Quaternion Calculus Notes

Sven Mikael Persson

February 15, 2013

1 Basic Algebra

Quaternions can be viewed in many ways, as a representation of 3D rotations, as a super-set of complex numbers where the imaginary part has a dimension of 3, as an element of a Lie Group (or special orthogonal group 3, or $SO(3)$), or even as a 4D vector. In these notes, we adopt the view that it is an element of a Lie Group, or $SO(3)$ when dealing with unit-quaternions. By extension, 3D rotations are also always represented by elements of $SO(3)$, and thus the space of $SO(3)$ and that of all possible 3D rotations are one and the same algebraic space. The view adopted here is also related to the view of quaternions as "super-complex" numbers if one makes one particular choice of rotationality of the imaginary space (one of four possible rotationalities, the one that corresponds with 3D rotations in a right-handed coordinate system). The view as "super-complex" numbers is elaborated better in Girard [Girard, 2007]. As for the view of quaternions as 4D vectors, this view is plain wrong in the author's opinion, and will thus not be discussed at all beyond this point in this paper.

1.1 Quaternions

A standard notation for quaternions is its decomposition into a scalar part $q_0 \in \mathbb{R}$ and a vector part $\mathbf{q}_v \in \mathbb{R}^3$ (or real and imaginary part, respectively), as so:

$$\mathbf{q} \equiv \left[q_0 \; \mathbf{q}_v^T\right]^T = [q_0 \; q_1 \; q_2 \; q_3]^T \tag{1}$$

which leads to the definition of the quaternion set $\mathbb{Q}$:

$$\mathbb{Q} : \{\mathbf{q} = \left[q_0 \; \mathbf{q}_v^T\right]^T \mid q_0 \in \mathbb{R} \;\wedge\; \mathbf{q}_v \in \mathbb{R}^3\} \tag{2}$$

In quaternionic algebra, the composition operator over quaternions is defined as:

$$\text{(composition)} \qquad \mathbf{q}\,\mathbf{p} = \begin{bmatrix} q_0 p_0 - \mathbf{q}_v^T \mathbf{p}_v \\ q_0 \mathbf{p}_v + p_0 \mathbf{q}_v + \mathbf{q}_v \times \mathbf{p}_v \end{bmatrix} \qquad \text{for} \quad \mathbf{q}, \mathbf{p} \in \mathbb{Q} \tag{3}$$

along with several fundamental operations, for $\mathbf{p} \in \mathbb{Q}$:

$$\text{(inversion)} \qquad \mathbf{p}^{-1} = \frac{1}{\|\mathbf{p}\|_2^2} \left[\; p_0 \quad -\mathbf{p}_v^T \;\right]^T \tag{4}$$

$$\text{(logarithm)} \qquad \log \mathbf{p} = \left[\; \log(\|\mathbf{p}\|_2) \quad \frac{\mathbf{p}_v^T}{\|\mathbf{p}_v\|_2} \cos^{-1}\left(\frac{p_0}{\|\mathbf{p}\|_2}\right) \;\right]^T \tag{5}$$

$$\text{(exponential)} \qquad \exp(\mathbf{p}) = \exp(p_0) \left[\; \cos(\|\mathbf{p}_v\|_2) \quad \frac{\mathbf{p}_v^T}{\|\mathbf{p}_v\|_2} \sin(\|\mathbf{p}_v\|_2) \;\right]^T \tag{6}$$

$$\text{(power)} \qquad \mathbf{p}^t = \exp(t \log \mathbf{p}) \tag{7}$$

1.2 Unit Quaternions

Unit quaternions, in $\bar{\mathbb{Q}}$, are those which lie in the quaternion set $\mathbb{Q}$ and satisfy the unit-norm constraint:

$$\bar{\mathbb{Q}} : \{\mathbf{q} \mid \mathbf{q} \in \mathbb{Q} \ \wedge \ \|\mathbf{q}\|_2^2 = q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1\} \tag{8}$$

or, equivalently expressed in terms of axis-angle:

$$\bar{\mathbb{Q}} : \{\mathbf{q} = \begin{bmatrix} \cos\phi \\ \sin\phi\,\mathbf{u} \end{bmatrix} \mid \phi \in \mathbb{R} \wedge \mathbf{u} \in \mathbb{R}^3 \wedge \|\mathbf{u}\|_2^2 = 1\} \tag{9}$$

Unit-quaternions, $\mathbf{q}, \mathbf{p} \in \bar{\mathbb{Q}}$, share the same algebra as quaternions, except that the operations can be simplified:

$$\text{(composition)} \qquad \mathbf{q}\,\mathbf{p} = \begin{bmatrix} q_0 p_0 - \mathbf{q}_v^T \mathbf{p}_v \\ q_0 \mathbf{p}_v + p_0 \mathbf{q}_v + \mathbf{q}_v \times \mathbf{p}_v \end{bmatrix} \tag{10}$$

$$\text{(inversion)} \qquad \mathbf{q}^{-1} = \begin{bmatrix} q_0 & -\mathbf{q}_v^T \end{bmatrix}^T \tag{11}$$

$$\text{(logarithm)} \qquad \log\mathbf{q} = \phi\mathbf{u} \tag{12}$$

$$\text{(exponential)} \qquad \exp(\phi\mathbf{u}) = \mathbf{q} \tag{13}$$

$$\text{(power)} \qquad \mathbf{q}^t = \exp(t\phi\mathbf{u}) \tag{14}$$

1.3 3D Rotations

As of the Euler theorem, any rotation in $\mathbb{R}^3$ can be represented by an angle of rotation $\theta \in \mathbb{R}$ about the axis of rotation $\mathbf{u} \in \mathbb{R}^3$. By letting $\phi = \frac{\theta}{2}$, the *Euler-Rodriguez parameters* $\mathbf{q}$ are defined, which are related to the rotation matrix $\mathbf{R}(\mathbf{q})$ as follows:

$$q_0 = \cos\frac{\theta}{2} \text{ and } \mathbf{q}_v = \sin\frac{\theta}{2}\mathbf{u} \tag{15}$$

$$\mathbf{R}(\mathbf{q}) = (q_0^2 - \mathbf{q}_v^T\mathbf{q}_v)\mathbf{I}_3 + 2\mathbf{q}_v\mathbf{q}_v^T + 2q_0[\mathbf{q}_v\times] \tag{16}$$

$$\mathbf{R}(\mathbf{q})\,\mathbf{v} = \mathbf{q}\,\mathbf{v}\,\mathbf{q}^{-1} \quad \Rightarrow \quad \mathbf{R}(\mathbf{q}_1)\mathbf{R}(\mathbf{q}_2) = \mathbf{R}(\mathbf{q}_1\mathbf{q}_2) \tag{17}$$

where $[\mathbf{q}_v\times]$ denotes the skew-symmetric matrix for vector $\mathbf{q}_v$ (also called the cross-product matrix). Note also, in Eq. (17), a slight abuse of notation that will carry through this paper, i.e., no distinction will be made between $\mathbb{R}^3$-vectors and *pure quaternions* (i.e., quaternions with zero scalar part).

Please refer to [Angeles, 2007a] and [Angeles, 2007b].

2 Quaternionic Identities

2.1 Exponential and Logarithmic Identities

Finally, a number of theorems related to transformations of quaternions are established.

Lemma 1. *Given two quaternions $\mathbf{q}, \mathbf{p} \in \mathbb{Q}$, the following is always true:*

$$\mathbf{q}\,\mathbf{p}\,\mathbf{q}^{-1} = \begin{bmatrix} p_0 \\ \mathbf{q}\,\mathbf{p}_v\,\mathbf{q}^{-1} \end{bmatrix} \tag{18}$$

Proof. Starting from the expression stated above, expanding as an addition and distributing the products gives the following simple steps:

$$\mathbf{q}\mathbf{p}\mathbf{q}^{-1} = \mathbf{q}\begin{bmatrix} p_0 \\ \mathbf{p}_v \end{bmatrix}\mathbf{q}^{-1} = \mathbf{q}\left(\begin{bmatrix} p_0 \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbf{p}_v \end{bmatrix}\right)\mathbf{q}^{-1} = \begin{bmatrix} p_0 \\ \mathbf{0} \end{bmatrix} + \mathbf{q}\begin{bmatrix} 0 \\ \mathbf{p}_v \end{bmatrix}\mathbf{q}^{-1} \tag{19}$$

and expanding the compositions:

$$\mathbf{q}\mathbf{p}\mathbf{q}^{-1} = \begin{bmatrix} p_0 \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} -\mathbf{q}_v^T\mathbf{p}_v \\ q_0\mathbf{p}_v + \mathbf{q}_v \times \mathbf{p}_v \end{bmatrix}\begin{bmatrix} q_0 \\ -\mathbf{q}_v \end{bmatrix}\frac{1}{\|\mathbf{q}\|_2^2} \tag{20}$$

$$\mathbf{q}\mathbf{p}\mathbf{q}^{-1} = \begin{bmatrix} p_0 \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} 0 \\ q_0(q_0\mathbf{p}_v + \mathbf{q}_v \times \mathbf{p}_v) + (\mathbf{q}_v^T\mathbf{p}_v)\mathbf{q}_v - (q_0\mathbf{p}_v + \mathbf{q}_v \times \mathbf{p}_v) \times \mathbf{q}_v \end{bmatrix}\frac{1}{\|\mathbf{q}\|_2^2} \tag{21}$$

which shows that the product $\mathbf{q}\mathbf{p}\mathbf{q}^{-1}$ leaves the scalar part of $\mathbf{p}$ unchanged through the transformation, leading to the conclusion:

$$\mathbf{q}\mathbf{p}\mathbf{q}^{-1} = \begin{bmatrix} p_0 \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbf{q}\mathbf{p}_v\mathbf{q}^{-1} \end{bmatrix} = \begin{bmatrix} p_0 \\ \mathbf{q}\mathbf{p}_v\mathbf{q}^{-1} \end{bmatrix} \tag{22}$$

□

Lemma (1) allows for the proof of the following theorem:

Theorem 2. *Given two quaternions $\mathbf{q}, \mathbf{p} \in \mathbb{Q}$, the following identity is true:*

$$\log(\mathbf{q}\mathbf{p}\mathbf{q}^{-1}) = \mathbf{q}\log(\mathbf{p})\mathbf{q}^{-1} \tag{23}$$

Proof. Using the expression of Lemma (1) and expanding the quaternionic logarithm:

$$\log(\mathbf{q}\mathbf{p}\mathbf{q}^{-1}) = \log\left(\begin{bmatrix} p_0 \\ \mathbf{q}\mathbf{p}_v\mathbf{q}^{-1} \end{bmatrix}\right) = \begin{bmatrix} \log\left(\sqrt{p_0^2 + \|\mathbf{q}\mathbf{p}_v\mathbf{q}^{-1}\|_2^2}\right) \\ \frac{\mathbf{q}\mathbf{p}_v\mathbf{q}^{-1}}{\|\mathbf{q}\mathbf{p}_v\mathbf{q}^{-1}\|_2}\cos^{-1}\left(\frac{p_0}{\sqrt{p_0^2+\|\mathbf{q}\mathbf{p}_v\mathbf{q}^{-1}\|_2^2}}\right) \end{bmatrix} \tag{24}$$

From basic quaternionic algebra, it is known that $\|\mathbf{q}\mathbf{p}_v\mathbf{q}^{-1}\|_2 = \|\mathbf{p}_v\|_2$, which leads to:

$$\log(\mathbf{q}\mathbf{p}\mathbf{q}^{-1}) = \begin{bmatrix} \log\left(\|\mathbf{p}\|_2\right) \\ \frac{\mathbf{q}\mathbf{p}_v\mathbf{q}^{-1}}{\|\mathbf{p}_v\|_2}\cos^{-1}\left(\frac{p_0}{\|\mathbf{p}\|_2}\right) \end{bmatrix} \tag{25}$$

and, again, using Lemma (1), the identity is obtained:

$$\log(\mathbf{q}\mathbf{p}\mathbf{q}^{-1}) = \mathbf{q}\begin{bmatrix} \log\left(\|\mathbf{p}\|_2\right) \\ \frac{\mathbf{p}_v}{\|\mathbf{p}_v\|_2}\cos^{-1}\left(\frac{p_0}{\|\mathbf{p}\|_2}\right) \end{bmatrix}\mathbf{q}^{-1} = \mathbf{q}\log(\mathbf{p})\mathbf{q}^{-1} \tag{26}$$

which completes the proof. □

Finally, a corollary of the above theorem is drawn:

Corollary 3. *Given two quaternions $\mathbf{q}, \mathbf{p} \in \mathbb{Q}$, the following identity is true:*

$$\exp(\mathbf{q}\mathbf{p}\mathbf{q}^{-1}) = \mathbf{q}\exp(\mathbf{p})\mathbf{q}^{-1} \tag{27}$$

Proof. Starting from the expression of the proposed identity:

$$\exp(\mathbf{q}\mathbf{p}_1\mathbf{q}^{-1}) = \mathbf{q}\exp(\mathbf{p}_2)\mathbf{q}^{-1} \tag{28}$$

applying the logarithm on either sides, and using Theorem (2), the following is obtained:

$$\mathbf{q}\mathbf{p}_1\mathbf{q}^{-1} = \log(\mathbf{q}\exp(\mathbf{p}_2)\mathbf{q}^{-1}) = \mathbf{q}\log(\exp(\mathbf{p}_2))\mathbf{q}^{-1} = \mathbf{q}\mathbf{p}_2\mathbf{q}^{-1} \tag{29}$$

which implies that $\mathbf{p}_1 = \mathbf{p}_2$, thus completing the proof. □

As a final note, obviously, unit-quaternions being a subset of general quaternions, Lemma (1), Theorem (2) and Corollary (3) all apply equally well to unit-quaternions.

2.2 Small Angle Approximation

It is also important to remind the reader that, since quaternion composition (or multiplication) is non-commutative, familiar identities of real or complex valued logarithms and exponentials do not apply in general (e.g. $e^{\mathbf{q}}e^{\mathbf{p}} \neq e^{\mathbf{q}+\mathbf{p}}$). However, there are two important exceptions: collinearity and small-angle approximation, which, respectively, are:

$$\mathbf{q}_v^T\mathbf{p}_v = \pm\|\mathbf{q}_v\|\|\mathbf{p}_v\| \qquad \Longleftrightarrow \qquad e^{\mathbf{q}}e^{\mathbf{p}} = e^{\mathbf{q}+\mathbf{p}} \tag{30}$$

$$\|\mathbf{p}_v\| \ll 1 \qquad \Rightarrow \qquad e^{\mathbf{q}}e^{\mathbf{p}} \approx e^{\mathbf{q}+\mathbf{p}} \tag{31}$$

3 Quaternionic Interpolations

If we want to discuss interpolations in the topology of $SO(3)$, we must first realize that there are a few fundamental operations needed when doing interpolations. First, we need an addition operator, which, in a linear vector-space, is simply the addition operator, but, in $SO(3)$, that operator is the quaternionic composition (or quaternion product) with a quaternion-difference represented by the local axis-angle vector $\mathbf{v} = \theta\mathbf{u}$. To make the correspondance clearer, we will define a new notation for the quaternion composition as follows:

$$\oplus : (\mathbf{q}, \mathbf{v}) \mapsto \mathbf{q}e^{\frac{1}{2}\mathbf{v}} \qquad \mathbf{q} \in \bar{\mathbb{Q}} \quad \text{and} \quad \mathbf{v} \in \mathbb{R}^3 \tag{32}$$

Second, we need a difference operator to obtain the difference between two quaternions. Continuing with the composition correspondance, the "difference" between two quaternions is really the axis-angle that is needed to go from one quaternion to another. Analogously, in a vector-space, we have $\mathbf{d} = \mathbf{v} - \mathbf{u}$ such that $\mathbf{v} = \mathbf{u} + \mathbf{d}$, in a quaternion space, we have $\mathbf{v} = \mathbf{q} \ominus \mathbf{p}$ such that $\mathbf{q} = \mathbf{p} \oplus \mathbf{v}$. Note that the ordering of the operands in that last expression is very critical as quaternionic composition is not commutative, and hence, following work on interpolations will require special care with respect to the ordering of the elements in the equations. We thus have:

$$\ominus : (\mathbf{q}, \mathbf{p}) \mapsto 2\log(\mathbf{p}^{-1}\mathbf{q}) \tag{33}$$

Finally, we need a "multiplication by a scalar" operator. For this, we can simply use a multiplication of the axis-angle quaternion-difference vector (since the scaling of a geodesic arc is a valid operation).

3.1 Linear Interpolation

If we first look at a linear interpolation in a vector space, like $\mathbb{R}^n$, we have:

$$\mathbf{v}(\eta) = \mathbf{v}_1 + \eta(\mathbf{v}_2 - \mathbf{v}_1) \tag{34}$$

Using our newly defined operators, we can directly express the above interpolation for the domain of quaternions, as follows:

$$\begin{aligned}\mathbf{q}(\eta) &= \mathbf{q}_1 \oplus \eta(\mathbf{q}_2 \ominus \mathbf{q}_1) &(35)\\ \mathbf{q}(\eta) &= \mathbf{q}_1 e^{\eta \log(\mathbf{q}_1^{-1}\mathbf{q}_2)} &(36)\\ \mathbf{q}(\eta) &= \mathbf{q}_1 (\mathbf{q}_1^{-1}\mathbf{q}_2)^{\eta} &(37)\end{aligned}$$

where the last expression above is found in literature as the *Spherical Linear Interpolation* method (or SLERP). This popular name comes from the fact that doing such an interpolation effectively travels along an arc of the 4D hyper-sphere formed by the set of unit-quaternions (or $SO(3)$), going from $\mathbf{q}_1$ to $\mathbf{q}_2$ with the parameter η. It is the opinion of the author that this name is rather unfortunate because it implies that unit-quaternions reside on a manifold of $\mathbb{R}^4$, rather than being in a distinct algebraic group. The fact remains, the SLERP method is simply a linear interpolation in $SO(3)$, nothing more.

4 Quaternionic Calculus

Following the definitions of the "circle-operators" $(\oplus, \ominus)$, we can now proceed to define a set of rules for performing derivatives, integrals and partial derivatives (and differentiation) on quaternions.

4.1 Derivatives

We will again draw from vector calculus by first expressing the definition of a derivative for vectors:

$$\frac{d}{dt}\mathbf{v}(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t}(\mathbf{v}(t + \Delta t) - \mathbf{v}(t)) \tag{38}$$

Analogously, we could say the following when dealing with quaternions:

$$\frac{d}{dt}\mathbf{q}(t) \equiv \lim_{\Delta t \to 0} \frac{1}{\Delta t}(\mathbf{q}(t + \Delta t) \ominus \mathbf{q}(t)) \tag{39}$$

which, after substitution of the operators, gives the following:

$$\frac{d}{dt}\mathbf{q}(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} 2\log(\mathbf{q}(t)^{-1}\mathbf{q}(t + \Delta t)) \tag{40}$$

The derivative of a unit-quaternion, as defined above, is indeed the angular velocity (in the local Lie Algebra, or "body-fixed frame"), and we can thus refer to it as:

$$\boldsymbol{\omega} \equiv \frac{d}{dt}\mathbf{q}(t) \tag{41}$$

where $\boldsymbol{\omega} \in \mathbb{R}^3$. The anti-derivative (or integral) follows in the next section.

4.1.1 Integral and Differentiation

We can reverse the previously defined derivative relationship by simply expressing it as follows:

$$\mathbf{q}(t) = \mathbf{q}(t_0) \oplus \int_{t_0}^{t} \boldsymbol{\omega}(\tau)\, d\tau \tag{42}$$

which we can use for differentiation, as follows:

$$\begin{aligned}
\mathbf{q}(t+\delta t) &= \int_{t}^{t+\delta t} \boldsymbol{\omega}(\tau)\, d\tau && (43)\\
\mathbf{q}(t+\delta t) \ominus \mathbf{q}(t) &= \int_{t}^{t+\delta t} \boldsymbol{\omega}(\tau)\, d\tau && (44)\\
\delta\mathbf{q}(t) &= \boldsymbol{\omega}(t)\delta t && (45)
\end{aligned}$$

4.1.2 Composition Rule

We can now proceed to define derivative rules for compositions of quaternions. Starting with this function:

$$\mathbf{f}(\mathbf{q}_1(t), \mathbf{q}_2(t)) = \mathbf{q}_1(t)\mathbf{q}_2(t) \tag{46}$$

we have:

$$\begin{aligned}
\frac{d}{dt}(\mathbf{q}_1(t)\mathbf{q}_2(t)) &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} 2\log(\mathbf{q}_2^{-1}(t)\, \mathbf{q}_1^{-1}(t)\, \mathbf{q}_1(t+\Delta t)\, \mathbf{q}_2(t+\Delta t)) && (47)\\
&= \lim_{\Delta t \to 0} \frac{1}{\Delta t} 2\log(\mathbf{q}_2^{-1}(t)\, e^{\frac{1}{2}\Delta t\, \boldsymbol{\omega}_1(t)}\, \mathbf{q}_2(t)\, e^{\frac{1}{2}\Delta t\, \boldsymbol{\omega}_2(t)}) && (48)
\end{aligned}$$

and, by using Theorem 2, we get:

$$\frac{d}{dt}(\mathbf{q}_1(t)\mathbf{q}_2(t)) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} 2\log(e^{\frac{1}{2}\Delta t\, \mathbf{q}_2^{-1}(t)\, \boldsymbol{\omega}_1(t)\, \mathbf{q}_2(t)} e^{\frac{1}{2}\Delta t\, \boldsymbol{\omega}_2(t)}) \tag{49}$$

and, with small angle approximation as $\frac{1}{2}\Delta t\, \boldsymbol{\omega}_2(t) \to 0$, we have:

$$\begin{aligned}
\frac{d}{dt}(\mathbf{q}_1(t)\mathbf{q}_2(t)) &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} 2\log(e^{\frac{1}{2}\Delta t\, \mathbf{q}_2^{-1}(t)\, \boldsymbol{\omega}_1(t)\, \mathbf{q}_2(t) + \frac{1}{2}\Delta t\, \boldsymbol{\omega}_2(t)}) && (50)\\
&= \lim_{\Delta t \to 0} (\mathbf{q}_2^{-1}(t)\, \boldsymbol{\omega}_1(t)\, \mathbf{q}_2(t) + \boldsymbol{\omega}_2(t)) && (51)\\
&= \mathbf{q}_2^{-1}(t)\, \boldsymbol{\omega}_1(t)\, \mathbf{q}_2(t) + \boldsymbol{\omega}_2(t) && (52)
\end{aligned}$$

4.1.3 Inversion Rule

When we have an inverse of a quaternion, its derivative becomes:

$$\begin{aligned}
\frac{d}{dt}(\mathbf{q}^{-1}(t)) &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} 2\log(\mathbf{q}(t)\, \mathbf{q}^{-1}(t+\Delta t)) && (53)\\
&= \lim_{\Delta t \to 0} \frac{1}{\Delta t} 2\log(\mathbf{q}(t)\, e^{\frac{-1}{2}\Delta t\, \boldsymbol{\omega}(t)}\, \mathbf{q}^{-1}(t)) && (54)
\end{aligned}$$

and, again, using Theorem 2, we get:

$$\frac{d}{dt}(\mathbf{q}^{-1}(t)) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} 2 \log(e^{\frac{-1}{2} \Delta t\, \mathbf{q}(t)\, \boldsymbol{\omega}(t)\, \mathbf{q}^{-1}(t)}) \tag{55}$$

$$= -\mathbf{q}(t)\, \boldsymbol{\omega}(t)\, \mathbf{q}^{-1}(t) \tag{56}$$

4.1.4 Difference Rule

Now, if you take the "difference" between two quaternions, as follows:

$$\mathbf{q}_1^{-1}(t)\, \mathbf{q}_2(t) \tag{57}$$

we can take its derivative and, using a similar derivation as with previous rules, we obtain:

$$\frac{d}{dt}(\mathbf{q}_1^{-1}(t)\, \mathbf{q}_2(t)) = \boldsymbol{\omega}_2(t) - \mathbf{q}_2^{-1}(t)\, \mathbf{q}_1(t)\, \boldsymbol{\omega}_1(t)\, \mathbf{q}_1^{-1}(t)\, \mathbf{q}_2(t) \tag{58}$$

The more interesting case is that of the $\ominus$ operator defined as:

$$\mathbf{f}(\mathbf{q}_1(t), \mathbf{q}_2(t)) = \mathbf{q}_2(t) \ominus \mathbf{q}_1(t) = 2 \log(\mathbf{q}_1^{-1}(t)\, \mathbf{q}_2(t)) \tag{59}$$

Because operator maps into the Lie Algebra of the "dividing" quaternion, its derivative cannot be found by a simple vector-space derivative. Instead, we will use the $\oplus$ operator to obtain a quaternion whose derivative we can take using our definition. So, let us define the following:

$$\mathbf{g}(\mathbf{q}_1(t), \mathbf{q}_2(t)) = 1 \oplus (\mathbf{q}_2(t) \ominus \mathbf{q}_1(t)) \tag{60}$$

then, we find the derivative as follows:

$$\frac{d}{dt}(1 \oplus (\mathbf{q}_2(t) \ominus \mathbf{q}_1(t))) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} 2 \log(e^{\frac{-1}{2}(\mathbf{q}_2(t) \ominus \mathbf{q}_1(t))} e^{\frac{1}{2}(\mathbf{q}_2(t+\Delta t) \ominus \mathbf{q}_1(t+\Delta t))}) \tag{61}$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} 2 \log(\mathbf{q}_2^{-1}(t)\mathbf{q}_1(t)\mathbf{q}_1^{-1}(t+\Delta t)\mathbf{q}_2(t+\Delta t)) \tag{62}$$

$$= \frac{d}{dt}(\mathbf{q}_1^{-1}(t)\, \mathbf{q}_2(t)) \tag{63}$$

$$= \boldsymbol{\omega}_2(t) - \mathbf{q}_2^{-1}(t)\, \mathbf{q}_1(t)\, \boldsymbol{\omega}_1(t)\, \mathbf{q}_1^{-1}(t)\, \mathbf{q}_2(t) \tag{64}$$

which is the expected result. Note that, trivially, we know that:

$$\frac{d}{dt}(\mathbf{q}_2(t) \ominus \mathbf{q}_1(t)) = \frac{d}{dt}(1 \oplus (\mathbf{q}_2(t) \ominus \mathbf{q}_1(t))) \tag{65}$$

because the real value 1 is the neutral element of quaternion composition.

4.2 Partial Derivatives and Chain Rules

Starting from the definition of the differentiation of a quaternion as a function of time, which is the following:

$$\delta \mathbf{q}(t) = \boldsymbol{\omega}(t) \delta t \tag{66}$$

Clearly, the differential of the quaternion, by our definition of the derivative, is a Lie Algebra element, i.e., in $\mathbb{R}^3$. Now, we seek a definition of a partial derivative which will allow for the following:

$$\delta \mathbf{f}(\mathbf{q}) = \frac{\partial \mathbf{f}}{\partial \mathbf{q}} \delta \mathbf{q} \tag{67}$$

meaning that the partial derivative, $\frac{\partial \mathbf{f}}{\partial \mathbf{q}}$, is necessarily a matrix (linear mapping) from the differential of $\mathbf{q}$ to the differential of $\mathbf{f}$, which are both $SO(3)$ Lie Algebra elements. Similarly, we could seek this:

$$\delta \mathbf{f}(\mathbf{q}_1, \mathbf{q}_2) = \frac{\partial \mathbf{f}}{\partial \mathbf{q}_1} \delta \mathbf{q}_1 + \frac{\partial \mathbf{f}}{\partial \mathbf{q}_2} \delta \mathbf{q}_2 \tag{68}$$

We will work our way there by examples. First, consider the following composition funciton:

$$\mathbf{f}(\mathbf{q}_1, \mathbf{q}_2) \equiv \mathbf{q}_1 \mathbf{q}_2 \tag{69}$$

When it is assumed that both parameters can be differentiated with respect to time, we already have an expression for the complete time-derivative of the function as follows:

$$\begin{aligned}
\frac{d}{dt}\mathbf{f}(\mathbf{q}_1(t), \mathbf{q}_2(t)) &= \boldsymbol{\omega}_2(t) + \mathbf{q}_2^{-1}(t)\boldsymbol{\omega}_1 \mathbf{q}_2 && (70)\\
\delta \mathbf{f}(\mathbf{q}_1(t), \mathbf{q}_2(t)) &= \boldsymbol{\omega}_2(t)\delta t + \mathbf{q}_2^{-1}(t)\boldsymbol{\omega}_1 \delta t \mathbf{q}_2 && (71)\\
\delta \mathbf{f}(\mathbf{q}_1(t), \mathbf{q}_2(t)) &= \delta \mathbf{q}_2 + \mathbf{q}_2^{-1}(t)\delta \mathbf{q}_1 \mathbf{q}_2 && (72)\\
\delta \mathbf{f}(\mathbf{q}_1, \mathbf{q}_2) &= \delta \mathbf{q}_2 + \mathbf{R}(\mathbf{q}_2^{-1})\delta \mathbf{q}_1 && (73)
\end{aligned}$$

So, clearly, we have, in this case:

$$\begin{aligned}
\frac{\partial}{\partial \mathbf{q}_1}\mathbf{f}(\mathbf{q}_1, \mathbf{q}_2) &= \mathbf{R}(\mathbf{q}_2^{-1}) && (74)\\
\frac{\partial}{\partial \mathbf{q}_2}\mathbf{f}(\mathbf{q}_1, \mathbf{q}_2) &= \mathbf{I}_{3x3} && (75)\\
\delta \mathbf{f}(\mathbf{q}_1, \mathbf{q}_2) &= \frac{\partial}{\partial \mathbf{q}_1}\mathbf{f}(\mathbf{q}_1, \mathbf{q}_2)\delta \mathbf{q}_1 + \frac{\partial}{\partial \mathbf{q}_2}\mathbf{f}(\mathbf{q}_1, \mathbf{q}_2)\delta \mathbf{q}_2 && (76)
\end{aligned}$$

Given that the differentials are essentially in $\mathbb{R}^3$, and that the partial derivatives are matrices, there is no reason for "special" definitions of the chain rule, i.e., one can apply the basic vector calculus chain rule. For strength of convictions, we can verify it with the following functions:

$$\begin{aligned}
\mathbf{g}(\mathbf{q}_0) &\equiv \mathbf{q}_0^{-1} && (77)\\
\mathbf{f}(\mathbf{q}_1, \mathbf{q}_2) &\equiv \mathbf{q}_1 \mathbf{q}_2 && (78)\\
\mathbf{h}(\mathbf{q}_0, \mathbf{q}_2) &\equiv \mathbf{f}(\mathbf{g}(\mathbf{q}_0), \mathbf{q}_2) = \mathbf{q}_0^{-1}\mathbf{q}_2 && (79)
\end{aligned}$$

From the previous section, we already have the differentials of those functions:

$$\begin{aligned}
\delta \mathbf{g}(\mathbf{q}_0) &= -\mathbf{R}(\mathbf{q}_0)\delta \mathbf{q}_0 && (80)\\
\delta \mathbf{f}(\mathbf{q}_1, \mathbf{q}_2) &= \delta \mathbf{q}_2 + \mathbf{R}(\mathbf{q}_2^{-1})\delta \mathbf{q}_1 && (81)\\
\delta \mathbf{h}(\mathbf{q}_0, \mathbf{q}_2) &= \delta \mathbf{q}_2 - \mathbf{R}(\mathbf{q}_2^{-1}\mathbf{q}_0)\delta \mathbf{q}_0 && (82)
\end{aligned}$$

If we apply the classic chain rule from vector-calculus, we should expect the following to be true:

$$\frac{\partial}{\partial \mathbf{q}_1}\mathbf{h}(\mathbf{q}_0,\mathbf{q}_2) = \frac{\partial}{\partial \mathbf{q}_1}\mathbf{f}(\mathbf{g}(\mathbf{q}_0),\mathbf{q}_2)\frac{\partial}{\partial \mathbf{q}_0}\mathbf{g}(\mathbf{q}_0) \tag{83}$$

$$\frac{\partial}{\partial \mathbf{q}_1}\mathbf{h}(\mathbf{q}_0,\mathbf{q}_2) = \mathbf{R}(\mathbf{q}_2^{-1})(-\mathbf{R}(\mathbf{q}_0)) \tag{84}$$

$$\frac{\partial}{\partial \mathbf{q}_1}\mathbf{h}(\mathbf{q}_0,\mathbf{q}_2) = -\mathbf{R}(\mathbf{q}_2^{-1}\mathbf{q}_0) \tag{85}$$

which is indeed what factors into the differential of $\mathbf{h}$ given above.

References

[Angeles, 2007a] Angeles, J. (2007a). *Fundamentals of Robotic Mechanical Systems: Theory, Methods, and Algorithms*, chapter 2, pages 27–79. Springer Science, 3rd edition.

[Angeles, 2007b] Angeles, J. (2007b). *Fundamentals of Robotic Mechanical Systems: Theory, Methods, and Algorithms*, chapter Appendix A, pages 507–512. Springer Science, 3rd edition.

[Girard, 2007] Girard, P. R. (2007). *Quaternions, Clifford algebras and relativistic physics*. Birkhauser Verlag AG.