Statistical mechanical models for stabiliser codes subject to correlated noise

Joint work with Steve Flammia (USyd/Yale)
Christopher Chubb

Centre for Engineered Quantum Systems
University of Sydney

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christopher.chubb@sydney.edu.au

christopherchubb.com

THE UNIVERSITY OF SYDNEY

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Statistical mechanical mapping

Stabiliser code & iid noise model	$\longrightarrow$	Disordered statistical mechanical model
Error-correcting threshold	$\longleftrightarrow$	Quenched phase transition
Maximum likelihood decoding	$\longleftrightarrow$	Calculating partition functions/free energy differences

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Our results

- Generalise mapping to arbitrary codes and correlated noise
- Show how to apply our mapping to circuit noise via the history code
- Numerically demonstrate that mild correlations can lower the threshold of the toric code considerably
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Independent case

Let $\llbracket A, B \rrbracket$ be the scalar commutator of two Paulis, $AB =: \llbracket A, B \rrbracket BA$.

For a stabiliser code generated by $\{S_k\}_k$, and an error Pauli E,

$$H_E(\vec{s}) := -\sum_i \sum_{\sigma \in \mathcal{P}_i} J_i(\sigma) \llbracket \sigma, E \rrbracket \prod_{k : \llbracket \sigma, S_k \rrbracket = -1} s_k$$

for some couplings $J_i(\sigma) \to \mathbb{R}$.

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Suppose we have an independent error model

$$\Pr(E) = \prod_i p_i(E_i).$$

Nishimori condition: $$\beta J_i(\sigma) = \frac{1}{4}\sum_{\tau\in\mathcal{P}} \log p_i(\tau)\,[\![\sigma,\tau]\!]\,,$$

Using the Fourier-like orthogonality relation $\frac{1}{4}\sum_\sigma [\![\sigma,\tau]\!] = \delta_{\tau,I}$ we get that

$$-\beta H_E(\vec{1}) = \sum_i \sum_\tau \log p_i(\tau) \frac{1}{4} \sum_{\sigma,\tau} [\![\sigma, E\tau]\!] = \sum_i \log p_i(E_i) = \log \Pr(E).$$

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$$Z_E = \sum_{\vec{s}} e^{-\beta H_E(\vec{s})} = \sum_S e^{-\beta H_{ES}(\vec{1})} = \sum_S \Pr(ES) = \Pr(\overline{E}).$$

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Error correction threshold as a quenched phase transition

Consider the free energy cost of a logical operator L,

$$\Delta_E(L) = -\frac{1}{\beta}\log Z_{EL} + \frac{1}{\beta} Z_E = \frac{1}{\beta}\log\frac{\Pr(E)}{\Pr(EL)}.$$

For a fixed error E

	Quantum code	Stat. mech. system
Below threshold	$\{\Pr(EL)\}_L$ peaked	$\Delta(L)\to\infty$ (in mean)
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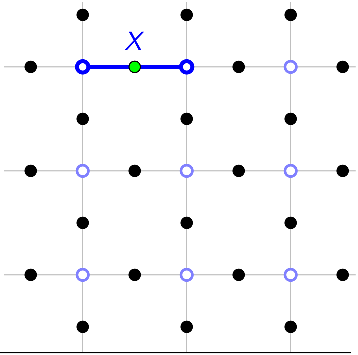
Toric code

Bit-flip → Random-bond Ising[1]

Indep. $X\&Z$ → 2×RBIM (uncoupled)

Depolarising → 2×RBIM (coupled)[2]



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Depolarising → Eight-vertex model [2]

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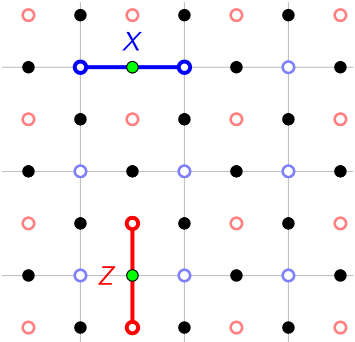

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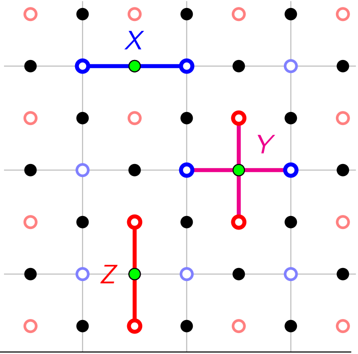

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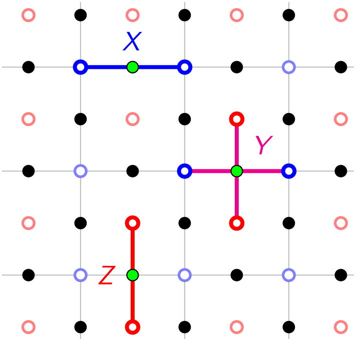

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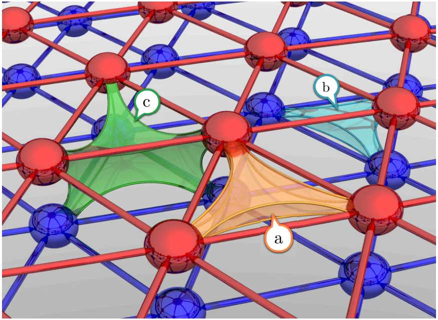

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Correlated case

The key point independence gave us was the ability to factor our noise model

$$\Pr(E) = \prod_i p_i(E_i).$$

We can generalise this to correlated models:

Factored distribution

An error model factors over regions $\{R_j\}_j$ if there exist $\phi_j : \mathcal{P}_{R_j} \to \mathbb{R}$ such that

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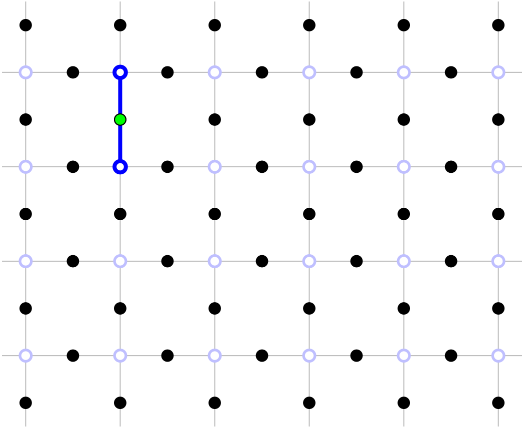
Toric code with correlated bit-flips

Correlations induce longer-range interactions



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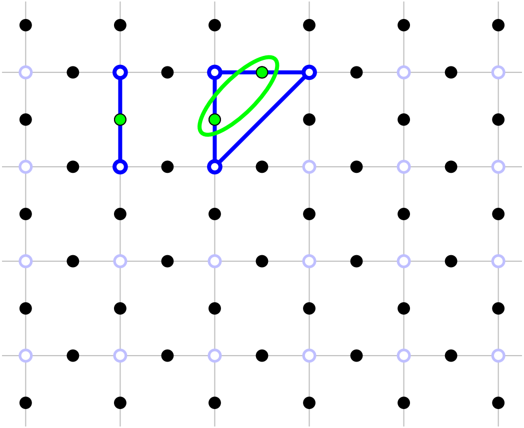

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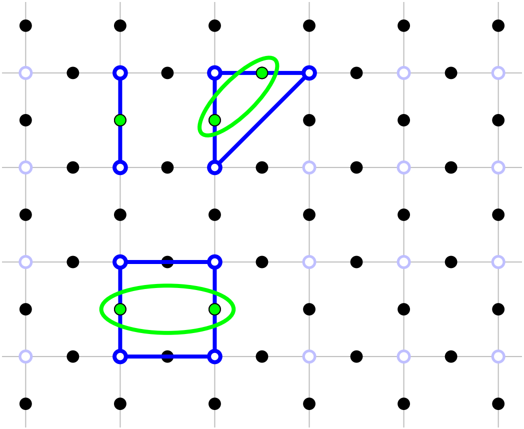

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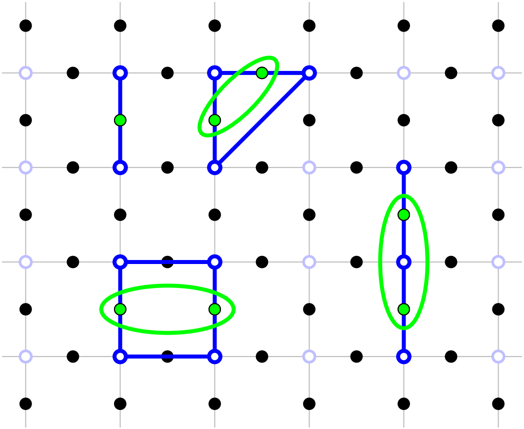
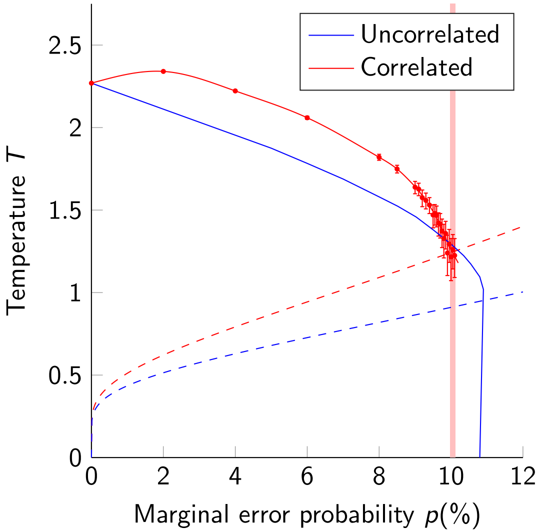
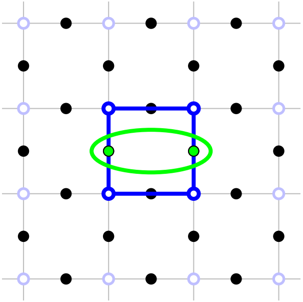


Monte Carlo simulations





Thresholds

Indep.[1]: $p_{\mathrm{th}} = 10.94(2)\%$
Corr.: $p_{\mathrm{th}} = 10.04(6)\%$

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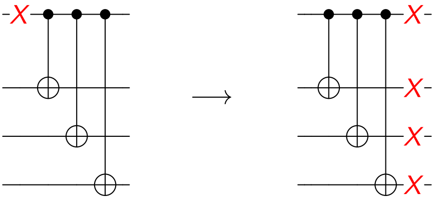
Circuit noise

Noise followed by ideal measurements is unrealistic. In reality, circuits will be faulty.

Applying measurement circuits will tend to spread around and correlate noise:



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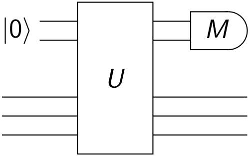
We will consider measurement circuits of the form



where U is a Clifford and M is a Pauli.

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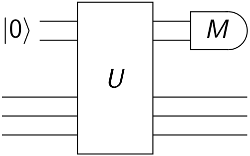

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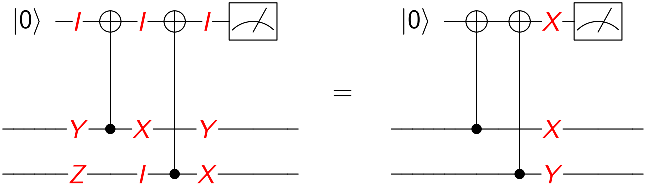

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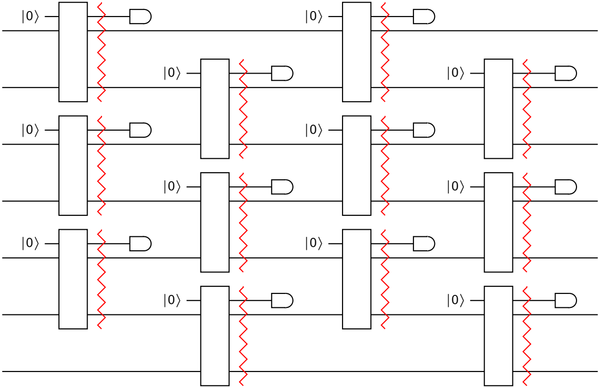

The distribution of the error history $(E_1, \ldots, E_T)$ factors.

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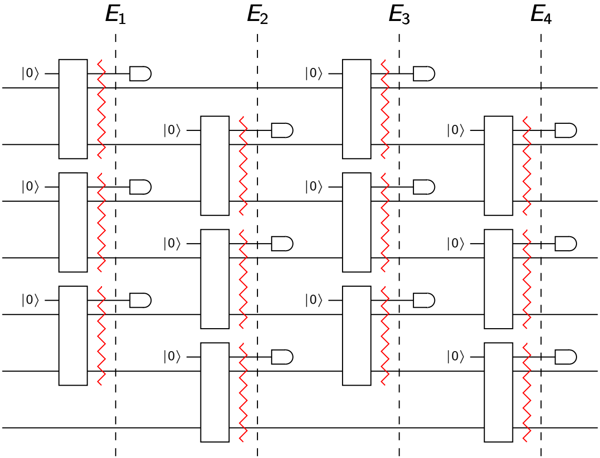

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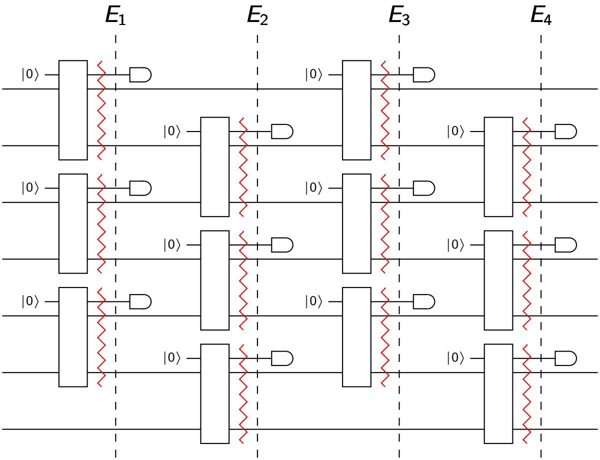

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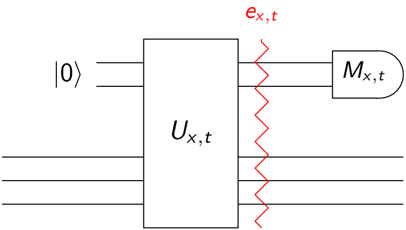

$(E_t)_t$ and $(e_{x,t})_{x,t}$ are 1-to-1, so

$$\Pr\Big((E_t)_t\Big) = \Pr\Big((e_{x,t})_{x,t}\Big).$$

If $e_{x,t} \sim p_{x,t}$ independently, then

$$\Pr\Big((E_t)_t\Big) = \prod_{x,t} p_{x,t}(e_{x,t}).$$

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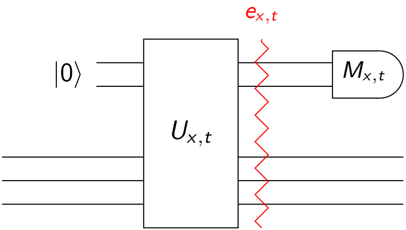

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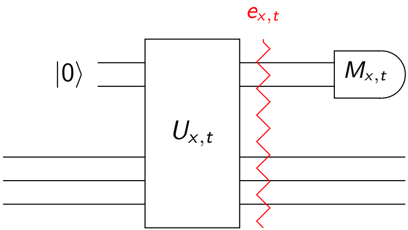

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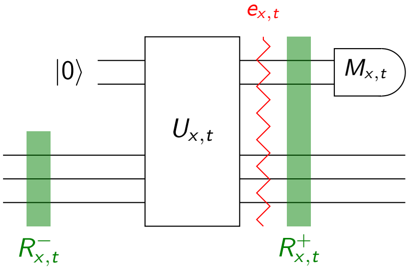

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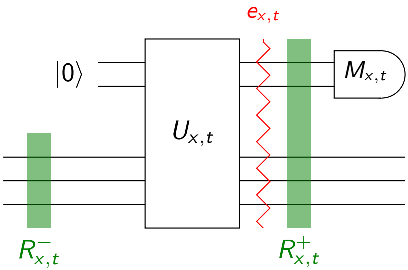

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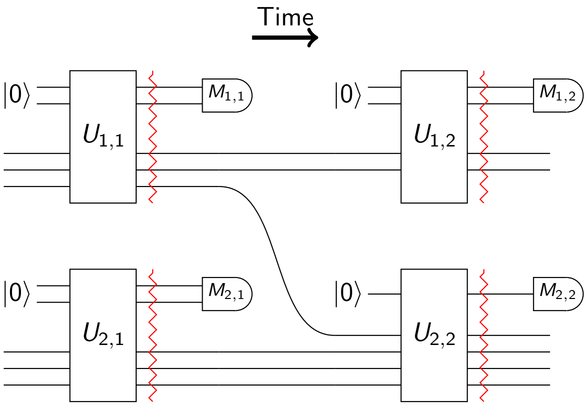
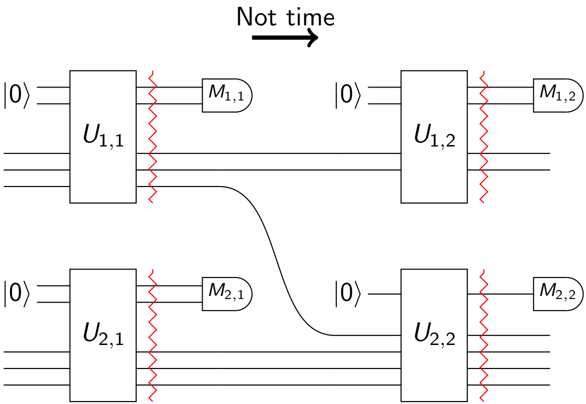
History code



- The error model is given by error history, which factor.
- Stabilisers are the measurements $(M_{x,t})_{x,t}$, and stabilisers are final time-slice.
- Logicals are logicals at final time-slice.

History code



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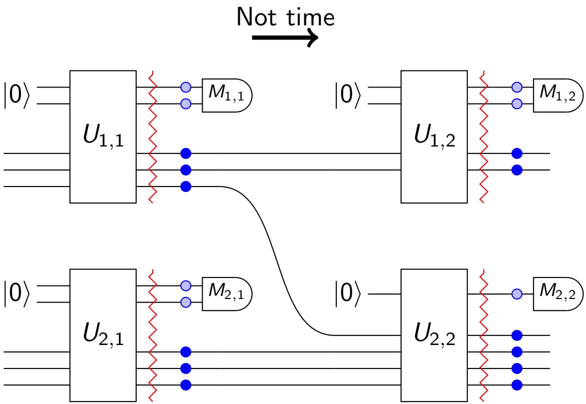

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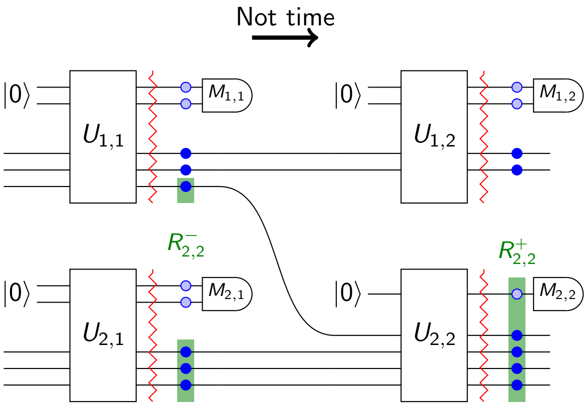

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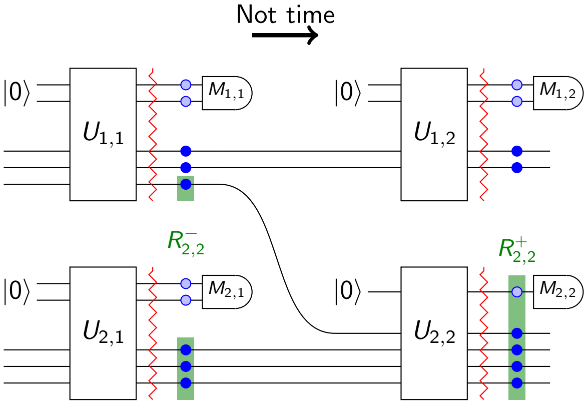

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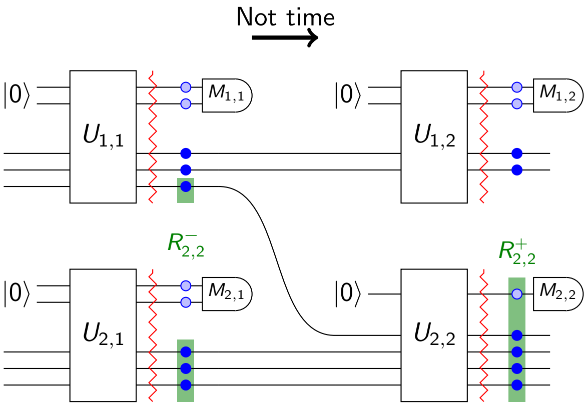

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Conclusions and further work

- Extended the stat. mech. mapping to correlated models
- Can apply stat. mech. mapping to circuit noise via the history code
- Stat. mech. mapping gives tensor network ML decoders

- Can we apply this to experimentally relevant correlated models?
- Can we use the decoders to understand to better understand the connection between correlation and the threshold (ongoing work).