Rotations using Quaternions

1 Rotation matrices

FRB = front-right-below = platform or body fixed coordinates

NED = north-east-down = local level coordinates

ENU = east-north-up = an alternate definition of local level coordinates

$$C^{\mathrm{FRB}}_{\mathrm{NED}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathrm{c}\phi & \mathrm{s}\phi \\ 0 & -\mathrm{s}\phi & \mathrm{c}\phi \end{bmatrix} \begin{bmatrix} \mathrm{c}\theta & 0 & -\mathrm{s}\theta \\ 0 & 1 & 0 \\ \mathrm{s}\theta & 0 & \mathrm{c}\theta \end{bmatrix} \begin{bmatrix} \mathrm{c}\psi & \mathrm{s}\psi & 0 \\ -\mathrm{s}\psi & \mathrm{c}\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \mathrm{c}\theta\,\mathrm{c}\psi & \mathrm{c}\theta\,\mathrm{s}\psi & -\mathrm{s}\theta \\ -\mathrm{c}\phi\,\mathrm{s}\psi + \mathrm{s}\phi\,\mathrm{s}\theta\,\mathrm{c}\psi & \mathrm{c}\phi\,\mathrm{c}\psi + \mathrm{s}\phi\,\mathrm{s}\theta\,\mathrm{s}\psi & \mathrm{s}\phi\,\mathrm{c}\theta \\ \mathrm{s}\phi\,\mathrm{s}\psi + \mathrm{c}\phi\,\mathrm{s}\theta\,\mathrm{c}\psi & -\mathrm{s}\phi\,\mathrm{c}\psi + \mathrm{c}\phi\,\mathrm{s}\theta\,\mathrm{s}\psi & \mathrm{c}\phi\,\mathrm{c}\theta \end{bmatrix}$$

$$\mathrm{C}^{\mathrm{NED}}_{\mathrm{FRB}} = \begin{bmatrix} \mathrm{c}\psi & -\mathrm{s}\psi & 0 \\ \mathrm{s}\psi & \mathrm{c}\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{c}\theta & 0 & \mathrm{s}\theta \\ 0 & 1 & 0 \\ -\mathrm{s}\theta & 0 & \mathrm{c}\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathrm{c}\phi & -\mathrm{s}\phi \\ 0 & \mathrm{s}\phi & \mathrm{c}\phi \end{bmatrix}$$

$$= \begin{bmatrix} \mathrm{c}\theta\,\mathrm{c}\psi & -\mathrm{c}\phi\,\mathrm{s}\psi + \mathrm{s}\phi\,\mathrm{s}\theta\,\mathrm{c}\psi & \mathrm{s}\phi\,\mathrm{s}\psi + \mathrm{c}\phi\, s\theta\,\mathrm{c}\psi \\ \mathrm{c}\theta\,\mathrm{s}\psi & \mathrm{c}\phi\,\mathrm{c}\psi + \mathrm{s}\phi\,\mathrm{s}\theta\,\mathrm{s}\psi & -\mathrm{s}\phi\,\mathrm{c}\psi + \mathrm{c}\phi\,\mathrm{s}\theta\,\mathrm{s}\psi \\ -\mathrm{s}\theta & \mathrm{s}\phi\,\mathrm{c}\theta & \mathrm{c}\phi\,\mathrm{c}\theta \end{bmatrix}$$

$$C^{\mathrm{NED}}_{\mathrm{ENU}} = C^{\mathrm{ENU}}_{\mathrm{NED}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

2 Quaternions

A quaternion can be thought of as a four element vector or as a sum:

$$q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = q_1\hat{\imath} + q_2\hat{\jmath} + q_3\hat{k} + q_4$$

where $\hat{\imath}$, and $\hat{k}$

$$\hat{\imath}\hat{\jmath} = \hat{k} \quad \hat{\jmath}\hat{k} = \hat{\imath} \quad \hat{k}\hat{\imath} = \hat{\jmath}$$

$$\hat{\jmath}\hat{\imath} = -\hat{k} \quad \hat{k}\hat{\jmath} = -\hat{\imath} \quad \hat{\imath}\hat{k} = -\hat{\jmath}$$

Quaternion multiplication (or composition) is defined as follows:

$$\begin{aligned}
a \cdot b &= \left(a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k} + a_4\right) \cdot \left(b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k} + b_4\right) \\
&= \left(a_1 b_4 + a_2 b_3 - a_3 b_2 + a_4 b_1\right)\hat{\imath} \\
&\quad + \left(-a_1 b_3 + a_2 b_4 + a_3 b_1 + a_4 b_2\right)\hat{\jmath} \\
&\quad + \left(a_1 b_2 - a_2 b_1 + a_3 b_4 + a_4 b_3\right)\hat{k} \\
&\quad - a_1 b_1 - a_2 b_2 - a_3 b_3 + a_4 b_4
\end{aligned}$$

$$\|q\| = \sqrt{q_1^2 + q_2^2 + q_3^2 + q_4^2}$$

Quaternion conjugation is defined by

$$q^* = \begin{bmatrix} -q_1 \\ -q_2 \\ -q_3 \\ q_4 \end{bmatrix} = -q_1\hat{\imath} - q_2\hat{\jmath} - q_3\hat{k} + q_4$$

The multiplicative inverse is given by

$$q^{-1} = \frac{q^*}{\|q\|^2}$$

Quaternion multiplication can also be performed using two equivalent matrix-vector forms:

$$a \cdot b = \begin{bmatrix} a_4 & -a_3 & a_2 & a_1 \\ a_3 & a_4 & -a_1 & a_2 \\ -a_2 & a_1 & a_4 & a_3 \\ -a_1 & -a_2 & -a_3 & a_4 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} b_4 & b_3 & -b_2 & b_1 \\ -b_3 & b_4 & b_1 & b_2 \\ b_2 & -b_1 & b_4 & b_3 \\ -b_1 & -b_2 & -b_3 & b_4 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$$

Quaternions obey the following properties:

$$(q^*)^* = q$$

$$\|q\|^2 = q^* \cdot q = q \cdot q^*$$

$$(p \cdot q)^* = q^* \cdot p^*$$

$$\|pq\| = \|p\| \, \|q\|$$

$$(p \cdot q)^{-1} = q^{-1} \cdot p^{-1}$$

3 Quaternions representing rotations

Quaternions can be used to represent rotations:

$$q = \begin{bmatrix} \mathbf{e} \sin\frac{\beta}{2} \\ \cos\frac{\beta}{2} \end{bmatrix} = \begin{bmatrix} e_x \sin\frac{\beta}{2} \\ e_y \sin\frac{\beta}{2} \\ e_z \sin\frac{\beta}{2} \\ \cos\frac{\beta}{2} \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$$

where $\mathbf{e} = (e_x, e_y, e_z)$ is the angle of rotation. A quaternion that represents a rotation must have unit length ($\|q\| = 1$). Also, note that $-q$ represents the same rotation as q.

A rotation from coordinate system x to coordinate system y is accomplished as follows:

$$
\begin{aligned}
y &= q \cdot x \cdot q^* \\
\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ 0 \end{bmatrix} &= \begin{bmatrix} q_4 & -q_3 & q_2 & q_1 \\ q_3 & q_4 & -q_1 & q_2 \\ -q_2 & q_1 & q_4 & q_3 \\ -q_1 & -q_2 & -q_3 & q_4 \end{bmatrix} \begin{bmatrix} q_4 & -q_3 & q_2 & -q_1 \\ q_3 & q_4 & -q_1 & -q_2 \\ -q_2 & q_1 & q_4 & -q_3 \\ q_1 & q_2 & q_3 & q_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \end{bmatrix} \\
&= \left[\begin{array}{ccc|c} q_1^2 - q_2^2 - q_3^2 + q_4^2 & 2(q_1q_2 - q_3q_4) & 2(q_1q_3 + q_2q_4) & 0 \\ 2(q_1q_2 + q_3q_4) & -q_1^2 + q_2^2 - q_3^2 + q_4^2 & 2(-q_1q_4 + q_2q_3) & 0 \\ 2(q_1q_3 - q_2q_4) & 2(q_1q_4 + q_2q_3) & -q_1^2 - q_2^2 + q_3^2 + q_4^2 & 0 \\ \hline 0 & 0 & 0 & q_1^2 + q_2^2 + q_3^2 + q_4^2 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \end{bmatrix}
\end{aligned}
$$

The rotation matrix C_x^y equivalent to q is the upper left 3×3 block from the equation above:

$$
C_x^y = \begin{bmatrix} q_1^2 - q_2^2 - q_3^2 + q_4^2 & 2(q_1q_2 - q_3q_4) & 2(q_1q_3 + q_2q_4) \\ 2(q_1q_2 + q_3q_4) & -q_1^2 + q_2^2 - q_3^2 + q_4^2 & 2(-q_1q_4 + q_2q_3) \\ 2(q_1q_3 - q_2q_4) & 2(q_1q_4 + q_2q_3) & -q_1^2 - q_2^2 + q_3^2 + q_4^2 \end{bmatrix}
$$

If q_x^y represents a rotation from x to y, and q_y^z represents a rotation from y to z, then the rotation from x to z is given by

$$q_x^z = q_y^z \cdot q_x^y.$$

4 Time derivative of a quaternion

Express a quaternion that is changing with time as a rotation followed by an infinitesimal rotation:

$$q(t + \Delta t) = (\Delta q) \cdot q(t)$$

An infinitesimal quaternion can be expressed as

$$
\Delta q = \begin{bmatrix} \mathbf{e} \sin \frac{\Delta\beta}{2} \\ \cos \frac{\Delta\beta}{2} \end{bmatrix} = \begin{bmatrix} \mathbf{e} \frac{\Delta\beta}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \boldsymbol{\omega} \Delta t \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\omega_x \Delta t \\ \frac{1}{2}\omega_y \Delta t \\ \frac{1}{2}\omega_z \Delta t \\ 1 \end{bmatrix},
$$

where we have used the fact that

$$\boldsymbol{\omega} = \lim_{\Delta t \to \infty} \frac{\Delta\beta}{\Delta t} \mathbf{e}.$$

$$q(t+\Delta t) = \begin{bmatrix} 1 & -\frac{1}{2}\omega_z\Delta t & \frac{1}{2}\omega_y\Delta t & \frac{1}{2}\omega_x\Delta t \\ \frac{1}{2}\omega_z\Delta t & 1 & -\frac{1}{2}\omega_x\Delta t & \frac{1}{2}\omega_y\Delta t \\ -\frac{1}{2}\omega_y\Delta t & \frac{1}{2}\omega_x\Delta t & 1 & \frac{1}{2}\omega_z\Delta t \\ -\frac{1}{2}\omega_x\Delta t & -\frac{1}{2}\omega_y\Delta t & -\frac{1}{2}\omega_z\Delta t & 1 \end{bmatrix} q(t)$$

$$\Psi(\boldsymbol{\omega}) \triangleq \begin{bmatrix} 0 & -\omega_z & \omega_y & \omega_x \\ \omega_z & 0 & -\omega_x & \omega_y \\ -\omega_y & \omega_x & 0 & \omega_z \\ -\omega_x & -\omega_y & -\omega_z & 0 \end{bmatrix}$$

$$q(t+\Delta t) = \left(I + \frac{1}{2}\Psi(\boldsymbol{\omega})\Delta t\right) q(t)$$

$$\frac{q(t+\Delta t) - q(t)}{\Delta t} = \frac{1}{2}\Psi(\boldsymbol{\omega})\, q(t)$$

Taking the limit as $\Delta t \to \infty$:

$$\dot{q}(t) = \frac{1}{2}\Psi(\boldsymbol{\omega})\, q(t)$$