Newton's Method

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Problem 1 Let $a, b \in \mathbb{R}$, with $b > 0$. An algorithm for division on early computers is based on the following idea: First, compute $b^{-1} = \frac{1}{b}$ by applying Newton's method to the function $f(x) = b - 1/x$. Afterwards, form the product $\frac{a}{b} = a \cdot b^{-1}$.
In this problem we'll work out the details of this idea:
1. Show that the Newton iteration is equivalent to the iteration:

$$x_{n+1} = x_n(2 - bx_n), n \leq 0.$$

2. Prove that this iteration converges if and only if $0 < x_0 < \frac{2}{b}$.
3. Make a plot and give an explanation which shows why this condition makes sense.
4. Implement this iteration and use it to compute $b^{-1} = 1/3$.

Solution.

1. $$f(x) = b - \frac{1}{x}, f'(x) = \frac{1}{x^2} \tag{1}$$

   We apply Newton's Method:

$$\begin{aligned} x_{k+1} &= x_k - \frac{f(x_k)}{f'(x_k)} \\ &= x_k - \frac{b - \frac{1}{x_k}}{\frac{1}{x_k^2}} \\ &= x_k - bx_k^2 + x_k \\ &= x_k(2 - bx_k) \end{aligned} \tag{2}$$

   Therefore, the Newton iteration is equivalent to the iteration:

$$x_{n+1} = x_n(2 - bx_n), n \geq 0 \tag{3}$$

2. Let's assume $x_0 = \frac{1}{b} + \epsilon$.
   Then

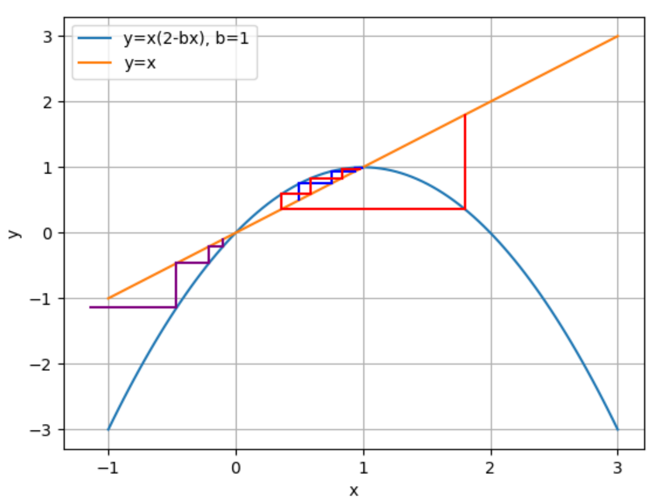
$$\begin{aligned} x_1 &= x_0(2 - bx_0) \\ &= (\frac{1}{b} + \epsilon)(2 - 1 - b\epsilon) \\ &= \frac{1}{b} - b\epsilon^2 \\ x_2 &= x_1(2 - bx_1) \\ &= (\frac{1}{b} - b\epsilon^2)(2 - 1 + b^2\epsilon^2) \\ &= \frac{1}{b} - b^3\epsilon^4 \end{aligned} \tag{4}$$

   By induction,

$$\begin{aligned} x_{k-1} &= \frac{1}{b} - b^{2^{k-1}-1}\epsilon^{2^{k-1}} \\ x_k &= x_{k-1}(2 - bx_{k-1}) \\ &= (\frac{1}{b} - b^{2^{k-1}-1}\epsilon^{2^{k-1}})(2 - 1 + b^{2^{k-1}}\epsilon^{2^{k-1}}) \\ &= \frac{1}{b} - b^{2^k-1}\epsilon^{2^k} \\ &= \frac{1}{b} - \frac{1}{b}(b\epsilon)^{2^k} \end{aligned} \tag{5}$$

   The iteration converges if and only if $-1 < b\epsilon < 1$, which is equivalent to $-1 < \epsilon < \frac{1}{b}$, and $0 < x_0 < \frac{2}{b}$.

3. The following graph shows the fixed point iteration of $x_{n+1} = x_n(2 - bx_n), n \geq 0$ when $b = 1$. In the graph, if $0 < x_0 < 2$, the iteration will converge to $f(\xi) = \xi = 1$. If $x_0 \leq 0$, x_k will become smaller and smaller, and thus the iteration diverges. If $x_0 \geq 2$, x_k will become bigger and bigger, and thus the iteration diverges.



4. We need to compute $b^{-1} = \frac{1}{3}$.
   $x_{n+1} = x_n(2 - bx_n)$.
   We choose $x_0 = 0.2$, which satisfies $0 < x_0 < \frac{2}{3}$.
   $x_1 = 0.28$.
   $x_2 = 0.3248$.
   $x_3 = 0.33311488$.
   $x_4 = 0.3333331901677568$.
   $x_5 = 0.3333333333332718$.
   $x_6 = 0.3333333333333333$.

   Python Implementation:
   ```
   X = [0.2]
   for i in range(1,7):
   X.append(X[i-1] * (2 - 3 * X[i-1]))
   print(X)
   ```

□

Problem 2 Let f be a twice continuously differentiable function ($f \in C^2$). We can assume that f is C^2 on all of $\mathbb{R}$ for simplicity. Let $\xi \in \mathbb{R}$ such that $f(\xi) = f'(\xi) = 0$. That is, f has a double root (or a root of multiplicity two) at ξ:
1. Show that in this case Newton's method is only linear convergent instead of quadratically convergent. Hint: study the proof of Newton's method and use the mean value theorem.
2. Show that if the Newton iteration is replaced with:

$$x_{n+1} = x_n - 2\frac{f(x_n)}{f'(x_n)}$$

quadratic convergence is recovered.

Solution.

1. Taylor Expansion with the Lagrange form of the remainder shows that

$$f(x_k) = f(\xi) + f'(\xi)(x_k - \xi) + \frac{f''(\eta)}{2}(x_k - \xi)^2 = \frac{f''(\eta)}{2}(x_k - \eta)^2$$

   for some η between ξ and x_k.

$$f'(x_k) = f'(\xi) + f''(\eta')(x_k - \eta') = f''(\eta')(x_k - \xi)$$

   for some η' between ξ and x_k.
   The Newton's iteration is as follows:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{f''(\eta)(x_k - \xi)}{2f''(\eta')}$$

$$\xi - x_{k+1} = \xi - x_k + \frac{f''(\eta)(x_k - \xi)}{2f''(\eta')}$$

Since η and η' converge to ξ,

$$\lim_{k\to\infty} \frac{|x_{k+1} - \xi|}{|x_k - \xi|} = \lim_{k\to\infty} (1 - \frac{f''(\eta)}{2f''(\eta')}) = \lim_{k\to\infty} (1 - \frac{f''(\xi)}{2f''(\xi)}) = \frac{1}{2}$$

Newton's method is only linear convergent.

2. The Newton's iteration is replaced with:

$$x_{n+1} = x_n - 2\frac{f(x_n)}{f'(x_n)}$$

Let

$$g(x) = x_n - 2\frac{f(x)}{f(x)}$$

We know that

$$f(x) = (x - \xi)^2 h(x)$$

where $h(\xi) \neq 0$.
Then

$$f'(x) = 2(x - \xi)h(x) + (x - \xi)^2 h'(x)$$

Then

$$g(x) = x - \frac{2(x - \xi)^2 h(x)}{2(x - \xi)h(x) + (x - \xi)^2 h'(x)}$$

Then

$$g'(x) = 1 - \frac{(2h(x) + (x - \xi)^2 h'(x))^2 + (2(x - \xi)h(x))(3h'(x) + (x - \xi)h'(x)))}{(2(x - \xi)h(x) + (x - \xi)^2 h'(x))^2}$$

So

$$g(\xi) = \xi$$
$$g'(\xi) = 0$$

Then

$$x_{k+1} = g(x_k) = g(\xi) + g'(\xi)(x_k - \xi) + \frac{g''(\eta)}{2}(x_k - \xi)^2 = \xi + \frac{g''(\eta)}{2}(x_k - \xi)^2$$

for some η between ξ and x_k.
Thus

$$x_{k+1} - \xi = \frac{g''(\eta)}{2}(x_k - \xi)^2$$
$$\frac{x_{k+1}}{(x_k - \xi)^2} - \xi = \frac{g''(\eta)}{2}$$

Therefore,

$$\lim_{k\to\infty} \frac{|x_{k+1} - \xi|}{|x_k - \xi|^2} = \frac{g''(\eta)}{2}$$

Quadratic convergence is recovered.

$\square$