Babylonian Algorithm

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Question 1

The Babylonian algorithm has the form $y_{k+1} = f(y_k)$, where $f = \frac{\frac{x}{y_k}+y_k}{2}$. We need to prove that if $y_0 > 0$, it converges to $\sqrt{x}$.

Solution.
Step 1: Show that f is a real-valued function, defined and continuous on a bounded closed interval $[\sqrt{x}, y_1]$, and $f(x) \in [\sqrt{x}, y_1]$ for all $x \in [\sqrt{x}, y_1]$.
$y_1 = \dfrac{\frac{x}{y_0} + y_0}{2} \geq \frac{1}{2} \cdot 2 \cdot \sqrt{\frac{x}{y_0} \cdot y_0} = \sqrt{x}$, since $a + b \geq 2\sqrt{ab} \iff (\sqrt{a} - \sqrt{b})^2 \geq 0 \ (a, b \geq 0)$
$y_2 = \dfrac{\frac{x}{y_1} + y_1}{2} \geq \frac{1}{2} \cdot 2 \cdot \sqrt{\frac{x}{y_1} \cdot y_1} = \sqrt{x}$
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Similarly, we have $y_3, y_4, ..., y_k \geq \sqrt{x}$.
$y_2 - y_1 = \dfrac{\frac{x}{y_1} + y_1}{2} - y_1 = \frac{1}{2} \cdot (\dfrac{x}{y_1} - y_1) \leq \frac{1}{2} \cdot (\dfrac{x}{\sqrt{x}} - \sqrt{x}) = 0$
thus $y_1 > y_2$.
Inductively, we have $y_1 > y_2 > ... > y_k \geq \sqrt{x}$.
Therefore, the domain of f is $[\sqrt{x}, y_1]$ (except for y_0 when $0 < y_0 < \sqrt{x}$). The range of f is also $[\sqrt{x}, y_1]$.
Therefore, f is a real-valued function, defined and continuous on a bounded closed interval $[\sqrt{x}, y_1]$, and $f(x) \in [\sqrt{x}, y_1]$ for all $x \in [\sqrt{x}, y_1]$.

Step 2: Show that f is a contraction on $[\sqrt{x}, y_1]$.
For all $m, n \in [\sqrt{x}, y_1]$,
$|f(m) - f(n)| = |\dfrac{\frac{x}{m} + m}{2} - \dfrac{\frac{x}{n} + n}{2}| = \frac{1}{2} \cdot |(\dfrac{x}{m} - \dfrac{x}{n}) + (m - n)| = \frac{1}{2} \cdot |m - n| \cdot |1 - \dfrac{x}{mn}|$
Since $m, n \in [\sqrt{x}, y_1]$,
we know that $mn \in [x, y_1^2]$, $\dfrac{x}{mn} \in [0, 1]$, $|1 - \dfrac{x}{mn}| \in [0, 1]$,
thus $|f(m) - f(n)| \leq \frac{1}{2} \cdot |m - n|$. $(L = \frac{1}{2})$
Therefore, f is a contraction on $[\sqrt{x}, y_1]$.

Step 3: Show that the Babylonian algorithm converges to $\sqrt{x}$ by Contraction Mapping Theorem.
From Step 1 and 2 we know that f is a real-valued function, defined and continuous on

a bounded closed interval $[\sqrt{x}, y_1]$, and $f(x) \in [\sqrt{x}, y_1]$ for all $x \in [\sqrt{x}, y_1]$, and f is a contraction on $[\sqrt{x}, y_1]$. By Contraction Mapping Theorem, f has a unique fixed point ε in the interval [a, b]. Thus the fixed point $\sqrt{x}$ is unique.
(If the fixed point $\sqrt{x}$ is not unique, suppose y_k is another fixed point, then $|f(y_k) - f(\sqrt{x})| = |y_k - \sqrt{x}| \leq \frac{1}{2}|y_k - \sqrt{x}|$) thus $y_k = x$, leading to a contradiction.)
Therefore, by Contraction Mapping Theorem, no matter what $y_0 > 0$ we choose, $y_1 \in [\sqrt{x}, y_1]$, the sequence (y_k) converges to $\sqrt{x}$ as $k \to \infty$ for any starting value. $\square$