Proceedings of the 2006 IEEE/RSJ
International Conference on Intelligent Robots and Systems
October 9 - 15, 2006, Beijing, China

Backstepping Control for a Quadrotor Helicopter

Tarek Madani and Abdelaziz Benallegue

Laboratoire d'Ingénierie des Systèmes de Versailles
10-12, avenue de l'Europe, 78140 Vélizy - FRANCE

E-mail: {madani and benalleg}@lisv.uvsq.fr

***Abstract*— This paper presents a nonlinear dynamic model for a quadrotor helicopter in a form suited for backstepping control design. Due to the under-actuated property of quadrotor helicopter, the controller can set the helicopter track three *Cartesian* positions (x, y, z) and the yaw angle to their desired values and stabilize the pitch and roll angles. The system has been presented into three interconnected subsystems. The first one representing the under-actuated subsystem, gives the dynamic relation of the horizontal positions (x, y) with the pitch and roll angles. The second fully-actuated subsystem gives the dynamics of the vertical position z and the yaw angle. The last subsystem gives the dynamics of the propeller forces. A backstepping control is presented to stabilize the whole system. The design methodology is based on the *Lyapunov* stability theory. Various simulations of the model show that the control law stabilizes a quadrotor with good tracking.**

I. INTRODUCTION

Unmanned Aerial Vehicles (UAV) have unmatched qualities that make them the only effective solution in specialized tasks where risks to pilots are high, where beyond normal human endurance is required, or where human presence is not necessary. UAVs are being used more and more in civilian applications such as traffic monitoring, recognition and surveillance vehicles, search and rescue operations [1]. They are highly capable to flown without an on-board pilot. These robotic aircrafts are often computerized and fully autonomous.

The first reported quadrotor helicopter (*Gyroplane No.1*) was built in 1907 by *the Breguet Brothers*. However, there was no means of control provided to the pilot other than a throttle for the engine to change the rotor speed, and the stability of the machine was found to be very poor. The machine was subsequently tethered so that it could move only vertically upward.

The automatic control of a quadrotor helicopter has attracted the attention of many researches in the past few years [2][3][4][5]. Generally, the control strategies are based on simplified models which have both a minimum number of states and minimum number of inputs. These reduced models should retain the main features that must be considered when designing control laws for real aerial vehicles.

In this work, we present the model of a quadrotor helicopter whose dynamical model is obtained via *Newton's* laws. A backstepping control strategy is proposed having in mind that the quadrotor can be seen as the interconnected of three subsystems: under-actuated subsystem, fully-actuated subsystem and propeller subsystem. The output of the under-actuated subsystem are the x and y motion. In order to control them, tilt angles (pitch and roll) need to be controlled. It appeared judicious to us to use a backstepping technique to solve this problem. The idea is to stabilize the system in two phases. Firstly, the positions (x, y) are controlled by a virtual control based on the tilt angles. Secondly, the tilt angles are controlled by varying the rotor speeds, thereby changing the lift forces. The backstepping technique will be used to control the z motion and the yaw angle of the fully-actuated subsystem.

The paper is organized as follows: in section II, a dynamic model for a quadrotor helicopter is developed. Based on this nonlinear model, we design in section III a backstepping control law. In section IV, some simulations are carried out to show the performance and stability of the proposed controller. Finally, in section V, our conclusions are presented.

II. DYNAMIC MODELING OF A QUADROTOR HELICOPTER

The quadrotor helicopter is shown in figure 1. The two pairs of rotors $(1, 3)$ and $(2, 4)$ turn in opposite direction in order to balance the moments and produce yaw motions as needed. On varying the rotor speeds altogether with the same quantity the lift forces will change affecting in this case the altitude z of the system. Yaw angle is obtained by speeding up the clockwise motors or slowing down depending on the desired angle direction. The motion direction according (x, y) axes depends on the sense of tilt angles (pitch and roll) whether they are positive or negative.

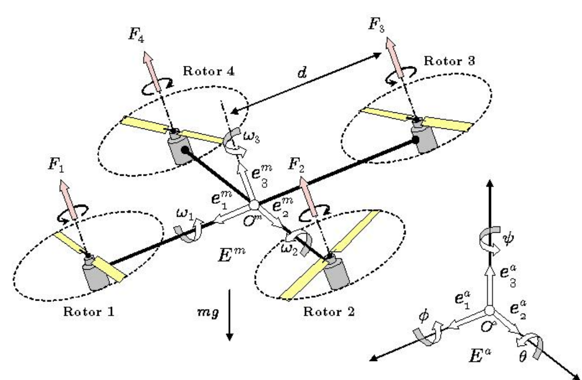

Fig. 1. Quadrotor helicopter

The equations describing the attitude and position of a quadrotor helicopter are basically those of a rotating rigid body with six degrees of freedom [6] [7]. They may be separated into kinematic equations and dynamic equations [8].

Let be two main reference frames (see Fig. 1):

- the earth fixed inertial reference frame E^a : $(O^a, \overrightarrow{e_1^a}, \overrightarrow{e_2^a}, \overrightarrow{e_3^a})$.
- the body fixed reference frame E^m : $(O^m, \overrightarrow{e_1^m}, \overrightarrow{e_2^m}, \overrightarrow{e_3^m})$ regidly attached to the aircraft.

Let the vector $\zeta \triangleq [x, y, z]^T$ and $\eta \triangleq [\phi, \theta, \psi]^T$ denote respectively *the altitude positions* and *the attitude angles* of the quadrotor (frame E^m) in the frame E^a relative to a fixed origin O^a. The attitude angles $\{\phi, \theta, \psi\}$ are respectively called pitch angle $(-\frac{\pi}{2} < \phi < \frac{\pi}{2})$, roll angle $(-\frac{\pi}{2} < \theta < \frac{\pi}{2})$ and yaw angle $(-\pi \leq \psi < \pi)$.

The quadrotor is restricted with the six degrees of freedom according to the reference frame E^m: Three translation velocities $V = [V_1, V_2, V_3]^T$ and three rotation velocities $\Omega = [\Omega_1, \Omega_2, \Omega_3]^T$. The relation existing between the velocities vectors (V, Ω) and $(\dot{\zeta}, \dot{\eta})$ are:

$$\left\{ \begin{array}{l} \dot{\zeta} = R_t V \\ \Omega = R_r \dot{\eta} \end{array} \right. \tag{1}$$

where R_t and R_r are respectively the transformation velocity matrix and the rotation velocity matrix between E^a and E^m such as:

$$R_t = \begin{bmatrix} C_\phi C_\psi & S_\phi S_\theta C_\psi - C_\phi S_\psi & C_\phi S_\theta C_\psi + S_\phi S_\psi \\ C_\theta S_\psi & S_\phi S_\theta S_\psi + C_\phi C_\psi & C_\phi S_\theta S_\psi - S_\phi C_\psi \\ -S_\phi & S_\phi C_\theta & C_\phi C_\theta \end{bmatrix} \tag{2}$$

and

$$R_r = \begin{bmatrix} 1 & 0 & -S_\theta \\ 0 & C_\phi & C_\theta S_\phi \\ 0 & -S_\phi & C_\phi C_\theta \end{bmatrix} \tag{3}$$

where $S_{(.)}$ and $C_{(.)}$ are the respective abbreviations of $\sin(.)$ and $\cos(.)$.

One can write $\dot{R}_t = R_t S(\Omega)$ where $S(\Omega)$ denotes the skew-symmetric matrix such that $S(\Omega)v = \Omega \times v$ for the vector cross-product $\times$ and any vector $v \in \mathbb{R}^3$. In other words, for a given vector Ω,the skew-symmetric matrix $S(\Omega)$ is defined as follows:

$$S(\Omega) = \begin{bmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{bmatrix} \tag{4}$$

The derivation of (1) with respect to time gives

$$\left\{ \begin{array}{l} \ddot{\zeta} = R_t \dot{V} + \dot{R}_t V = R_t \dot{V} + R_t S(\Omega) V = R_t(\dot{V} + \Omega \times V) \\ \dot{\Omega} = R_r \ddot{\eta} + \left(\frac{\partial R_r}{\partial \phi} \dot{\phi} + \frac{\partial R_r}{\partial \theta} \dot{\theta} \right) \dot{\eta} \end{array} \right. \tag{5}$$

Using the *Newton's* laws in the reference frame E^m, about the quadrotor helicopter subjected to forces $\sum F_{ext}$ and moments $\sum T_{ext}$ applied to the epicenter, one obtains the dynamic equation motions:

$$\left\{ \begin{array}{l} \sum F_{ext} = m\dot{V} + \Omega \times (mV) \\ \sum T_{ext} = I_T \dot{\Omega} + \Omega \times (I_T \Omega) \end{array} \right. \tag{6}$$

where m and $I_T = diag[I_x, I_y, I_z]$ are respectively the mass and the total inertia matrix of helicopter, $\sum F_{ext}$ and $\sum T_{ext}$ includes the external forces/torques developed in the epicenter of a quadrotor according to the direction of the reference frame E^m, such as:

$$\left\{ \begin{array}{l} \sum F_{ext} = F - F_{aero} - F_{grav} \\ \sum T_{ext} = T - T_{aero} \end{array} \right. \tag{7}$$

where the forces $\{F, F_{aero}, F_{grav}\}$ and the torques $\{T, T_{aero}\}$ are explained in the table I where $G = [0, 0, g]^T$ is the gravity vector $(g = 9.81 m.s^{-2})$, $\{K_t, K_r\}$ are two diagonal aerodynamic friction matrices.

Model	**Source**
$F = [0, 0, F_3]^T$ $T = [T_1, T_2, T_3]^T$	propeller system
$F_{aero} = K_t V$ $T_{aero} = K_r \Omega$	aerodynamic friction
$F_{grav} = m R_t^T G$	gravity effect

TABLE I

MAIN PHYSICAL EFFECTS ACTING ON A QUADROTOR

The forces F and torques T produced by the propeller system of a quadrotor are:

$$F = \begin{bmatrix} 0 \\ 0 \\ \sum_{i=1}^4 F_i \end{bmatrix}, \ T = \begin{bmatrix} d\ (F_2 - F_4) \\ d\ (F_3 - F_1) \\ c \sum_{i=1}^4 (-1)^{i+1} F_i \end{bmatrix} \tag{8}$$

where d is the distance from the epicenter of a quadrotor to the rotor axes and $c > 0$ is the drag factor.

Using (5), (6) and (7) allows to give the equation of the dynamics of rotation of the quadrotor expressed in the reference frame E^a :

$$\left\{ \begin{array}{ll} F = m R_t^T \ddot{\zeta} + K_t R_t^T \dot{\zeta} + m R_t^T G & \\ T = & I_T R_r \ddot{\eta} + I_T \left(\frac{\partial R_r}{\partial \phi} \dot{\phi} + \frac{\partial R_r}{\partial \phi} \dot{\theta} \right) \dot{\eta} \\ & + K_r R_r \dot{\eta} + (R_r \dot{\eta}) \times (I_T R_r \dot{\eta}) \end{array} \right. \tag{9}$$

The dynamic model (9) of the quadrotor helicopter has six outputs $\{x, y, z, \phi, \theta, \psi\}$ while it has only four independent inputs. Therefore the quadrotor is an under-actuated system. We are not able to control all of the states at the same time. A possible combination of controlled outputs can be $\{x, y, z, \psi\}$ in order to track the desired positions, more to an arbitrary heading and stabilize the other two angles, which introduces stable zero dynamics into the system [5]. A good controller should be able to reach a desired position and a desired yaw angle while guaranteeing stability of the pitch and roll angles.

III. BACKSTEPPING CONTROL DESIGN

The main objective of this section is to design a backstepping controller of a quadrotor helicopter ensuring that the position $\{x(t), y(t), z(t), \psi(t)\}$ tracks the desired trajectory $\{x_d(t), y_d(t), z_d(t), \psi_d(t)\}$ asymptotically. This can be done in two phases:

- The dynamic model is written in an appropriate form. Indeed, it is divided into three subsystems: an under-actuated subsystem, a fully-actuated subsystem and a propeller subsystem.
- Synthesis of the control law via backstepping technique in seven steps.

Let $u = [\dot{F}_1, \dot{F}_2, \dot{F}_3, \dot{F}_4]^T$ the control input. The dynamics equations (9) can be rewritten in a state-space form according to the following state vectors:

$$x_1 = \begin{bmatrix} x \\ y \end{bmatrix}, x_3 = \begin{bmatrix} \phi \\ \theta \end{bmatrix}, x_5 = \begin{bmatrix} \psi \\ z \end{bmatrix}, \quad x_7 = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix}.$$
$$x_2 = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}, x_4 = \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \end{bmatrix}, x_6 = \begin{bmatrix} \dot{\psi} \\ \dot{z} \end{bmatrix}, \tag{10}$$

We obtain the stat-space equations of an under-actuated subsystem S_1, a fully-actuated subsystem S_2 and a propeller subsystem S_3:

$$\begin{aligned} S_1 &: \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = f_0(x_2, x_3, x_5, x_6) + g_0(x_5, x_7)\varphi_0(x_3) \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = f_1(x_3, x_4, x_6, x_7) + g_1(x_3)\varphi_1(x_7) \end{cases} \\ S_2 &: \begin{cases} \dot{x}_5 = x_6 \\ \dot{x}_6 = f_2(x_3, x_4, x_6, x_7) + g_2(x_3)\varphi_2(x_7) \end{cases} \\ S_3 &: \{ \ \dot{x}_7 = u \end{aligned} \tag{11}$$

where the matrices $g_i \ (i = 0, 1, 2)$ are[1]

$$g_0 = \frac{\sum_{i=1}^4 F_i}{m} \begin{bmatrix} S_\psi & C_\psi \\ -C_\psi & S_\psi \end{bmatrix}, \ g_1 = \begin{bmatrix} \frac{1}{I_x} & \frac{1}{I_y} S_\phi T_\theta \\ 0 & \frac{1}{I_y} C_\phi \end{bmatrix}$$
$$g_2 = \begin{bmatrix} \frac{1}{I_z} C_\phi Se_\theta & 0 \\ 0 & \frac{1}{m} C_\phi C_\theta \end{bmatrix} \tag{12}$$

the vectors $\varphi_i \ (i = 0, 1, 2)$ are

$$\varphi_0 = \begin{bmatrix} S_\phi \\ C_\phi S_\theta \end{bmatrix}, \ \varphi_1 = \begin{bmatrix} d(F_2 - F_4) \\ d(F_3 - F_1) \end{bmatrix}$$
$$\varphi_2 = \begin{bmatrix} c(F_1 - F_2 + F_3 - F_4) \\ F_1 + F_2 + F_3 + F_4 \end{bmatrix} \tag{13}$$

and the vectors $f_i \ (i = 0, 1, 2)$ are

$$f_0 = \begin{bmatrix} f_x \\ f_y \end{bmatrix}, \ f_1 = \begin{bmatrix} f_\phi \\ f_\theta \end{bmatrix}, \ f_2 = \begin{bmatrix} f_\psi \\ f_z \end{bmatrix} \tag{14}$$

with

$$\begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = -\frac{1}{m} R_t K_t R_t^T \dot{\zeta} - G \tag{15}$$
$$\begin{aligned} \begin{bmatrix} f_\phi \\ f_\theta \\ f_\psi \end{bmatrix} = & -(I_T R_r)^{-1} [\ I_T (\tfrac{\partial R_r}{\partial \phi} \dot{\phi} + \tfrac{\partial R_r}{\partial \phi} \dot{\theta}) \dot{\eta} \\ & -K_r R_r \dot{\eta} - (R_r \dot{\eta}) \times (I_T R_r \dot{\eta})] \\ & + \begin{bmatrix} \frac{c}{I_z} C_\phi T_\theta \sum_{i=1}^4 (-1)^{i+1} F_i \\ -\frac{c}{I_z} S_\phi \sum_{i=1}^4 (-1)^{i+1} F_i \\ \frac{d}{I_y} S_\phi Se_\theta (F_3 - F_1) \end{bmatrix} \end{aligned}$$

[1] T_θ and $Se_{(.)}$ are respectively the abbreviations of $\tan(.)$ and $\frac{1}{\cos(.)}$

Using the backstepping approach [9], one can synthesize the control law forcing the subsystems S_1, S_2 and S_3 to follow the desired trajectories. In this purpose, we will build the control law by the seven following steps:

- **Step 1:** For the first step we consider the virtual system

$$\dot{x}_1 = v_1 \tag{16}$$

Let the first tracking error

$$z_1 = x_{1d} - x_1 \tag{17}$$

and considering the *Lyapunov* function positive definite

$$V_1 = \frac{1}{2} z_1^T z_1 \tag{18}$$

Its time derivative is

$$\dot{V}_1 = z_1^T \dot{z}_1 = z_1^T (\dot{x}_{1d} - \dot{x}_1) = z_1^T (\dot{x}_{1d} - v_1) \tag{19}$$

The stabilization of z_1 can be obtained by introducing a first virtual control input v_1:

$$v_1 = A_1 z_1 + \dot{x}_{1d} \tag{20}$$

with $A_1 \in \mathbb{R}^{2\times 2}$ is a positive definite matrix.

The equation (19) is then $\dot{V}_1 = -z_1^T A_1 z_1 < 0$.

Step 2: For the second step we consider the following new virtual system

$$\dot{x}_2 = f_0(x_2, x_3, x_5, x_6) + g_0(x_5, x_7) v_2 \tag{21}$$

where v_2 is a second virtual control input.

Let us proceed to variable change by making

$$z_2 = v_1 - x_2 = A_1 z_1 + \underbrace{\dot{x}_{1d} - \dot{x}_1}_{\dot{z}_1} \tag{22}$$

hence $\dot{z}_1 = -A_1 z_1 + z_2$.

For the second step we consider the augmented *Lyapunov* function:

$$V_2 = \frac{1}{2} \sum\nolimits_{i=1}^2 z_i^T z_i \tag{23}$$

The time derivative of (23) is

$$\begin{aligned} \dot{V}_2 &= z_1^T \dot{z}_1 + z_2^T \dot{z}_2 = z_1^T (-A_1 z_1 + z_2) + z_2^T (\dot{v}_1 - \dot{x}_2) \\ &= -z_1^T A_1 z_1 + z_2^T (z_1 + \dot{v}_1 - f_0 - g_0 v_2) \end{aligned} \tag{24}$$

The stabilisation of z_2 can be obtained by introducing the following augmented virtual control

$$v_2 = g_0^{-1} (z_1 + A_2 z_2 + \dot{v}_1 - f_0) \tag{25}$$

with $A_2 \in \mathbb{R}^{2\times 2}$ is a positive definite matrix.

Remark 1: It is worthy noting that the determinant of the matrix g_0 is $\left(\frac{1}{m} \sum_{i=1}^4 F_i\right)^2 > 0$, if $\sum_{i=1}^4 F_i \neq 0$. Therefore, the matrix g_0 is nonsingular in general operation condition because $\sum_{i=1}^4 F_i$ represents the total thrust on the body in the z-axis and is generally nonzero to overcome the gravity.

By using the equation (24) is comes $\dot{V}_2 = -\sum_{i=1}^2 z_i^T A_i z_i < 0$.

- **Step 3:** For the third step we consider the virtual system

$$\dot{x}_3 = v_3 \tag{26}$$

Let

$$z_3 = v_2 - \varphi_0(x_3) = g_0^{-1}(z_1 + A_2 z_2 + \dot{v}_1 - f_0) - \varphi_0(x_3) \quad (27)$$

then

$$g_0 z_3 = z_1 + A_2 z_2 + \underbrace{\dot{v}_1 - g_0\varphi_0}_{\dot{z}_2}$$

hence $\dot{z}_2 = -z_1 - A_2 z_2 + g_0 z_3$.

For the third step we consider the *Lyapunov* function:

$$V_3 = \frac{1}{2}\sum\nolimits_{i=1}^{3} z_i^T z_i \quad (28)$$

The derivatives with respect to time of (28) is

$$\begin{aligned} \dot{V}_3 &= z_1^T \dot{z}_1 + z_2^T \dot{z}_2 + z_3^T \dot{z}_3 \quad (29)\\ &= z_1^T(-A_1 z_1 + z_2) + z_2^T(-z_1 - A_2 z_2 + g_0 z_3) \\ &\quad + z_3^T(\dot{v}_2 - \dot{\varphi}_0) \\ &= -z_1^T A_1 z_1 - z_2^T A_2 z_2 + z_3^T(g_0^T z_2 + \dot{v}_2 - J_0 v_3) \end{aligned}$$

where J_0 is the *Jacobian* matrix of φ_0 such as:

$$J_0 = \frac{\partial \varphi_0(x_3)}{\partial x_3} = \begin{bmatrix} C_\phi & 0 \\ -S_\phi S_\theta & C_\phi C_\theta \end{bmatrix} \quad (30)$$

Remark 2: It should be noted that the determinant of the jacobian matrix $J_0(x_3)$ is $C_\phi^2 C_\theta$. Therefore, $J_0(x_3)$ is nonsingular when $(-\frac{\pi}{2} < \phi < \frac{\pi}{2})$ and $(-\frac{\pi}{2} < \theta < \frac{\pi}{2})$, which is satisfied generally.

The stabilization of z_3 can be obtained by introducing a virtual control:

$$v_3 = J_0^{-1}(g_0^T z_2 + A_3 z_3 + \dot{v}_2) \quad (31)$$

with $A_3 \in \mathbb{R}^{2\times 2}$ is a positive definite matrix.

Using (29), we get $\dot{V}_3 = -\sum_{i=1}^{3} z_i^T A_i z_i < 0$.

- **Step 4:** we consider the final virtual system of an under-actuated subsystem S_1:

$$\dot{x}_4 = f_1(x_3, x_4, x_6, x_7) + g_1(x_3) v_4 \quad (32)$$

Putting

$$z_4 = v_3 - x_4 = J_0^{-1}(g_0^T z_2 + A_3 z_3 + \dot{v}_2) - x_4 \quad (33)$$

we get

$$J_0 z_4 = g_0^T z_2 + A_3 z_3 + \underbrace{\dot{v}_2 - J_0 x_4}_{\dot{z}_3} \quad (34)$$

hence $\dot{z}_3 = -g_0^T z_2 - A_3 z_3 + J_0 z_4$.

The global *Lyapunov* function of a subsystem S_1 is

$$V_4 = \frac{1}{2}\sum\nolimits_{i=1}^{4} z_i^T z_i \quad (35)$$

Its time derivative is

$$\begin{aligned} \dot{V}_4 &= z_1^T \dot{z}_1 + z_2^T \dot{z}_2 + z_3^T \dot{z}_3 + z_4^T \dot{z}_4 \quad (36)\\ &= z_1^T(-A_1 z_1 + z_2) + z_2^T(-z_1 - A_2 z_2 + g_0 z_3) \\ &\quad + z_3^T(-g_0^T z_2 - A_3 z_3 + J_0 z_4) + z_4^T(\dot{v}_3 - \dot{x}_4) \\ &= -z_1^T A_1 z_1 - z_2^T A_2 z_2 - z_3^T A_3 z_3 \\ &\quad + z_4^T(J_0^T z_3 + \dot{v}_3 - f_1 - g_1 v_4) \end{aligned}$$

The stabilization of subsystem S_1 can be obtained by introducing a following virtual control law:

$$v_4 = g_1^{-1}(J_0^T z_3 + A_4 z_4 + \dot{v}_3 - f_1) \quad (37)$$

with $A_4 \in \mathbb{R}^{2\times 2}$ is a positive definite matrix.

Remark 3: Since the condition $(-\frac{\pi}{2} < \phi < \frac{\pi}{2})$ is generally satisfied, then the matrix g_1 given by (12) is nonsingular.

While introducing the control (37) in equation (36) one obtains $\dot{V}_4 = -\sum_{i=1}^{4} z_i^T A_i z_i < 0$. Consequently, the subsystem S_1 is asymptotically stable with the virtual control inputs v_1, v_2, v_3 and v_4.

After having stabilized an under-actuated subsystem S_1, there remains to us the stabilization of a fully-actuated subsystem S_2. To carry out this task, we will devote the two next steps.

- **Step 5:** In this step we consider the virtual system

$$\dot{x}_5 = v_5 \quad (38)$$

Let the tracking error

$$z_5 = x_{5d} - x_5 \quad (39)$$

and the *Lyapunov* function:

$$V_5 = \frac{1}{2} z_5^T z_5 \quad (40)$$

The time derivative of (40) is

$$\dot{V}_5 = z_5^T \dot{z}_5 = z_5^T(\dot{x}_{5d} - \dot{x}_5) = z_5^T(\dot{x}_{5d} - v_5) \quad (41)$$

The stabilization of z_5 can be obtained by introducing a virtual control:

$$v_5 = A_5 z_5 + \dot{x}_{5d} \quad (42)$$

with $A_5 \in \mathbb{R}^{2\times 2}$ is a positive definite matrix.

By using (42) in (41) one obtains $\dot{V}_5 = -z_5^T A_5 z_5 < 0$.

- **Step 6:** For this step we consider the system

$$\dot{x}_6 = f_2(x_3, x_4, x_6, x_7) + g_2(x_3) v_6 \quad (43)$$

Let

$$z_6 = v_5 - x_6 = A_5 z_5 + \underbrace{\dot{x}_{5d} - \dot{x}_5}_{\dot{z}_5} \quad (44)$$

Then $\dot{z}_5 = -A_5 z_5 + z_6$.

The augmented *Lyapunov* function for this step is

$$V_6 = \frac{1}{2}(z_5^T z_5 + z_6^T z_6) \quad (45)$$

Its time derivative is

$$\begin{aligned} \dot{V}_6 &= z_5^T \dot{z}_5 + z_6^T \dot{z}_6 = z_5^T(-A_5 z_5 + z_6) + z_6^T(\dot{v}_5 - \dot{x}_6) \\ &= -z_5^T A_5 z_5 + z_6^T(z_5 + \dot{v}_5 - f_2 - g_2 v_6) \quad (46) \end{aligned}$$

The stabilization of S_2 can be obtained by introducing a following virtual control:

$$v_6 = g_2^{-1}(z_5 + A_6 z_6 + \dot{v}_5 - f_2) \quad (47)$$

with $A_6 \in \mathbb{R}^{2\times 2}$ is a positive definite matrix.

Remark 4: Knowing that $(-\frac{\pi}{2} < \phi < \frac{\pi}{2})$ and $(-\frac{\pi}{2} < \theta < \frac{\pi}{2})$, it is easy to show in (12) that the matrix g_2 is nonsingular.

The introducing the control (47) in equation (46) one obtains $\dot{V}_6 = -z_5^T A_5 z_5 - z_6^T A_6 z_6 < 0$. One can thus conclude that the subsystem S_2 is asymptotically stable.

Now, in the following step, we will develop the real control which stabilizes the whole system.

- **Step 7:** For the final step we consider the propeller subsystem

$$\dot{x}_7 = u \tag{48}$$

Let

$$\begin{aligned} z_7 &= \left[\begin{array}{c} v_4 - \varphi_1(x_7) \\ v_6 - \varphi_2(x_7) \end{array} \right] \\ &= \left[\begin{array}{c} g_1^{-1}(J_0^T z_3 + A_4 z_4 + \underbrace{\dot{v}_3 - f_1 - g_1\varphi_1}_{\dot{z}_4}) \\ g_2^{-1}(z_5 + A_6 z_6 + \underbrace{\dot{v}_5 - f_2 - g_2\varphi_2}_{\dot{z}_6}) \end{array} \right] \end{aligned} \tag{49}$$

hence $\dot{z}_4 = g_1^* z_7 - J_0^T z_3 - A_4 z_4$ and $\dot{z}_6 = g_2^* z_7 - z_5 - A_6 z_6$ where $g_1^* = [g_1, 0_{2\times 2}]$ and $g_2^* = [0_{2\times 2}, g_2]$ with $0_{2\times 2}$ is a null matrix in $\mathbb{R}^{2\times 2}$.

The *Lyapunov* function candidate of the whole system $\{S_1, S_2, S_3\}$ is

$$V_7 = \frac{1}{2}\sum\nolimits_{i=1}^{7} z_i^T z_i \tag{50}$$

Its time derivative is given

$$\begin{aligned} \dot{V}_7 = &\ \sum\nolimits_{i=1}^{7} z_i^T \dot{z}_i \\ = &\ z_1^T(-A_1 z_1 + z_2) + z_2^T(-z_1 - A_2 z_2 + g_0 z_3) \\ &+ z_3^T(-g_0^T z_2 - A_3 z_3 + J_0 z_4) \\ &+ z_4^T(-J_0^T z_3 - A_4 z_4 + g_1^* z_7) + z_5^T(-A_5 z_5 + z_6) \\ &+ z_6^T(-z_5 - A_6 z_6 + g_2^* z_7) + z_7^T\left(\left[\begin{array}{c} \dot{v}_4 \\ \dot{v}_6 \end{array}\right] - \left[\begin{array}{c} \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{array}\right]\right) \\ = &\ -\sum\nolimits_{i=1}^{6} z_i^T A_i z_i + z_7^T\left(\left[\begin{array}{cc} g_1 & 0_{2\times 2} \\ 0_{2\times 2} & g_2 \end{array}\right]^T \left[\begin{array}{c} z_4 \\ z_6 \end{array}\right] \right. \\ &\left. + \left[\begin{array}{c} \dot{v}_4 \\ \dot{v}_6 \end{array}\right] - \left[\begin{array}{c} J_1 \\ J_2 \end{array}\right] \underbrace{\dot{x}_7}_{u}\right) \end{aligned} \tag{51}$$

where $J_1 = \frac{\partial \varphi_1(x_7)}{\partial x_7}$ and $J_2 = \frac{\partial \varphi_2(x_7)}{\partial x_7}$ are the *Jacobian* matrices of φ_1 and φ_2 such as:

$$J_1 = \left[\begin{array}{cccc} 0 & d & 0 & -d \\ -d & 0 & d & 0 \end{array}\right], J_2 = \left[\begin{array}{cccc} c & -c & c & -c \\ 1 & 1 & 1 & 1 \end{array}\right] \tag{52}$$

Therefore, the stabilization of the whole system can be obtained by introducing a following control law:

$$\begin{aligned} u = \left[\begin{array}{c} J_1 \\ J_2 \end{array}\right]^{-1} &\left(\left[\begin{array}{cc} g_1 & 0_{2\times 2} \\ 0_{2\times 2} & g_2 \end{array}\right]^T \left[\begin{array}{c} z_4 \\ z_6 \end{array}\right] \right. \\ &\left. + \left[\begin{array}{c} \dot{v}_4 \\ \dot{v}_6 \end{array}\right] + A_7 z_7\right) \end{aligned} \tag{53}$$

with $A_7 \in \mathbb{R}^{4\times 4}$ is a positive definite matrix.

Remark 5: It should be noted that the determinant of the matrix $\left[\begin{array}{c} J_1 \\ J_2 \end{array}\right]$ is $8cd^2$. Therefore, this matrix is nonsingular when $c > 0$ and $d > 0$.

While introducing the control (53) in equation (51) one obtains $\dot{V}_7 = -\sum_{i=1}^{7} z_i^T A_i z_i < 0$. Consequently, by using (17, 20, 22, 25, 27, 31, 33, 37, 39, 42, 44, 47 and 53), the whole system (11) is asymptotically stable with the following control law:

$$\left\{\begin{aligned} v_1 &= A_1(x_{1d} - x_1) + \dot{x}_{1d} \\ v_2 &= g_0^{-1}[(x_{1d} - x_1) + A_2(v_1 - x_2) + \dot{v}_1 - f_0] \\ v_3 &= J_0^{-1}[g_0^T(v_1 - x_2) + A_3(v_2 - \varphi_0) + \dot{v}_2] \\ v_4 &= g_1^{-1}[J_0^T(v_2 - \varphi_0) + A_4(v_3 - x_4) + \dot{v}_3 - f_1] \\ v_5 &= A_5(x_{5d} - x_5) + \dot{x}_{5d} \\ v_6 &= g_2^{-1}[(x_{5d} - x_5) + A_6(v_5 - x_6) + \dot{v}_5 - f_2] \\ u &= \left[\begin{array}{c} J_1 \\ J_2 \end{array}\right]^{-1} \left(\left[\begin{array}{cc} g_1 & 0_{2\times 2} \\ 0_{2\times 2} & g_2 \end{array}\right]^T \left[\begin{array}{c} v_3 - x_4 \\ v_5 - x_6 \end{array}\right] \right. \\ &\quad \left. + \left[\begin{array}{c} \dot{v}_4 \\ \dot{v}_6 \end{array}\right] + A_7 \left[\begin{array}{c} v_4 - \varphi_1 \\ v_6 - \varphi_2 \end{array}\right]\right) \end{aligned}\right. \tag{54}$$

IV. SIMULATION RESULT

In order to verify the effectiveness and the efficiency of the proposed backstepping control law, an application to quadrotor helicopter is conducted by simulation using *Rung-Kutta's* method with variable step. The nominal parameters for quadrotor helicopter are: $m = 2kg$, $I_x = I_y = I_z/2 = 1.2416 Nm.s^2/rad$, $d = 0.2m$, $c = 0.01m$, $g = 9.81m/s^2$, $K_t = diag[10^{-2}, 10^{-2}, 10^{-2}]N.s/m$ and $K_r = diag[10^{-3}, 10^{-3}, 10^{-3}]Nm.s/rad$. The following controller parameters are used: $A_1 = \ldots = A_6 = diag[3, 3]$ and $A_7 = diag[3, 3, 3, 3]$. The proposed backstepping control law (54) requires the knowledge of $\dot{v}_1$, $\dot{v}_2$, $\dot{v}_3$, $\dot{v}_4$, $\dot{v}_5$ and $\dot{v}_6$. In order to avoid analytical derivation difficulties, we estimate them by using the following finite difference time approximation:

$$\dot{v}_i = \frac{\Delta v_i}{\Delta t} \quad \text{for } i = 1, \ldots, 6 \tag{55}$$

where Δv_i is the change in input value and Δt is the change in time since the previous simulation time step. The helicopter is initially in hover flight and the initial conditions are: $x_1(0) = \ldots = x_6(0) = [0, 0]^T$ and $x_7(0) = \frac{mg}{4}[1, 1, 1, 1]^T$. The reference trajectory chosen for $x_d(t)$, $y_d(t)$, $z_d(t)$ and $\psi_d(t)$ is that of the step response of the following transfer function:

$$H(s) = \frac{1}{(s+1)^6} \tag{56}$$

with s is the *Laplace* variable.

Figures 2 and 3 show the positions and the tracking errors of a quadrotor helicopter respectively. It can be seen the good tracking of the desired trajectory. Moreover, we can notice in figure 2 an optimization of tilt angles (pitch and roll) and consequently the use of minimal energy. The behavior of all virtual controls is given in figure 4. We note that their evolutions are smooth and derivable. The obtained input control signals (see figure 5) are acceptable and physically realizable.

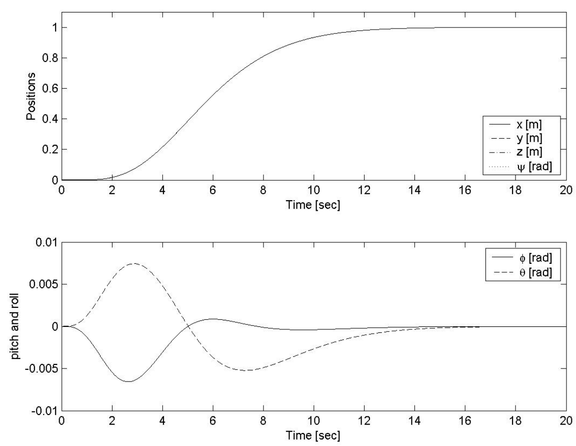

Fig. 2. Position outputs

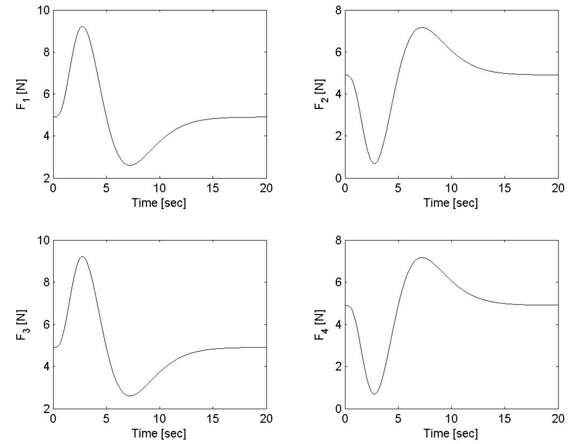

Fig. 5. Force control inputs

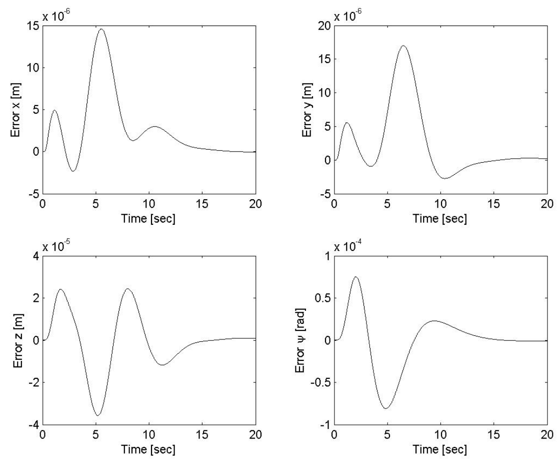

Fig. 3. Tracking errors for x, y, z and ψ

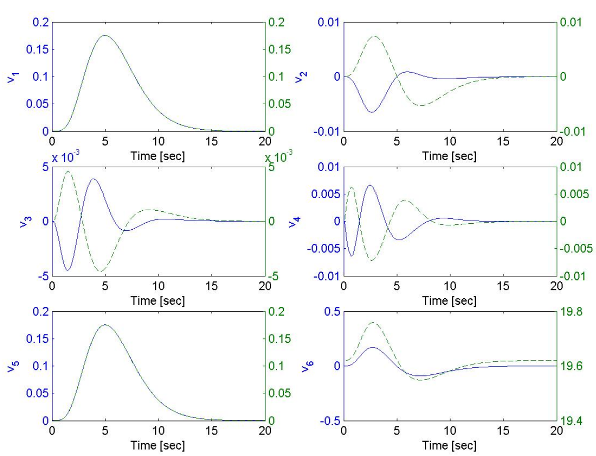

Fig. 4. Evolution of the virtual controls: solid line with the left y-axis denote the first element and the dashed line with the right y-axis denote the second element.

V. Conclusion

In this paper, we have presented the dynamic modeling of quadrotor helicopter and introduced a new approach of a backstepping control. This process is under-actuated system because it has six outputs while it has only four inputs. The whole system of the quadrotor helicopter is divided into three subsystems: an under-actuated subsystem, fully-actuated subsystem and a propeller subsystem. A backstepping control algorithm is proposed to stabilize the whole system and is able to drive a quadrotor to the desired trajectory of *Cartesian* position and yaw angle. The simulation results show the good performance of the proposed control approach.

References

[1] O. Shakernia, Y. Ma, T. J. Koo and S. S. Sastry, ”Landing an Unmanned Air Vehicle: Vision-based Motion Estimation and Nonlinear Control”, Asian J. Control, vol. 1, no. 3, 1999.

[2] A. Mokhtari, A. Benallegue and A. Belaidi, ”Polynomial Linear Quadratic Gaussian ans Sliding Mode Observer for a Quadrotor Unmanned Aerial Vehicle”, Journal of Robotics and Mechatronics, vol. 17, no. 4, pp. 483-495, 2005.

[3] S. Bouabdallah and R. Siegwart, ”Backstepping and Sliding-mode Techniques Applied to an Indoor Micro Quadrotor”, Proceedings of the 2005 IEEE International Conference on Robotics and Automation, pp. 2259-2264, 2005.

[4] P. Castillo, A. Dzul and R. Lozano, ”Real-Time Stabilization and Tracking of Four-Rotor Mini Rotorcraft”, IEEE Transactions on Control Systems Technology, vo. 12, no. 4, pp. 510-516, 2004.

[5] E. Altug, J. P. Ostrowski and C. J. Taylor,”Quadrotor Control using Dual Cameral Visual Feedback”, Proceedings of the 2003 IEEE International Conference on Robotics and Automation, vol. 3, pp. 4294-4299, 2003.

[6] M. Vukobratovic, *Applied Dynamics of Manipulation Robots: Modelling, Analysis and Examples*, Berlin: Springer-Verlag, 1989.

[7] S. B. V. Gomes and J. J. Jr. G.Ramas, ”Airship dynamic modelling for autonomous operation”, IEEE International Conference on Robotics and Automation, 1998.

[8] H. Nijmeijer and AJ Van Der Shaft, *Nonlinear Dynamic Control Systems,* Springer Verlag, 1990.

[9] I. Fantoni and R. Lozano, *Non-linear control for underactuated mechanical systems*, Springer, 2002.