Assignment C

Common Notations

We use the following notation:

S_t	:	Stock price at time t
σ	:	Volatility of the Stock (assumed constant)
r	:	Interest rate
T	:	Time to option expiration (in years)
K	:	Strike price
W_t	:	Brownian motion process
$N(0,1)$	:	Standard normal distribution
$(A-B)^+$	:	Max(A-B, 0)

Generating Random Inputs

- Where applicable, you may use the function shown below to generate input values randomly:

```
float random_data(float low, float hi)
{
    float r = (float)rand() / (float)RAND_MAX;
    return low + r * (hi - low);
}
```

- For example, you may use this function to populate the input matrices in Assignment 1, and initialize the input values in option pricing assignments.

Measuring Running Time using Chrono

- We use a timer to measure the execution times of a program.
- We have many choices to get such timing measurements. Code snippet below shows how to use chrono in the C++ Standard Library.

```
#include <chrono>

using namespace std::chrono;

int main()
{
   high_resolution_clock::time_point t1 =
       high_resolution_clock::now();

   do_the_works();

   high_resolution_clock::time_point t2 =
      high_resolution_clock::now();

   std::cout << "Elapsed time: " <<
      duration_cast<milliseconds>(t2 - t1).count() << " ms";

}
```

- Defined in chrono header.

Assignment C(C/C++)

- Due: June 24 by 6 PM.
- Write a function to price European style Call options using a binomial tree.
- Jarrow-Rudd tree is explained below but you may use any other type (e.g. CRR) of your choice.
- Measure time taken to price 1 million options.
- You are not required to use techniques such as vectorization/multithreading for this assignment.
- Aim of this assignment is to get the students to think about performance and set the stage for week 1 lecture.
- As long as anyone makes a genuine and an honest attempt to solve this problem one will get full points for this assignment, even if the solution is not complete.

The Jarrow-Rudd Binomial Tree

- We will use a binomial tree to simulate the paths of the underlying stochastic process.
- Jarrow-Rudd tree assumes the random value (z=N(0, 1)) takes +1 or -1 with equal (risk neutral) probability of 1/2 at every node in the tree.
- If we know the stock price at time t, Equations 1 and 2 give us possible values for the stock prices at time $t + \Delta t$.

Stock Price Process

- This leads to stock price at time $t + \Delta t$:

$$z = 1 : S^U_{t+\Delta t} = S_t \exp^{((r-\sigma^2/2)\Delta t + \sigma\sqrt{\Delta} t)} \tag{1}$$

$$z = -1 : S^D_{t+\Delta t} = S_t \exp^{((r-\sigma^2/2)\Delta t - \sigma\sqrt{\Delta} t)} \tag{2}$$

$$\begin{tikzcd}
 & S^U_{t+\Delta t} \\
S_t \arrow[ur] \arrow[dr] & \\
 & S^D_{t+\Delta t}
\end{tikzcd}$$

- We refer to these two stock prices as $S^U_{t+\Delta t}$ and $S^D_{t+\Delta t}$ to mean the stock prices at *up* and *down* nodes respectively, at time $t + \Delta t$.

- We simulate how the stock prices would evolve on the tree using Equations 1 and 2:
  - divide time to expiration, T, into equal N parts such that $\Delta t = T/N$
  - discretize W_t such that

$$\Delta W_t = \sqrt{\Delta t} N(0,1) \tag{3}$$

    and assume the underlying Brownian motion can only move a fixed amount up or down at each time step.
    (In the case of Jarrow-Rudd tree, the random variable N(0, 1) given in Equation 3 takes +1 or -1 with an equal probability of 1/2.)
  - we start from the left of the tree, i.e. $t = 0$ and build the binary tree for the stock price process at every Δt increment

- A full binomial tree with 3 times steps (N=3) is shown next.

$$
\begin{tikzcd}
 & & & S^{UUU}_{3\Delta t=T} \\
 & & S^{UU}_{2\Delta t} \arrow[ur] \arrow[dr] & \\
 & S^{U}_{\Delta t} \arrow[ur] \arrow[dr] & & S^{UUD}_{3\Delta t=T} \\
S_0 \arrow[ur] \arrow[dr] & & S^{UD}_{2\Delta t} \arrow[ur] \arrow[dr] & \\
 & S^{D}_{\Delta t} \arrow[ur] \arrow[dr] & & S^{UDD}_{3\Delta t=T} \\
 & & S^{DD}_{2\Delta t} \arrow[ur] \arrow[dr] & \\
 & & & S^{DDD}_{3\Delta t=T}
\end{tikzcd}
$$

- Such a construction yields a binomial tree.
- This tree recombines by construction.
- Recombining means: if you start from a node at time t, an *up* move at time t followed by a *down* move at time $t+\Delta t$ and a *down* move at time t followed by an *up* move at time $t+\Delta t$ end up at the same node at time $t+2\Delta t$
  $S_{\Delta t}^{D} = S_0 \exp^{((r-\sigma^2/2)\Delta t - \sigma\sqrt{\Delta} t)}$
  $S_{\Delta t}^{DU} = S_{\Delta t}^{D} \exp^{((r-\sigma^2/2)\Delta t + \sigma\sqrt{\Delta} t)} = S_0 \exp^{2(r-\sigma^2/2)\Delta t} = S_{\Delta t}^{UD}$
- Recombining trees do not grow very large when we increase the number of time steps.
- Note:
  1. The arrows show the direction of *path simulation*.
  2. Superscripts D, DD, U, UU etc. denote how many levels the stock price has gone up or down from the initial value.
     U denotes an up movement; D denotes a down movement.

Pricing an European Option

We use the stock prices at each node on the tree to calculate the option price:

- The option payoff at time t=T, i.e. $C(S_T)$, is given by the options payoff function.
  - Call: $C(S_T) = max(S_T - K, 0)$
  - Put: $C(S_T) = max(K - S_T, 0)$
- The option payoff at time t, $C(S_t)$, is given by the discounted expectation of the option payoffs at time $t + \Delta t$:

$$C_t = [p_u C^U_{t+\Delta t} + p_d C^D_{t+\Delta t}] exp^{-r\Delta t} \qquad p_u = p_d = 0.5 \quad (4)$$

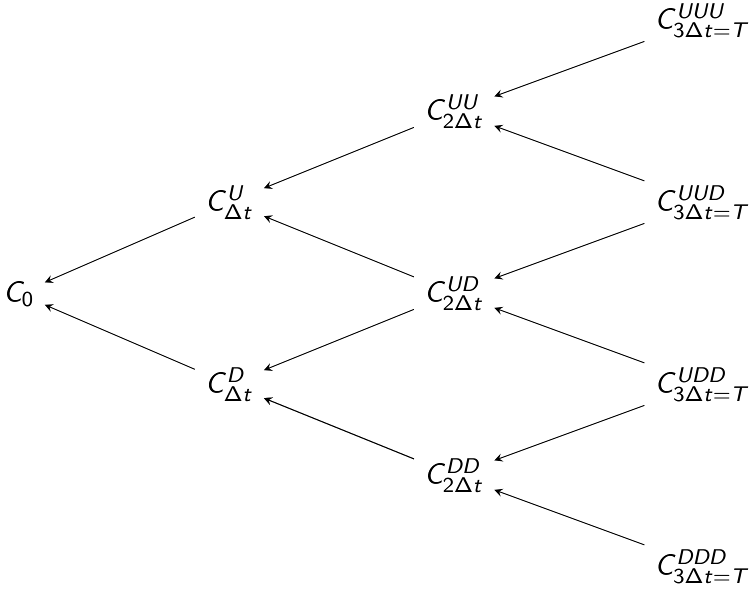
$$\begin{tikzcd}
 & C^U_{t+\Delta t} \arrow[dl] \\
C_t & \\
 & C^D_{t+\Delta t} \arrow[ul]
\end{tikzcd}$$

Pricing an European Option

- Risk neutral probabilities associated with each path (p_u and p_d) is 0.5 (by construction) for Jarrow-Rudd trees.
- Starting from the right ($t = T$) back-propagate the option prices to find the price of the option at time zero.
- The price of option at time t, i.e. C_t is given by the discounted expectation of prices at time $t + \Delta t$, given in Equation 4.

  Notes:
  1. The arrows show the direction of the calculation.
  2. Superscripts D, DD, U, UU etc. correspond to the stock price levels.
  3. 0.5 Risk neutral probability value in Equation 4 applies to Jarrow-Rudd trees only. Other types of trees may have different risk neutral probabilities.

- A full binomial tree with 3 time steps (N=3) is shown below:



- The option price at time zero ($t = 0$) is given by C_0.