Approximate symmetries of Hamiltonians

Joint work with Steve Flammia
arXiv:1608.02600

Christopher Thomas Chubb

Centre for Engineered Quantum Systems
University of Sydney

Caltech 2017

physics.usyd.edu.au/~cchubb/

Provable properties of many-body systems

For gapped 1D systems we can say quite a lot such as the area law[1], and can efficiently find ground states[2].

In 2D little is known, but there are many heuristics (QMC, Snaked MPS, PEPS, 2D-MERA etc.). Unless P = NP we can't find ground states in 2D[3].

Instead of directly finding ground states, can we instead look for indirect signals of topological order?

Can we show these indirect signals via unconditional certificates?

---

[1] Hastings, arXiv:0705.2024

[2] Provably: LVV, arXiv:1307.5143. Heuristic: DMRG, TEBD, TDVP, etc. arXiv:1603.03039.

[3] F. Barahona, doi:10/c9hm7h.

Provable properties of many-body systems

For gapped 1D systems we can say quite a lot such as the area law[1], and can efficiently find ground states[2].

In 2D little is known, but there are many heuristics (QMC, Snaked MPS, PEPS, 2D-MERA etc.). Unless P = NP we can't find ground states in 2D[3].

Instead of directly finding ground states, can we instead look for indirect signals of topological order?

Can we show these indirect signals via unconditional certificates?

---

[1] Hastings, arXiv:0705.2024

[2] Provably: LVV, arXiv:1307.5143. Heuristic: DMRG, TEBD, TDVP, etc. arXiv:1603.03039.

[3] F. Barahona, doi:10/c9hm7h.

Provable properties of many-body systems

For gapped 1D systems we can say quite a lot such as the area law[1], and can efficiently find ground states[2].

In 2D little is known, but there are many heuristics (QMC, Snaked MPS, PEPS, 2D-MERA etc.). Unless P = NP we can't find ground states in 2D[3].

Instead of directly finding ground states, can we instead look for indirect signals of topological order?

Can we show these indirect signals via unconditional certificates?

[1] Hastings, arXiv:0705.2024
[2] Provably: LVV, arXiv:1307.5143. Heuristic: DMRG, TEBD, TDVP, etc. arXiv:1603.03039.
[3] F. Barahona, doi:10/c9hm7h.

Provable properties of many-body systems

For gapped 1D systems we can say quite a lot such as the area law[1], and can efficiently find ground states[2].

In 2D little is known, but there are many heuristics (QMC, Snaked MPS, PEPS, 2D-MERA etc.). Unless P = NP we can't find ground states in 2D[3].

Instead of directly finding ground states, can we instead look for indirect signals of topological order?

Can we show these indirect signals via unconditional certificates?

[1] Hastings, arXiv:0705.2024
[2] Provably: LVV, arXiv:1307.5143. Heuristic: DMRG, TEBD, TDVP, etc. arXiv:1603.03039.
[3] F. Barahona, doi:10/c9hm7h.

Ground space degeneracy

Why?

- Crude signal of topological order[4]
- Code size of a quantum code[5]

How?

- Find orthogonal ground states (DMRG etc.)
- Indirect certificates such as symmetries

[4] Wen and Niu, doi:10/cm4d3f. Wen, doi:10/frd4gr. Bridgeman et.al., arXiv:1603.02275.
[5] A. Kitaev, cond-mat/0506438.

Ground space degeneracy

Why?

- Crude signal of topological order[4]
- Code size of a quantum code[5]

How?

- Find orthogonal ground states (DMRG etc.)
- Indirect certificates such as symmetries

[4] Wen and Niu, doi:10/cm4d3f. Wen, doi:10/frd4gr. Bridgeman et.al., arXiv:1603.02275.
[5] A. Kitaev, cond-mat/0506438.

Ground space degeneracy

Why?

- Crude signal of topological order[4]
- Code size of a quantum code[5]

How?

- Find orthogonal ground states (DMRG etc.)
- Indirect certificates such as symmetries

[4] Wen and Niu, doi:10/cm4d3f. Wen, doi:10/frd4gr. Bridgeman et.al., arXiv:1603.02275.
[5] A. Kitaev, cond-mat/0506438.

Symmetries

For a system with a Hamiltonian H, an exact symmetry S is a unitary with

$$\text{Exact symmetry :} \qquad [H, S] = 0$$

We will consider relaxing this to

$$\text{Approximate symmetry :} \qquad \|[H, S]\| \leq \epsilon$$

Symmetries

For a system with a Hamiltonian H, an exact symmetry S is a unitary with

$$\text{Exact symmetry :} \qquad [H,S]=0$$

We will consider relaxing this to

$$\text{Approximate symmetry :} \qquad \|[H,S]\| \leq \epsilon$$

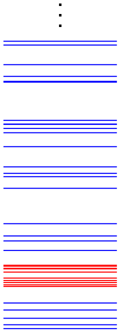
Degeneracy

H is a finite-dimensional Hamiltonian with a band which is:

- Exactly degenerate
- The ground space
- Gapped

In the full paper we consider general gapped bands.

$\sigma(H) =$

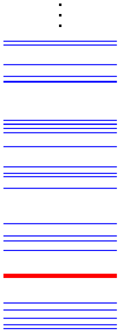


Degeneracy

H is a finite-dimensional Hamiltonian with a band which is:

- Exactly degenerate
- The ground space
- Gapped

In the full paper we consider general gapped bands.

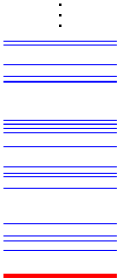
$\sigma(H) =$ 

Degeneracy

H is a finite-dimensional Hamiltonian with a band which is:

- Exactly degenerate
- The ground space
- Gapped

In the full paper we consider general gapped bands.

$\sigma(H) =$ 

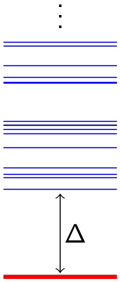
Degeneracy

H is a finite-dimensional Hamiltonian with a band which is:

- Exactly degenerate
- The ground space
- Gapped

In the full paper we consider general gapped bands.

$\sigma(H) =$



Example 1: Classical Ising Model

$$H = -\sum_i s_i s_{i+1} \qquad s_i \in \{-1, +1\}.$$

The Ising model has:

- A symmetry

$$F : s_i \to -s_i \quad \forall i.$$

- Two non-symmetric ground states

$$\vec{s}_1 = (-1, -1, \dots) \qquad \xleftrightarrow{F} \qquad \vec{s}_2 = (+1, +1, \dots)$$

Do symmetries always map us between multiple ground states?

Example 1: Classical Ising Model

$$H = -\sum_i s_i s_{i+1} \qquad s_i \in \{-1, +1\}.$$

The Ising model has:

- A symmetry

$$F : s_i \to -s_i \;\; \forall i.$$

- Two non-symmetric ground states

$$\vec{s}_1 = (-1, -1, \dots) \qquad \xleftrightarrow{F} \qquad \vec{s}_2 = (+1, +1, \dots)$$

Do symmetries always map us between multiple ground states?

Example 1: Classical Ising Model

$$H = -\sum_i s_i s_{i+1} \qquad s_i \in \{-1, +1\}.$$

The Ising model has:

- A symmetry

$$F : s_i \to -s_i \ \ \forall i.$$

- Two non-symmetric ground states

$$\vec{s}_1 = (-1, -1, \dots) \qquad \xleftrightarrow{F} \qquad \vec{s}_2 = (+1, +1, \dots)$$

Do symmetries always map us between multiple ground states?

Example 1: Classical Ising Model

$$H = -\sum_i s_i s_{i+1} \qquad s_i \in \{-1, +1\}.$$

The Ising model has:

- A symmetry

$$F : s_i \to -s_i \ \ \forall i.$$

- Two <u>non-symmetric</u> ground states

$$\vec{s}_1 = (-1, -1, \dots) \qquad \xleftrightarrow{F} \qquad \vec{s}_2 = (+1, +1, \dots)$$

Do symmetries always map us between multiple ground states?

Example 2: Cluster State Model

$$H = -\sum_i Z_{i-1} X_i Z_{i+1}$$

The Cluster State model has:

- Two commuting symmetries

$$S_1 = \prod_{i \text{ odd}} X_i, \qquad S_2 = \prod_{i \text{ even}} X_i.$$

- One symmetric ground state

Commuting symmetries cannot certify degeneracy.

Example 2: Cluster State Model

$$H = -\sum_i Z_{i-1} X_i Z_{i+1}$$

The Cluster State model has:

- Two commuting symmetries

$$S_1 = \prod_{i \text{ odd}} X_i, \qquad S_2 = \prod_{i \text{ even}} X_i.$$

- One symmetric ground state

Commuting symmetries cannot certify degeneracy.

Example 2: Cluster State Model

$$H = -\sum_i Z_{i-1} X_i Z_{i+1}$$

The Cluster State model has:

- Two commuting symmetries

$$S_1 = \prod_{i \text{ odd}} X_i, \qquad S_2 = \prod_{i \text{ even}} X_i.$$

- One symmetric ground state

Commuting symmetries cannot certify degeneracy.

Example 2: Cluster State Model

$$H = -\sum_i Z_{i-1} X_i Z_{i+1}$$

The Cluster State model has:

- Two <u>commuting</u> symmetries

$$S_1 = \prod_{i \text{ odd}} X_i, \qquad S_2 = \prod_{i \text{ even}} X_i.$$

- One <u>symmetric</u> ground state

Commuting symmetries <u>cannot</u> certify degeneracy.

Example 3: Quantum Ising Model

$$H = -\sum_i Z_i Z_{i+1}$$

The Quantum Ising model has:

- Two non-commuting symmetries

$$\bar{X} = \prod_i X_i \qquad \bar{Z} = Z_1.$$

- Two-dimensional ground space

$$G = \mathrm{Span}\{|000\ldots\rangle, |111\ldots\rangle\}.$$

Non-commuting symmetries can witness degeneracy.

Example 3: Quantum Ising Model

$$H = -\sum_i Z_i Z_{i+1}$$

The Quantum Ising model has:

- Two non-commuting symmetries

$$\bar{X} = \prod_i X_i \qquad \bar{Z} = Z_1.$$

- Two-dimensional ground space

$$G = \mathrm{Span}\{|000\ldots\rangle, |111\ldots\rangle\}.$$

Non-commuting symmetries can witness degeneracy.

Example 3: Quantum Ising Model

$$H = -\sum_i Z_i Z_{i+1}$$

The Quantum Ising model has:

- Two non-commuting symmetries

$$\bar{X} = \prod_i X_i \qquad \bar{Z} = Z_1.$$

- Two-dimensional ground space

$$G = \text{Span}\{|000\dots\rangle, |111\dots\rangle\}.$$

Non-commuting symmetries can witness degeneracy.

Example 3: Quantum Ising Model

$$H = -\sum_i Z_i Z_{i+1}$$

The Quantum Ising model has:

- Two <u>non-commuting</u> symmetries

$$\bar{X} = \prod_i X_i \qquad \bar{Z} = Z_1.$$

- <u>Two-dimensional</u> ground space

$$G = \mathrm{Span}\{|000\ldots\rangle, |111\ldots\rangle\}.$$

Non-commuting symmetries <u>can</u> witness degeneracy.

Twisted commutation

To certify degeneracy we want non-commuting symmetries. Specifically we will consider symmetries that twisted commute

$$[U, V]_\eta := UV - \eta VU = 0.$$

for some $|\eta| = 1$. We will also let $\eta := e^{2\pi i\alpha}$, using α and η interchangeably.

We are also going to consider relaxing this

$$\|[U, V]_\eta\| \leq \delta.$$

Twisted commutation

To certify degeneracy we want non-commuting symmetries. Specifically we will consider symmetries that twisted commute

$$[U, V]_\eta := UV - \eta VU = 0.$$

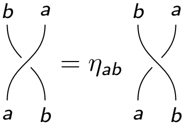
for some $|\eta| = 1$. We will also let $\eta := e^{2\pi i\alpha}$, using α and η interchangeably.

We are also going to consider relaxing this

$$\|[U, V]_\eta\| \leq \delta.$$

Twisted commutation and Abelian anyons

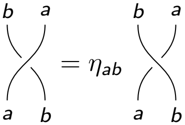
Braiding two Abelian anyons incurs an overall phase factor.



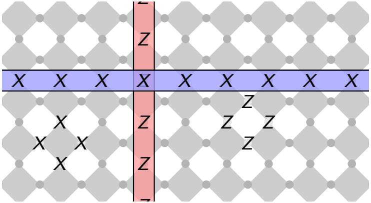
The operators that correspond to moving these anyons around should therefore twisted commute.

Twisted commutation and Abelian anyons

Braiding two Abelian anyons incurs an overall phase factor.



The operators that correspond to moving these anyons around should therefore twisted commute.



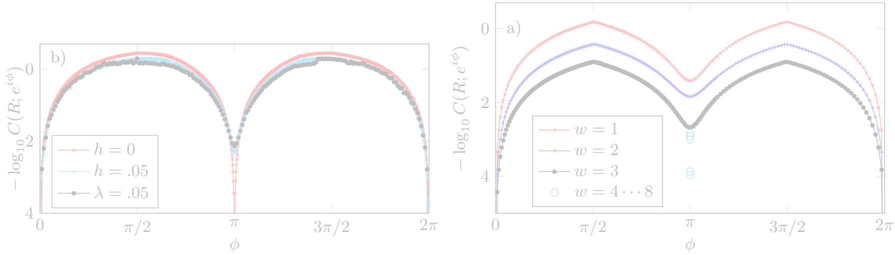
Relation to recent numerics

J. Bridgeman et.al.[6] numerically search for these operators by optimising a cost function, which takes the form

$$C(L, R; \eta) := \left\|[L, H]\right\|_2^2 + \left\|[R, H]\right\|_2^2 + \left\|[L, R]_\eta\right\|_2^2.$$

Seeing low cost for a given phase η is signal of a corresponding anyon.

For both the (perturbed) toric code (b) and honeycomb model (a), we see a signals of a $\mathbb{Z}_2$ anyon at $\phi = \pi$ ($\eta := e^{i\phi}$).



[6] arXiv:1603.02275

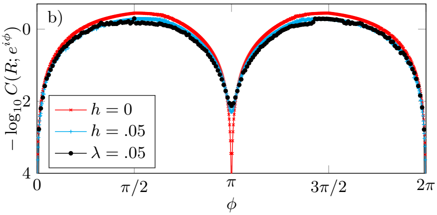
Relation to recent numerics

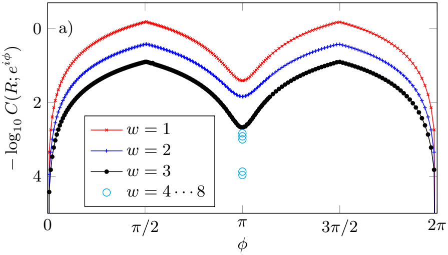
J. Bridgeman et.al.[6] numerically search for these operators by optimising a cost function, which takes the form

$$C(L,R;\eta) := \left\|[L,H]\right\|_2^2 + \left\|[R,H]\right\|_2^2 + \left\|[L,R]_\eta\right\|_2^2.$$

Seeing low cost for a given phase η is signal of a corresponding anyon.

For both the (perturbed) toric code (b) and honeycomb model (a), we see a signals of a $\mathbb{Z}_2$ anyon at $\phi = \pi$ ($\eta := e^{i\phi}$).





[6] arXiv:1603.02275

Restriction to groundspace

Lemma

For any approximate symmetry

$$\|[U, H]\| \leq \epsilon,$$

there exists a unitary u on the ground space, which approximates the action of U

$$\|u - \Pi U \Pi\| \leq 3\epsilon/\Delta,$$

where Π is the ground space projector.

This allows us to restrict to the ground space with low distortion

$$\|[U, H]\|, \|[V, H]\| \leq \epsilon, \ \|[U, V]_\eta\| \leq \delta \qquad \Longrightarrow \qquad \|[u, v]_\eta\| \leq \delta' := \delta + 12\epsilon/\Delta.$$

A similar result holds for any unitarily invariant norm, with slightly different constants.

Restriction to groundspace

Lemma

For any approximate symmetry

$$\|[U, H]\| \leq \epsilon,$$

there exists a unitary u on the ground space, which approximates the action of U

$$\|u - \Pi U \Pi\| \leq 3\epsilon/\Delta,$$

where Π is the ground space projector.

This allows us to restrict to the ground space with low distortion

$$\|[U, H]\|, \|[V, H]\| \leq \epsilon, \ \|[U, V]_\eta\| \leq \delta \qquad \Longrightarrow \qquad \|[u, v]_\eta\| \leq \delta' := \delta + 12\epsilon/\Delta.$$

A similar result holds for any unitarily invariant norm, with slightly different constants.

Dimension certification

We will now used twisted commutation to certify dimensionality in three ways:

1. Using a simple argument, we will show a dimension lower bound for a single pair
2. We will show how this argument extends to multiple pairs
3. We will give improved single pair bounds, which are tight

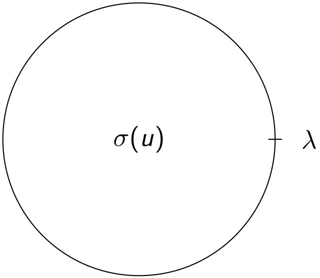
Stone-Von Neumann Theorem

Theorem

If η is a dth root of unity, then $[u, v]_\eta = 0$ implies that $\dim(u)$ is a multiple of d.

Consider the action of v on a λ-eigenvector of u:

$$u\left(v\left|\lambda\right\rangle\right) = \eta v u\left|\lambda\right\rangle = \lambda\eta\left(v\left|\lambda\right\rangle\right)$$



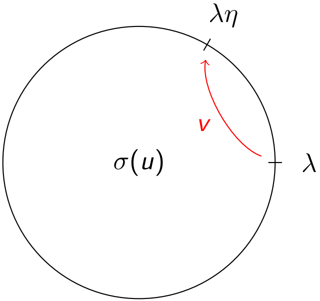
Stone-Von Neumann Theorem

Theorem

If η is a dth root of unity, then $[u,v]_\eta = 0$ implies that $\dim(u)$ is a multiple of d.

Consider the action of v on a λ-eigenvector of u:

$$u\left(v\left|\lambda\right\rangle\right) = \eta v u \left|\lambda\right\rangle = \lambda\eta\left(v\left|\lambda\right\rangle\right)$$



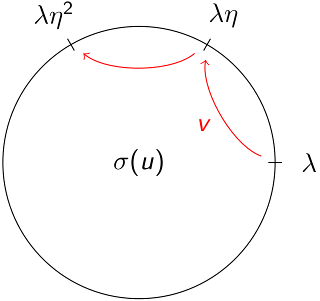
Stone-Von Neumann Theorem

Theorem

If η is a dth root of unity, then $[u, v]_\eta = 0$ implies that $\dim(u)$ is a multiple of d.

Consider the action of v on a λ-eigenvector of u:

$$u\left(v\left|\lambda\right\rangle\right) = \eta v u\left|\lambda\right\rangle = \lambda\eta\left(v\left|\lambda\right\rangle\right)$$



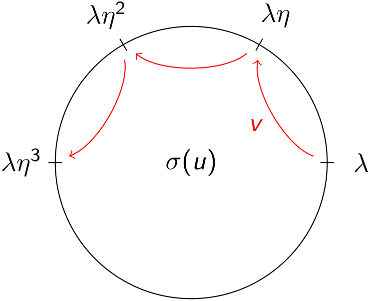
Stone-Von Neumann Theorem

Theorem

If η is a dth root of unity, then $[u,v]_\eta = 0$ implies that $\dim(u)$ is a multiple of d.

Consider the action of v on a λ-eigenvector of u:

$$u\left(v\left|\lambda\right\rangle\right) = \eta v u\left|\lambda\right\rangle = \lambda\eta\left(v\left|\lambda\right\rangle\right)$$



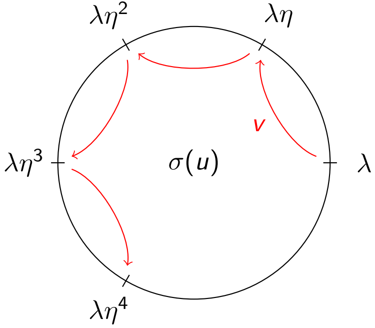
Stone-Von Neumann Theorem

Theorem

If η is a dth root of unity, then $[u, v]_\eta = 0$ implies that $\dim(u)$ is a multiple of d.

Consider the action of v on a λ-eigenvector of u:

$$u\,(v\,|\lambda\rangle) = \eta v u\,|\lambda\rangle = \lambda\eta\,(v\,|\lambda\rangle)$$



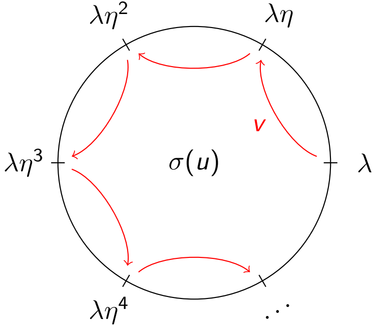
Stone-Von Neumann Theorem

Theorem

If η is a dth root of unity, then $[u,v]_\eta = 0$ implies that $\dim(u)$ is a multiple of d.

Consider the action of v on a λ-eigenvector of u:

$$u\left(v\left|\lambda\right\rangle\right) = \eta v u \left|\lambda\right\rangle = \lambda\eta\left(v\left|\lambda\right\rangle\right)$$



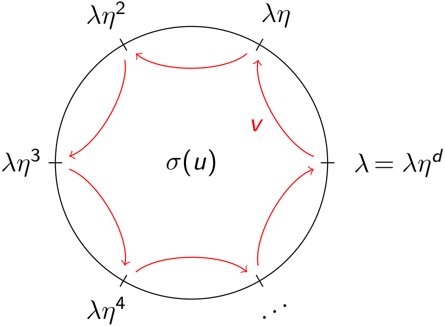
Stone-Von Neumann Theorem

Theorem

If η is a dth root of unity, then $[u, v]_\eta = 0$ implies that $\dim(u)$ is a multiple of d.

Consider the action of v on a λ-eigenvector of u:

$$u\left(v\left|\lambda\right\rangle\right) = \eta v u \left|\lambda\right\rangle = \lambda\eta\left(v\left|\lambda\right\rangle\right)$$



Approximate case

If we relax

$$[u, v]_\eta = 0 \qquad \to \qquad \|[u, v]_\eta\| \le \delta'$$

then once again starting from λ-eigenvector and considering the action of v:

$$|\left\langle\lambda\middle|v^\dagger uv\middle|\lambda\right\rangle - \lambda\eta| \le \delta'$$

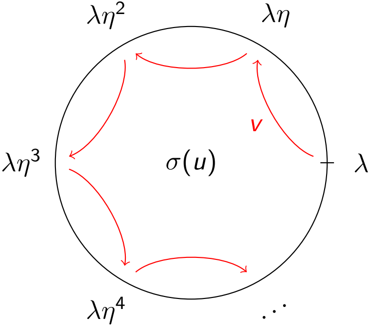
Approximate case

If we relax

$$[u, v]_\eta = 0 \qquad \to \qquad \|[u, v]_\eta\| \leq \delta'$$

then once again starting from λ-eigenvector and considering the action of v:

$$|\langle\lambda|v^\dagger uv|\lambda\rangle - \lambda\eta| \leq \delta'$$



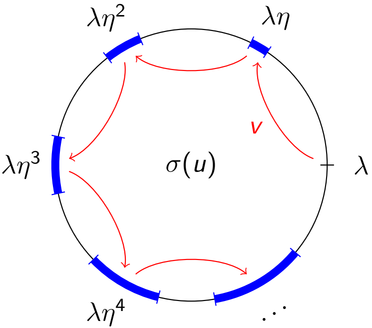
Approximate case

If we relax

$$[u, v]_\eta = 0 \qquad \rightarrow \qquad \|[u, v]_\eta\| \leq \delta'$$

then once again starting from λ-eigenvector and considering the action of v:

$$|\langle\lambda|v^\dagger uv|\lambda\rangle - \lambda\eta| \leq \delta'$$

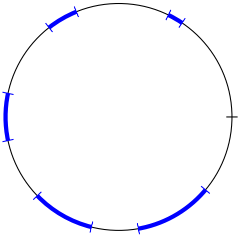


Full and partial certification

For η a dth root of unity

$$\|[u,v]_\eta\| < \frac{2}{d-1}\left[1-\cos\pi/d\right] \sim \frac{1}{d^3}$$

implies all arcs are non-overlapping, and so the degeneracy is at least d.



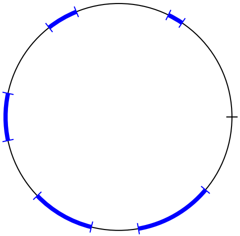
As the error increases however, the degeneracy that can be certified lowers.

Full and partial certification

For η a dth root of unity

$$\|[u,v]_\eta\| < \frac{2}{d-1}\left[1-\cos\pi/d\right] \sim \frac{1}{d^3}$$

implies all arcs are non-overlapping, and so the degeneracy is at least d.



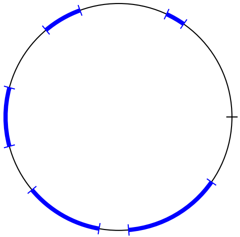
As the error increases however, the degeneracy that can be certified lowers.

Full and partial certification

For η a dth root of unity

$$\|[u,v]_\eta\| < \frac{2}{d-1}\left[1-\cos\pi/d\right] \sim \frac{1}{d^3}$$

implies all arcs are non-overlapping, and so the degeneracy is at least d.



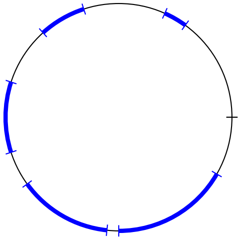
As the error increases however, the degeneracy that can be certified lowers.

Full and partial certification

For η a dth root of unity

$$\|[u, v]_\eta\| < \frac{2}{d-1}\left[1 - \cos \pi / d\right] \sim \frac{1}{d^3}$$

implies all arcs are non-overlapping, and so the degeneracy is at least d.



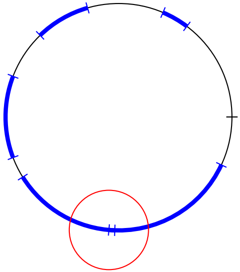
As the error increases however, the degeneracy that can be certified lowers.

Full and partial certification

For η a dth root of unity

$$\|[u, v]_\eta\| < \frac{2}{d-1}\left[1 - \cos \pi/d\right] \sim \frac{1}{d^3}$$

implies all arcs are non-overlapping, and so the degeneracy is at least d.



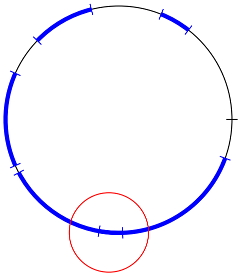
As the error increases however, the degeneracy that can be certified lowers.

Full and partial certification

For η a dth root of unity

$$\|[u,v]_\eta\| < \frac{2}{d-1}\left[1-\cos\pi/d\right] \sim \frac{1}{d^3}$$

implies all arcs are non-overlapping, and so the degeneracy is at least d.



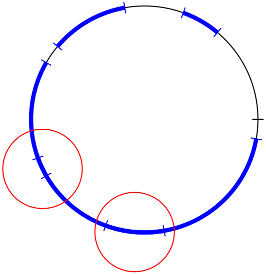
As the error increases however, the degeneracy that can be certified lowers.

Full and partial certification

For η a dth root of unity

$$\|[u, v]_\eta\| < \frac{2}{d-1}\left[1 - \cos \pi/d\right] \sim \frac{1}{d^3}$$

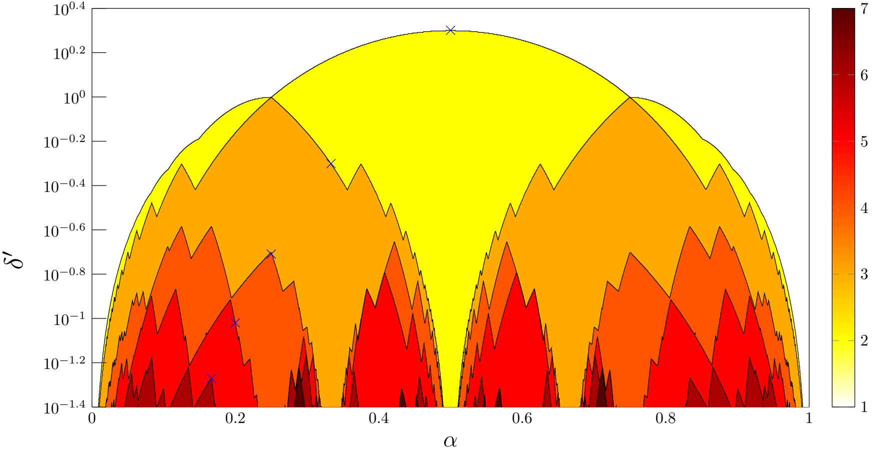
implies all arcs are non-overlapping, and so the degeneracy is at least d.



As the error increases however, the degeneracy that can be certified lowers.

Certifiable degeneracy

If we have $\|[u, v]_\eta\| \leq \delta'$ for $\eta := e^{2i\pi\alpha}$, then the degeneracy we can certify is



Multiple pairs

Suppose we have multiple pairs of symmetries $(u_1, v_1), \ldots, (u_k, v_k)$ with twisted commutation relations

$$\left\| [u_i, v_i]_{1/d_i} \right\| \leq \delta \qquad \forall i,$$

for some integers $d_1, \ldots, d_k$, and commutation relations

$$\left\| [u_i, u_j] \right\| \leq \delta^2 \qquad \text{and} \qquad \left\| [u_i, v_j] \right\| \leq \delta \qquad \forall i \neq j.$$

Multiple pairs

For a single pair, we found approximate eigenvectors, characterised by an approximate eigenvalue λ as

$$\left|\langle\lambda|u|\lambda\rangle - \lambda\right| \leq \zeta.$$

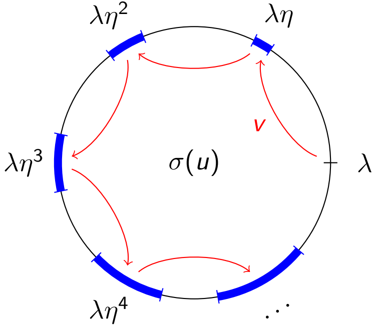
We found that eigenvalues must exist in arcs

Multiple pairs

For a single pair, we found approximate eigenvectors, characterised by an approximate eigenvalue λ as

$$\left|\langle\lambda|u|\lambda\rangle - \lambda\right| \leq \zeta.$$

We found that eigenvalues must exist in arcs



Multiple pairs

Here we will find approximate shared eigenvectors of $\{u_1, \ldots, u_k\}$, labelled by a vector of approximate eigenvalues $(\lambda_1, \ldots, \lambda_k)$ as

$$\left|\langle \lambda_1, \ldots, \lambda_k | u_i | \lambda_1, \ldots, \lambda_k \rangle - \lambda_i\right| \leq \zeta_i \qquad \forall i.$$

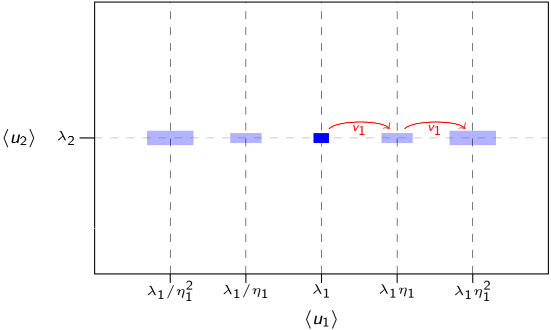
We now find that the vectors $(\lambda_1, \ldots, \lambda_k)$ lie in rectangles on an k-torus.

Multiple pairs

Here we will find approximate shared eigenvectors of $\{u_1, \ldots, u_k\}$, labelled by a vector of approximate eigenvalues $(\lambda_1, \ldots, \lambda_k)$ as

$$\left|\langle \lambda_1, \ldots, \lambda_k | u_i | \lambda_1, \ldots, \lambda_k \rangle - \lambda_i\right| \leq \zeta_i \qquad \forall i.$$

We now find that the vectors $(\lambda_1, \ldots, \lambda_k)$ lie in rectangles on an k-torus.

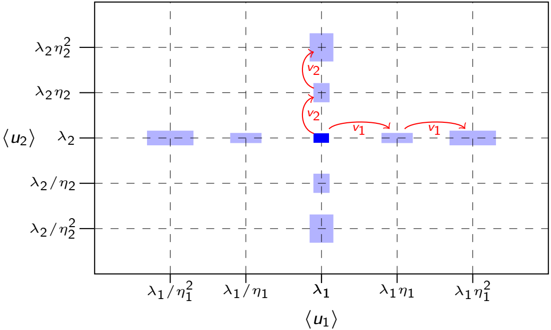


Multiple pairs

Here we will find approximate shared eigenvectors of $\{u_1, \ldots, u_k\}$, labelled by a vector of approximate eigenvalues $(\lambda_1, \ldots, \lambda_k)$ as

$$\left|\langle \lambda_1, \ldots, \lambda_k | u_i | \lambda_1, \ldots, \lambda_k \rangle - \lambda_i \right| \leq \zeta_i \qquad \forall i.$$

We now find that the vectors $(\lambda_1, \ldots, \lambda_k)$ lie in rectangles on an k-torus.

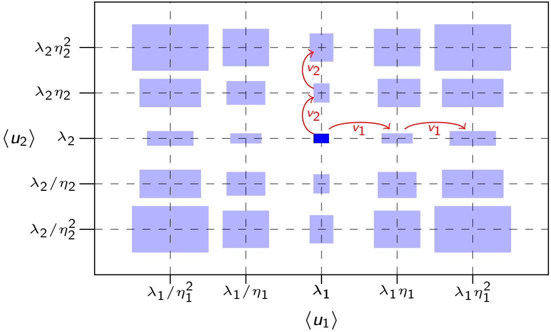


Multiple pairs

Here we will find approximate shared eigenvectors of $\{u_1, \ldots, u_k\}$, labelled by a vector of approximate eigenvalues $(\lambda_1, \ldots, \lambda_k)$ as

$$\left|\langle\lambda_1, \ldots, \lambda_k|u_i|\lambda_1, \ldots, \lambda_k\rangle - \lambda_i\right| \leq \zeta_i \qquad \forall i.$$

We now find that the vectors $(\lambda_1, \ldots, \lambda_k)$ lie in rectangles on an k-torus.



Multiple pairs

If $d_i = d$ for all i, then a full certification

$$\dim \geq d^k$$

requires

$$\delta \leq \frac{1}{d^2\left(d^k + kd\right)\left(d^k - 1\right)} \sim d^{-(2k+2)}.$$

Minimum twisted commutator

For the case of a single pair of symmetries, we want an improved bound that is tight, and not just a lower bound.

To do this we will invert the relationship between twisted commutation and dimensionality.

Specifically, we will look at minimum twisted commutation value in a given dimension g

$$\Lambda_{\alpha,g} := \min_{u,v \in U(g)} \left\| [u,v]_\eta \right\|$$

We also bound similar minima for all Ky Fan-Schatten norms $\|\cdot\|_{k,p}$ for $p \geq 2$.

Minimum twisted commutator

For the case of a single pair of symmetries, we want an improved bound that is tight, and not just a lower bound.

To do this we will invert the relationship between twisted commutation and dimensionality.

Specifically, we will look at minimum twisted commutation value in a given dimension g

$$\Lambda_{\alpha,g} := \min_{u,v \in U(g)} \left\| [u, v]_\eta \right\|$$

We also bound similar minima for all Ky Fan-Schatten norms $\| \cdot \|_{k,p}$ for $p \geq 2$.

Minimum twisted commutator

For the case of a single pair of symmetries, we want an improved bound that is tight, and not just a lower bound.

To do this we will invert the relationship between twisted commutation and dimensionality.

Specifically, we will look at minimum twisted commutation value in a given dimension g

$$\Lambda_{\alpha,g} := \min_{u,v\in U(g)} \left\| [u,v]_\eta \right\|$$

We also bound similar minima for all Ky Fan-Schatten norms $\|\cdot\|_{k,p}$ for $p \geq 2$.

Why the minimum twisted commutator?

$$\Lambda_{\alpha,g} := \min_{u,v \in U(g)} \left\| [u,v]_\eta \right\|$$

Suppose we have two operators u and v, and all we can compute is a twisted commutator value with

$$\left\| [u,v]_\eta \right\| < \Lambda_{\alpha,g}$$

for some g, then we can say that

$$\dim(u) = dim(v) \neq g.$$

Minimum twisted commutator

We start by bounding the operator norm $\|\cdot\|$ by the Frobenius norm $\|\cdot\|_2$ as

$$\left\|[u,v]_\eta\right\| \geq \left\|[u,v]_\eta\right\|_2 / \sqrt{g}.$$

Next we use the a key result from spectral perturbation theory.

Theorem (Wielandt-Hoffman theorem)

For normal matrices a and b, the Frobenius distance is lower bounded by the spectral Frobenius distance

$$\|a-b\|_2 \geq \min_{\sigma\in S_g} \left\|\sigma[\vec{\lambda}(a)] - \vec{\lambda}(b)\right\|_2.$$

Here $\vec{\lambda}(x)$ denotes a vector of the eigenvalues of x, and S_g the permutation group.

Minimum twisted commutator

We start by bounding the operator norm $\|\cdot\|$ by the Frobenius norm $\|\cdot\|_2$ as

$$\left\|[u,v]_\eta\right\| \geq \left\|[u,v]_\eta\right\|_2 / \sqrt{g}.$$

Next we use the a key result from spectral perturbation theory.

Theorem (Wielandt-Hoffman theorem)

For normal matrices a and b, the Frobenius distance is lower bounded by the spectral Frobenius distance

$$\|a-b\|_2 \geq \min_{\sigma\in S_g} \left\|\sigma[\vec{\lambda}(a)] - \vec{\lambda}(b)\right\|_2.$$

Here $\vec{\lambda}(x)$ denotes a vector of the eigenvalues of x, and S_g the permutation group.

Minimum twisted commutator

If we let $\{e^{i\theta_j}\}_j$ be the eigenvalues of u, then this gives us that

$$
\begin{aligned}
\sqrt{g}\left\|[u,v]_\eta\right\| &= \sqrt{g}\|uv-\eta vu\| \\
&= \|uv-\eta vu\|_2 \\
&= \|v^\dagger uv-\eta u\|_2 \\
&\geq \min_\sigma \left\|\sigma[\vec{\lambda}(v^\dagger uv)]-\vec{\lambda}(\eta u)\right\|_2 \\
&\geq \min_\sigma \left\|\sigma[\vec{\lambda}(u)]-\eta\vec{\lambda}(u)\right\|_2 \\
&= \min_\sigma \sqrt{\sum_{j=1}^{g}\left|e^{i\theta_{\sigma(j)}}-\eta e^{i\theta_j}\right|^2} \\
&= 2\min_\sigma \sqrt{\sum_{j=1}^{g}\sin^2\left(\frac{\theta_{\sigma(j)}-\theta_j-2\pi\alpha}{2}\right)}.
\end{aligned}
$$

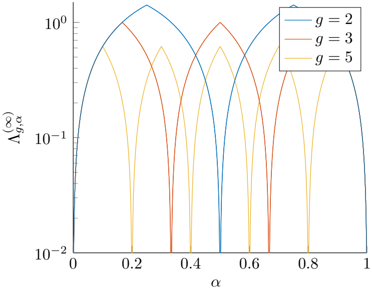
Minimum twisted commutator

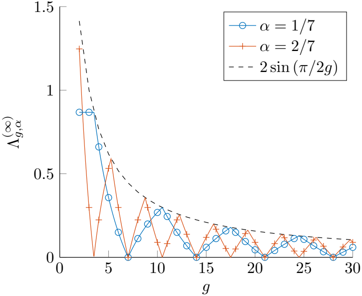
Minimising this quantity we find that

$$\left\|[u,v]_\eta\right\| \geq 2\sin\left(\pi\left|\frac{\lfloor g\alpha\rceil - g\alpha}{g}\right|\right),$$

which is saturated by appropriate powers of the g-dimensional Paulis

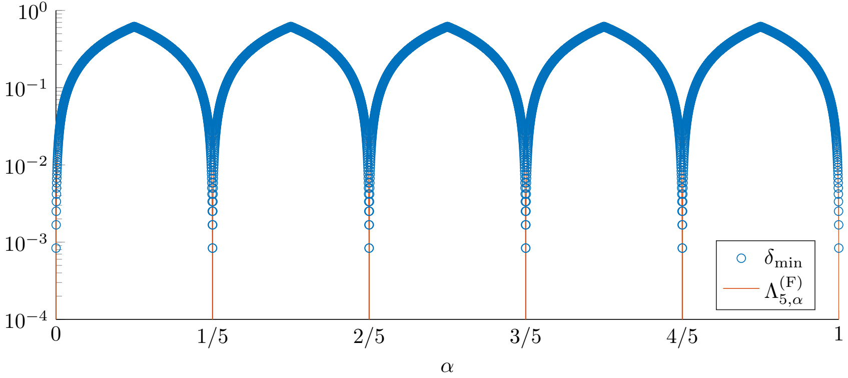
$$u = X_g, \qquad v = Z_g^{\lfloor g\alpha\rceil}.$$





Relation back to numerics

For exactly solvable models ($\mathbb{Z}_d$ quantum double), the minimum TCV matches the numerics[7], for example in the $\mathbb{Z}_5$ toric code:



[7] J. Bridgeman et.al., arXiv:1603.02275.

Conclusion and further work

- Non-commuting symmetries can serve as certificates of degeneracy
- Twisted commutation gives provable degeneracy certification
- These certifcates are robust to approximate commutation relations

- Can we bridge the gap between numerics and analytics for non-exactly solvable models? What about non-Abelian anyons?
- What happens if we take in to account locality?
- Do these arguments generalise to symmetry protected topological order?
- Can this be extended to more general approximate representations?
- Do provably efficient algorithms for calculating these certificates exist?

ArXiv: 1608.02600