Robotics II
Day 10: Control Systems I

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1. Linearization and Laplace transform the Equation of Motion for an inverted pendulum.

<u>Equation of motion for an inverted pendulum</u>

$$\begin{pmatrix} 0 \\ \mathrm{f} \end{pmatrix} = \begin{pmatrix} \mathrm{I} + \mathrm{m}_\theta {l_g}^2 & \mathrm{m}_\theta l_g cos(\theta) \\ \mathrm{m}_\theta l_g cos(\theta) & \mathrm{m_x} + \mathrm{m}_\theta \end{pmatrix} \begin{pmatrix} \ddot{\theta} \\ \ddot{x} \end{pmatrix} - \begin{pmatrix} 0 \\ \mathrm{m}_\theta l_g sin(\theta)\, \dot{\theta}^2 \end{pmatrix} - \begin{pmatrix} \mathrm{m}_\theta g l_g sin(\theta) \\ 0 \end{pmatrix}$$

<u>Linearization</u>

$$\begin{cases} \mathrm{f} = (\mathrm{m_x} + \mathrm{m}_\theta)\ddot{x} + \mathrm{m}_\theta l_g \ddot{\theta} \\ 0 = \mathrm{m}_\theta l_g \ddot{x} + (\mathrm{I} + \mathrm{m}_\theta {l_g}^2)\ddot{\theta} - \mathrm{m}_\theta g l_g \theta \end{cases}$$

<u>Laplace transforming</u>

$$\begin{cases} \mathrm{F(s)} = (\mathrm{m_x} + \mathrm{m}_\theta) s^2 \mathrm{X(s)} + \mathrm{m}_\theta l_g s^2 \theta(\mathrm{s}) \\ 0 = \mathrm{m}_\theta l_g s^2 \mathrm{X(s)} + (\mathrm{I} + \mathrm{m}_\theta {l_g}^2) s^2 \theta(\mathrm{s}) - \mathrm{m}_\theta g l_g \theta(\mathrm{s}) \end{cases}$$

2. Derive the transfer function of the inverted pendulum.

$$\mathrm{G(s)} = \frac{\theta(\mathrm{s})}{\mathrm{F(s)}}$$

$$\mathrm{G(s)} = \frac{1}{-\left[\frac{\mathrm{I}(\mathrm{m_x}+\mathrm{m}_\theta)}{\mathrm{m}_\theta l_g} + \mathrm{m_x} l_g\right] s^2 + (\mathrm{m_x} + \mathrm{m}_\theta)\mathrm{g}}$$

3. Stability Analysis.

Characteristic Equation,

$$-\left[\frac{\mathrm{I}(\mathrm{m_x}+\mathrm{m}_\theta)}{\mathrm{m}_\theta l_g} + \mathrm{m_x} l_g\right] s^2 + (\mathrm{m_x} + \mathrm{m}_\theta)\mathrm{g} = 0$$

The characteristic equation has positive real roots, hence it is <u>unstable</u>.

<u>Hurwitz stability criterion</u>

$$[H] = \begin{bmatrix} 0 & 0 & 0 \\ -\left[\frac{\mathrm{I}(\mathrm{m_x}+\mathrm{m}_\theta)}{\mathrm{m}_\theta l_g} + \mathrm{m_x} l_g\right] & (\mathrm{m_x} + \mathrm{m}_\theta)\mathrm{g} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[H_1] = [0]$$
$$[H_1] = 0$$

$$[H_2] = \begin{bmatrix} 0 & 0 \\ -\left[\dfrac{\mathrm{I}(\mathrm{m_x} + \mathrm{m_\theta})}{\mathrm{m_\theta} l_g} + \mathrm{m_x} l_g\right] & (\mathrm{m_x} + \mathrm{m_\theta})\mathrm{g} \end{bmatrix}$$
$$[H_2] = 0$$

$$[H_3] = \begin{bmatrix} 0 & 0 & 0 \\ -\left[\dfrac{\mathrm{I}(\mathrm{m_x} + \mathrm{m_\theta})}{\mathrm{m_\theta} l_g} + \mathrm{m_x} l_g\right] & (\mathrm{m_x} + \mathrm{m_\theta})\mathrm{g} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$[H_3] = 0$$

Since all the determinants are non-positive, the system is unstable.