Polynomial-time degenerate ground state approximation of gapped 1D Hamiltonians

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Summary

We present a ground state approximation algorithm which extends known provably efficient methods[1, 2] to degenerate systems. For a 1D gapped and local spin system of constant degeneracy, our algorithm generates an orthonormal set of approximate ground states. For a system of n qudits with spectral gap ϵ, this algorithm yields states with ground space overlap at least $1 - \eta$ in run-time T where

$$T = \begin{cases} n^{\mathcal{O}(1/\epsilon)} & \text{for } 1/\eta = n^{\mathcal{O}(1)} \\ n^{O(1)} & \text{for } 1/\eta = n^{o(1)} \end{cases}$$

As such the gapped 1D LOCAL HAMILTONIAN problem remains in P in the presence of degeneracy.

Entanglement

States of arbitrary entanglement cannot be efficiently represented in general. The level of entanglement exhibited by approximate ground states of certain systems is however limited.

Background: Area Law

For a general state the entanglement entropy of any region can scale with the volume. For the ground state of a gapped and local systems however it is conjectured[3] instead to scale at most with the area (see Figure 1) of that region, known as an *area law*. This result has been proven for 1D systems[4], and was recently extended to degenerate systems[5], allowing for efficient state representation[6] in the form of Matrix Product States.
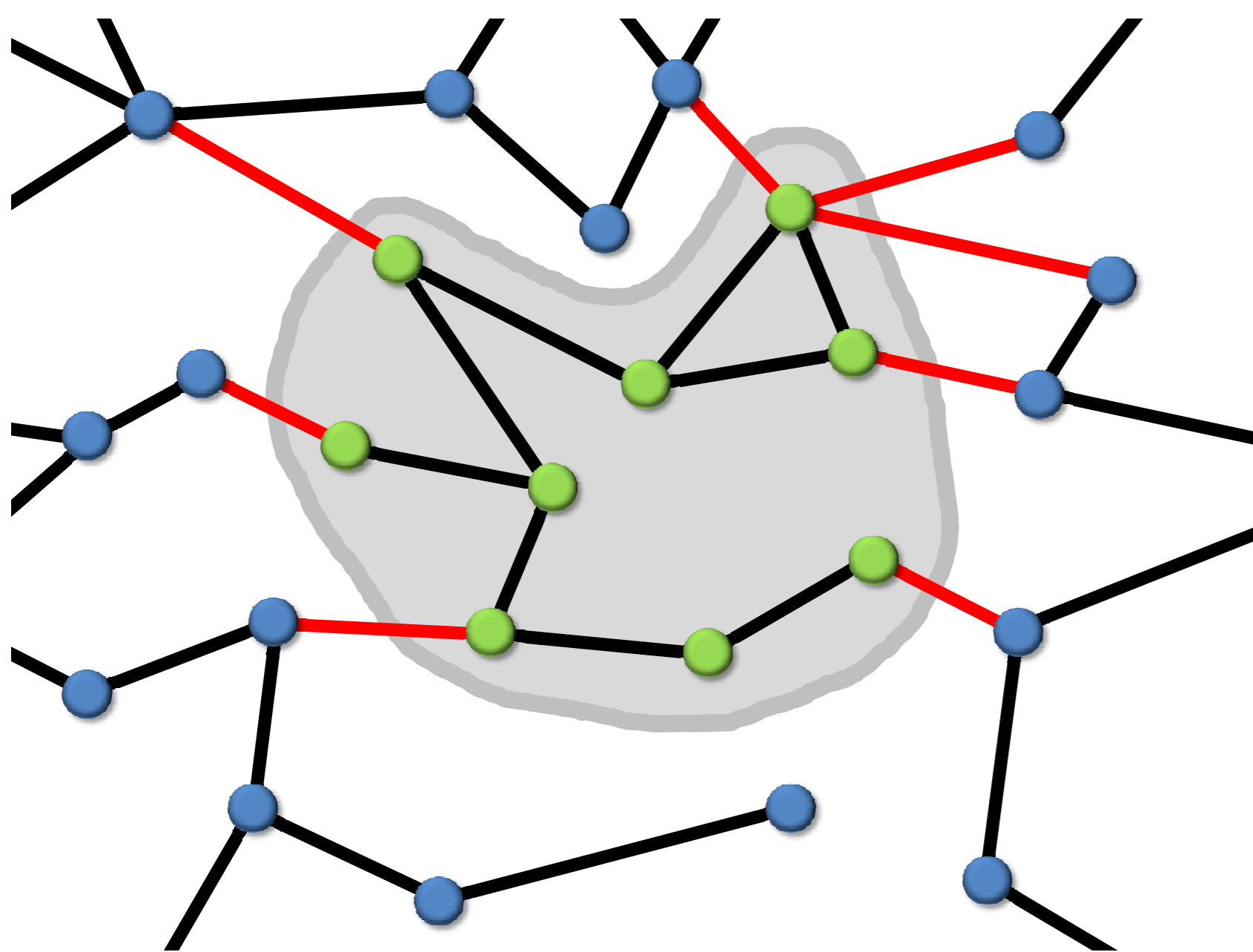

Figure 1: For a lattice region (grey), the volume is given by the contained vertices (green) and area by the intersecting edges (red).

Ground state approximation

A naïve method of calculating ground states is to solve the convex programs

$$\min \quad \mathrm{Tr}(H\sigma_i) \tag{1}$$
$$\text{where} \quad \langle\Gamma_1|\sigma_i|\Gamma_1\rangle = \cdots = \langle\Gamma_{i-1}|\sigma_i|\Gamma_{i-1}\rangle = 0, \tag{2}$$
$$\sigma_i \geq 0, \ \mathrm{Tr}\,\sigma_i = 1.$$

where $|\Gamma_i\rangle$ is the leading eigenvector of σ_i. As the Hilbert space is exponentially large this cannot be efficiently solved however. Instead we wish to find a small subspace – guaranteed to contain low-energy states – on which this program can be efficiently solved. Our algorithm constructs such a space in the form of an *n-viable set*.

Definition: Viable Set

A set S of states is i-viable if:
- States in S are defined on i qudits.
- There exists a set of orthonormal witness states with large ground space overlap, whose reduced state on the first i qudits is supported on $\mathrm{Span}(S)$.

Algorithm Sketch

We construct the desired n-viable set by inducting on i. Expanding a $(i-1)$-viable set to a i-viable set – of the same size and error – is done in three steps:

1) Extension: Tensor product the viable set with a basis on the ith qudit, causing the size to grow whilst the error is preserved.

2) Trimming: Use convex optimisations to remove locally high energy states, bringing the cardinality back down at some error cost.

3) Error Reduction: Using approximate ground state projectors, bring down the error at some size cost.

The most complicated step to generalise is trimming. The idea behind trimming is to discard locally high energy states through a series of convex programs; local versions of Program (1).
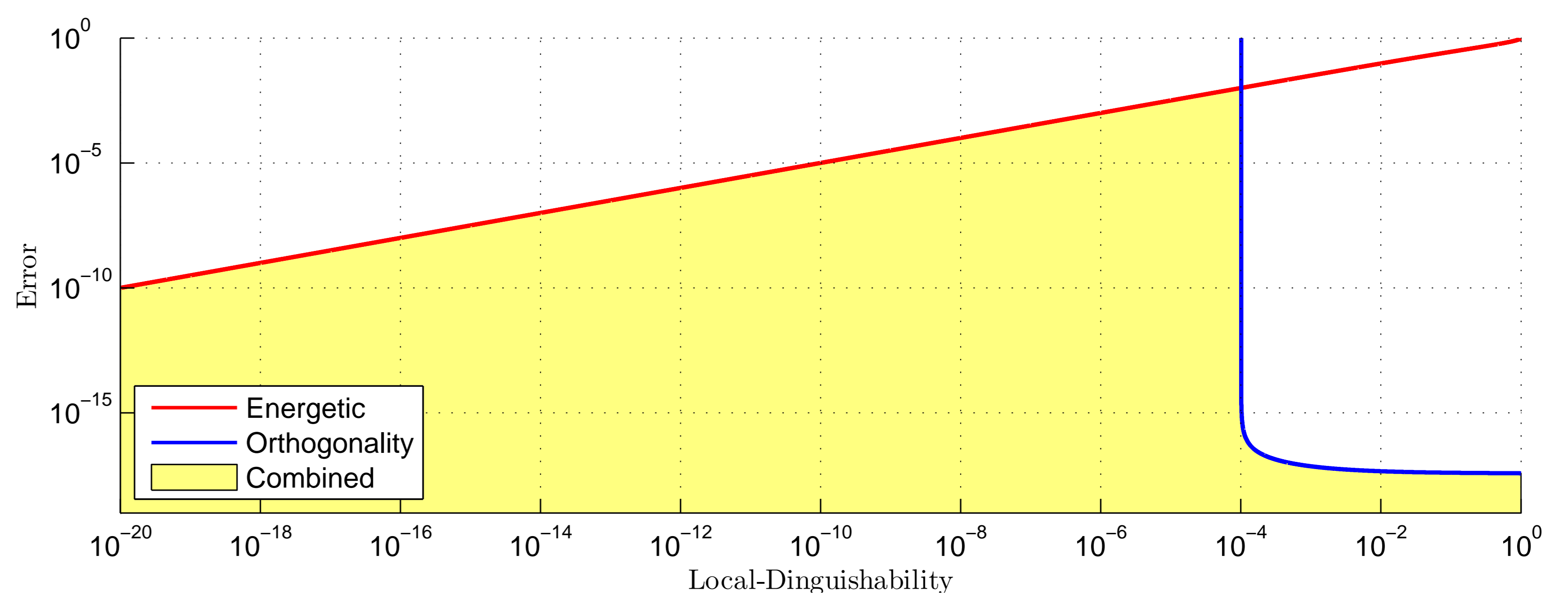

Figure 2: The error levels of the second witness from both the energetic and orthogonality trimming programs, (3) and (4) respectively.

Degenerate trimming

There are two main goals when trimming in the degenerate algorithm:

Goal 1: Keep the error of the witnesses low.
Goal 2: Ensure there exist a full set of witnesses.

Using *approximate decoupling* we can achieve the first goal by simply optimising the local part of the Hamiltonian H_L. Reliably meeting the second goal however requires a new technique.

Distinguishability

On the whole system generating a full set of ground states can be ensured by imposing the orthogonality condition Constraint (2), there is no local analogue of this however. Considering the viable sets which states can witness, there exist a natural notion of local-distinguishability (LD). Orthogonality can be considered a notion of global-distinguishability, so we might hope LD is a local analogue. Orthogonality however does not necessarily imply LD, an example in which orthogonal states manifest both extremes of LD is given below.

Examples: Local Distinguishability

Consider the two orthogonal 3-qubit states:
$$|\psi_1\rangle = |000\rangle \qquad |\psi_2\rangle = |010\rangle .$$
When reduced to the first qubit alone both states are identical (minimal LD)
$$\rho_1^{(1)} = \rho_2^{(1)} = |0\rangle\langle0| ,$$
however reducing first *two* qubits yields orthogonal states (maximal LD)
$$\rho_1^{(1,2)} = |00\rangle\langle00| \quad \perp \quad \rho_2^{(1,2)} = |01\rangle\langle01| .$$

Orthogonalisation Process

The solution to the problem of LD is a second convex program, which allows us to meet both goals. First we solve the original *energetic trimming program*:

$$\min \quad \mathrm{Tr}(H_L\sigma_1) \tag{3}$$
$$\text{where} \quad \sigma_1 \geq 0, \ \mathrm{Tr}\,\sigma_1 = 1.$$

This will allow us to meet Goal 1, giving one low energy witness. As for the second witness, required to meet Goal 2, we solve the *orthogonality trimming program*:

$$\min \quad \mathrm{Tr}(P\sigma_2) \tag{4}$$
$$\text{where} \quad \mathrm{Tr}(H_L\sigma_2) \leq \mathrm{Tr}(H_L\sigma_1) + \text{small error} ,$$
$$\sigma_2 \geq 0, \ \mathrm{Tr}\,\sigma_2 = 1.$$

where P is the projector onto the viable set produced by the energetic program (3). The idea is to construct a second witness, with an energy not much higher, and minimal overlap.

Result

In the low-LD limit it turns out that the energetic process actually does generate a second witness of low error, in the high-LD limit however it is the orthogonality program which does so (see Figure 2). By combining the solutions of both we can ensure that the error stays below a given threshold independent of the LD. This allows us to generalise the viable set construction to the degenerate case, and thus to approximate non-unique ground states efficiently.

References

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