{bmatrix} \dfrac{\partial f_1(x_1, x_2, x_3)}{\partial x_1} & \dfrac{\partial f_1(x_1, x_2, x_3)}{\partial x_2} & \dfrac{\partial f_1(x_1, x_2, x_3)}{\partial x_3} \\ \dfrac{\partial f_2(x_1, x_2, x_3)}{\partial x_1} & \dfrac{\partial f_2(x_1, x_2, x_3)}{\partial x_2} & \dfrac{\partial f_2(x_1, x_2, x_3)}{\partial x_3} \end{bmatrix} \\
&= \begin{bmatrix} \dfrac{\partial \frac{x_1}{x_2}}{\partial x_1} & \dfrac{\partial \frac{x_1}{x_2}}{\partial x_2} & \dfrac{\partial \frac{x_1}{x_2}}{\partial x_3} \\ \dfrac{\partial \cos^2(x_1 x_3)}{\partial x_1} & \dfrac{\partial \cos^2(x_1 x_3)}{\partial x_2} & \dfrac{\partial \cos^2(x_1 x_3)}{\partial x_3} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{x_2} & -\frac{x_1}{x_2^2} & 0 \\ -2x_3 \cos(x_1 x_3) \sin(x_1 x_3) & 0 & -2x_1 \cos(x_1 x_3) \sin(x_1 x_3) \end{bmatrix}
\end{aligned}
$$

□

Question 2: Existence of Jacobians

Under what conditions is a Jacobian for some function f well-defined, i.e. what conditions must f satisfy in order for its Jacobian to exist?

Solution.

For a Jacobian to be well-defined for some function f, the function must meet the following conditions:

1. The function $f : \mathbb{R}^n \to \mathbb{R}^m$ must be a mapping from an n-dimensional input space to an m-dimensional output space. In other words, $f(x_1, x_2, \ldots, x_n) = [f_1(x_1, x_2, \ldots, x_n), f_2(x_1, x_2, \ldots, x_n), \ldots, f_m(x_1, x_2, \ldots, x_n)]$.
2. The first-order partial derivatives with respect to each input variable must exist at the point where the Jacobian is evaluated.

□

Question 3: Honey, I Hit a Singularity

Suppose we compute the Jacobian J of the kinematics equations for a manipulator end-effector with joint parameters $q \in \mathbb{R}^n$, and we find that there exist valid configurations q' of our robot such that $\det(J) = 0$ when J is evaluated at q'.

1. How can we physically interpret this condition? What happens to the manipulator at the configurations q'?

   Solution.

   At configurations q' where $\det(J) = 0$, the Jacobian matrix is singular and will result in a loss of degrees of freedom (DOF) for the end-effector. Physically, this condition can be interpreted in the following ways:

   (a) Loss of controllability: In singular configurations, there may exist directions in which the end-effector cannot move, regardless of the joint velocities. This is because the rank of the Jacobian matrix is less than the full rank, indicating that some of the rows/columns of the matrix are linearly dependent.

   (b) Infinite velocities: The manipulator may need to apply an infinite joint velocity to achieve a desired end-effector velocity. This is clearly not possible in practice, and as a result, the manipulator will not be able to achieve the desired motion.

   (c) Possible mechanical instability: The manipulator may be mechanically unstable or approaching a mechanical limit. For example, a robot arm may be fully extended or folded in on itself, leading to reduced dexterity and a higher likelihood of encountering singularities.

   □

2. What happens to the numerical methods for inverse kinematics that we discussed in class (the Newton-Raphson method and the gradient descent-based approach) if an update yields joint parameters that have this property? Can either method recover from an update like this?

   Solution.

   Both methods can be negatively affected by singular configurations. In the Newton-Raphson method, each updates involves computing the inverse of the Jacobian (J^{-1}) or the pseudo-inverse ($J^{\dagger}$). When the Jacobian becomes singular, its inverse does not exist, which can cause the method to fail. However, the method might still recover from this if the method is using the pseudo-inverse, which is more robust to singularities.

   When the determinant of the Jacobian matrix is zero, the Jacobian $J(q)$ loses rank and has a null space with a non-zero dimension. The gradient $\nabla_q E(q) = J(q)^T(f(q) - x^{\star})$ may or may not be zero. If the error vector $(f(q) - x^{\star})$ is aligned with the null space of the Jacobian, the gradient would be zero. The gradient descent algorithm would not be able to make progress, making it difficult to recover from the singularity. If the error vector is not entirely within the null space of the Jacobian, the gradient will be non-zero. In this case, the gradient descent algorithm can still make progress, but the ill-conditioning of the Jacobian may cause slow convergence or instability. $\square$