Hermite Interpolation

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Problem 1. Let $x_0 < x_1$, $f \in \mathbb{C}^1([x_0, x_1])$, and let $p \in \mathbb{P}_3$ be a cubic polynomial such that:

$$p^{(i)}(x_j) = f^{(i)}(x_j), \quad i = 0, 1, \quad j = 0, 1. \tag{1}$$

Show that p exists and is unique. (Hint: for uniqueness, assume that there are two possibilities: $p, q \in \mathbb{P}_3$ such that $p \neq q$. What can you say about the roots of $p - q$?)

Solution.
Existence:
$p(t) = p(x_0)\phi_0(t) + p(x_1)\phi_1(t) + p'(x_0)\psi_0(t) + p'(x_1)\psi_1(t)$
where
$\phi_0(t) = \frac{(t-x_1)^2[(x_0-x_1)+2(x_0-t)]}{(x_0-x_1)^3}$
$\phi_1(t) = \frac{(t-x_0)^2[(x_1-x_0)+2(x_1-t)]}{(x_0-x_1)^3}$
$\psi_0(t) = \frac{(t-x_0)(t-x_1)^2}{(x_0-x_1)^2}$
$\psi_1(t) = \frac{(t-x_0)^2(t-x_1)}{(x_0-x_1)^2}$

Uniqueness:
Assume that there are two possibilities: $p, q \in \mathbb{P}_3$ such that $p \neq q$.
By assumption, $p(x_0) = f(x_0)$, $q(x_0) = f(x_0)$. Thus $p(x_0) - q(x_0) = 0$. Thus x_0 is a root of $p - q$.
By assumption, $p'(x_0) = f'(x_0)$, $q'(x_0) = f'(x_0)$. Thus $p'(x_0) - q'(x_0) = 0$. Thus $(p - q)'(x_0) = 0$. Thus x_0 is a local extremum of $p - q$.
Similarly, x_1 is a root, and also a local extremum of $p - q$.
Since $p - q \in \mathbb{P}_3$, $p - q$ has at most 2 local extrema. So x_0 and x_1 are the only two local extremas of $p - q$. However, x_0 and x_1 are also roots of $p - q$, so there exists at least 1 local extremum between them. So there is a contradiction.
Therefore, the assumption is false. p is unique.

□

Problem 2. Prenter: Problem 3, Page 56. Quintic Hermite polynomials p solve the interpolation problem:

$$p^{(i)}(x_j) = f^{(i)}(x_j), \quad i = 0, 1, 2 \quad j = 0, 1. \tag{2}$$

Show that p exists and is unique.

Solution.
Assume that $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5$.
Solving for p is equivalent to solving the following equation.

$$\begin{bmatrix} 1 & x_0 & {x_0}^2 & {x_0}^3 & {x_0}^4 & {x_0}^5 \\ 1 & x_1 & {x_1}^2 & {x_1}^3 & {x_1}^4 & {x_1}^5 \\ 0 & 1 & 2x_0 & 3x_0^2 & 4x_0^3 & 5x_0^4 \\ 0 & 1 & 2x_1 & 3x_1^2 & 4x_1^3 & 5x_1^4 \\ 0 & 0 & 2 & 6x_0 & 12x_0^2 & 20x_0^3 \\ 0 & 0 & 2 & 6x_1 & 12x_1^2 & 20x_1^3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ f'(x_0) \\ f'(x_1) \\ f''(x_0) \\ f''(x_1) \end{bmatrix}$$

The determinant of the matrix is $4(x_0 - x_1)^9$. So given $x_0 \neq x_1$, the determinant is not 0. This indicates that the equation is solving and has unique solution.
Therefore, p exists and is unique.

$\square$

Problem 3. Prenter: Problem 4, Page 60.
(a) Construct the cardinal basis for the piecewise quintic Hermite polynomials $H_5(\pi)$.
(b) What is the dimension of $H_5(\pi)$? Prove it.

Solution.
(a) We have that $p(x) = f(x_0)H_{00}(x) + f(x_1)H_{01}(x) + f'(x_0)H_{10}(x) + f(x_1)H_{11}(x) + f''(x_0)H_{20}(x) + f''(x_1)H_{21}(x)$ where

$$p^{(i)}(x_j) = f^{(i)}(x_j), \quad i = 0, 1, 2 \quad j = 0, 1. \tag{3}$$

and the cardinal basis is $\{H_{00}, H_{01}, H_{10}, H_{11}, H_{20}, H_{21}\}$.
We need:
$H_{00}(x_0) = 1, H_{00}(x_1) = 0, H'_{00}(x_0) = 0, H'_{00}(x_1) = 0, H''_{00}(x_0) = 0, H''_{00}(x_1) = 0.$
$H_{01}(x_0) = 0, H_{01}(x_1) = 1, H'_{01}(x_0) = 0, H'_{01}(x_1) = 0, H''_{01}(x_0) = 0, H''_{01}(x_1) = 0.$
$H_{10}(x_0) = 0, H_{10}(x_1) = 0, H'_{10}(x_0) = 1, H'_{10}(x_1) = 0, H''_{10}(x_0) = 0, H''_{10}(x_1) = 0.$
$H_{11}(x_0) = 0, H_{11}(x_1) = 0, H'_{11}(x_0) = 0, H'_{11}(x_1) = 1, H''_{11}(x_0) = 0, H''_{11}(x_1) = 0.$
$H_{20}(x_0) = 0, H_{20}(x_1) = 0, H'_{20}(x_0) = 0, H'_{20}(x_1) = 0, H''_{20}(x_0) = 1, H''_{20}(x_1) = 0.$
$H_{21}(x_0) = 0, H_{21}(x_1) = 0, H'_{21}(x_0) = 0, H'_{21}(x_1) = 0, H''_{21}(x_0) = 0, H''_{21}(x_1) = 1.$

The above is hard to solve, so we use the following method:
Assume that $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5$.
We know:

$$\begin{bmatrix} 1 & x_0 & {x_0}^2 & {x_0}^3 & {x_0}^4 & {x_0}^5 \\ 1 & x_1 & {x_1}^2 & {x_1}^3 & {x_1}^4 & {x_1}^5 \\ 0 & 1 & 2x_0 & 3x_0^2 & 4x_0^3 & 5x_0^4 \\ 0 & 1 & 2x_1 & 3x_1^2 & 4x_1^3 & 5x_1^4 \\ 0 & 0 & 2 & 6x_0 & 12x_0^2 & 20x_0^3 \\ 0 & 0 & 2 & 6x_1 & 12x_1^2 & 20x_1^3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ f'(x_0) \\ f'(x_1) \\ f''(x_0) \\ f''(x_1) \end{bmatrix}$$

So

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} 1 & x_0 & {x_0}^2 & {x_0}^3 & {x_0}^4 & {x_0}^5 \\ 1 & x_1 & {x_1}^2 & {x_1}^3 & {x_1}^4 & {x_1}^5 \\ 0 & 1 & 2x_0 & 3x_0^2 & 4x_0^3 & 5x_0^4 \\ 0 & 1 & 2x_1 & 3x_1^2 & 4x_1^3 & 5x_1^4 \\ 0 & 0 & 2 & 6x_0 & 12x_0^2 & 20x_0^3 \\ 0 & 0 & 2 & 6x_1 & 12x_1^2 & 20x_1^3 \end{bmatrix}^{-1} \begin{bmatrix} f(x_0) \\ f(x_1) \\ f'(x_0) \\ f'(x_1) \\ f''(x_0) \\ f''(x_1) \end{bmatrix} =$$

$$\begin{bmatrix} \frac{-x_1^3(10x_0^2-5x_0x_1+x_1^2)}{(x_0-x_1)^5} & \frac{x_0^3(x_0^2-5x_0x_1+10x_1^2)}{(x_0-x_1)^5} & \frac{x_0x_1^3(4x_0-x_1)}{(x_0-x_1)^4} & \frac{x_0^3x_1(4x_1-x_0)}{(x_0-x_1)^4} & \frac{-x_0^2x_1^3}{2(x_0-x_1)^3} & \frac{x_0^3x_1^2}{2(x_0-x_1)^3} \\ \frac{30x_0^2x_1^2}{(x_0-x_1)^5} & \frac{-30x_0^2x_1^2}{(x_0-x_1)^5} & \frac{-x_1^2(2x_0+x_1)(6x_0-x_1)}{(x_0-x_1)^4} & \frac{x_0^2(x_0+2x_1)(x_0-6x_1)}{(x_0-x_1)^4} & \frac{x_0x_1^2(3x_0+2x_1)}{2(x_0-x_1)^3} & \frac{x_0^2x_1(2x_0+3x_1)}{2(x_0-x_1)^3} \\ \frac{-30x_0x_1(x_0+x_1)}{(x_0-x_1)^5} & \frac{30x_0x_1(x_0+x_1)}{(x_0-x_1)^5} & \frac{6x_0x_1(2x_0+3x_1)}{(x_0-x_1)^4} & \frac{6x_0x_1(3x_0+2x_1)}{(x_0-x_1)^4} & \frac{-x_1(3x_0^2+6x_0x_1+x_1^2)}{2(x_0-x_1)^3} & \frac{x_0(x_0^2+6x_0x_1+3x_1^2)}{2(x_0-x_1)^3} \\ \frac{10(x_0^2+4x_0x_1+x_1^2)}{(x_0-x_1)^5} & \frac{-10(x_0^2+4x_0x_1+x_1^2)}{(x_0-x_1)^5} & \frac{-2(2x_0^2+10x_0x_1+3x_1^2)}{(x_0-x_1)^4} & \frac{-2(3x_0^2+10x_0x_1+2x_1^2)}{(x_0-x_1)^4} & \frac{x_0^2+6x_0x_1+3x_1^2)}{2(x_0-x_1)^3} & \frac{-3x_0^2-6x_0x_1+x_1^2)}{2(x_0-x_1)^3} \\ \frac{-15(x_0+x_1)}{(x_0-x_1)^5} & \frac{15(x_0+x_1)}{(x_0-x_1)^5} & \frac{7x_0+8x_1}{(x_0-x_1)^4} & \frac{8x_0+7x_1}{(x_0-x_1)^4} & \frac{-2x_0-3x_1}{2(x_0-x_1)^3} & \frac{3x_0+2x_1}{2(x_0-x_1)^3} \\ \frac{6}{(x_0-x_1)^5} & \frac{-6}{(x_0-x_1)^5} & \frac{-3}{(x_0-x_1)^4} & \frac{3}{(x_0-x_1)^4} & \frac{1}{2(x_0-x_1)^3} & \frac{-1}{2(x_0-x_1)^3} \end{bmatrix}$$

$$\begin{bmatrix} f(x_0) \\ f(x_1) \\ f'(x_0) \\ f'(x_1) \\ f''(x_0) \\ f''(x_1) \end{bmatrix}$$

So $p(x) = [1 - \frac{(x-x_0)^3}{(x_1-x_0)^3} + 3\frac{(x-x_0)^3(x-x_1)}{(x_1-x_0)^4} - 6\frac{(x-x_0)^3(x-x_1)^2}{(x_1-x_0)^5}] \cdot f(x_0) + [\frac{(x-x_0)^3}{(x_1-x_0)^3} - 3\frac{(x-x_0)^3(x-x_1)}{(x_1-x_0)^4} + 6\frac{(x-x_0)^3(x-x_1)^2}{(x_1-x_0)^5}] \cdot f(x_1) + \frac{(x_1-x_0)^4(x-x_1)-(x_1-x_0)^2(x-x_0)^3+2(x_1-x_0)(x-x_0)^3(x-x_1)-3(x-x_0)^3(x-x_1)^3}{(x_1-x_0)^4} \cdot f'(x_0) + \frac{(x_1-x_0)(x-x_0)^3(x-x_1)-3(x-x_0)^3(x-x_1)^2}{(x_1-x_0)^4} \cdot f'(x_1) + \frac{-2(x-x_0)(x-x_1)^3-3(x-x_0)^2(x-x_1)^2}{2(x_1-x_0)^3} \cdot f''(x_0) + \frac{2(x-x_0)^3(x-x_1)+3(x-x_0)^2(x-x_1)^2}{2(x_1-x_0)^3} \cdot f''(x_1)$.

Therefore, the cardinal basis is $\{H_{00}, H_{01}, H_{10}, H_{11}, H_{20}, H_{21}\}$ where
$H_{00} = 1 - \frac{(x-x_0)^3}{(x_1-x_0)^3} + 3\frac{(x-x_0)^3(x-x_1)}{(x_1-x_0)^4} - 6\frac{(x-x_0)^3(x-x_1)^2}{(x_1-x_0)^5}$
$H_{01} = \frac{(x-x_0)^3}{(x_1-x_0)^3} - 3\frac{(x-x_0)^3(x-x_1)}{(x_1-x_0)^4} + 6\frac{(x-x_0)^3(x-x_1)^2}{(x_1-x_0)^5}$
$H_{10} = \frac{(x_1-x_0)^4(x-x_1)-(x_1-x_0)^2(x-x_0)^3+2(x_1-x_0)(x-x_0)^3(x-x_1)-3(x-x_0)^3(x-x_1)^3}{(x_1-x_0)^4}$
$H_{11} = \frac{(x_1-x_0)(x-x_0)^3(x-x_1)-3(x-x_0)^3(x-x_1)^2}{(x_1-x_0)^4}$
$H_{20} = \frac{-2(x-x_0)(x-x_1)^3-3(x-x_0)^2(x-x_1)^2}{2(x_1-x_0)^3}$
$H_{21} = \frac{2(x-x_0)^3(x-x_1)+3(x-x_0)^2(x-x_1)^2}{2(x_1-x_0)^3}$.

(b) The dimension of $H_5(\pi)$ is $3n+3$ (in the $n=1$ case is 6).
There are $3n+3$ H_{ij}'s, we only need to prove that they are linearly independent.
Assume that they are not linearly independent, so that $\Sigma_{i=0}^{2}\Sigma_{j=0}^{n} H_{ij} = 0$.
But we know that for $x = x_0$, $H_{00}(x_0) = 1, H_{10}(x_1) = 0, H_{20}(x_0) = 0, H_{01}(x_0) = 0, H_{11}(x_1) = 0, H_{21}(x_0) = 0 \ldots$
Thus, $\Sigma_{i=0}^{2}\Sigma_{j=0}^{n} H_{ij}(x_0) = 1 \neq 0$.
Therefore, H_{ij}'s are linearly independent.

$\square$

Problem 4. Consider the quintic Hermite interpolation problem on $[x_0, x_1]$ for the function $f(x) =$

$x^8 - 1$. That is, find $p(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \alpha_4 x^4 + \alpha_5 x^5$ such that $p(x_i) = f(x_i)$, $p'(x_i) = f'(x_i)$, and $p''(x_i) = f''(x_i) (i = 0, 1)$:

1. Compute $p(x)$ by solving the "Vandermonde system" for this interpolation problem (i.e., solving $V\alpha = f$ for α and write p in the monomial basis).

2. Compute p(x) in the cardinal basis. (Hint: use the result of part (b) of Problem 3 in this homework).

3. Estimate $max_{x_0 \le x \le x_1} |f(x) - p(x)|$. Let $x_0 = 0$ and $x_1 = h$, and rewrite expression in terms of h. Do the same but with $x_0 = 1$ and $x_1 = 1 + h$. As $h \to 0$, which expression goes to zero faster? What can you conclude?

4. Let $p(x)$ be the degree 3 Lagrange interpolant for this problem (with uniformly spaced nodes). Solve the previous problem but for this choice of p. What do you observe and what can you conclude?

Solution.

1. $p(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \alpha_4 x^4 + \alpha_5 x^5$.

$$\begin{bmatrix} 1 & x_0 & {x_0}^2 & {x_0}^3 & {x_0}^4 & {x_0}^5 \\ 1 & x_1 & {x_1}^2 & {x_1}^3 & {x_1}^4 & {x_1}^5 \\ 0 & 1 & 2x_0 & 3x_0^2 & 4x_0^3 & 5x_0^4 \\ 0 & 1 & 2x_1 & 3x_1^2 & 4x_1^3 & 5x_1^4 \\ 0 & 0 & 2 & 6x_0 & 12x_0^2 & 20x_0^3 \\ 0 & 0 & 2 & 6x_1 & 12x_1^2 & 20x_1^3 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{bmatrix} = \begin{bmatrix} x_0^8 - 1 \\ x_1^8 - 1 \\ 8x_0^7 \\ 8x_1^7 \\ 56x_0^6 \\ 56x_1^6 \end{bmatrix}$$

So

$$\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{bmatrix} = \begin{bmatrix} 1 & x_0 & {x_0}^2 & {x_0}^3 & {x_0}^4 & {x_0}^5 \\ 1 & x_1 & {x_1}^2 & {x_1}^3 & {x_1}^4 & {x_1}^5 \\ 0 & 1 & 2x_0 & 3x_0^2 & 4x_0^3 & 5x_0^4 \\ 0 & 1 & 2x_1 & 3x_1^2 & 4x_1^3 & 5x_1^4 \\ 0 & 0 & 2 & 6x_0 & 12x_0^2 & 20x_0^3 \\ 0 & 0 & 2 & 6x_1 & 12x_1^2 & 20x_1^3 \end{bmatrix}^{-1} \begin{bmatrix} x_0^8 - 1 \\ x_1^8 - 1 \\ 8x_0^7 \\ 8x_1^7 \\ 56x_0^6 \\ 56x_1^6 \end{bmatrix} =$$

$$\begin{bmatrix} \frac{-x_1^3(10x_0^2-5x_0x_1+x_1^2)}{(x_0-x_1)^5} & \frac{x_0^3(x_0^2-5x_0x_1+10x_1^2)}{(x_0-x_1)^5} & \frac{x_0x_1^3(4x_0-x_1)}{(x_0-x_1)^4} & \frac{x_0^3x_1(4x_1-x_0)}{(x_0-x_1)^4} & \frac{-x_0^2x_1^3}{2(x_0-x_1)^3} & \frac{x_0^3x_1^2}{2(x_0-x_1)^3} \\ \frac{30x_0^2x_1^2}{(x_0-x_1)^5} & \frac{-30x_0^2x_1^2}{(x_0-x_1)^5} & \frac{-x_1^2(2x_0+x_1)(6x_0-x_1)}{(x_0-x_1)^4} & \frac{x_0^2(x_0+2x_1)(x_0-6x_1)}{(x_0-x_1)^4} & \frac{x_0x_1^2(3x_0+2x_1)}{2(x_0-x_1)^3} & \frac{x_0^2x_1(2x_0+3x_1)}{2(x_0-x_1)^3} \\ \frac{-30x_0x_1(x_0+x_1)}{(x_0-x_1)^5} & \frac{30x_0x_1(x_0+x_1)}{(x_0-x_1)^5} & \frac{6x_0x_1(2x_0+3x_1)}{(x_0-x_1)^4} & \frac{6x_0x_1(3x_0+2x_1)}{(x_0-x_1)^4} & \frac{-x_1(3x_0^2+6x_0x_1+x_1^2)}{2(x_0-x_1)^3} & \frac{x_0(x_0^2+6x_0x_1+3x_1^2)}{2(x_0-x_1)^3} \\ \frac{10(x_0^2+4x_0x_1+x_1^2)}{(x_0-x_1)^5} & \frac{-10(x_0^2+4x_0x_1+x_1^2)}{(x_0-x_1)^5} & \frac{-2(2x_0^2+10x_0x_1+3x_1^2)}{(x_0-x_1)^4} & \frac{-2(3x_0^2+10x_0x_1+2x_1^2)}{(x_0-x_1)^4} & \frac{x_0^2+6x_0x_1+3x_1^2)}{2(x_0-x_1)^3} & \frac{-3x_0^2-6x_0x_1+x_1^2)}{2(x_0-x_1)^3} \\ \frac{-15(x_0+x_1)}{(x_0-x_1)^5} & \frac{15(x_0+x_1)}{(x_0-x_1)^5} & \frac{7x_0+8x_1}{(x_0-x_1)^4} & \frac{8x_0+7x_1}{(x_0-x_1)^4} & \frac{-2x_0-3x_1}{2(x_0-x_1)^3} & \frac{3x_0+2x_1}{2(x_0-x_1)^3} \\ \frac{6}{(x_0-x_1)^5} & \frac{-6}{(x_0-x_1)^5} & \frac{-3}{(x_0-x_1)^4} & \frac{3}{(x_0-x_1)^4} & \frac{1}{2(x_0-x_1)^3} & \frac{-1}{2(x_0-x_1)^3} \end{bmatrix}$$

$$\begin{bmatrix} x_0^8 - 1 \\ x_1^8 - 1 \\ 8x_0^7 \\ 8x_1^7 \\ 56x_0^6 \\ 56x_1^6 \end{bmatrix} = \begin{bmatrix} -6x_0^5x_1^3 - 9x_0^4x_1^4 - 6x_0^3x_1^5 - 1 \\ x_0^2x_1^2(18x_0^3 + 42x_0^2x_1 + 42x_0x_1^2 + 18x_1^3) \\ -2x_0x_1(9x_0^4 + 36x_0^3x_1 + 50x_0x_1 + 36x_0x_1^3 + 9x_1^4) \\ 6x_0^5 + 54x_0^4x_1 + 108x_0^3x_1^2 + 108x_0^2x_1^3 + 54x_0x_1^4 + 6x_1^5 \\ -15x_0^4 - 54x_0^3x_1 - 72x_0^2x_1^2 - 54x_0x_1^3 - 15x_1^4 \\ 10x_0^3 + 18x_0^2x_1 + 18x_0x_1^2 + 10x_1^3 \end{bmatrix}$$

So $p(x) = -6x_0^5x_1^3 - 9x_0^4x_1^4 - 6x_0^3x_1^5 - 1 + [x_0^2x_1^2(18x_0^3 + 42x_0^2x_1 + 42x_0x_1^2 + 18x_1^3)]x + [-2x_0x_1(9x_0^4 + 36x_0^3x_1 + 50x_0x_1 + 36x_0x_1^3 + 9x_1^4)]x^2 + [6x_0^5 + 54x_0^4x_1 + 108x_0^3x_1^2 + 108x_0^2x_1^3 + 54x_0x_1^4 + 6x_1^5]x^3[-15x_0^4 - 54x_0^3x_1 - 72x_0^2x_1^2 - 54x_0x_1^3 - 15x_1^4]x^4 + [10x_0^3 + 18x_0^2x_1 + 18x_0x_1^2 + 10x_1^3]x^5$

2. The result of part (b) of Problem 3 gives the cardinal basis $\{H_{00}, H_{01}, H_{10}, H_{11}, H_{20}, H_{21}\}$ where
$H_{00} = 1 - \frac{(x-x_0)^3}{(x_1-x_0)^3} + 3\frac{(x-x_0)^3(x-x_1)}{(x_1-x_0)^4} - 6\frac{(x-x_0)^3(x-x_1)^2}{(x_1-x_0)^5}$
$H_{01} = \frac{(x-x_0)^3}{(x_1-x_0)^3} - 3\frac{(x-x_0)^3(x-x_1)}{(x_1-x_0)^4} + 6\frac{(x-x_0)^3(x-x_1)^2}{(x_1-x_0)^5}$
$H_{10} = \frac{(x_1-x_0)^4(x-x_1)-(x_1-x_0)^2(x-x_0)^3+2(x_1-x_0)(x-x_0)^3(x-x_1)-3(x-x_0)^3(x-x_1)^3}{(x_1-x_0)^4}$
$H_{11} = \frac{(x_1-x_0)(x-x_0)^3(x-x_1)-3(x-x_0)^3(x-x_1)^2}{(x_1-x_0)^4}$
$H_{20} = \frac{-2(x-x_0)(x-x_1)^3-3(x-x_0)^2(x-x_1)^2}{2(x_1-x_0)^3}$
$H_{21} = \frac{2(x-x_0)^3(x-x_1)+3(x-x_0)^2(x-x_1)^2}{2(x_1-x_0)^3}$.

And we know that
$f_(x_0) = x_0^8 - 1$
$f_(x_1) = x_1^8 - 1$
$f'(x_0) = 8x_0^7$
$f'(x_1) = 8x_1^7$
$f''(x_0) = 56x_0^6$
$f''(x_1) = 56x_1^6$.

So $p(x) = [1 - \frac{(x-x_0)^3}{(x_1-x_0)^3} + 3\frac{(x-x_0)^3(x-x_1)}{(x_1-x_0)^4} - 6\frac{(x-x_0)^3(x-x_1)^2}{(x_1-x_0)^5}] \cdot (x_0^8 - 1) + [\frac{(x-x_0)^3}{(x_1-x_0)^3} - 3\frac{(x-x_0)^3(x-x_1)}{(x_1-x_0)^4} + 6\frac{(x-x_0)^3(x-x_1)^2}{(x_1-x_0)^5}] \cdot (x_1^8 - 1) + \frac{(x_1-x_0)^4(x-x_1)-(x_1-x_0)^2(x-x_0)^3+2(x_1-x_0)(x-x_0)^3(x-x_1)-3(x-x_0)^3(x-x_1)^3}{(x_1-x_0)^4} \cdot 8x_0^7 + \frac{(x_1-x_0)(x-x_0)^3(x-x_1)-3(x-x_0)^3(x-x_1)^2}{(x_1-x_0)^4} \cdot 8x_1^7 + \frac{-2(x-x_0)(x-x_1)^3-3(x-x_0)^2(x-x_1)^2}{2(x_1-x_0)^3} \cdot 56x_0^6 + \frac{2(x-x_0)^3(x-x_1)+3(x-x_0)^2(x-x_1)^2}{2(x_1-x_0)^3} \cdot 56x_1^6$.

3. $f(x) = x^8 - 1, f'(x) = 8x^7, f''(x) = 56x^6, f^{(3)}(x) = 336x^5, f^{(4)}(x) = 1680x^4, f^{(5)}(x) = 6720x^3, f^{(6)}(x) = 20160x^2$

$|f(x) - p(x)| = \frac{f^{(6)}(\xi)}{6!}(x - x_0)^3(x - x_1)^3 = 28\xi^2(x - x_0)^3(x - x_1)^3$ where $\xi \in [x_0, x_1]$.

When $x_0 = 0, x_1 = h$, $max_{x_0 \le x \le x_1}|f(x) - p(x)| = max_{x_0 \le x \le x_1} 28h^2x^3(x - h)^3 = 28h^2(\frac{h}{2})^6 = \frac{7}{16}h^8$.

When $x_1 = 0, x_1 = 1 + h$, $max_{x_0 \le x \le x_1}|f(x) - p(x)| = max_{x_0 \le x \le x_1} 28(1 + h)^2(x - 1)^3(x - 1 - h)^3 = \frac{7}{16}(1 + h)^2(\frac{h}{2})^6 = \frac{7}{16}h^6(1 + h)^2$.

The first expression goes to zero faster.

4. $|f(x) - p(x)| = \frac{f^{(4)}(\xi)}{4!}(x - x_0)(x - x_a)(x - x_b)(x - x_1) = 1680\xi^4(x - x_0)(x - x_a)(x - x_b)(x - x_1)$ where $\xi \in [x_0, x_1]$.

When $x_0 = 0, x_1 = h$, $max_{x_0 \le x \le x_1}|f(x) - p(x)| = 1680h^4(\frac{h}{2})^2(\frac{h}{6})^2 = \frac{35}{3}h^8$.

When $x_0 = 1, x_1 = 1 + h$, $max_{x_0 \le x \le x_1}|f(x) - p(x)| = 1680(1 + h)^4(\frac{h}{2})^2(\frac{h}{6})^2 = \frac{35}{3}h^4(1 + h)^4$.

The first expression goes to zero faster.

$\square$