Approximation by Orthogonal Polynomials

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Problem 1 Let $n = 5$ and apply the Gram-Schmidt process to $m_0, ..., m_5$ to get a new set of orthogonal polynomials $p_0, ..., p_5$ such that $(p_i, p_j) = \delta_{ij}$.

Solution.

$q_0(x) = 1$

$q_1(x) = x - \frac{(x,1)}{(1,1)} \cdot 1 = x$

$q_2(x) = x^2 - \frac{(x^2,1)}{(1,1)} \cdot 1 - \frac{(x^2,x)}{(x,x)} \cdot x = x^2 - \frac{1}{3}$

$q_3(x) = x^3 - \frac{(x^3,1)}{(1,1)} \cdot 1 - \frac{(x^3,x)}{(x,x)} \cdot x - \frac{(x^3, x^2 - \frac{1}{3})}{(x^2 - \frac{1}{3}, x^2 - \frac{1}{3})} \cdot (x^2 - \frac{1}{3}) = x^3 - \frac{3}{5}x$

$q_4(x) = x^4 - \frac{(x^4,1)}{(1,1)} \cdot 1 - \frac{(x^4,x)}{(x,x)} \cdot x - \frac{(x^4, x^2 - \frac{1}{3})}{(x^2 - \frac{1}{3}, x^2 - \frac{1}{3})} \cdot (x^2 - \frac{1}{3}) - \frac{(x^4, x^3 - \frac{3}{5}x)}{(x^3 - \frac{3}{5}x, x^3 - \frac{3}{5}x)} \cdot (x^3 - \frac{3}{5}x) = x^4 - \frac{6}{7}x^2 + \frac{3}{35}$

$q_5(x) = x^5 - \frac{(x^5,1)}{(1,1)} \cdot 1 - \frac{(x^5,x)}{(x,x)} \cdot x - \frac{(x^5, x^2 - \frac{1}{3})}{(x^2 - \frac{1}{3}, x^2 - \frac{1}{3})} \cdot (x^2 - \frac{1}{3}) - \frac{(x^5, x^3 - \frac{3}{5}x)}{(x^3 - \frac{3}{5}x, x^3 - \frac{3}{5}x)} \cdot (x^3 - \frac{3}{5}x) - \frac{(x^5, x^4 - \frac{6}{7}x^2 + \frac{3}{35})}{(x^4 - \frac{6}{7}x^2 + \frac{3}{35}, x^4 - \frac{6}{7}x^2 + \frac{3}{35})} \cdot (x^4 - \frac{6}{7}x^2 + \frac{3}{35}) = x^5 - \frac{10}{9}x^3 + \frac{5}{21}x$

Then we normalize the $q_i(x)$ above:

$p_0(x) = \frac{u_0(x)}{\sqrt{(u_0(x), u_0(x))}} = \frac{1}{\sqrt{2}}$

$p_1(x) = \frac{u_1(x)}{\sqrt{(u_1(x), u_1(x))}} = \sqrt{\frac{3}{2}}x$

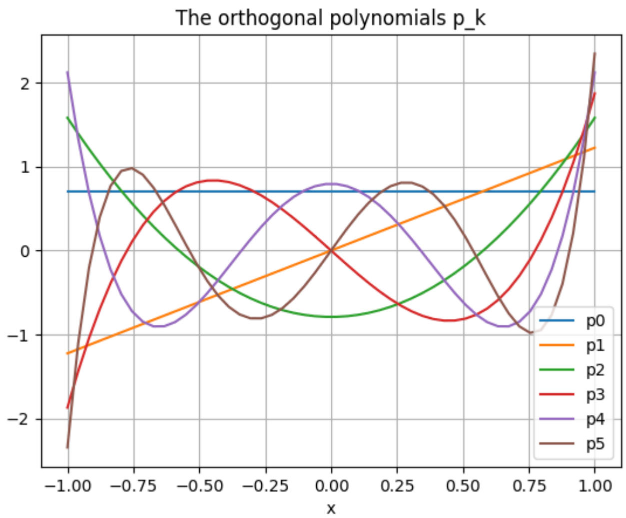
$p_2(x) = \frac{u_2(x)}{\sqrt{(u_2(x), u_2(x))}} = \sqrt{\frac{45}{8}}(x^2 - \frac{1}{3})$

$$p_3(x) = \frac{u_3(x)}{\sqrt{(u_3(x), u_3(x))}} = \sqrt{\frac{175}{8}}(x^3 - \frac{3}{5})$$
$$p_4(x) = \frac{u_4(x)}{\sqrt{(u_4(x), u_4(x))}} = \sqrt{\frac{11025}{128}}(x^4 - \frac{6}{7}x^2 + \frac{3}{35})$$
$$p_5(x) = \frac{u_5(x)}{\sqrt{(u_5(x), u_5(x))}} = \sqrt{\frac{43659}{128}}(x^5 - \frac{10}{9}x^3 + \frac{5}{21}x)$$

□

Problem 2 Plot $p_0, ..., p_5$ on $[-1, 1]$.

Solution.



□

Problem 3 Find the set of coefficients $c_0, ..., c_n$ such that:

$$||f - \sum_{i=0}^{n} c_i p_i||^2$$

is minimized.

Solution.

$$\begin{aligned} & ||f - \Sigma_{i=0}^{n} c_i p_i||^2 \\ & = (f - \Sigma_{i=0}^{n} c_i p_i, f - \Sigma_{i=0}^{n} c_i p_i) \\ & = (f, f) - \Sigma_{i=0}^{n}(c_i^2 - 2 \cdot c_i \cdot (f, p_i)) \\ & = (f, f) - \Sigma_{i=0}^{n}(f, p_i)^2 + \Sigma_{i=0}^{n}(c_i - (f, p_i))^2 \end{aligned} \tag{1}$$

To minimize $||f - \Sigma_{i=0}^{n} c_i p_i||^2$, we need to minimize $\Sigma_{i=0}^{n}(c_i - (f, p_i))^2$. So we can choose $c_i = (f, p_i)$.
Hence, the set of coefficients is $c_i = (f, p_i)$.

□

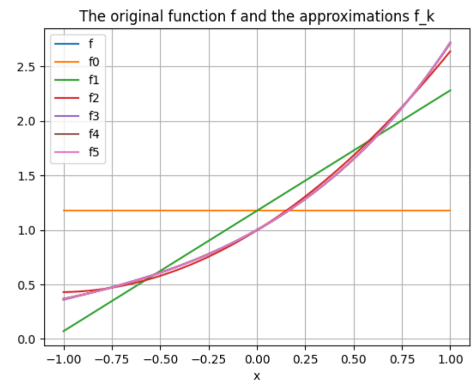
Problem 4 Pick three square-integrable functions defined on $[-1, 1]$ which are not polynomials. They do not need to be smooth or even continuous. Now, for $n = 5$, compute the coefficients $c_0, ..., c_5$ for each function f and form the approximation:

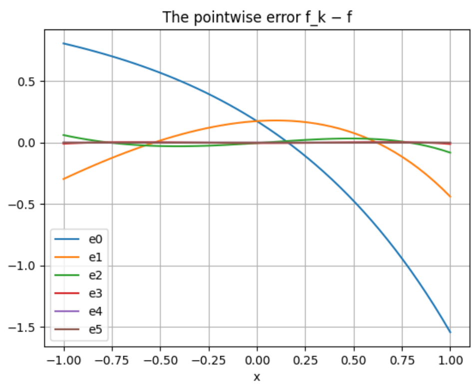
$$\hat{f}_k(x) = \sum_{k=0}^{5} c_k p_k(x)$$

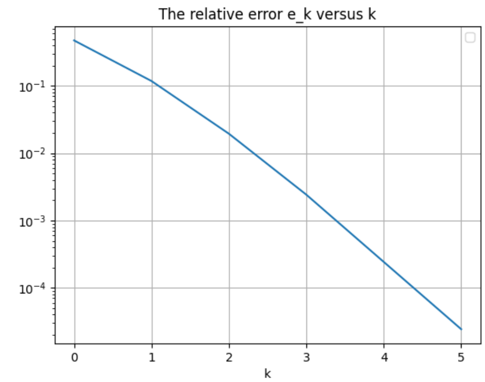
and the relative error:

$$e_k = \frac{||\hat{f}_k - f||}{|||f|}$$

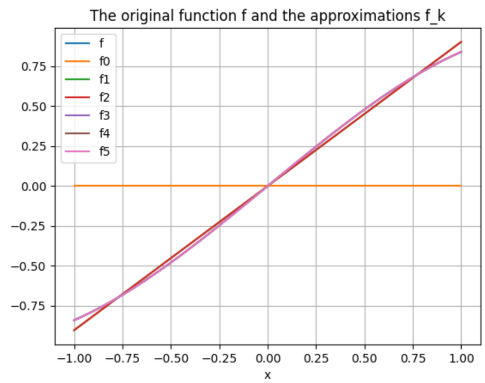
Solution.
Function: $f = e^x$
$c_0 = 1.6619854665670049$
$c_1 = 0.9011169177071356$
$c_2 = 0.22630166550703368$
$c_3 = 0.03766012028576332$
$c_4 = 0.004697606458717772$
$c_5 = 0.000468865078058156$

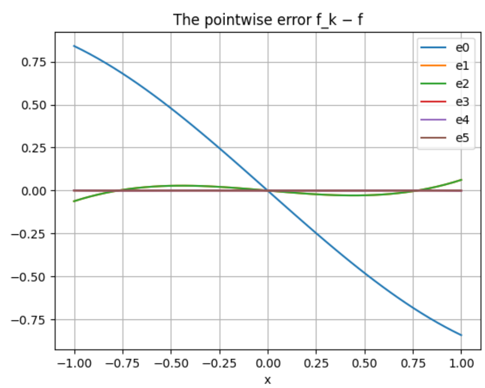


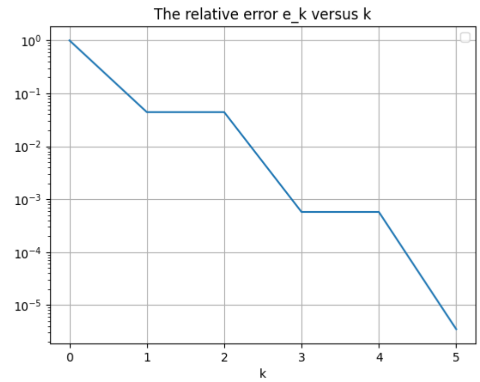




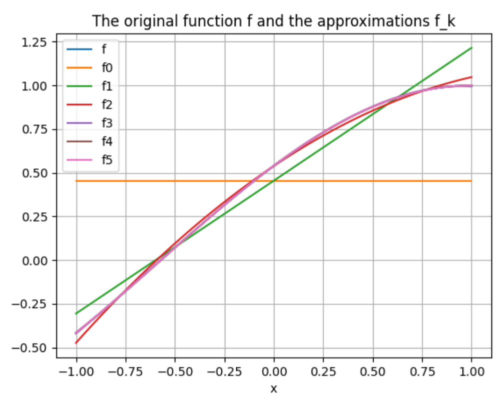
Function: $f = sin(x)$
$c_0 = 0.0$
$c_1 = 0.737709589932675$
$c_2 = 0.0$
$c_3 = -0.03369954074512352$
$c_4 = 0.0$
$c_5 = 0.0004341503382063461$

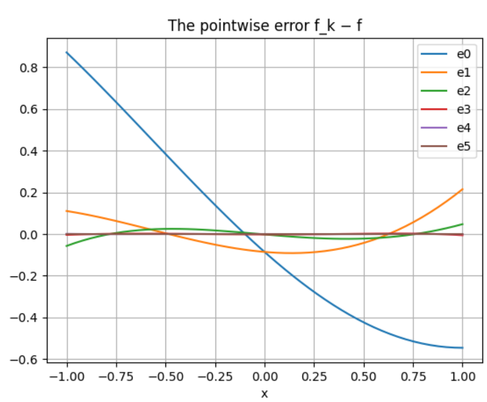


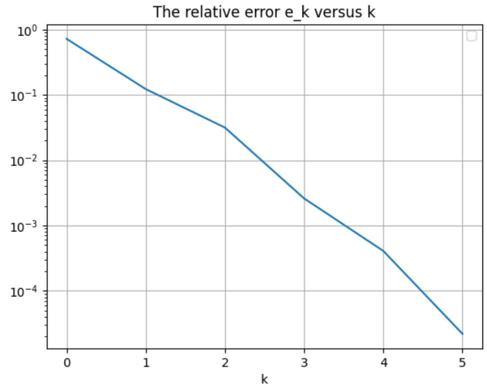




Function: $f = cos(x-1)$
$c_0 = 0.6429703766233398$
$c_1 = 0.6207612151428781$
$c_2 = -0.10599221589151814$
$c_3 = -0.028357185738372953$
$c_4 = 0.002317560177512029$
$c_5 = 0.00036532491264536227$







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