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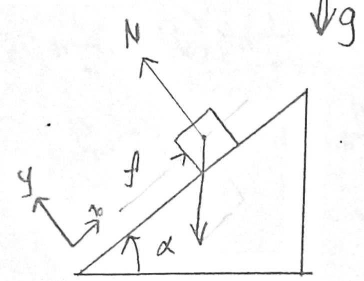
Exercise 1



Newton - Euler method

$$\Sigma F = ma$$

$$f - mg\sin\alpha = m\ddot{x}$$

Lagrangian

$$q_1 = x, \dot{q}_1 = \dot{x}, \quad Q = f$$

$$K = \frac{1}{2}m\dot{x}^2, \quad V = mgx\sin\alpha$$

$$L = \frac{1}{2}m\dot{x}^2 - mgx\sin\alpha$$

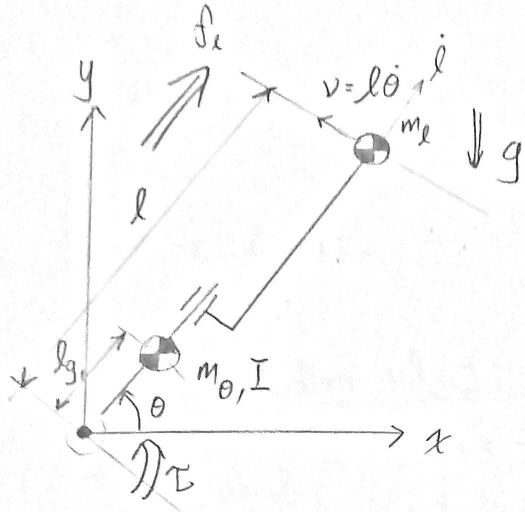
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i$$

$$m\ddot{x} - (-mg\sin\alpha) = f$$

EOM, $f - mg\sin\alpha = m\ddot{x}$

$$f = m\ddot{x} + mg\sin\alpha$$

Exercise 2



Lagrangian

$$\begin{cases} q_1 = \theta,\ q_2 = l,\ Q_1 = \tau \\ \dot{q}_1 = \dot{\theta},\ \dot{q}_2 = \dot{l},\ Q_2 = f_l \end{cases}$$

$$T_{rot.} = \frac{1}{2} I \dot{\theta}^2 + \frac{1}{2} m_l (l\dot{\theta})^2$$

$$T_{trans.} = \frac{1}{2} m_l \dot{l}^2$$

$$T = \frac{1}{2} I \dot{\theta}^2 + \frac{1}{2} m_l (l\dot{\theta})^2 + \frac{1}{2} m_l \dot{l}^2$$

$$V = (m_\theta l_{g_1} + m_l l) g \sin\theta$$

$$\mathcal{L} = T - V$$

q_1, $$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = \tau$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = I\dot{\theta} + m_l l^2 \dot{\theta}$$

$$\frac{d}{dt}(\partial_{\dot{\theta}} \mathcal{L}) = I\ddot{\theta} + 2m_l l \dot{l} \dot{\theta} + m_l l^2 \ddot{\theta}$$

$$= (I + m_l l^2)\ddot{\theta} + 2m_l l \dot{l} \dot{\theta}$$

$$-\partial_\theta \mathcal{L} = (m_\theta l_{g_1} + m_l l) g \cos\theta$$

$$(I + m_l l^2)\ddot{\theta} + 2m_l l \dot{l} \dot{\theta} + (m_\theta l_{g_1} + m_l l) g\cos\theta = \tau \;//$$

q_2, $$\frac{d}{dt}(\partial_{\dot{l}} \mathcal{L}) - \partial_l \mathcal{L} = f_l$$

$$\partial_{\dot{l}} \mathcal{L} = m_l \dot{l}$$

$$\frac{d}{dt}(\partial_{\dot{l}} \mathcal{L}) = m_l \ddot{l}$$

$$-\partial_l \mathcal{L} = m_l g \sin\theta - m_l \dot{\theta}^2 l$$

$$-m_l l \dot{\theta}^2 + m_l \ddot{l} + m_l g \sin\theta = f \;//$$

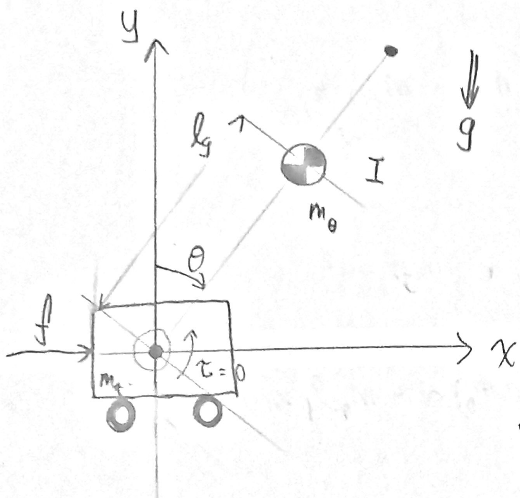
EOM,

$$\tau - 2m_l \dot{l} l \dot{\theta} - (m_\theta l_{g_1} + m_l l) g\cos\theta = (I + m_l l^2)\ddot{\theta}$$

$$f - m_l g \sin\theta + m_l l \dot{\theta}^2 = m_l \ddot{l}$$

$$\begin{cases} \tau = (I + m_l l^2)\ddot{\theta} + 2m_l l \dot{l} \dot{\theta} + (m_\theta l_{g_1} + m_l l) g\cos\theta \\ f = m_l \ddot{l} - m_l l \dot{\theta}^2 + m_l g \sin\theta \end{cases}$$

ercise 3



Lagrangian

$$\begin{cases} q_1 = \theta, & \dot q_1 = \dot\theta, & Q_1 = -\tau \\ q_2 = x, & \dot q_2 = \dot x, & Q_2 = f \end{cases}$$

$$T_{trans} = \frac{1}{2} m_x \dot x^2 + \frac{1}{2} m_\theta \left[(\dot x + l_g \dot\theta \cos\theta)^2 + (-l_g \dot\theta \sin\theta)^2 \right]$$

$$T_{rot} = \frac{1}{2} I \dot\theta^2$$

$$T = \frac{1}{2} I \dot\theta^2 + \frac{1}{2}(m_x + m_\theta)\dot x^2 + \frac{1}{2} m_\theta (l_g \dot\theta)^2 + m_\theta l_g \cos\theta \, \dot x \dot\theta$$

$$V = mg l_g \cos\theta$$

q_1,

$$\partial_{\dot\theta} L = I\dot\theta + m_\theta l_g^2 \dot\theta + m_\theta l_g \dot x \cos\theta$$

$$\frac{d}{dt}(\partial_{\dot\theta} L) = I\ddot\theta + m_\theta l_g^2 \ddot\theta + m_\theta l_g \ddot x \cos\theta - m_\theta l_g \dot x \dot\theta \sin\theta$$

$$\partial_\theta L = -m_\theta l_g \dot x \dot\theta \sin\theta + mg l_g \sin\theta$$

$$I\ddot\theta + m_\theta l_g \ddot\theta + m_\theta l_g \ddot x \cos\theta - m_\theta l_g \dot x \dot\theta \sin\theta + m_\theta l_g \dot x \dot\theta \sin\theta - mg l_g \sin\theta = -\tau$$

$$-\tau + mg l_g \sin\theta = (I + m_\theta l_g^2)\ddot\theta + m_\theta l_g \cos\theta \, \ddot x$$

$$mg l_g \sin\theta = (I + m_\theta l_g^2)\ddot\theta + m_\theta l_g \cos\theta \, \ddot x \quad //$$

q_2,

$$\partial_{\dot x} L = (m_x + m_\theta)\dot x + m_\theta l_g \cos\theta \, \dot\theta$$

$$\frac{d}{dt}(\partial_{\dot x} L) = (m_x + m_\theta)\ddot x + m_\theta l_g \cos\theta \, \ddot\theta - m_\theta l_g \sin\theta \, \dot\theta^2$$

$$\partial_x L = 0$$

$$f + m_\theta l_g \sin\theta \, \dot\theta^2 = (m_x + m_\theta)\ddot x + m_\theta l_g \cos\theta \, \ddot\theta \quad //$$

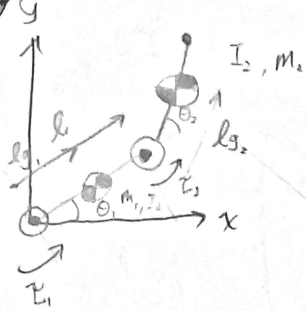
EOM

$$m g l_g \sin\theta = (I + m_\theta l_g^2)\ddot{\theta} + m_\theta l_g \cos\theta\, \ddot{x}$$

$$f + m_\theta l_g \sin\theta\, \dot{\theta}^2 = m_\theta l_g \cos\theta\, \ddot{\theta} + (m_x + m_\theta)\ddot{x}$$

$$\begin{cases} 0 = (I + m_\theta l_g^2)\ddot{\theta} + m_\theta l_g \cos\theta\, \ddot{x} - m_\theta g l_g \sin\theta \\ f = m_\theta l_g \cos\theta\, \ddot{\theta} + (m_x + m_\theta)\ddot{x} - m_\theta l_g \sin\theta\, \dot{\theta}^2 \end{cases}$$

ase 4



$$\begin{cases} q_1 = \theta_1, & \dot q_1 = \dot\theta_1, & Q_1 = \tau_1 \\ q_2 = \theta_2, & \dot q_2 = \dot\theta_2, & Q_2 = \tau_2 \end{cases}$$

$$T_1 = \frac{1}{2}(I_1 + m_1 l_{g_1}^2)\dot\theta_1^2$$

$$T_2 = \frac{1}{2}(I_2 + m_2 l_{g_2}^2)\dot\theta_2^2 + \frac{1}{2} m_2 \left[(l_1^2 + l_{g_2}^2 + 2 l_1 l_{g_2}\cos\theta_2)\dot\theta_1^2\right] + m_2\left[(l_{g_2}^2 + l_1 l_{g_2}\cos\theta_2)\dot\theta_1\dot\theta_2\right]$$

$$r_2 = l_1 e^{j\theta_1} + l_{g_2} e^{j(\theta_1+\theta_2)}$$

$$\dot r_2 = j\dot\theta_1 l_1 e^{j\theta_1} + j(\dot\theta_1 + \dot\theta_2) l_{g_2} e^{j(\theta_1+\theta_2)}$$

$$= \left[\dot\theta_1 l_1 + (\dot\theta_1+\dot\theta_2) l_{g_2} e^{j\theta_2}\right] j e^{j\theta_1}$$

$$|\dot r_2|^2 = \left[\dot\theta_1 l_1 + (\dot\theta_1+\dot\theta_2) l_{g_2} e^{j\theta_2}\right] \cdot \left[\dot\theta_1 l_1 + (\dot\theta_1+\dot\theta_2) l_{g_2} e^{-j\theta_2}\right]$$

$$= (\dot\theta_1 l_1)^2 + (\dot\theta_1+\dot\theta_2)^2 l_{g_2}^2 + \dot\theta_1(\dot\theta_1+\dot\theta_2) l_1 l_{g_2}\left[e^{j\theta_2} + e^{-j\theta_2}\right]$$

$$= (\dot\theta_1 l_1)^2 + (\dot\theta_1+\dot\theta_2)^2 l_{g_2}^2 + 2\dot\theta_1(\dot\theta_1+\dot\theta_2) l_1 l_{g_2}\cos\theta_2$$

$$= \dot\theta_1^2(l_1^2 + l_{g_2}^2 + 2 l_1 l_{g_2}\cos\theta_2) + \dot\theta_2^2 l_{g_2}^2 + 2\dot\theta_1\dot\theta_2 \cdot (l_{g_2}^2 + l_1 l_g \cos\theta_2)$$

$$V_1 = m_1 g l_{g_1} \sin\theta_1$$

$$V_2 = m_2 g\left[l_1 \sin\theta_1 + l_{g_2}\sin(\theta_1+\theta_2)\right]$$

$$\partial_{\dot\theta_1} \mathcal{L} = (I_1 + m_1 l_{g_1}^2)\dot\theta_1 + m_2\left[(l_1^2 + l_{g_2}^2 + 2 l_1 l_{g_2}\cos\theta_2)\right]\dot\theta_1 + m_2\left[(l_{g_2}^2 + l_1 l_{g_2}\cos\theta_2)\right]\dot\theta_2$$

$$\frac{d}{dt}(\partial_{\dot\theta_1}\mathcal{L}) = (I_1 + m_1 l_{g_1}^2)\ddot\theta_1 + m_2\left[(l_1^2 + l_2^2 + 2 l_1 l_g \cos\theta_2)\right]\ddot\theta_1 + m_2\left[-2 l_1 l_{g_2}\sin\theta_2\right]\dot\theta_1\dot\theta_2 + m_2\left[(l_{g_2}^2 + l_1 l_{g_2}\cos\theta_2)\right]\ddot\theta_2 + m_2\left[-l_1 l_{g_2}\sin\theta_2\right]\dot\theta_2^2$$

$$\partial_{\theta_1}\mathcal{L} = -m_1 g l_{g_1}\cos\theta_1 - m_2 g l_1 \cos\theta_1 - m_2 g l_{g_2}\cos(\theta_1+\theta_2)$$

$$\partial_{\dot\theta_2}\mathcal{L} = (I_2 + m_2 l_{g_2}^2)\dot\theta_2 + m_2(l_{g_2}^2 + l_1 l_{g_2}\cos\theta_2)\dot\theta_1$$

$$\frac{d}{dt}(\partial_{\dot\theta_2}\mathcal{L}) = (I_2 + m_2 l_{g_2}^2)\ddot\theta_2 \quad m_2(l_{g_2}^2 + l_1 l_{g_2}\cos\theta_2)\ddot\theta_1 + m_2(-l_1 l_{g_2}\sin\theta_2)\dot\theta_1\dot\theta_2$$

$$\partial_{\theta_2}\mathcal{L} = -m_2 l_1 l_{g_2}\sin\theta_2 \dot\theta_1^2 - m_2 l_1 l_{g_2}\sin\theta_2 \dot\theta_1\dot\theta_2 - m_2 g l_{g_2}\cos(\theta_1+\theta_2)$$

EOM

$$\begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} = \begin{bmatrix} \left(I_1 + m_1 lg_1^2 + m_2(l_1^2 + lg_2^2 + 2 l_1 lg_2 \cos\theta_2)\right) & m_2(lg_2^2 + l_1 lg_2 \cos\theta_2) \\ m_2(lg_2^2 + l_1 lg_2 \cos\theta_2) & I_2 + m_2 lg_2^2 \end{bmatrix} \begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix}$$

$$+ m_2 l_1 lg_2 \sin\theta_2 \begin{pmatrix} -\dot{\theta}_2^2 \\ \dot{\theta}_1^2 \end{pmatrix} + 2 m_2 l_1 lg_2 \begin{pmatrix} -\dot{\theta}_1 \dot{\theta}_2 \\ 0 \end{pmatrix}$$

$$+ \begin{pmatrix} m_1 g lg_1 \cos\theta_1 + m_2 g l_1 \cos\theta_1 + m_2 g lg_2 \cos(\theta_1 + \theta_2) \\ m_2 g lg_2 \cos(\theta_1 + \theta_2) \end{pmatrix}$$