# Approximate symmetries of Hamiltonians

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**Abstract**

We explore the relationship between approximate symmetries of a gapped Hamiltonian and the structure of its ground space. We start by showing that approximate symmetry operators—unitary operators whose commutators with the Hamiltonian have norms that are sufficiently small—which possess certain mutual commutation relations can be restricted to the ground space with low distortion. We generalize the Stone-von Neumann theorem to matrices that *approximately* satisfy the canonical (Heisenberg-Weyl-type) commutation relations, and use this to show that approximate symmetry operators can certify the degeneracy of the ground space even though they only approximately form a group. Importantly, the notions of "approximate" and "small" are all independent of the dimension of the ambient Hilbert space, and depend only on the degeneracy in the ground space. Our analysis additionally holds for any gapped band of sufficiently small width in the excited spectrum of the Hamiltonian, and we discuss applications of these ideas to topological quantum phases of matter and topological quantum error correcting codes. Finally, in our analysis we also provide an exponential improvement upon bounds concerning the existence of shared approximate eigenvectors of approximately commuting operators which may be of independent interest.

## 1 Introduction

Given a quantum system described by a Hamiltonian $H$, a symmetry is simply an operator that exactly commutes with $H$. Often symmetries are restricted to be hermitian or unitary. $H$ can be simultaneously block diagonalized with the symmetry operator, and the degeneracy within these blocks is determined by the symmetry. In a system that possesses exact symmetries, a sufficiently weak perturbation will preserve the number of states of any band gapped away from the rest of the spectrum, but the symmetries will generally become only approximate.

In this work we consider a natural converse to this: suppose we know that a system has some approximate symmetries and a gapped band (possibly the ground space band). Can we "unperturb" the symmetries into exact symmetries within a given band? Can we also use the approximate group structure of the approximate symmetries to count the degeneracy within the band? We answer these questions in the affirmative, giving quantitive bounds on when such a procedure can be performed, and thus when such approximate symmetries can be used as certificates of ground space degeneracy.

A related area of mathematical research with a long and rich history is the relationship between the properties of approximately and exactly commuting matrices. An exemplary problem which dates back as far as the 1950s [1–5] is whether a pair of approximately commuting matrices lie near an exactly commuting pair, i.e. whether there exists a dimension independent $\delta > 0$ for each $\epsilon > 0$ such that

$$\|[H,S]\| \leq \delta \qquad \Longrightarrow \qquad \exists \tilde{H},\tilde{S} : [\tilde{H},\tilde{S}] = 0, \text{ where } \|H-\tilde{H}\|, \|S-\tilde{S}\| \leq \epsilon.$$

Interpreting $H$ as the Hamiltonian and $S$ as the action of a symmetry, this problem can be interpreted as whether approximate symmetries are necessarily near exact symmetries of a perturbed system. It has been shown that just such a theorem holds if all matrices are Hermitian [6–8]. Another physical consequence of this is that a pair commuting observables can be approximately simultaneously measured [8].

For discrete symmetries, described by unitary matrices, the above is known to be generally false [9]. This is due to a K-theoretic obstruction [10–12], though it is true if this obstruction vanishes [6, 13], or under the assumption of a spectral gap [14]. Imposing a form of self-duality analogous to time-reversal symmetry, a connection to fermionic systems can be made [15]. In this case the relevant K-theoretic obstruction reducing to the spin Chern number of some system, highlighting a link between the fields of topologically ordered quantum systems [16] and approximately commuting matrices.

Here we will consider Hamiltonians $H$ with multiple non-commuting approximate symmetries, and establish a connection to the ground space degeneracy. The form of non-commutation we will consider will involve *twisted* commutation relations.

**Definition 1** (Twisted commutator). *For $\alpha \in [0,1)$, the* twisted commutator *is defined as*

$$[X,Y]_\alpha := XY - e^{2i\pi\alpha}YX.$$

*We will refer to $\alpha$ as the* twisting parameter*, and for some unitarily invariant norm $\|\!|\cdot|\!\|$ we will refer to $\|\!|[\cdot,\cdot]_\alpha|\!\|$ as the* twisted commutation value*. When considering a pair of operators in tandem that each have a low twisted commutation value we will refer to as a* twisted pair*.*

We remark that the $\alpha = 0$ and $\alpha = 1/2$ cases correspond to the commutator and anti-commutator respectively.

While commuting operators exist in all dimensions, twisted commuting operators only exist in some dimensions, depending on the twisting parameter. For example, no $\alpha \neq 0$ twisted commutator can non-trivially vanish in a one-dimensional space. If we restrict to unitary operators, the Stone-von Neumann Theorem[1] [17,18] classifies the dimensions in which twisted commutation can occur. In this paper we will generalize this connection into the regime of *approximately* twisted commuting operators.

**Theorem 1** (Stone-von Neumann theorem). *Given $\alpha = p/q$ with $p,q$ coprime, then unitary operators $X$ and $Y$ which exactly twisted commute as*

$$[X,Y]_\alpha = 0$$

*only exist in dimensions which are multiples of $q$.*

Suppose we have a physical system with Hamiltonian $H$. Let $\Pi$ be the orthogonal projector onto the ground space, and $\bar{\Pi} := I - \Pi$. For simplicity, take the ground state energy of $H$ to be zero, such that $\Pi H = 0$. As well as this, we will assume that the excited states are gapped away from the ground space, such that they all have an energy at least $\Delta$, i.e. $H \geq \Delta\bar{\Pi}$. For such a system there exist two notions of symmetry we will discuss.

**Definition 2** (Symmetry). *Let $U$ be a unitary operator. We will refer to it as a* ground symmetry *if it commutes with the ground space projector*

$$[U,\Pi] = 0,$$

*and as an $\epsilon$-*approximate symmetry *if it approximately commutes with the Hamiltonian with respect to a given unitarily invariant norm*

$$\|\!|[U,H]|\!\| \leq \epsilon.$$

The error thresholds we are going to consider will depend on the spectral gap $\Delta$ of the system in question. One way to improve the scaling with the gap would be to consider symmetries defined not by commutation with the Hamiltonian, but by commutation with functions of the Hamiltonian. For example we could consider commutation with an (unnormalized) Gibbs state

$$\left|\!\left|\!\left|\left[U, e^{-\beta H}\right]\right|\!\right|\!\right| \leq \epsilon.$$

Such a symmetry can be seen to be an $\epsilon$-approximate symmetry of $H' = I - e^{-\beta H}$, which shares a ground space with $H$ and has a gap of $1 - e^{-\beta\Delta}$. If we have some control over the temperature, such as in Monte Carlo simulations, then this gives a trade-off we can use to improve the gap scaling. If for example we set $\beta = \ln(2)/\Delta$, then we get a fixed gap of $1/2$. A similar analysis could be performed with any monotonic function of $H$.

## 1.1 Results

In Section 2 we will explore the relationship between approximate and ground symmetries, showing that an approximate symmetry is always near a ground symmetry. Extending this to the case of multiple symmetries, we will see that approximate symmetries can be restricted to the ground space with low distortion, implying the existence of operators on the ground space with certain twisted commutation relations.

**Theorem 2** (Restriction to the ground space). *If there are two $\epsilon$-approximate symmetries $U$ and $V$ which approximately twisted commute*

$$\|\!|[U,V]_\alpha|\!\| \leq \delta,$$

*there exists unitaries $u$ and $v$ acting on the ground space with gap $\Delta$ which also approximately twisted commute*

$$\|\!|[u,v]_\alpha|\!\| \leq \delta + 24\epsilon/\Delta.$$

[1] As usually stated, the Stone-von Neumann theorem is much more general than Theorem 1. As we will only be concerned with finite-dimensional operators, and unconcerned with uniqueness, this form will suffice for our purposes.

Note that for simplicity we will henceforth take the band in consideration to be an exactly degenerate ground space. We will see however that our proof will rely not on the bound $H \geq \Delta\bar{\Pi}$, but on its relaxation $H^2 \geq \Delta^2\bar{\Pi}$, meaning that the band could be anywhere in the spectrum, so long as it is gapped on both sides by at least $\Delta$. Furthermore we can take $w := \|H\Pi\| > 0$ such that our band has non-zero width. By considering the new Hamiltonian $H' := H - H\Pi$, we get that our restricted result holds for more general bands once the necessary changes have been made.

**Corollary 3** (Restriction to a general band)**.** *If there are two $\epsilon$-approximate symmetries $U$ and $V$ which approximately twisted commute*

$$\|[U, V]_\alpha\| \leq \delta,$$

*there exists unitaries $u$ and $v$ acting on band of gap $\Delta$ and width $w$ which also approximately twisted commute*

$$\|[u, v]_\alpha\| \leq \delta + 24(\epsilon + w)/\Delta.$$

Now that we have restricted our symmetries down to the ground space, by studying the relationship between dimensionality and approximate twisted commutation, we can hope to use these twisted symmetries as witnesses of ground space degeneracy. As above, we will henceforth adhere to the convention of upper case letters denoting operators which act on the system as a whole, and lower case operators which only act on the ground space.

In Section 3 we start by giving a proof of Theorem 1, and consider generalizing this argument to the case of *approximately* twisted commuting operators. We consider a twisted pair of unitaries, and construct states which can be used to lower bound the number of eigenvalues these operators possess. By doing so we will show that if these operators have a sufficiently low twisted commutation value, then a lower bound on their degeneracy can be inferred.

**Theorem 4.** *If $u$ and $v$ are unitaries such that for some $d \in \mathbb{N}$*

$$\left\|[u, v]_{1/d}\right\| < \frac{2}{d-1}\left[1 - \cos\pi/d\right],$$

*then the dimension of each operator is at least $d$.*

While we do not have a closed form bound on the twisted commutation value required to certify other dimensions ($d \neq 1/\alpha$), in Appendix B we discuss an algorithm to determine which degeneracies are certified by twisted pairs of given parameters. Using this we will plot the dimension that can be certified as a function of both the twisting parameter and the corresponding twisted commutation value.

In Appendix A we strengthen existing results on shared approximate eigenvectors for approximately commuting operators when a normality condition is introduced, exponentially improving the dimension dependence of the bounds relative to known results [19]. Using this, in Section 3.3 we consider extending this procedure to the case of two pairs of twisted commuting unitaries. Here we will once again construct a set of ground states, showing that for sufficient parameters that they are linearly independent. Using this we can obtain a similar dimensionality lower bound.

**Theorem 5.** *If $u_1$, $u_2$, $v_1$ and $v_2$ are unitaries such that they satisfy the commutation relations*

$$\|[u_1, u_2]\| \leq \gamma \qquad \|[u_1, v_2]\| \leq \delta \qquad \|[u_2, v_1]\| \leq \delta$$

*and twisted commutation relations*

$$\left\|[u_1, v_1]_{1/d_1}\right\| \leq \delta \qquad \left\|[u_2, v_2]_{1/d_2}\right\| \leq \delta$$

*with $d_1 \leq d_2$ and*

$$\sqrt{\gamma}d_1 d_2 + (d_1 + d_2)\delta < \frac{\sin^2(\pi/2d_1)}{(d_1 d_2 - 1)^2},$$

*then the dimension of each operator is at least $d_1 d_2$.*

In Section 4 we provide a more comprehensive analysis for the case of a single twisted pair. Leveraging results from spectral perturbation theory, we find an explicit closed form for the minimum twisted commutation value.

**Theorem 6** (Minimum twisted commutation value)**.** *Suppose that $u$ and $v$ are $g$-dimensional unitaries, then*

$$\left\|[u, v]_\alpha\right\|_p \geq 2g^{1/p}\sin\left(\pi\left|\frac{\lfloor g\alpha\rceil - g\alpha}{g}\right|\right)$$

*for $p \geq 2$. Moreover this bound is tight, in that sense that there exist $g$-dimensional unitaries which saturate the above bound and only depend on $\lfloor g\alpha\rceil$, the nearest integer to $g\alpha$.*

For a given twisted pair, all dimensions for which the twisted commutation value falls below this minimum can therefore be ruled out as valid dimensions. As this bound is not monotonic as a function of $g$, it not only provides a lower bound, but a more rich classification of which degeneracies are disallowed.

After giving proofs of the main results outlined above in Sections 2, 3, and 4, we turn to broader discussion and applications of these ideas. Section 5 is devoted to discussion of future directions for this work that add the additional constraint that the Hamiltonian is *local*, and we discuss the relationship to the notions of topological order and topological quantum codes. In particular we how recent numerical methods for studying quantum many-body systems [20] could leverage the bounds presented here to provide certificates of the topological degeneracy of certain quantum systems.

## 2 Restriction to the ground space

In this section we will make precise the notion that approximate can be utilized as proxies of ground symmetries. We first establish a relationship between approximate symmetries and the ground symmetries that they imply. Then we consider operators with approximate twisted commutation relations, and we show that these can also be restricted faithfully to the ground space with low distortion.

Constructing a ground symmetry from an approximate symmetry will come in two steps. First we will pinch the symmetry $U$ with respect to $\Pi$, giving an operator $P$ for which $[P, \Pi] = 0$. While this will render $P$ no longer unitary, we will see that it will still be *approximately* unitary. We will then show that approximate unitaries are nearly unitary. Using this we will construct a nearby unitary $\tilde{U}$ that retains commutation with the ground space projector, thus constituting a ground symmetry.

To bound the change in our symmetry that is caused by pinching, we will start by bounding the off-diagonal blocks of $U$ with respect to $\Pi$.

**Lemma 2.1** (Small off-diagonal blocks)**.** *If $U$ is an $\epsilon$-approximate symmetry, then off-diagonal blocks of $U$ with respect to $\Pi$ have bounded norms, in particular $\left|\!\left|\!\left|\bar{\Pi}U\Pi\right|\!\right|\!\right| \le \epsilon/\Delta$ and $\left|\!\left|\!\left|\Pi U\bar{\Pi}\right|\!\right|\!\right| \le \epsilon/\Delta$.*

*Proof.* The Hamiltonian $H$ is a positive operator, and we have in particular $H \ge \Delta\bar{\Pi} \ge 0$, which in turn implies that $H^2 \ge \Delta^2\bar{\Pi}$. Then the modulus of the off-diagonal block of $U$ is bounded in the semidefinite operator ordering as follows:

$$\begin{aligned}
\Delta^2 \left|\bar{\Pi}U\Pi\right|^2 &= (\Delta\bar{\Pi}U\Pi)^\dagger(\Delta\bar{\Pi}U\Pi) \\
&= \Pi U^\dagger \cdot \Delta^2\bar{\Pi} \cdot U\Pi \\
&\le \Pi U^\dagger \cdot H^2 \cdot U\Pi \\
&= (HU\Pi)^\dagger(HU\Pi) \\
&= \left|HU\Pi\right|^2 .
\end{aligned}$$

By the min-max theorem, this ordering on the moduli implies that the singular values of $\Delta\bar{\Pi}U\Pi$ are, listed in order, each upper bounded by those of $HU\Pi$. As unitarily invariant norms are monotonic functions of the singular values, this allows us to conclude that $\Delta\left|\!\left|\!\left|\bar{\Pi}U\Pi\right|\!\right|\!\right| \le \left|\!\left|\!\left|HU\Pi\right|\!\right|\!\right|$.

We can bound this in turn by the commutator:

$$\begin{aligned}
\Delta \cdot \left|\!\left|\!\left|\bar{\Pi}U\Pi\right|\!\right|\!\right| &\le \left|\!\left|\!\left|HU\Pi\right|\!\right|\!\right| \\
&= \left|\!\left|\!\left|(HU - UH)\,\Pi\right|\!\right|\!\right| \\
&\le \left|\!\left|\!\left|[H, U]\,\Pi\right|\!\right|\!\right| \\
&\le \left|\!\left|\!\left|[H, U]\right|\!\right|\!\right| \\
&\le \epsilon
\end{aligned}$$

The same argument can be used for $\Pi U\bar{\Pi}$. ∎

Using this we can now construct a ground symmetry $\tilde{U}$ by pinching $U$ with respect to $\Pi$, and enforcing unitarity.

**Lemma 2.2** (Approximate symmetries are nearly ground symmetries)**.** *For an $\epsilon$-approximate symmetry $U$, there exists a ground symmetry $\tilde{U}$ such that*

$$\|U - \tilde{U}\| \le 6\epsilon/\Delta.$$

*Proof.* Firstly, pinch the operator $U$ into the diagonal blocks $P := \Pi U \Pi + \bar{\Pi} U \bar{\Pi}$. Clearly $[P, \Pi] = 0$ by construction, but $P$ is no longer unitarity. Using Lemma 2.1 we can see that $P$ is close to $U$.

$$\|U - P\| = \left\|\Pi U \bar{\Pi} + \bar{\Pi} U \Pi\right\| \leq \left\|\Pi U \bar{\Pi}\right\| + \left\|\bar{\Pi} U \Pi\right\| \leq 2\epsilon/\Delta$$

We note in passing that $|U - P|$ is block diagonal, so if $\|\cdot\|$ were a Schatten $p$-norm, then using the identity $\|A \oplus B\|_p^p = \|A\|_p^p + \|B\|_p^p$, the factor of 2 acquired here can be strengthened to $2^{1/p}$, disappearing entirely in the operator norm.

The operator $P$ is contractive by the pinching lemma, so we can use the fact that $U$ is unitary to show that $P$ is approximately unitary.

$$\begin{aligned}
\left\|I - PP^\dagger\right\| &= \left\|UU^\dagger - PP^\dagger\right\| \\
&= \left\|(U - P)P^\dagger + U(U^\dagger - P^\dagger)\right\| \\
&\leq \|U - P\| \cdot \|P\| + \|U - P\| \\
&\leq 2\|U - P\| \\
&\leq 4\epsilon/\Delta\,.
\end{aligned}$$

Next we will see that approximate unitaries are near unitaries. To see this, take the singular value decomposition $P = LSR^\dagger$, and let $\tilde{U} := LR^\dagger$. By the unitary invariance of the norm, the distance between $P$ and $\tilde{U}$ is entirely controlled by the singular values $S$. By non-negativity of singular values and contractivity of $P$, we have the operator inequality $0 \leq S \leq 1$, which implies $S^2 \leq S$ and thus $I - S \leq I - S^2$. Using this we can see that $\tilde{U}$ is close to $P$,

$$\begin{aligned}
\left\|\tilde{U} - P\right\| &= \left\|L(I - S)R^\dagger\right\| \\
&= \left\|I - S\right\| \\
&\leq \left\|I - S^2\right\| \\
&= \left\|L(I - S^2)L^\dagger\right\| \\
&= \left\|I - PP^\dagger\right\| \\
&\leq 4\epsilon/\Delta\,.
\end{aligned}$$

It should be noted that their exists an ordering on the singular values, corresponding to performing the SVD block-wise, such that the block structure of $P$ with respect to $\Pi$ is retained in $\tilde{U}$, and so $[\tilde{U}, \Pi] = 0$ forming a ground symmetry as desired.

Applying the triangle inequality we get the final bound,

$$\|U - \tilde{U}\| \leq \|U - P\| + \|\tilde{U} - P\| \leq 6\epsilon/\Delta\,.$$

□

We will now consider how the existence of nearby ground symmetries allows twisted commutation relations of approximate symmetries to be pulled down into the ground space.

**Theorem 2** (Restriction to the ground band)**.** *For two $\epsilon$-approximate symmetries $U$ and $V$ which approximately twisted commute*

$$\|[U, V]_\alpha\| \leq \delta,$$

*there exists unitaries $u$ and $v$ acting on the ground space which also approximately twisted commute as*

$$\|[u, v]_\alpha\| \leq \delta + 24\epsilon/\Delta.$$

*Proof.* Consider a $\tilde{U}$ and $\tilde{V}$ given by applying Lemma 2.2 to $U$ and $V$ respectively, such that

$$\|U - \tilde{U}\|,\ \|V - \tilde{V}\| \leq 6\epsilon/\Delta.$$

Using this we can see that the twisted commutator cannot grow much:

$$\begin{aligned}
\left\|[\tilde{U}, \tilde{V}]_\alpha - [U, V]_\alpha\right\| &= \left\|\tilde{U}\tilde{V} - e^{2\pi i\alpha}\tilde{V}\tilde{U} - UV + e^{2\pi i\alpha}VU\right\| \\
&= \left\|\left[\left(\tilde{U} - U\right)V + \tilde{U}\left(\tilde{V} - V\right)\right] + e^{2\pi i\alpha}\left[\left(\tilde{V} - V\right)U + \tilde{V}\left(\tilde{U} - U\right)\right]\right\| \\
&\leq \left\|\left(\tilde{U} - U\right)V\right\| + \left\|\tilde{U}\left(\tilde{V} - V\right)\right\| + \left\|\left(\tilde{V} - V\right)U\right\| + \left\|\tilde{V}\left(\tilde{U} - U\right)\right\| \\
&= 2\left\|\tilde{U} - U\right\| + 2\left\|\tilde{V} - V\right\| \\
&\leq 24\epsilon/\Delta
\end{aligned}$$

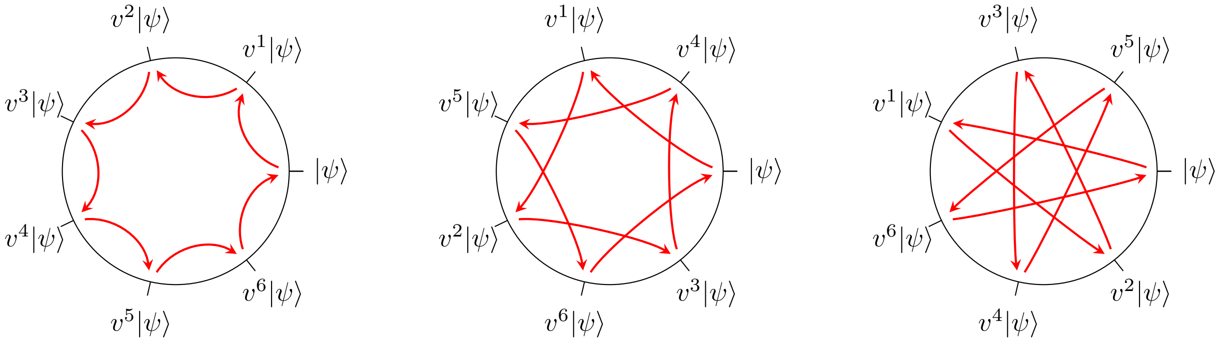


Figure 1: The action of powers of $v$ on an eigenvector $|\psi\rangle$ of $u$. On the left $[u,v]_{1/7}=0$, in the centre $[u,v]_{2/7}=0$, and on the right $[u,v]_{3/7}=0$. Here the position of the circle represents the corresponding eigenvalue of $u$.

So we conclude

$$\left\|\left|[\tilde{U},\tilde{V}]_\alpha\right\|\right| \leq \delta + 24\epsilon/\Delta.$$

Next let $u$ and $v$ be the ground space restrictions of $\tilde{U}$ and $\tilde{V}$. As each is a ground symmetry, $u$ and $v$ must be unitary. The twisted commutator value of the ground symmetries also carries through to these restrictions:

$$\begin{aligned}
\left|\!\left|\!\left|[u,v]_\alpha\right|\!\right|\!\right| &= \left|\!\left|\!\left|[u\oplus 0, v\oplus 0]_\alpha\right|\!\right|\!\right| \\
&= \left|\!\left|\!\left|[\Pi\tilde{U}\Pi,\Pi\tilde{V}\Pi]_\alpha\right|\!\right|\!\right| \\
&= \left|\!\left|\!\left|\Pi[\tilde{U},\tilde{V}]_\alpha\right|\!\right|\!\right| \\
&\leq \left|\!\left|\!\left|[\tilde{U},\tilde{V}]_\alpha\right|\!\right|\!\right| \\
&\leq \delta + 24\epsilon/\Delta.
\end{aligned}$$

Note that if we had a set of more than two unitaries, this additive growth in the twisted commutation value would hold equally for every pair separately. □

# 3 Degeneracy lower bounds

In this section we show how twisted pairs of unitary operators can be used to give lower bounds on the degeneracy of the ground space. We start by considering an exact twisted pair and the Stone-von Neumann theorem. We will then show how this argument can be generalized to approximate twisted pairs, and show how a lower bound on the degeneracy follows from a specific inequality on the value of the twisted commutator. Finally we will see how this can also be extended to more general collections of twisted commuting operators through the example case of two twisted pairs that are approximately mutually commuting.

## 3.1 Stone-von Neumann Theorem

Consider a $u$ and $v$ which exactly twisted commute, so that $uv = e^{2i\pi\alpha}vu$. Let $(\lambda, |\psi\rangle)$ be an eigenpair of $u$. Using the twisted commutation relation, we see that $|\psi'\rangle := v|\psi\rangle$ forms a $\lambda e^{2i\pi\alpha}$-eigenvector. It follows that $v$ forms an isomorphism between the $\lambda$ and $\lambda e^{2i\pi\alpha}$-eigenspaces of $u$, which allows us to conclude that their dimensions must be the same. Carrying this argument forward, we can see that any eigenspaces whose eigenvalues differ by any power of $e^{2i\pi\alpha}$ must also be isomorphic.

Suppose we take $\alpha\in\mathbb{Q}$, with $\alpha = p/q$ with $p$, $q$ coprime. As we can see in Fig. 1, a simple divisibility argument implies that the eigenspaces come in isomorphic multiples of $q$, which therefore implies that the overall dimension of $u$ and $v$ is a multiple of $q$ also.

We now generalize this connection between the twisted commutator and the spectrum of one of the operators to allow for only approximate twisted commutation.

## 3.2 One twisted pair

Let us first extend the above argument to the case of a single approximate twisted pair. For simplicity, we consider the case where $\alpha = p/q$ with $p = 1$ and $q = d$, so we therefore have the twisting parameter be $\eta := e^{2i\pi/d}$. This is not much of a restriction since if $p > 1$ we can replace $v$ with $v^{\bar{p}}$ where is the modular multiplicative inverse of $p$ such that $\bar{p}p = 1 \bmod q$ and then apply the results of the $p = 1$ case. Under this substitution the twisted commutator will grow by at most a factor of $\lfloor q/2 \rfloor$. However, in Appendix B we will show an alternative method that in fact works for arbitrary $\alpha \in \mathbb{R}$ and gives tighter bounds than this simple reduction. We also consider without loss of generality the case where $u$ has at least one $+1$ eigenvalue, which can always be achieved by redefining $u$ by multiplying by a complex unit phase factor.

Suppose we have two unitaries $u$ and $v$ such that

$$\left\| [u, v]_{1/d} \right\| = \|uv - \eta vu\| \leq \delta.$$

Our results will show that these operators must, for sufficiently small $\delta$, be at least $d$-dimensional. To do this we will explicitly show that $u$ has at least $d$ distinct eigenvalues.

Let $|\psi\rangle$ be the $+1$ eigenvector of $u$, such that $u|\psi\rangle = |\psi\rangle$. Consider the orbit of $|\psi\rangle$ under $v$, i.e. the states $|j\rangle := v^j|\psi\rangle$ for $j = -\lfloor \frac{d-1}{2} \rfloor, \ldots, \lceil \frac{d-1}{2} \rceil$. These vectors are precisely the vectors depicted in Figure 1. We first show that these are approximate eigenstates of $u$.

**Lemma 3.1** (Change in expectation value: One pair)**.** *The expectation value of $u$ with respect to $|j\rangle$ is approximately $\eta^j$, specifically*

$$\left| \langle j|u|j\rangle - \eta^j \right| \leq |j|\, \delta.$$

*Proof.* This follows from the twisted commutator of $u$ and $v$ being small. By expanding the commutator and applying the triangle inequality we can see that $\|uv - \eta vu\| \leq \delta$ implies $\left\| uv^j - \eta^j v^j u \right\| \leq |j|\, \delta$.

$$\begin{aligned}
\left\| uv^j - \eta^j v^j u \right\| &= \left\| \left(uv^j - \eta vuv^{j-1}\right) + \left(\eta vuv^{j-1} - \eta^2 v^2 uv^{j-2}\right) + \cdots + \left(\eta^{j-1} v^{j-1} uv - \eta^j v^j u\right) \right\| \\
&\leq \left\| uv^j - \eta vuv^{j-1} \right\| + \left\| \eta vuv^{j-1} - \eta^2 v^2 uv^{j-2} \right\| + \cdots + \left\| \eta^{j-1} v^{j-1} uv - \eta^j v^j u \right\| \\
&= \left\| (uv - \eta vu)\, v^{j-1} \right\| + \left\| \eta v\, (uv - \eta vu)\, v^{j-2} \right\| + \cdots + \left\| \eta^{j-1} v^{j-1} \,(uv - \eta vu) \right\| \\
&= |j| \cdot \|uv - \eta vu\|
\end{aligned}$$

From this we can see that the expectation value of $|j\rangle$ lies close to $\eta^j$:

$$\begin{aligned}
|j|\, \delta &\geq \left\| uv^j - \eta^j v^j u \right\| \\
&= \left\| v^{-j} uv^j - \eta^j u \right\| \\
&\geq \left| \langle \psi| \left[ v^{-j} uv^j - \eta^j u \right] |\psi\rangle \right| \\
&\geq \left| \langle \psi| v^{-j} uv^j |\psi\rangle - \eta^j \, \langle \psi|u|\psi\rangle \right| \\
&= \left| \langle j|u|j\rangle - \eta^j \right|.
\end{aligned}$$

□

So we can see that the $\{|j\rangle\}$ form a set of vectors with expectation values distributed approximately evenly around the unit circle, much like the states in the $\delta = 0$ case as seen in Fig. 1. To relate these states to the dimensions of $u$ and $v$, we will now show that there must exist an eigenvalue of $u$ near the expectation value of each state.

**Lemma 3.2** (Existence of eigenvalues)**.** *If there exists a state $|x\rangle$ such that*

$$\left| \langle x|u|x\rangle - e^{i\theta} \right| \leq \zeta$$

*then $u$ possesses an eigenvalue $e^{i\phi}$ such that*

$$|\phi - \theta| \leq \cos^{-1}(1 - \zeta).$$

*Proof.* The bound on the expectation value with respect to $u$ implies

$$\mathrm{Re} \left\langle x \middle| e^{-i\theta} u \middle| x \right\rangle \geq 1 - \zeta.$$

As this expectation value is a convex combination of the eigenvalues of $u$, all of which lie on the unit circle, there must exists an eigenvalue of $e^{-i\theta} u$ with real value at least $1 - \zeta$. This in turn implies that $u$ possesses an eigenvalue $e^{i\phi}$ such that

$$\mathrm{Re}\, e^{i(\phi - \theta)} = \cos(\phi - \theta) \geq 1 - \zeta.$$

□

Combining the two above lemmas, we can lower bound the number of distinct eigenvalues of $u$.

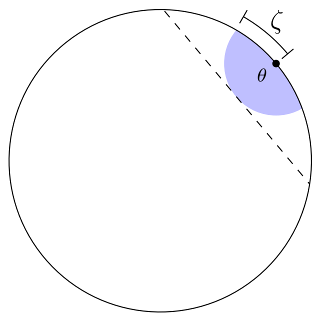


Figure 2: Lemma 3.2 gives that if there exists an expectation value in the blue region, there must exist an eigenvalue within the minor segment indicated by the dotted line.

**Theorem 3.** *If $u$ and $v$ are unitaries such that*

$$\left\|[u,v]_{1/d}\right\| < \frac{2}{d-1}\Big[1-\cos\pi/d\Big],$$

*then the dimension of each operator is at least $d$.*

*Proof.* From Lemma 3.1 we know that $\left|\langle j|u|j\rangle - e^{2i\pi j/d}\right| \leq |j|\,\delta$. Applying Lemma 3.2 we therefore get that $u$ must have a corresponding eigenvalue $e^{i\phi_j}$ where

$$|\phi_j - 2j\pi/d| \leq \cos^{-1}\left(1-|j|\,\delta\right).$$

As such we can see that each eigenvalue is within some error of a $d$th root of unity.

Next we want to find a bound for $\delta$ which ensures that these eigenvalues must be distinct; that is, the error regions for distinct eigenvalues should be disjoint. To do this we need $|\phi_j-\phi_k|>0$ for all $j\neq k$. Taking the worst case over $j \neq k$:

$$\begin{aligned}|\phi_j-\phi_k| &= \left|\frac{2\pi}{d}(j-k)+\left(\phi_j-\frac{2j\pi}{d}\right)-\left(\phi_k-\frac{2k\pi}{d}\right)\right| \\ &\geq \frac{2\pi}{d}\,|j-k| - \left|\phi_j-\frac{2j\pi}{d}\right| - \left|\phi_k-\frac{2k\pi}{d}\right| \\ &\geq \frac{2\pi}{d} - \cos^{-1}\left(1-\left\lceil\frac{d-1}{2}\right\rceil\delta\right) - \cos^{-1}\left(1-\left\lfloor\frac{d-1}{2}\right\rfloor\delta\right).\end{aligned}$$

Here the last line follows from the fact that $j$ and $k$ cannot both saturate the worst-case distance of $\lceil\frac{d-1}{2}\rceil$. Therefore, the worst case can be chose without loss of generality to be $j=\lceil\frac{d-1}{2}\rceil$ and $k=-\lfloor\frac{d-1}{2}\rfloor$. Using the concavity of $\cos^{-1}(z)$ over $z\in[0,1]$, we can loosen this to

$$|\phi_j-\phi_k| \geq \frac{2\pi}{d} - 2\cos^{-1}\left(1-\frac{d-1}{2}\,\delta\right).$$

Clearly this step is trivial for odd $d$.

Thus we get that a sufficient condition for all of the eigenvalues to be distinct is that the right-hand side of this inequality is strictly positive, and therefore we have the equivalent condition

$$\cos^{-1}\left(1-\frac{d-1}{2}\,\delta\right) < \frac{\pi}{d}.$$

Rearranging, we find the specified bound on $\delta$ of

$$\delta < \frac{2}{d-1}\Big[1-\cos(\pi/d)\Big].$$

□

Above we have only considered the case $d=1/\alpha$, similar analysis could be performed for bounds required to certify dimensions $d'\neq 1/\alpha$. In Appendix B we describe an algorithm for calculating which dimensions can be certified for an arbitrary pair of parameters $\alpha$ and $\delta$ — running this algorithm gives Fig. 3.

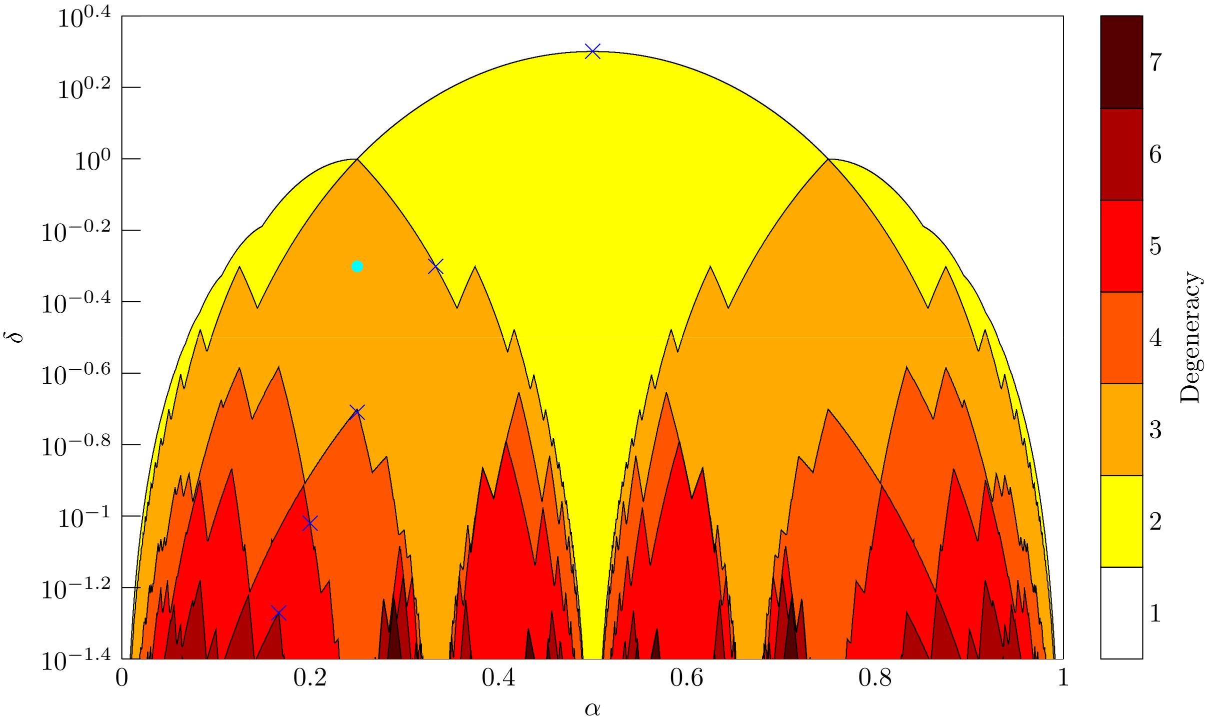


Figure 3: The dimension that can be certified, as a function of the twisted commutator value and twisting parameter. The blue crosses indicate the level of certification considered in Section 3.2. The algorithm for calculating this figure is demonstrated in Appendix B, by considering how the certification is calculated at the turquoise dot ($\alpha = 1/4$ and $\delta = 1/2$).

### 3.3 Two twisted pairs

Next we are going to argue that the above analysis can be extended to more general collections of twisted commuting symmetries. By way of example, we are going to consider the case of two twisted pairs

$$\left\| [u_1, v_1]_{1/d_1} \right\| \leq \delta, \qquad \left\| [u_2, v_2]_{1/d_2} \right\| \leq \delta,$$

each of which approximately commute

$$\|[u_1, u_2]\| \leq \gamma, \qquad \|[u_1, v_2]\| \leq \delta, \qquad \|[u_2, v_1]\| \leq \delta.$$

The equivalent of Stone-von Neumann theorem laid our in Section 3.1 gives that for $\gamma = \delta = 0$, the dimension of such operators must be a multiple of $d_1 d_2$. We are going to give bounds on $\gamma$ and $\delta$ below which we can prove the dimension to be at least $d_1 d_2$.

Previously we bounded the dimension from below by bounding the number of distinct eigenvalues. This is possible because these eigenvalues imply the existence of an orthonormal set of associated eigenvectors. As $u_1$ and $u_2$ do not commute, they will not necessarily possess an orthonormal set of shared eigenvectors. Instead we will have to address these vectors more directly, constructing *approximate shared eigenvectors* and proving their linear independence. First we will see that the approximate commutation of $u_1$ and $u_2$ can be used to demonstrate the existence of such a vector.

The existence of approximate shared eigenvectors of approximately commuting matrices was first proven in generality by Bernstein in Ref. [19]. Whilst Bernstein considers potentially non-normal matrices, in our case both $u_1$ and $u_2$ are unitary. In Appendix A we leverage this additional structure to exponentially tighten the bounds on the approximate shared eigenvectors. One of the relevant bounds considered in Appendix A gives the following immediate corollary.

**Corollary 7** (Approximate eigenvector)**.** *There exists a vector $|\psi\rangle$ such that, after rephasing $u_1$ and $u_2$, it is an approximate shared $+1$-eigenvector of both, namely that*

$$\|u_1|\psi\rangle - |\psi\rangle\|, \|u_2|\psi\rangle - |\psi\rangle\| \leq \sqrt{\gamma} d_1 d_2/2.$$

*Proof.* Given an assumption that the dimension is at most $d_1 d_2$, this is a direct application of Theorem A.2, which we consider in detail in Appendix A. □

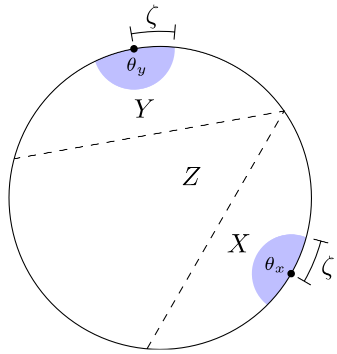


Figure 4: A disc representing the expectation values of vectors with respect to $w$, as well as the three regions $X$, $Y$, $Z$ into which the disc is divided. The expectation values with respect to $|x\rangle$ and $|y\rangle$ lie in each of the blue regions.

As in the case of a single pair, we will then consider the orbit of this vector under the action of products of $v_1$ and $v_2$. Let $|i,j\rangle := v_1^i v_2^j |\psi\rangle$ for $i = -\left\lfloor \frac{d_1-1}{2} \right\rfloor, \ldots, \left\lceil \frac{d_1-1}{2} \right\rceil$ and $j = -\left\lfloor \frac{d_2-1}{2} \right\rfloor, \ldots, \left\lceil \frac{d_2-1}{2} \right\rceil$. For convenience once again let $\eta_i := e^{2i\pi/d_i}$.

**Lemma 3.3** (Change in expectation value: two pair)**.** *The states $|i,j\rangle$ are shared approximate eigenstates of $u_1$ and $u_2$. Specifically their approximate eigenvalues are the corresponding powers of $\eta_1$ and $\eta_2$*

$$\left|\langle i,j|u_1|i,j\rangle - \eta_1^i\right|, \left|\langle i,j|u_2|i,j\rangle - \eta_2^j\right| \le \sqrt{\gamma} d_1 d_2/2 + (|i| + |j|)\,\delta.$$

*Proof.* From Corollary 7 we have

$$|\langle\psi|u_1|\psi\rangle - 1| \le \sqrt{\gamma} d_1 d_2/2.$$

Applying an argument similar yo that in Lemma 3.1 to how this expectation changes under the action of $v_1$, we get

$$\left|\langle i,0|u_1|i,0\rangle - \eta_1^i \langle\psi|u_1|\psi\rangle\right| \le |i|\,\delta.$$

Applying it again for the actions under $v_2$ gives

$$|\langle i,j|u_1|i,j\rangle - \langle i,0|u_1|i,0\rangle| \le |j|\,\delta.$$

Applying the triangle inequality and merging the three above inequalities gives the stated bound. A similar argument can be performed for $u_2$. □

In the single pair case, we used the expectation values to imply the existence of nearby eigenvalues. Due to the lack of a shared eigenbasis of $u_1$ and $u_2$, we cannot do the same.

The reason that a set of distinct eigenvalues lower bounds the dimension is that, for normal operators such as unitaries, the eigenvalues imply the existence of an orthonormal eigenbasis. Instead of proving the existence of such vectors indirectly through the eigenvalues, we could instead prove our vectors $\{|i,j\rangle\}$ to be linearly independent — this is the approach we will take.

To this end, we will start by showing two approximate eigenvectors of a unitary with inconsistent expectation values are approximately orthogonal.

**Lemma 3.4** (Low overlap)**.** *If two normalized vectors $|x\rangle$ and $|y\rangle$ have expectation values with some unitary $w$ such that*

$$\left|\langle x|w|x\rangle - e^{i\theta_x}\right| \le \zeta \qquad \text{and} \qquad \left|\langle y|w|y\rangle - e^{i\theta_y}\right| \le \zeta$$

*then the two vectors have a bounded overlap*

$$|\langle x|y\rangle| \le \sqrt{2\zeta}\left|\csc\left(\frac{\theta_y - \theta_x}{4}\right)\right|.$$

*Proof.* Firstly, let $w' := e^{-i\theta_x} w$ and $\theta := \theta_y - \theta_x$. Next consider splitting the unit circle into three arcs $X$, $Y$, and $Z$. We let $X$ and $Y$ be centered on $\theta_x$ and $\theta_y$ respectively, and define them to be the largest possible regions such that they remain disjoint. We define $Z$ to be the remaining region, as shown in Fig. 4. Note that any linear combination of eigenvectors whose eigenvalues lie in $X$ will have an expectation value in the segment subtended by $X$, and similar for $Y$.

Now split $|x\rangle$ into two components

$$|x\rangle = \sqrt{1-\lambda_x}|x_X\rangle + \sqrt{\lambda_x}|x_{YZ}\rangle,$$

where $|x_X\rangle$ is in the span of eigenvectors with values in $X$, and $|x_{YZ}\rangle$ similar for $Y \cup Z$. As such we have that

$$\operatorname{Re}\langle x_{YZ}|w'|x_{YZ}\rangle \le \cos(\theta/2) \le \operatorname{Re}\langle x_X|w'|x_X\rangle \le 1.$$

Next we use the bound on the expectation value.

$$\begin{aligned}
\zeta &\ge |\langle x|w'|x\rangle - 1| \\
&\ge 1 - \operatorname{Re}\langle x|w'|x\rangle \\
&= 1 - (1-\lambda_x)\operatorname{Re}\langle x_X|w'|x_X\rangle - \lambda_x \operatorname{Re}\langle x_{YZ}|w'|x_{YZ}\rangle \\
&\ge 1 - (1-\lambda_x) - \lambda_x \cos(\theta/2) \\
&= 2\lambda_x \sin^2(\theta/4)
\end{aligned}$$

Thus we conclude that $\lambda_x \le (\zeta/2)\csc^2(\theta/4)$. Similarly if we were to have decomposed $|y\rangle$ into parts contained in $Y$ and $XZ$ as $|y\rangle = \sqrt{1-\lambda_y}|y_Y\rangle + \sqrt{\lambda_y}|y_{XZ}\rangle$ then $\lambda_y \le (\zeta/2)\csc^2(\theta/4)$.

Further decomposing

$$|x_{YZ}\rangle = \cos\varphi_x|x_Y\rangle + \sin\varphi_x|x_Z\rangle \qquad |y_{XZ}\rangle = \cos\varphi_y|y_X\rangle + \sin\varphi_y|y_Z\rangle\,,$$

then the inner product has the form

$$\begin{aligned}
|\langle x|y\rangle| &= \left|\sqrt{1-\lambda_x}\sqrt{\lambda_y}\cos\varphi_y\,\langle x_X|y_X\rangle + \sqrt{\lambda_x}\sqrt{1-\lambda_y}\cos\varphi_x\,\langle x_Y|y_Y\rangle + \sqrt{\lambda_x}\sqrt{\lambda_y}\sin\varphi_x\sin\varphi_y\,\langle x_Z|y_Z\rangle\right| \\
&\le \sqrt{1-\lambda_x}\sqrt{\lambda_y}\cos\varphi_y + \sqrt{\lambda_x}\sqrt{1-\lambda_y}\cos\varphi_x + \sqrt{\lambda_x}\sqrt{\lambda_y}\sin\varphi_x\sin\varphi_y.
\end{aligned}$$

Using the identity $|A\cos\phi + B\sin\phi|^2 \le |A|^2 + |B|^2$, we can maximize over $\varphi_x$ to get

$$|\langle x|y\rangle| \le \sqrt{1-\lambda_x}\sqrt{\lambda_y}\cos\varphi_y + \sqrt{\lambda_x}\sqrt{1-\lambda_y\cos^2\varphi_y}.$$

Using $\cos\varphi_y \le 1$, we can simplify this bound to

$$|\langle x|y\rangle| \le \sqrt{\lambda_y} + \sqrt{\lambda_x}.$$

Applying the $\zeta$-dependent bounds on the $\lambda$ values, we get the stated bounds. □

Now that we have a way of bounding the overlap between our vectors, we need to determine how low this overlap needs to be before linear independence can be ensured.

**Lemma 3.5** (Overlap threshold)**.** *Take a set of normalized vectors $S = \{|v_i\rangle\}$ for $1 \le i \le n$. If the pairwise overlap between any two vectors is bounded $|\langle v_i|v_j\rangle| < 1/(n-1)$ for $i \ne j$, then $S$ is linearly independent.*

*Proof.* Let $G$ be the Gram matrix associated with $S$. As each of the vectors is normalized $G_{ii} = 1$ for all $i$. As all of the non-diagonal entries are strictly modulus-bounded by $1/(n-1)$, this matrix is strictly diagonally dominant, i.e.

$$|G_{ii}| > \sum_{j\ne i}|G_{ij}| \quad \text{for all } i.$$

From the Geshgorin circle theorem, such matrices are non-singular and therefore full rank. This implies that the Gram matrix is full rank, and therefore $S$ is linearly independent.

Note that this analysis is tight, i.e. if $\langle v_i|v_j\rangle = -1/(n-1)$ for all $i \ne j$ then $G$ is singular and $\sum_i |v_i\rangle = 0$. By considering the eigenvectors of such a Gram matrix, a set of vectors satisfying this can be backed out. □

Given this bound, we can finally find the condition for our vectors to be linearly independent and therefore lower bound the dimension of the space in which they reside.

**Theorem 4.** *If $u_1$, $u_2$, $v_1$ and $v_2$ are unitaries such that they satisfy the commutation relations*

$$\|[u_1,u_2]\| \le \gamma \qquad \|[u_1,v_2]\| \le \delta \qquad \|[u_2,v_1]\| \le \delta$$

*and twisted commutation relations*

$$\left\|[u_1,v_1]_{1/d_1}\right\| \le \delta \qquad \left\|[u_2,v_2]_{1/d_2}\right\| \le \delta$$

*with $d_1 \le d_2$ and*

$$\sqrt{\gamma}d_1d_2 + (d_1+d_2)\delta < \frac{\sin^2(\pi/2d_1)}{(d_1d_2-1)^2},$$

*then the dimension of each operator is at least $d_1d_2$.*

*Proof.* From Lemma 3.3 we have that our vectors have expectation values bounded near powers of $\eta_1$ and $\eta_2$

$$\left|\langle i,j|u_1|i,j\rangle - \eta_1^i\right|, \ \left|\langle i,j|u_2|i,j\rangle - \eta_2^j\right| \leq \sqrt{\gamma} d_1 d_2/2 + (|i|+|j|)\,\delta.$$

Take a pair of vectors $|i,j\rangle$ and $|i',j'\rangle$ such that $i \neq i'$. Applying Lemma 3.4 with $w = u_1$ we get that their overlap is bounded as

$$|\langle i,j|i',j'\rangle|^2 \leq \left[\sqrt{\gamma} d_1 d_2 + 2\max\{|i|+|j|\,, |i'|+|j'|\}\delta\right] \cdot \csc^2\left(\frac{\pi(i-i')}{2d_1}\right).$$

Combining this with a similar argument for $u_2$, and assuming $d_1 \leq d_2$, we get that for $(i,j) \neq (i',j')$

$$|\langle i,j|i',j'\rangle|^2 \leq \left[\sqrt{\gamma} d_1 d_2 + (d_1+d_2)\delta\right] \csc^2\left(\frac{\pi}{2d_1}\right)$$

Thus we can see that

$$\left[\sqrt{\gamma} d_1 d_2 + (d_1+d_2)\delta\right] \csc^2\left(\frac{\pi}{2d_1}\right) < \frac{1}{(d_1 d_2 - 1)^2}\,.$$

implies $|\langle i,j|i',j'\rangle| < 1/(d_1 d_2 - 1)$ for all $(i,j) \neq (i',j')$. By Lemma 3.5 this means that the collection of vectors $\{|i,j\rangle\}_{i,j}$ are linearly independent, constructively proving the dimensionality of the operators in question to be at least $d_1 d_2$. Rearranging this gives the specified bound. □

# 4 Minimum twisted commutation value

In the previous section we considered finding lower bounds on the dimensions of nearly twisting commuting operators. In the exact case, the Stone-von Neumann theorem (c.f. Theorem 1) tell us that unitaries $x$ and $y$ for which

$$[x,y]_{1/d} = 0$$

are not only *at least* $d$-dimensional, but are *a multiple of* $d$-dimensional. We might therefore hope for a more comprehensive understanding of twisted commutation that provides more information than simply a lower bound on the dimension. In this section we will consider the twisted commutator in the Shatten norms $\|\!|\cdot|\!\| := \|\cdot\|_p$ with $p \geq 2$, and find the minimum possible twisted commutator value as a function of dimension.

**Definition 3** (Minimum twisted commutator value)**.** *We define* $\Lambda^{(p)}_{g,\alpha}$ *to be the* minimum twisted commutator value*, with respect to the Shatten* $p$*-norm, over all unitary matrices of dimension* $g$*, to be the quantity*

$$\Lambda^{(p)}_{g,\alpha} := \min_{u,v \in U(g)} \left\|[u,v]_\alpha\right\|_p .$$

In this language, the Stone-von Neumann theorem gives that $\Lambda^{(p)}_{g,\alpha} = 0$ if and only if $g\alpha \in \mathbb{Z}$. If we had an understanding of the values of $\Lambda^{(p)}_{g,\alpha}$ where $g\alpha \notin \mathbb{Z}$, then we could use twisted commutation value as a way of certifying dimension. In particular, if one thinks of $\alpha$ as fixed, and one knows the value $\left\|[u,v]_\alpha\right\|_p$ to be less than $\Lambda^{(p)}_{g,\alpha}$ for certain dimensions $g$, then these certain dimensions are ruled out as possible dimensions of $u$ and $v$. In this section we will explicitly evaluate $\Lambda^{(p)}_{g,\alpha}$.

To lower bound $\Lambda^{(p)}_{g,\alpha}$, we will utilize techniques from spectral perturbation theory to bound a related quantity known as the *spectral distance*. By considering a family of operators which twisted commute, we will furthermore show this bound to be tight.

**Definition 4** (spectral distance)**.** *The* spectral $p$-distance $d_p(a,b)$ *between two matrices* $a$ *and* $b$ *is the* $p$*-norm of the vector containing the differences between eigenvalues of the two matrices, minimized over all possible orderings. If we let* $\lambda(x)$ *denote the vector of eigenvalues of a* $g \times g$ *matrix* $x$ *then algebraically*

$$d_p(a,b) := \min_{\sigma \in S_g} \left\|\sigma\left[\lambda(a)\right] - \lambda(b)\right\|_p = \min_{\sigma \in S_g} \left(\sum_{j=1}^{g} \left|\lambda_{\sigma(j)}(a) - \lambda_j(b)\right|^p\right)^{1/p},$$

*where the minimization is over all elements* $\sigma$ *of the permutation group* $S_g$ *on* $g$ *symbols.*

## 4.1 Frobenius spectral bound

Before attacking the spectral distance, we are first going to restrict ourselves to the case of the Frobenius norm ($p = 2$), where we shall denote the norm by $\|\cdot\|_F$, the corresponding spectral distance by $d_F(\cdot,\cdot)$, and the twisted commutator minimum by $\Lambda^{(F)}_{g,\alpha}$. In this special case, the spectral distance between two normal matrices is bounded by their norm difference. Once again let $\eta := e^{2i\pi\alpha}$.

**Lemma 4.1** (Wielandt-Hoffman inequality [21])**.** *For normal matrices $a$ and $b$, $d_F(a,b) \le \|a-b\|_F$.*

Applying Wielandt-Hoffman to $\Lambda^{(F)}_{g,\alpha}$ we see that the corresponding spectral distance provides a lower bound,

$$\Lambda^{(F)}_{g,\alpha} = \min_{u,v\in U(d)} \left\|v^\dagger uv - \eta u\right\|_F \ge \min_{u,v\in U(g)} d_F(v^\dagger uv, \eta u) = \min_{u\in U(g)} d_F(u,\eta u)\,.$$

Though $\left\|v^\dagger uv - \eta u\right\|_F$ depended on both $u$ and $v$, $d_F(u,\eta u)$ depends only on the spectrum of $u$, making for a much simpler optimization. This inequality will turn out to be tight.

Denote the eigenvalues of $u$ by $\{e^{i\theta_j}\}$, then the spectral distance in question is given by

$$d_F^2(u,\eta u) := \min_{\sigma\in S_g} \sum_{j=1}^{g} \left|e^{i\theta_{\sigma(j)}} - e^{i(\theta_j+2\pi\alpha)}\right|^2 = \min_{\sigma\in S_g} \sum_{j=1}^{g} 4\sin^2\left(\frac{\theta_{\sigma(j)}-\theta_j-2\pi\alpha}{2}\right).$$

Define $f(\sigma;\theta_1,\dots,\theta_g)$ to be the argument of the above optimization

$$f(\sigma;\theta_1,\dots,\theta_g) := \sum_{j=1}^{g} 4\sin^2\left(\frac{\theta_{\sigma(j)}-\theta_j-2\pi\alpha}{2}\right) \tag{1}$$

such that $d_F^2(u,\eta u) = \min_\sigma f(\sigma;\theta_1,\dots,\theta_g)$ . The optimization of $d_F^2(u,\eta u)$ can therefore be reduced to an optimization of $f(\sigma;\theta_1,\dots,\theta_g)$.

We can now break the optimization of $f$ down into two parts. First we will show that for any assignment of permutation and angles, there exists a cyclic permutation, and adjusted angles, for which the value of $f$ is the same. This will allow us to consider a minimizing permutation which has only a single cycle without loss of generality. Secondly we shall see that, for such a cyclic permutation, the set of angles which minimize $f$ are those that are equally distributed around the unit circle. Given these, we will find an explicit minimum for $f$, and thus for $d_F(u,\eta u)$.

**Lemma 4.2** (Reduction to cyclic permutations)**.** *For a given multi-cycle permutation $\sigma$ and set of angles $\{\theta_j\}$, there exists a cyclic permutation $\sigma'$ and set of adjusted angles $\{\theta'_j\}$ such that*

$$f(\sigma;\theta_1,\dots,\theta_g) = f(\sigma';\theta'_1,\dots,\theta'_g).$$

*Proof.* Firstly, our indices can be reordered such that the cycles of $\sigma$ are contiguous, i.e. in cycle notation

$$\sigma = (1\ \dots\ k_1-1)\,(k_1\ \dots\ k_2-1)\dots(k_n\ \dots\ g),$$

for some $1 < k_1 \cdots < k_n \le g$. (Note that the result is trivially true if $g = 1$, so we restrict to $g > 1$.) As $f$ only depends on the difference between angles whose indices are within the same cycle of $\sigma$, if we shift all the angles within the same cycle by the same amount, the value of $f$ will not change. For example if we take the change of angle

$$\theta'_j := \begin{cases} \theta_j - \theta_1 & 1\le j<k_1 \\ \theta_j-\theta_{k_1} & k_1\le j<k_2 \\ \quad\vdots & \\ \theta_j-\theta_{k_n} & k_n\le j\le g. \end{cases}$$

then $f(\sigma;\theta_1,\dots,\theta_g) = f(\sigma;\theta'_1,\dots,\theta'_g)$. Notice that $\theta'_1 = \theta'_{k_1} = \dots = \theta'_{k_n} = 0$ by construction.

We now wish to merge the permutation $\sigma$ into a single cyclic permutation

$$\sigma' := (1\ \dots\ g). \tag{2}$$

To do this, the only entries of the permutation which need to be changed are those at the end of each cycle.

$$\begin{aligned} \sigma(k_1-1) = 1 \quad &\to \quad \sigma'(k_1-1) = k_1 \\ \sigma(k_2-1) = k_1 \quad &\to \quad \sigma'(k_2-1) = k_2 \\ \vdots \qquad &\qquad\qquad \vdots \\ \sigma(g) = k_n \quad &\to \quad \sigma'(g) = 1\,. \end{aligned}$$

By definition of the adjusted angles however, the only indices that change are those for which the angles have already been made identical in the previous step, i.e. $\theta'_{\sigma(j)} = \theta'_{\sigma'(j)}$ for all $j$. As $f$ only depends on $\sigma$ through how it acts on the angles, this means that this doesn't alter the value of $f$, therefore $f(\sigma;\theta'_1,\dots,\theta'_g) = f(\sigma';\theta'_1,\dots,\theta'_g)$.

Combining these two arguments, we see that arbitrary permutations can be fused into a single cyclic permutation, with no change in the value of $f$. □

Now that we have addressed the nature of the optimal permutation, namely showing that it can be taken to be cyclic, we turn out attention to the optimal angles.

**Lemma 4.3.** *For a given single-cycle permutation $\sigma$, the sets of angles which optimize $f$, as defined in Eq. (1), correspond to those evenly distributed around the unit circle, and the difference between adjacent angles $\theta_j$ and $\theta_{\sigma(j)}$ is $2\pi\lfloor d\alpha\rceil/g$, where $\lfloor\cdot\rceil$ denotes integer rounding. Moreover the corresponding minimal value of $f$ is*

$$\min_{\{\theta_j\}_j} f(\sigma;\theta_1,\dots,\theta_g) = 2\sqrt{g}\sin\left(\pi\left|\frac{\lfloor d\alpha\rceil - g\alpha}{g}\right|\right).$$

*Proof.* Denote both of the terms[2] in $f$ which non-trivially depend non-trivially on by $f_j(\theta_j)$. Using the double angle formula and the auxiliary angle method, we can reduce the $\theta_j$ dependence to a single sinusoidal term.

$$\begin{aligned} f_j(\theta_j) &= 4\sin^2\left(\frac{\theta_j - \theta_{\sigma(j)} - 2\pi\alpha}{2}\right) + 4\sin^2\left(\frac{\theta_{\sigma^{-1}(j)} - \theta_j - 2\pi\alpha}{2}\right) \\ &= 4 - 4\cos\left(2\pi\alpha + \frac{\theta_{\sigma(j)} - \theta_{\sigma^{-1}(j)}}{2}\right)\cos\left(\theta_j - \frac{\theta_{\sigma(j)} + \theta_{\sigma^{-1}(j)}}{2}\right). \end{aligned}$$

We can therefore see that the optimal $\theta_j$, leaving all other angles fixed, satisfies

$$\theta_j = \left(\theta_{\sigma(j)} + \theta_{\sigma^{-1}(j)}\right)/2 \mod \pi.$$

This implies that $\theta_{\sigma(j)} - \theta_j = \theta_j - \theta_{\sigma^{-1}(j)} \mod 2\pi$, i.e. $\theta_j$ lies in at the 'midpoint' of its neighbors, as described by $\sigma$. By inducting the above argument we find that $\theta_{\sigma(j)} - \theta_j = \theta_{\sigma(k)} - \theta_k \mod 2\pi$ for all $j,k$ meaning that all adjacent angles are equally spaced around the unit circle. This means that if we have $g$ angles, and label our indices such that $\sigma(j) = j+1 \mod g$, then for some fixed integer $m$, the optimal angles are of the form

$$\theta_j = \theta_1 + 2\pi m(j-1)/g \mod 2\pi. \tag{3}$$

The only free parameter left now is $m$, the spacing between adjacent points. Plugging Eq. (3) into the definition of $f$ we find

$$f(\sigma;\theta_1,\dots,\theta_g) = 2\sqrt{g}\left|\sin\left(\pi\left[m/g - \alpha\right]\right)\right|.$$

This is in turn minimized for $m = \lfloor g\alpha\rceil$, giving the stated spacing and minima. □

As this minimum of $f$ is independent of the permutation $\sigma$, we get an overall minimum for $f$ for free.

**Corollary 8.** *The minimum twisted commutator value (Definition 3) in the Frobenius norm $\Lambda^{(F)}_{g,\alpha}$ is lower bounded*

$$\Lambda^{(F)}_{g,\alpha} \geq 2\sqrt{g}\sin\left(\pi\left|\frac{\lfloor g\alpha\rceil - g\alpha}{g}\right|\right).$$

*Proof.* This result can be seen by recalling that the definition of $f$ in Eq. (1) gives that

$$\min_{u\in U(g)} d_F(u,\eta u) = \min_{\sigma,\{\theta_j\}_j} f(\sigma,\theta_1,\dots,\theta_g).$$

As Lemma 4.2 tells us that we can consider cyclic permutations without loss of generality, we can apply the minimum found in Lemma 4.3, giving

$$\min_{u\in U(g)} d_F(u,\eta u) = 2\sqrt{g}\sin\left(\pi\left|\frac{\lfloor g\alpha\rceil - g\alpha}{g}\right|\right).$$

Applying the Wielandt-Hoffman theorem (Lemma 4.1), we get that the above minimum spectral distance lower bounds the twisted commutator in the Frobenius norm, as required. □

[2] In saying there are two such terms we have assumed $g \geq 3$. If $g = 1$ the lemma is trivial ($f$ is constant), and if $g = 2$ then we have double counted in $f_j(\theta_j)$, but our analysis of its minimum remains valid.

## 4.2 Higher norms and tightness

With the above bound in hand, we now turn our attention to tightness. A canonical family of operators which exhibit twisted commutation is that of the *generalized Pauli operators*, also known as Sylvester's *clock and shift matrices*

$$C := \sum_j \omega^{j-1} |j\rangle\langle j|, \qquad\qquad S := \sum_j |j \oplus 1\rangle\langle j|$$

where $\omega = e^{2i\pi/g}$ is a primitive $g$th root of unity, and $\oplus$ denotes addition modulo $g$. As $S$ simply cyclically permutes the eigenbasis of $C$, we can see that $S^\dagger C S = \omega C$, or $[C, S]_{1/g} = 0$. By taking appropriate powers these operators can also yield pairs which twisted commute with a phase that is any power of $\omega$, specifically we see $\left[C, S^k\right]_{k/g} = 0$. Suppose we take such a pair and evaluate the twisted commutator at an arbitrary phase $\eta = e^{2i\pi\alpha}$. We then find,

$$\begin{aligned}
\left\| \left[C, S^k\right]_\alpha \right\|_F &= \left\| C S^k - \eta S^k C \right\|_F \\
&= \left\| (1 - \omega^k \eta) C S^k \right\|_F \\
&= \sqrt{g} \left| 1 - \omega^k \eta \right| \\
&= 2\sqrt{g} \left| \sin\!\left(\pi \left(\alpha + k/g\right)\right) \right|.
\end{aligned}$$

If we now take $k = -\lfloor g\alpha \rceil$, then we saturate Corollary 8, proving tightness of the bound on $\Lambda^{(F)}_{g,\alpha}$, allowing us to conclude

$$\Lambda^{(F)}_{g,\alpha} = 2\sqrt{g} \sin\left(\pi \left| \frac{\lfloor g\alpha \rceil - g\alpha}{g} \right| \right).$$

For the above optimizations we restricted ourself to the $p = 2$ case of the Frobenius norm. The nature of the minimizers found allows us to pull this analysis up into minima for the $p > 2$ Shatten norms as well.

**Theorem 5** (Minimum twisted commutation value)**.** *Suppose that $u$ and $v$ are $g$-dimensional unitaries, then*

$$\left\| [u, v]_\alpha \right\|_p \geq 2 g^{1/p} \sin\left(\pi \left| \frac{\lfloor g\alpha \rceil - g\alpha}{g} \right| \right).$$

*Moreover this bound is tight, in that sense that there exist $g$-dimensional unitaries which saturate the above bound and only depend on $\lfloor g\alpha \rceil$.*

*Proof.* By the equivalence of Shatten norms, the minimum Frobenius norm will also imply a lower bound for other $p$-norms as well. Specifically for $p \geq 2$ we have

$$\|M\|_p \geq g^{1/p - 1/2} \|M\|_F.$$

Applying these to the definition of $\Lambda^{(p)}_{g,\alpha}$, this bound gives that $\Lambda^{(p)}_{g,\alpha} \geq g^{1/p-1/2} \Lambda^{(F)}_{g,\alpha}$ for $p \geq 2$. It turns out that this inequality is saturated by matrices $M$ with flat spectra, i.e. those proportional to unitaries. It so happens that the clock and shift operators considered demonstrate tightness by having a twisted commutator with precisely this property, and therefore also saturate and demonstrate the tightness of the induced $p \geq 2$ bound. We therefore conclude that

$$\Lambda^{(p)}_{g,\alpha} = g^{1/p-1/2} \Lambda^{(F)}_{g,\alpha} = 2 g^{1/p} \sin\left(\pi \left| \frac{\lfloor g\alpha \rceil - g\alpha}{g} \right| \right).$$

□

# 5 Applications and open questions

We now discuss several avenues for improvements, generalizations, refinements, and applications of these ideas.

## 5.1 Local Hamiltonians

In this paper the only assumption we made about our Hamiltonian $H$ was the presence of a spectral gap. A natural additional structure to impose is that $H$ be a many-body Hamiltonian: decompose our Hilbert space into a tensor product of many smaller Hilbert spaces, and let our Hamiltonian take the form

$$H = \sum_k h_k$$

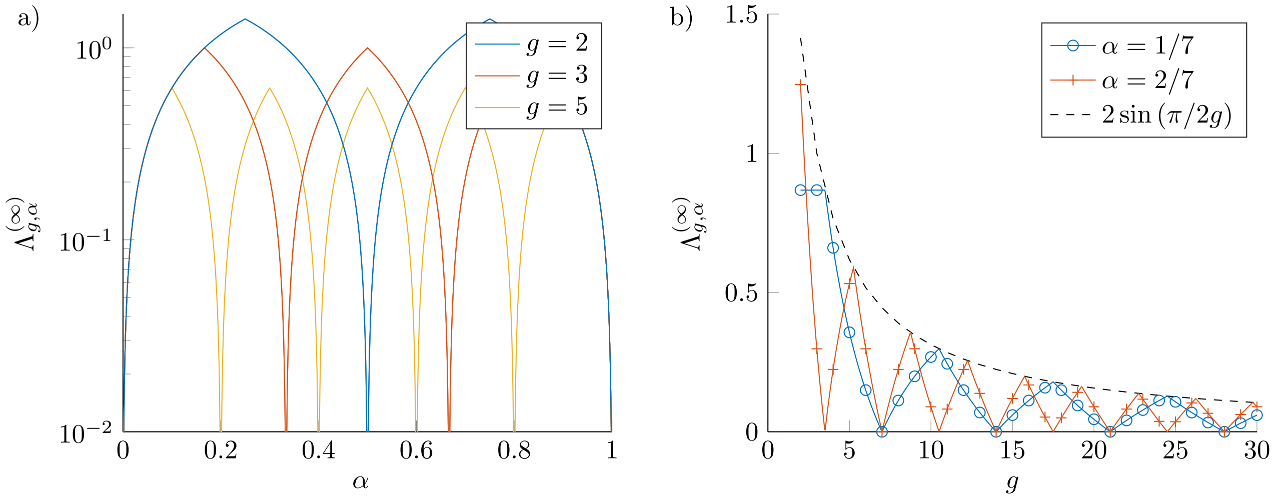


Figure 5: The twisted commutator value minimum (in the operator norm) $\Lambda^{(\infty)}_{g,\alpha}$. a) The dependence on the twisting parameter $\alpha$ for a few fixed dimensions $g$. The presence of roots at multiples of $1/g$ are those predicted by Theorem 1. b) Now fixing the twisting parameter $\alpha$, the dependence on the dimension $g$ is shown. Note that $g$ can only take integer values, indicated by the circles and pluses, with the continuous lines simply intended to guide the eye. The dotted black line indicates an $\alpha$-independent upper bound on $\Lambda^{(\infty)}_{g,\alpha}$ given by applying the bound $|x - \lfloor x \rceil| \leq 1/2$.

where each term $h_k$ acts non-trivially on a constant number of these tensor factor spaces. Additional to this we could also impose that the factors on which it acts are geometrically local as well. Under this special case it may be that either the bounds on degeneracy certification might be able to be improved, or we might be able to prove the existence of degeneracy witnesses with additional structure, e.g. such witnesses might act in a geometrically local fashion.

### 5.2 Topologically ordered systems

While the notions of approximate symmetry and degeneracy of a ground band are both robust to small perturbations, naïvely one can only consider perturbations of a strength no larger than the gap. For topologically ordered systems [16] however, we can afford much larger perturbations under certain locality assumptions.

Under the influence of local perturbations, the low-energy band structure, most notably the ground space degeneracy, is robust even if the overall strength of the perturbation is extensive [22]. Moreover, any symmetries which witnesses this degeneracy can be quasi-adiabatically continued [23] into approximate symmetries which witness the degeneracy of the ground band in the perturbed system. It is in this sense that the existence of degeneracy witnesses can be considered robust to even rather strong perturbations, at least for the ground band.

The family of *abelian quantum double models* possess symmetries supported on quasi-1D regions which satisfy twisted commutation relations related to the braid and fusion rules of the underlying anyons [24]. More general models such as non-abelian/twisted quantum doubles [24–26], and Levin-Wen string net models [27] are all believed to possess symmetries which satisfy more general commutation-like relations based on more general notions of commutation. One possible example is the twist product [28] which only commutes the two operators on part of the system, braiding them together.

$$\Big(\sum_i A_i \otimes A_i'\Big) \infty \Big(\sum_j B_j \otimes B_j'\Big) := \sum_{ij} A_i B_j \otimes B_j' A_i'.$$

An obvious extension of this work is to take various properties of these underlying systems implied by this commutation-like relations, and see if they too carry through into the regime of *approximate* relations.

In a recent paper, Bridgeman et. al. sought to classify the phases of 2D topologically ordered spin systems belonging to the same phase as abelian quantum doubles [20]. This was done by numerically optimizing twisted pairs of symmetries. This optimization was done over a tensor network [29, 30] ansatz of quasi-1D operators known as matrix product operators. For two operators $L$ and $R$, supported on intersecting quasi-1D regions, the cost function takes the form

$$C(L, R; \alpha) \propto \epsilon_L^2 + \epsilon_R^2 + \delta^2$$

where $\epsilon_L := \|[L, H]\|_F$, $\epsilon_R := \|[R, H]\|_F$, and $\delta = \|[L, R]_\alpha\|_F$.

Minimizing $C(L, R; \alpha)$ over $L$ and $R$ for a fixed $\alpha$, they found that in the abelian quantum doubles the minimizers were unitary, and that both $\epsilon_L$ and $\epsilon_R$ vanish to within numerical accuracy, leaving only the twisted

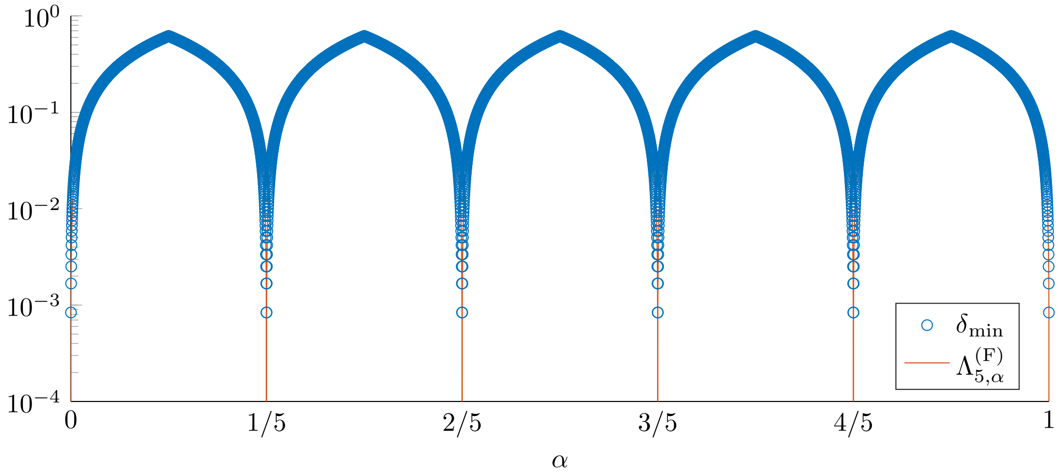


Figure 6: The twisted commutator value for ribbon operators on the $\mathbb{Z}_5$ quantum double model, as calculated using the algorithm of [20], as compared to the minimum possible twisted commutator value in 5-dimensions. Note that the difference between the two plots is no more than $3 \times 10^{-13}$.

commutator $\delta$. By observing the values of $\alpha$ for which the minimum cost is low, they hoped to classify the topological phases of the underlying Hamiltonian. By Theorem 2 we know that, at least to within numerical accuracy, the ribbon operators found restrict down to ground symmetries with the same twisted commutation relations. In Fig. 6 we compare, for the $\mathbb{Z}_5$ quantum double model, their numerically obtained values of this twisted commutator $\delta_{\min}$ with the minimal possible twisted commutator $\Lambda^{(F)}_{5,\alpha}$, showing close agreement and lending support to the efficacy of this numerical method.

### 5.3 Quantum codes

One class of systems for which twisted commuting symmetries play a special role are quantum codes, in which they can be interpreted as logical operators [31–33]. For a quantum code encoding $N$ codewords, the logical algebra must correspond to $\mathrm{Mat}_N(\mathbb{C})$, which necessarily contains a pair of operators $X$ and $Z$ such that $[X, Z]_{1/N} = 0$; indeed the algebra generated by any two such operators $X$ and $Z$ is itself $\mathrm{Mat}_N(\mathbb{C})$.

While the existence of logical operators for which $\alpha = 1/N$ twisted commute can be ensured, we might only see and expect operators with twisted commutations characteristic of smaller ground spaces. Though the logical algebra is given by $\mathrm{Mat}_N(\mathbb{C})$, this space often naturally decomposes into a tensor product decomposition: the logical qudits. By geometrically restricting where on the system the operators can act, we can often restrict which factors the logical operators have nontrivial commutation relations with. This is the case for celebrated examples such as the toric code [24]. This can be seen above in Fig. 6, where the logical operators are restricted to string-like regions that are only sensitive to one $\mathrm{Mat}_5(\mathbb{C})$ factor of the larger $\mathrm{Mat}_{25}(\mathbb{C})$ logical algebra; one of the two 5-level qudits. In the same way that Ref. [20] sought to use the existence of twisted commuting symmetries to classify topological phases, how this existence varies with respect to the geometry imposed on these operators might provide a tool to probe what portion of the logical algebra is accessible on certain regions.

In the language of quantum codes, our results can be interpreted as bounds below which approximate logical operators imply the existence of a certain number of code words. A possible avenue for future work is whether there exists bounds below which not only can the number of codestates be bounded, but reliable encoding, decoding, and error correction can all be performed with these approximate logical operators. Understanding when information stored in such states is approximately preserved, as opposed to exactly preserved [34], could have interesting applications in approximate quantum error correction.

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# A Approximate shared eigenvectors for almost commuting matrices

In this section we will show that for two approximately commuting matrices, an approximate shared eigenvector exists. This problem has been considered before by Bernstein [19], who showed the following result.

**Theorem 9** (Bernstein [19])**.** *Take $A$ and $B$ to be complex matrices of dimension $n \geq 2$. If $\|B\| \leq 1$, and for some $\delta > 0$ we have*

$$\|[A, B]\| \leq \frac{\delta^n(1-\delta)}{1-\delta^{n-1}},$$

*then for each eigenvalue $\lambda$ of $A$, there exists a $\mu$ and normalized $|x\rangle$ such that*

$$\|A|x\rangle - \lambda|x\rangle\|, \|B|x\rangle - \mu|x\rangle\| \leq \delta.$$

Notice above the required bound on the commutator for a given $\delta$ scales exponentially in $n$. Below we will improve this dimension scaling by adding the additional assumption that one of the matrices is normal, allowing us to bring this down to a $1/n^2$ dependence. First we will state the more general result, which only requires one of the matrices to be normal. After proving this, we will then give the some additional cases where the second matrix is either unitary or Hermitian and the output is required to satisfy some additional constraints.

Before we prove this, we will need two simple lemmas. Take $A$ and $B$ to be $n \times n$ matrices. Let $A$ be normal, with an eigenvalue decomposition $A = \sum_i \lambda_i |i\rangle\langle i|$. Given this, we will see that we can upper bound the off-diagonal terms of $B$ in this eigenbasis of $A$.

**Lemma A.1** (Off-diagonal bound)**.** *If* $\|[A,B]\| \leq \epsilon$ *then*

$$|\lambda_i - \lambda_j|\,|\langle i|B|j\rangle| \leq \epsilon.$$

*Proof.* Using the fact that the operator norm dominates any component of a matrix, we can simply evaluate the relevant component of the commutator:

$$\begin{aligned}\epsilon &\geq \big\|[A,B]\big\| \\ &\geq |\langle i|\,[A,B]\,|j\rangle| \\ &= |\langle i|\,[AB-BA]\,|j\rangle| \\ &= |\lambda_i - \lambda_j| \cdot |\langle i|B|j\rangle|\ .\end{aligned}$$

□

Next take $\lambda$ to be a specific eigenvalue of $A$. Let $I_0$ be the singleton set containing the index corresponding to $\lambda$, or all these indices if $\lambda$ is degenerate. Define $I_k$ to be all the indices whose eigenvalues are within some radius $r > 0$ in the complex plane (to be chosen later) of those in $I_{k-1}$, i.e.

$$I_k := \{i \,|\, \exists j \in I_{k-1} : |\lambda_i - \lambda_j| \leq r\}\,.$$

Clearly this sequence becomes fixed after at most $n$ terms, and so let $I := I_n$ be this fixed point. Intuitively $I$ can be thought of as the indices corresponding to eigenvalues which form a cluster around $\lambda$ where every eigenvalue in the cluster is linked to at least one other by a disk of radius $r$ in the complex plane.

By construction this set has two properties we require. First it is bounded away from any other index,

$$i \in I, j \notin I \implies |\lambda_i - \lambda_j| > r$$

Second, because all of the eigenvalues corresponding to elements in $I$ have nearby neighbors in $I$, this means that the diameter of the disk containing all of the eigenvalues in $I$ has a diameter bounded by at most $nr$,

$$i \in I \implies |\lambda_i - \lambda| \leq nr\,.$$

Next let $V$ be the space spanned by the eigenvectors whose indices lies in $I$,

$$V = \mathrm{Span}\,\{|i\rangle \,|\, i \in I\}\,.$$

Denote the orthogonal complement of $V$ by $\bar{V}$, and decompose both matrices into blocks by $V \oplus \bar{V}$ as

$$A = \begin{pmatrix} A_V & \\ & A_{\bar{V}} \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} B_{VV} & B_{V\bar{V}} \\ B_{\bar{V}V} & B_{\bar{V}\bar{V}} \end{pmatrix}.$$

**Lemma A.2.**

$$\|B_{\bar{V}V}\| < n\epsilon/2r$$

*Proof.* In the eigenbasis of $A$, the components of $B_{\bar{V}V}$ correspond to $\langle i|B|j\rangle$ for $i \notin I$, $j \in I$. By construction of $I$ we have that $|\lambda_i - \lambda_j| > r$, and so by Lemma A.1

$$|\langle i|B|j\rangle| \leq \frac{\epsilon}{|\lambda_i - \lambda_j|} < \frac{\epsilon}{r}.$$

This implies therefore that $\|B_{\bar{V}V}\|_{\max} < \epsilon/r$. Using the fact that the operator norm exceeds the max-norm by at most the square root of the number of elements, we get

$$\|B_{\bar{V}V}\| \leq \|B_{\bar{V}V}\|_{\max} \times \sqrt{\dim V \times \dim \bar{V}}.$$

Given that $\dim V + \dim \bar{V} = n$, we have that $\dim V \times \dim \bar{V} \leq n^2/4$, and so

$$\|B_{\bar{V}V}\| < n\epsilon/2r\,.$$

□

**Theorem A.1** (Fixed eigenvalue)**.** *Suppose that $A$ and $B$ are $n$-dimensional matrices, such that $A$ is normal and $\|[A,B]\| \leq \epsilon$. For a fixed $\lambda$ which is an eigenvalue of $A$, there exists a normalized $|u\rangle$ and $\mu$ such that*

$$\|A|u\rangle - \lambda|u\rangle\|\,, \|B|u\rangle - \mu|u\rangle\| \leq n\sqrt{\epsilon/2}.$$

*Proof.* Take $|u\rangle$ to be a right eigenvector of $B_{VV}$ (contained within $V$), of eigenvalue $\mu$. This then gives that the relevant errors with respect to $A$ and $B$ behave as:

$$\begin{aligned} \|A|u\rangle - \lambda|u\rangle\| &= \|A_V|u\rangle - \lambda|u\rangle\| \\ &= \|(A_V - \lambda)|u\rangle\| \\ &\leq \|A_V - \lambda\| \\ &\leq nr \end{aligned} \qquad \begin{aligned} \|B|u\rangle - \mu|u\rangle\| &= \|B_{VV}|u\rangle - \mu|u\rangle + B_{\bar{V}V}|u\rangle\| \\ &= \|B_{\bar{V}V}|u\rangle\| \\ &\leq \|B_{\bar{V}V}\| \\ &< n\epsilon/2r\,. \end{aligned}$$

Here the inequality for the $A$ matrix comes from the diameter bound on the disk containing the eigenvalues in $I$, and the inequality for the $B$ matrix comes from Lemma A.2.

Next we can pick an $r$ for which the maximum of these two terms is minimized, specifically $r = \sqrt{\epsilon/2}$. Substituting this into each bound gives the overall bound of $n\sqrt{\epsilon/2}$. ☐

In general $\mu$ is not an eigenvalue of $B$. We can impose that this value have unit norm in the unitary case, or impose it to be an eigenvalue of $B$ in the Hermitian case, at the additional cost of a factor of $\sqrt{2}$.

**Corollary 10** (Unitary case)**.** *Suppose that $A$ and $B$ are $n$-dimensional matrices, such that $A$ is normal, $B$ is unitary, and $\|[A,B]\| \leq \epsilon$. For a fixed $\lambda$ which is an eigenvalue of $A$, there exists a vector $|u\rangle$ and $\mu$ with $|\mu| = 1$, such that*

$$\|A|u\rangle - \lambda|u\rangle\|\,, \|B|u\rangle - \mu|u\rangle\| \leq n\sqrt{\epsilon}.$$

*Proof.* The proof is similar to above. Suppose we took $|u\rangle$ to be a right eigenvector of $B_{VV}$ with eigenvalue $\mu'$, which since $B$ is unitary implies that $|\mu'| \leq 1$. Let $\mu = \mu'/|\mu'|$ such that $|\mu - \mu'| = 1 - |\mu'|$. Requiring that $B$ is a unitary matrix implies

$$B_{VV}^\dagger B_{VV} + B_{\bar{V}V}^\dagger B_{\bar{V}V} = I_V\,.$$

As $|u\rangle$ is a $\mu'$-eigenvector of $B_{VV}$, we can upper bound the difference $|\mu - \mu'|$:

$$\begin{aligned} |\mu - \mu'| &= 1 - |\mu'| \\ &\leq 1 - |\mu'|^2 \\ &= \langle u|\left[I_V - B_{VV}^\dagger B_{VV}\right]|u\rangle \\ &\leq \left\|I_V - B_{VV}^\dagger B_{VV}\right\| \\ &= \left\|B_{\bar{V}V}^\dagger B_{\bar{V}V}\right\| \\ &= \left\|B_{\bar{V}V}\right\|^2 \\ &\leq \|B_{\bar{V}V}\| \\ &< n\epsilon/2r\,. \end{aligned}$$

The cost associated with $B$ now becomes

$$\begin{aligned} \|B|u\rangle - \mu|u\rangle\| &\leq \|B|u\rangle - \mu'|u\rangle\| + |\mu - \mu'| \\ &< n\epsilon/2r + n\epsilon/2r \\ &= n\epsilon/r\,. \end{aligned}$$

The bound of $\|A|u\rangle - \lambda|u\rangle\| \leq nr$ carries over unchanged. Once again minimaxing over $r$ ($r = \sqrt{\epsilon}$), we get the stated bound. ☐

**Corollary 11** (Hermitian case)**.** *Suppose that $A$ and $B$ are $n$-dimensional matrices, such that $A$ is normal, $B$ is Hermitian, and $\|[A,B]\| \leq \epsilon$. For a fixed $\lambda$ which is an eigenvalue of $A$, there exists a vector $|u\rangle$ and $\mu$ which is an eigenvalue of $B$, such that*

$$\|A|u\rangle - \lambda|u\rangle\|\,, \|B|u\rangle - \mu|u\rangle\| \leq n\sqrt{\epsilon}.$$

*Proof.* Once again, let $|u\rangle$ be an eigenvector of $B_{VV}$ of value $\mu'$. Next let $B'$ be the pinching of $B$ to $V \oplus \bar{V}$

$$B' = \begin{pmatrix} B_{VV} & 0 \\ 0 & B_{\bar{V}\bar{V}} \end{pmatrix}.$$

First we note that $B$ and $B'$ are close in the operator norm, as

$$\begin{aligned} \|B - B'\| &= \|B_{\bar{V}V} + B_{V\bar{V}}\| \\ &= \max\{\|B_{\bar{V}V}\|\,, \|B_{V\bar{V}}\|\} \\ &\leq n\epsilon/2r\,. \end{aligned}$$

By definition $\mu'$ was a eigenvalue of $B_{VV}$, and therefore of $B'$. Weyl's inequality [35] gives that there exists an eigenvalue $\mu$ of $B$ that is within the operator norm distance of $\mu'$, i.e.

$$|\mu - \mu'| \leq n\epsilon/2r.$$

As with the unitary case, this $|\mu - \mu'|$ also induces a factor of two in the cost associated with $B$, and therefore gives the same final errors bounds. $\square$

In all of the above analysis, we have considered $\lambda$ to be fixed. If instead we allow $\lambda$ to be *any* eigenvalue of $A$, then we can get a factor of 2 improvement in our bounds.

**Theorem A.2** (Free eigenvalue)**.** *Suppose that $A$ and $B$ are $n$-dimensional matrices, such that $A$ is normal and $\|[A, B]\| \leq \epsilon$. There exists a $\lambda$ which is an eigenvalue of $A$, and $|u\rangle$ and $\mu$ such that*

$$\|A|u\rangle - \lambda|u\rangle\|, \|B|u\rangle - \mu|u\rangle\| \leq n\sqrt{\epsilon}/2.$$

*Once again if $B$ is unitary and we require $\mu$ to have unit norm, or if $B$ is Hermitian and we require $\mu$ to be an eigenvalue, then the errors grow to*

$$\|A|u\rangle - \lambda|u\rangle\|, \|B|u\rangle - \mu|u\rangle\| \leq n\sqrt{\epsilon/2}.$$

*Proof.* The proof here is once again similar to Theorem A.1. Suppose we construct a set $I$ as before, from a cluster of eigenvalues with each element of $I$ within $r$ of at least one other element of $I$. The distance between any two eigenvalues with indices in $I$ is at most $nr$. There also exist eigenvalues with indices in $I$ such that all other such eigenvalues lie within $nr/2$ as follows. Consider the graph $G$ whose vertices are the points $\lambda_i \in I$ and where $(\lambda_i, \lambda_j)$ is an edge iff $|\lambda_i - \lambda_j| \leq r$. This graph had diameter less than $n$ (the maximum number of point in $I$). Consider a maximum-length path in this graph, and choose the vertex corresponding to a midpoint of this path. In the complex plane, by the triangle inequality the eigenvalue associated to this midpoint vertex is at most distance $nr/2$ away from the other eigenvalues.

Let $\lambda$ be such a midpoint eigenvalue. This choice allows us to improve the error bound on the $A$ approximate eigenvector by a factor of two,

$$\|A|u\rangle - \lambda|u\rangle\| \leq nr/2\,.$$

Here the analysis works the same as previously. This gives an overall factor of $\sqrt{2}$ improvement,

$$\|A|u\rangle - \lambda|u\rangle\|, \|B|u\rangle - \mu|u\rangle\| \leq n\sqrt{\epsilon}/2\,.$$

Using the same techniques as Corollaries 10 and 11 we can impose $\mu$ to be either unit norm or an eigenvalue respectively, growing the error associated with $B$ to

$$\|B|u\rangle - \mu|u\rangle\| \leq n\epsilon/r.$$

This factor of two carries through to a $\sqrt{2}$ in the final error, giving the stated final bound of $n\sqrt{\epsilon/2}$. $\square$

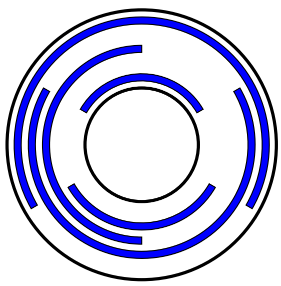
# B An algorithm for the certifiable degeneracy of a twisted pair

In this appendix we sketch how, for a given pair of parameters $\alpha$ and $\delta$, we can calculate the minimum possible dimension of unitaries $u$ and $v$ such that $\|[u, v]_\alpha\| \leq \delta$. Lemmas 3.1 and 3.2 give that for all $j \in \mathbb{Z}$, there exists an eigenvalue $e^{i\phi_j}$ of $u$ such that

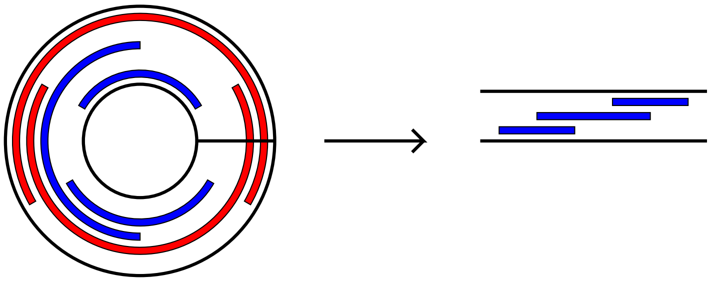
$$|\phi_j - 2\pi\alpha j| \leq \cos^{-1}(1 - |j|\,\delta).$$

The question now is to find the minimum number of eigenvalues such that at least one lies in each of the above arcs. This is known as the *transversal number*, and can be efficiently calculated by a greedy algorithm [36]. We now sketch this algorithm for the example parameters $\alpha = 1/4$ and $\delta = 1/2$ (indicated by the turquoise dot in Fig. 3).

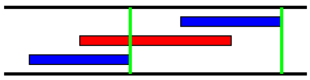
The first thing to note is that these arcs are trivial for $\delta\,|j| \geq 2$, in that they are the entire unit circle. For this reason we need only consider a finite number of arcs for $j = -\lfloor 2/\delta \rfloor, \ldots, \lfloor 2/\delta \rfloor$. In our case this corresponds $j = -3, \ldots, 3$. Below we have drawn these non-trivial arcs, omitting the trivial $j = 0$ arc.



Next we note that the $j = 0$ arc is simply a point, implying that $u$ must contain a $+1$ eigenvalue. Given this, any arc containing $+1$ can be thrown away (indicated in red below). Because of this, we can now unfold our arcs on a circle into intervals on a line.



We then take the intervals to be sorted by end-point. Considering each interval in order, we place an eigenvalue at the end of each interval as necessary, indicated as a green line below. Note that any interval which already contains an included eigenvalue when we arrive at it can be ignored, indicated by the red interval below.



Including the already found eigenvalue at $+1$, this gives us the minimum number of points necessary to satisfy each arc.