CSCI-UA 480 Robot Intelligence Homework 3

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Question 1: Vector Identities

Let $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$ be 3 vectors and let $\cdot$ and $\times$ denote the dot and cross products in $\mathbb{R}^3$, respectively. Verify the identities below.

(a) $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$

Solution.

$$\begin{aligned}
\vec{a} \cdot (\vec{b} \times \vec{c}) &= (a_1 i + a_2 j + a_3 k) \cdot ((b_1 i + b_2 j + b_3 k) \times (c_1 i + c_2 j + c_3 k)) \\
&= (a_1 i + a_2 j + a_3 k) \cdot ((b_2 c_3 - b_3 c_2) i + (b_3 c_1 - b_1 c_3) j + (b_1 c_2 - b_2 c_1) k) \\
&= a_1 (b_2 c_3 - b_3 c_2) + a_2 (b_3 c_1 - b_1 c_3) + a_3 (b_1 c_2 - b_2 c_1) \\
&= (a_2 b_3 - a_3 b_2) c_1 + (a_3 b_1 - a_1 b_3) c_2 + (a_1 b_2 - a_2 b_1) c_3 \\
&= ((a_2 b_3 - a_3 b_2) i + (a_3 b_1 - a_1 b_3) j + (a_1 b_2 - a_2 b_1) k) \cdot (c_1 i + c_2 j + c_3 k) \\
&= ((a_1 i + a_2 j + a_3 k) \times ((b_1 i + b_2 j + b_3 k)) \times (c_1 i + c_2 j + c_3 k)) \\
&= (\vec{a} \times \vec{b}) \cdot \vec{c}
\end{aligned}$$

□

(b) $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

Solution.

$$\begin{aligned}
\vec{a} \times (\vec{b} \times \vec{c}) =& (a_1 i + a_2 j + a_3 k) \times ((b_1 i + b_2 j + b_3 k) \times (c_1 i + c_2 j + c_3 k)) \\
=& (a_1 i + a_2 j + a_3 k) \times ((b_2 c_3 - b_3 c_2) i + (b_3 c_1 - b_1 c_3) j + (b_1 c_2 - b_2 c_1) k) \\
=& (a_2 (b_1 c_2 - b_2 c_1) - a_3 (b_3 c_1 - b_1 c_3)) i + \\
& (a_3 (b_2 c_3 - b_3 c_2) - a_1 (b_1 c_2 - b_2 c_1)) j + \\
& (a_1 (b_3 c_1 - b_1 c_3) - a_2 (b_2 c_3 - b_3 c_2)) k \\
=& (a_2 b_1 c_2 + a_3 b_1 c_3 - a_2 b_2 c_1 - a_3 b_3 c_1) i + \\
& (a_3 b_2 c_3 + a_1 b_2 c_1 - a_3 b_3 c_2 - a_1 b_1 c_2) j + \\
& (a_1 b_3 c_1 + a_2 b_3 c_2 - a_1 b_1 c_3 - a_2 b_2 c_3) k \\
=& (a_1 b_1 c_1 + a_2 b_1 c_2 + a_3 b_1 c_3) i + (a_1 b_2 c_1 + a_2 b_2 c_2 + a_3 b_2 c_3) j + (a_1 b_3 c_1 + a_2 b_3 c_2 + a_3 b_3 c_3) k - \\
& (a_1 b_1 c_1 + a_2 b_2 c_1 + a_3 b_3 c_1) i - (a_1 b_1 c_2 + a_2 b_2 c_2 + a_3 b_3 c_2) j - (a_1 b_1 c_3 + a_2 b_2 c_3 + a_3 b_3 c_3) k \\
=& (a_1 c_1 + a_2 c_2 + a_3 c_3)(b_1 i + b_2 j + b_3 k) - (a_1 b_1 + a_2 b_2 + a_3 b_3)(c_1 i + c_2 j + c_3 k) \\
=& (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}
\end{aligned}$$

□

Question 2: Rigid Bodies

(a) Describe the configurations of each of the everyday objects below as a combination of links and joint.

(1) A cabinet.

(2) A bicycle.

(3) A skateboard.

Solution.

(1) A cabinet might consist of a rectangular box-shaped base with hinged doors on top. The base and the doors can be viewed as links, and they are connected by revolute joints, which allow the doors to rotate around a fixed axis.

(2) The main structural component of the bicycle is the frame, which can be considered a single link. The fork connects the front wheel to the frame with a revolute joint, while the handlebars are attached to the fork with another revolute joint. The pedals are attached to the crankset with a prismatic joint, and the wheels are connected to the frame with cylindrical joints. The seat is attached to the frame with a prismatic joint that allows it to move up and down.

(3) The main components of a skateboard are the deck, trucks, and wheels. The deck is the flat board that the rider stands on, and it is connected to the trucks with prismatic joints that allow for movement in the vertical direction. The trucks are the components that attach the wheels to the deck, and they consist of a baseplate, hanger, and kingpin connected with a combination of revolute, cylindrical, and spherical joints that allow for rotation and movement in multiple directions. The wheels are attached to the trucks with a cylindrical joint, and they rotate freely to allow the skateboard to move forward and turn.

□

(b) Let $T(\vec{v}) = \boldsymbol{R}\vec{v} + \vec{t}$ be a 3-D rigid body transformation, where $\boldsymbol{R}$ is a rotation matrix. Consider a pair of arbitrary 3-D vectors, $\vec{u}, \vec{w} \in \mathbb{R}^3$. Prove that $T(\cdot)$ preserves Euclidean distance, i.e. $||T(\vec{u}) - T(\vec{w})||^2 = ||\vec{u} - \vec{w}||^2$.

Solution.

To prove that $T(\cdot)$ preserves Euclidean distance, we need to show that:

$||T(\vec{u}) - T(\vec{w})||^2 = (\boldsymbol{R}\vec{u} + \vec{t} - \boldsymbol{R}\vec{w} - \vec{t})^T(\boldsymbol{R}\vec{u} + \vec{t} - \boldsymbol{R}\vec{w} - \vec{t}) = ||\vec{u} - \vec{w}||^2$

Expanding the above expression, we have:

$(\boldsymbol{R}\vec{u} + \vec{t} - \boldsymbol{R}\vec{w} - \vec{t})^T(\boldsymbol{R}\vec{u} + \vec{t} - \boldsymbol{R}\vec{w} - \vec{t}) = \vec{u}^T\boldsymbol{R}^T\boldsymbol{R}\vec{u} - 2\vec{u}^T\boldsymbol{R}^T\boldsymbol{R}\vec{w} + \vec{w}^T\boldsymbol{R}^T\boldsymbol{R}\vec{w}$

Since $\boldsymbol{R}$ is a rotation matrix, it satisfies $\boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{R}\boldsymbol{R}^T = \boldsymbol{I}$, where $\boldsymbol{I}$ is the identity matrix. Substituting this into the above expression, we get:

$\vec{u}^T\vec{u} - 2\vec{u}^T\boldsymbol{R}^T\boldsymbol{R}\vec{w} + \vec{w}^T\vec{w} = \vec{u}^T\vec{u} - 2\vec{u}^T\vec{w} + \vec{w}^T\vec{w} = ||\vec{u} - \vec{w}||^2$

Therefore, $T(\cdot)$ preserves Euclidean distance. □

(c) A rigid body moving in $\mathbb{R}^2$ has three degrees of freedom (two from translation and one from rotation), while a rigid body in $\mathbb{R}^3$ has six degrees of freedom (three from translation and three from rotation).

Prove that for general $n \in N \geq 2$, an n-dimensional rigid body has $\frac{n}{2}(n+1)$ degrees of freedom. How many of these are due to rotation? How many are due to translation?

Solution.

Translation involves n degrees of freedom, corresponding to the displacement along the n axises.

Rotation involves $\frac{n(n-1)}{2}$ degrees of freedom. To fully specify the rotation, we need to specify the angle of rotation θ, and the axis of rotation, which is an $(n-2)$-dimensional subspace of $\mathbb{R}^n$. The axis of rotation can be specified by $(n-2)$ independent parameters, which correspond to the directions of two linearly independent vectors that span the subspace. The angle of rotation is one additional parameter. Therefore, rotation involves $(n-2)+1 = n-1$ degrees of freedom for each vector in the rigid body. Since there are n vectors in the rigid body, the total number of degrees of freedom due to rotation is $\frac{n(n-1)}{2}$.

Putting together the degrees of freedom due to translation and rotation, we get:

Degrees of freedom $= n + \frac{n(n-1)}{2} = \frac{n}{2}(n+1)$ □

Question 3: Rotations

1. Prove that the set of all 3-D rotation matrices form a algebraic group under matrix multiplication. Namely, given two arbitrary rotation matrices $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$, demonstrate that each of the following properties hold.

   (a) **Closure**: $\boldsymbol{R}_2' = \boldsymbol{R}_1\boldsymbol{R}_2$ is a valid rotation matrix.

   (b) **Identity**: there exists a valid rotation matrix e such that $\boldsymbol{R}_1 e = e\boldsymbol{R}_1 = \boldsymbol{R}_1$ and $\boldsymbol{R}_2 e = e\boldsymbol{R}_2 = \boldsymbol{R}_2$.

   (c) **Inverse**: there exist rotation matrices $\boldsymbol{R}_1^{-1}$, $\boldsymbol{R}_2^{-1}$ such that $\boldsymbol{R}_1\boldsymbol{R}_1^{-1} = e = \boldsymbol{R}_2\boldsymbol{R}_2^{-1}$.

   Recall that a 3×3 matrix $\boldsymbol{R}$ is a valid rotation matrix if and only if $\boldsymbol{R}^T = \boldsymbol{R}^{-1}$ and $\det(\boldsymbol{R}^T) = 1$.

   An algebraic group is called *Abelian* if the group operation is commutative. Is the group formed by 3-D rotation matrices under matrix multiplication *Abelian*? Why, or why not?

   Solution.

   (a) **Closure**: For valid rotation matrices $\boldsymbol{R}_1, \boldsymbol{R}_2$, we have

   $$(\boldsymbol{R}_2')^T(\boldsymbol{R}_2') = (\boldsymbol{R}_1\boldsymbol{R}_2)^T(\boldsymbol{R}_1\boldsymbol{R}_2) = \boldsymbol{R}_2^T\boldsymbol{R}_1^T\boldsymbol{R}_1\boldsymbol{R}_2 = \boldsymbol{R}_2^T(\boldsymbol{R}_1^T\boldsymbol{R}_1)\boldsymbol{R}_2 = \boldsymbol{R}_2^T\boldsymbol{R}_2 = \boldsymbol{I}$$

   $$\det((\boldsymbol{R}_2')^T) = \det(\boldsymbol{R}_2') = \det(\boldsymbol{R}_1\boldsymbol{R}_2) = \det(\boldsymbol{R}_1)\det(\boldsymbol{R}_2) = 1\cdot 1 = 1$$

   Therefore, $\boldsymbol{R}_2' = \boldsymbol{R}_1\boldsymbol{R}_2$ is a valid rotation matrix.

   (b) **Identity**: The 3×3 identity matrix $\boldsymbol{I}$ is a valid rotation matrix since $\boldsymbol{I}^T = \boldsymbol{I}^{-1}$ and $\det(\boldsymbol{I}) = 1$. For valid rotation matrices $\boldsymbol{R}_1, \boldsymbol{R}_2$, we have $\boldsymbol{R}_1 I = I\boldsymbol{R}_1 = \boldsymbol{R}_1$ and $\boldsymbol{R}_2 I = I\boldsymbol{R}_2 = \boldsymbol{R}_2$.

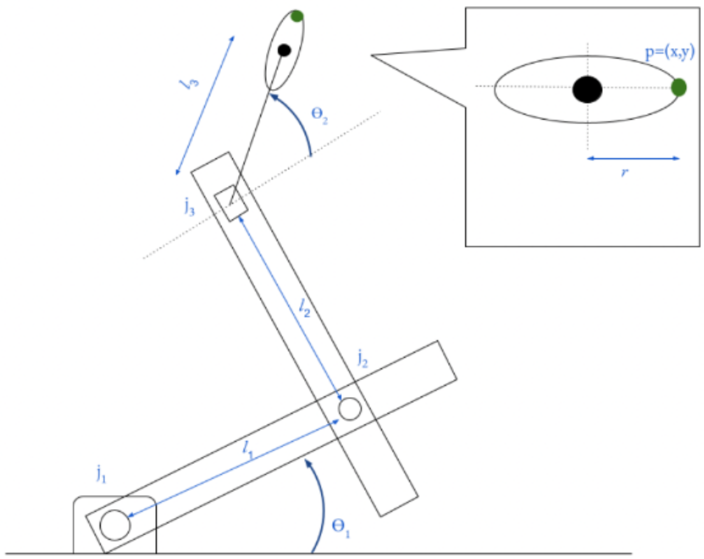
   (c) **Inverse**: For valid rotation matrices $\boldsymbol{R}_1, \boldsymbol{R}_2$, we have $\boldsymbol{R}_1^T = \boldsymbol{R}_1^{-1}$ and $\boldsymbol{R}_2^T = \boldsymbol{R}_2^{-1}$. Therefore,

   $$\boldsymbol{R}_1\boldsymbol{R}_1^{-1} = \boldsymbol{R}_1\boldsymbol{R}_1^T = e = \boldsymbol{R}_2\boldsymbol{R}_2^T = \boldsymbol{R}_2\boldsymbol{R}_2^{-1}$$

   where e is the 3×3 identity matrix I.

   □

2. Consider the robot shown on the following page. Express the position and orientation of the green point $p = (x, y)$ relative to the base of the arm in terms of joint angles and link parameters. Note that joint j_2 is a *prismatic* joint, able to extend the link by sliding up and down, affecting the height of l_2.



Solution.

The first joint: $(l_1 \cos\theta_1, l_1 \sin\theta_1)$

The second joint: $(l_1 \cos\theta_1 - l_2 \sin\theta_1, l_1 \sin\theta_1 + l_2 \cos\theta_1)$

The green point: $(l_1 \cos\theta_1 - l_2 \sin\theta_1 + (l_3 + r)\cos(\theta_1 + \theta_2), l_1 \sin\theta_1 + l_2 \cos\theta_1 + (l_3 + r)\sin(\theta_1 + \theta_2))$ □