Contents

1 Equations of Motion

1.1 Reference

Gravity force in inertial frame

$$\mathbf{g} = -g\mathbf{k}$$

Rotor force in body frame

$$\mathbf{F}_{R,B} = k\left(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4\right)\mathbf{k}$$

Rotor force in inertial frame

$$\mathbf{F}_R = \mathbf{R}(\mathbf{q})\mathbf{F}_{R,B}(\gamma)$$

Drag force in body frame

$$\mathbf{F}_{D,B}(\dot{\mathbf{x}}_B)$$

$$\dot{\mathbf{x}}_B = \mathbf{R}^{-1}\dot{\mathbf{x}}$$

Drag force in inertial frame

$$\mathbf{F}_D = \mathbf{R}(\mathbf{q})\mathbf{F}_{D,B}(\dot{\mathbf{x}}_B)$$

1.2 Translation

$$m\ddot{\mathbf{x}} = m\mathbf{g} + \mathbf{F}_R + \mathbf{F}_D$$

$$\ddot{\mathbf{x}} = \mathbf{g} + \frac{1}{m}\mathbf{F}_R + \frac{1}{m}\mathbf{F}_D$$

$$\ddot{\mathbf{x}} = \mathbf{g} + \frac{1}{m}\mathbf{R}\mathbf{F}_{R,B} + \frac{1}{m}\mathbf{R}\mathbf{F}_{D,B}$$

1.3 Rotation

$$\mathbf{I}\dot{\boldsymbol{\omega}} = \tau - \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega})$$

$$\dot{\boldsymbol{\omega}} = \mathbf{I}^{-1}\left(\boldsymbol{\tau} - \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega})\right)$$

$$I = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$

$$\boldsymbol{\tau}_B = \begin{bmatrix} Lk\left(\gamma_1 - \gamma_3\right) \\ Lk\left(\gamma_2 - \gamma_4\right) \\ b\left(\gamma_1 - \gamma_2 + \gamma_3 - \gamma_4\right) \end{bmatrix} = \begin{bmatrix} Lk & 0 & -Lk & 0 \\ 0 & Lk & 0 & -Lk \\ b & -b & b & -b \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{bmatrix}$$

1.3.1 Quaternion

$$\dot{\mathbf{q}} = \mathbf{A}_3\boldsymbol{\omega}$$

$$\ddot{\mathbf{q}} = \dot{\mathbf{A}}_3\boldsymbol{\omega} + \mathbf{A}_3\dot{\boldsymbol{\omega}}$$

$$\ddot{\mathbf{q}} = \dot{\mathbf{A}}_3\boldsymbol{\omega} + \mathbf{A}_3\mathbf{I}^{-1}\left(\boldsymbol{\tau} - \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega})\right)$$

1.3.2 Euler

$$\boldsymbol{\omega} = \begin{bmatrix} 1 & 0 & -s_{\boldsymbol{\theta}} \\ 0 & c_\phi & c_{\boldsymbol{\theta}} s_\phi \\ 0 & -s_\phi & c_{\boldsymbol{\theta}} s_\phi \end{bmatrix} \dot{\boldsymbol{\theta}} = \mathbf{A}_5^{-1}\dot{\boldsymbol{\theta}}$$

$$\dot{\boldsymbol{\theta}} = \mathbf{A}_5\boldsymbol{\omega}$$

$$\ddot{\boldsymbol{\theta}} = \dot{\mathbf{A}}_5\boldsymbol{\omega} + \mathbf{A}_5\dot{\boldsymbol{\omega}}$$

$$\ddot{\boldsymbol{\theta}} = \dot{\mathbf{A}}_5\boldsymbol{\omega} + \mathbf{A}_5\left(\mathbf{I}^{-1}\left(\boldsymbol{\tau} - \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega})\right)\right)$$

$$\ddot{\boldsymbol{\theta}} = \mathbf{S}^{-1}\left(\mathbf{I}^{-1}\left(\boldsymbol{\tau} - \boldsymbol{\omega} \times (I\boldsymbol{\omega})\right) - \dot{\mathbf{S}}\dot{\boldsymbol{\theta}}\right) \tag{1}$$

2 Control

2.1 Attitude

2.1.1 Quaternion

$$\mathbf{e}_1 = \mathbf{q}_{ref}\mathbf{q}^*$$

$$\mathbf{e}_2 = \boldsymbol{\omega}_{ref} - \boldsymbol{\omega}$$

2.2 Position

$$\mathbf{e}_5 = \mathbf{x}_{ref} - \mathbf{x}$$

$$\dot{\mathbf{e}}_5 = \dot{\mathbf{x}}_{ref} - \dot{\mathbf{x}}$$

$$\dot{\mathbf{e}}_5 = \dot{\mathbf{x}}_{ref} - \mathbf{v}$$

$$\mathbf{v}_{ref} = \mathbf{C}_5\mathbf{e}_5 + \dot{\mathbf{x}}_{ref} + \boldsymbol{\Lambda}_5\boldsymbol{\chi}_5$$

$$\dot{\mathbf{v}}_{ref} = \mathbf{C}_5\dot{\mathbf{e}}_5 + \ddot{\mathbf{x}}_{ref} + \boldsymbol{\Lambda}_5\mathbf{e}_5$$

$$\mathbf{e}_6 = \mathbf{v}_{ref} - \mathbf{v}$$

$$\mathbf{e}_6 = \mathbf{C}_5\mathbf{e}_5 + \dot{\mathbf{x}}_{ref} + \boldsymbol{\Lambda}_5\boldsymbol{\chi}_5 - \mathbf{v}$$

$$\dot{\mathbf{e}}_6 = \dot{\mathbf{v}}_{ref} - \dot{\mathbf{v}} = \dot{\mathbf{v}}_{ref} - \mathbf{a}$$

$$\dot{\mathbf{e}}_6 = \mathbf{C}_5\dot{\mathbf{e}}_5 + \ddot{\mathbf{x}}_{ref} + \boldsymbol{\Lambda}_5\mathbf{e}_5 - \ddot{\mathbf{x}}$$

$$\dot{\mathbf{x}}_{ref} = \mathbf{v}_{ref} - \mathbf{C}_5\mathbf{e}_5 - \boldsymbol{\Lambda}_5\boldsymbol{\chi}_5$$

$$\dot{\mathbf{x}}_{ref} = \mathbf{e}_6 + \mathbf{v} - \mathbf{C}_5\mathbf{e}_5 - \boldsymbol{\Lambda}_5\boldsymbol{\chi}_5$$

$$\dot{\mathbf{e}}_5 = \mathbf{e}_6 - \mathbf{C}_5\mathbf{e}_5 - \boldsymbol{\Lambda}_5\boldsymbol{\chi}_5$$

$$\dot{\mathbf{e}}_6 = \mathbf{f}_6(\mathbf{e}_5, \mathbf{e}_6)$$

$$\mathbf{f}_6(\mathbf{e}_5, \mathbf{e}_6) = \mathbf{C}_5\left(\mathbf{e}_6 - \mathbf{C}_5\mathbf{e}_5 - \boldsymbol{\Lambda}_5\boldsymbol{\chi}_5\right) + \ddot{\mathbf{x}}_{ref} + \boldsymbol{\Lambda}_5\mathbf{e}_5 - \ddot{\mathbf{x}}$$

$$\ddot{\mathbf{x}} = -\mathbf{f}_6(\mathbf{e}_5, \mathbf{e}_6) + \mathbf{C}_5\left(\mathbf{e}_6 - \mathbf{C}_5\mathbf{e}_5 - \mathbf{\Lambda}_5\boldsymbol{\chi}_5\right) + \ddot{\mathbf{x}}_{ref} + \mathbf{\Lambda}_5\mathbf{e}_5$$

$$\mathbf{F}_R = m\left(-\mathbf{C}_7\mathbf{e}_5 - \mathbf{C}_8\mathbf{e}_6 + \mathbf{C}_5\left(\mathbf{e}_6 - \mathbf{C}_5\mathbf{e}_5 - \mathbf{\Lambda}_5\boldsymbol{\chi}_5\right) + \ddot{\mathbf{x}}_{ref} + \mathbf{\Lambda}_5\mathbf{e}_5 - \mathbf{g} - \frac{1}{m}\mathbf{F}_D\right)$$

$$\mathbf{F}_R = m\left(\left(-\mathbf{C}_7 - \mathbf{C}_5^2 + \mathbf{\Lambda}_5\right)\mathbf{e}_5 + \left(\mathbf{C}_8 + \mathbf{C}_5\right)\mathbf{e}_6 - \mathbf{C}_5\mathbf{\Lambda}_5\boldsymbol{\chi}_5 + \ddot{\mathbf{x}}_{ref} - \mathbf{g} - \frac{1}{m}\mathbf{F}_D\right)$$

$$\begin{aligned}\mathbf{F}_R = m\,\Big(&\left(-\mathbf{C}_7 - \mathbf{C}_5^2 + \mathbf{\Lambda}_5\right)\mathbf{e}_5\\ &+\left(-\mathbf{C}_8 + \mathbf{C}_5\right)\left(\mathbf{C}_5\mathbf{e}_5 + \dot{\mathbf{x}}_{ref} + \mathbf{\Lambda}_5\boldsymbol{\chi}_5 - \mathbf{v}\right)\\ &- \mathbf{C}_5\mathbf{\Lambda}_5\boldsymbol{\chi}_5 + \ddot{\mathbf{x}}_{ref} - \mathbf{g} - \frac{1}{m}\mathbf{F}_D\Big)\end{aligned}$$

$$\begin{aligned}\frac{1}{m}\mathbf{F}_R = &\left(-\mathbf{C}_7 - \mathbf{C}_5^2 + \mathbf{\Lambda}_5 + \left(-\mathbf{C}_8 + \mathbf{C}_5\right)\mathbf{C}_5\right)\mathbf{x}_{ref}\\ &-\left(-\mathbf{C}_7 - \mathbf{C}_5^2 + \mathbf{\Lambda}_5 + \left(-\mathbf{C}_8 + \mathbf{C}_5\right)\mathbf{C}_5\right)\mathbf{x}\\ &+\left(-\mathbf{C}_8 + \mathbf{C}_5\right)\dot{\mathbf{x}}_{ref}\\ &+\left(-\mathbf{C}_8\right)\mathbf{\Lambda}_5\boldsymbol{\chi}_5\\ &-\left(-\mathbf{C}_8 + \mathbf{C}_5\right)\mathbf{v}\\ &+\ddot{\mathbf{x}}_{ref} - \mathbf{g} - \frac{1}{m}\mathbf{F}_D\end{aligned}$$

$$\begin{aligned}0 = &+\mathbf{C}_{11}\boldsymbol{\chi}_5\\ &+\mathbf{C}_9(\mathbf{x}_{ref} - \mathbf{x})\\ &+\mathbf{C}_{10}(\dot{\mathbf{x}}_{ref} - \mathbf{v})\\ &+\ddot{\mathbf{x}}_{ref} - \ddot{\mathbf{x}}\end{aligned}$$

2.3 Dynamics of Acceleration

Instead of assuming $\ddot{\mathbf{x}}$ is our control input, we can continue the above process. We guess that the result will be

$$\begin{aligned}0 = &+\mathbf{C}_9\boldsymbol{\chi}_5\\ &+\mathbf{C}_{10}(\mathbf{x}_{ref} - \mathbf{x})\\ &+\mathbf{C}_{11}(\dot{\mathbf{x}}_{ref} - \mathbf{v})\\ &+\mathbf{C}_{12}(\ddot{\mathbf{x}}_{ref} - \mathbf{a})\\ &+\dddot{\mathbf{x}}_{ref} - \dot{\mathbf{a}}\end{aligned}$$

$$\mathbf{a} = \mathbf{a}_g + \mathbf{a}_R$$

$$\dot{\mathbf{a}} = \dot{\mathbf{a}}_R$$

$$\mathbf{a}_R = \mathbf{q}^* \mathbf{a}_{RB} \mathbf{q}$$

In general, if q rotates from y to x frame

$$\mathbf{y} = \mathbf{q}^* \mathbf{x} \mathbf{q}$$

$$\dot{\mathbf{y}} = \dot{\mathbf{q}}^* \mathbf{x} \mathbf{q} + \mathbf{q}^* \dot{\mathbf{x}} \mathbf{q} + \mathbf{q}^* \mathbf{x} \dot{\mathbf{q}}$$

$$\dot{\mathbf{y}} = \mathbf{q}^* \boldsymbol{\omega}^* \mathbf{x} \mathbf{q} + \mathbf{q}^* \dot{\mathbf{x}} \mathbf{q} + \mathbf{q}^* \mathbf{x} \boldsymbol{\omega} \mathbf{q}$$

$$\dot{\mathbf{y}} = \mathbf{q}^* \left(\boldsymbol{\omega}^* \mathbf{x} + \dot{\mathbf{x}} + \mathbf{x} \boldsymbol{\omega} \right) \mathbf{q}$$

$$\mathbf{q} \dot{\mathbf{y}} \mathbf{q}^* = \boldsymbol{\omega}^* \mathbf{x} + \dot{\mathbf{x}} + \mathbf{x} \boldsymbol{\omega}$$

In our case

$$\mathbf{a}_{RB} = \begin{bmatrix} 0 \\ 0 \\ a_{RB} \\ 0 \end{bmatrix}$$

which leads to the simple result

$$\boldsymbol{\omega}^* \mathbf{x} + \dot{\mathbf{x}} + \mathbf{x} \boldsymbol{\omega} = \begin{bmatrix} 0 \\ -\omega_y a_{RB} \\ \omega_x a_{RB} \\ \dot{a}_{RB} \end{bmatrix}$$

Notice that ω_z does not appear because it has no effect on the translation of the vehicle. In order to solve for ω_x, ω_y, and a_{RB}, we need a relationship between a_{RB} and its derivative. For example, we could use forward differencing.

2.3.1 Transfer Function

$$\begin{aligned} \ddot{x} = \Big(& c_9 y - c_9 x + c_{10} \dot{y} - c_{10} \dot{x} \\ & + c_{11} \int y - c_{11} \int x \\ & + \ddot{y} - g - \frac{F_D}{m} \Big) \end{aligned}$$

$$
\begin{aligned}
s^2X - sx(0) - \dot{x}(0) = (\, & c_9Y - c_9X \\
& + c_{10}sY - c_{10}y(0) - c_{10}sX + c_{10}x(0) \\
& + c_{11}\frac{Y}{s} - c_{11}\int x \\
& + \ddot{y} - g - \frac{F_D}{m}\bigg)
\end{aligned}
$$