Contents

1 Equations of Motion

1.1 Reference

Gravity force in inertial frame

$$\mathbf{g} = -g\mathbf{k}$$

Rotor force in body frame

$$\mathbf{F}_{R,B} = k\left(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4\right)\mathbf{k}$$

Rotor force in inertial frame

$$\mathbf{F}_R = \mathbf{R}(\mathbf{q})\mathbf{F}_{R,B}(\gamma)$$

Drag force in body frame

$$\mathbf{F}_{D,B}(\dot{\mathbf{x}}_B)$$

$$\dot{\mathbf{x}}_B = \mathbf{R}^{-1}\dot{\mathbf{x}}$$

Drag force in inertial frame

$$\mathbf{F}_D = \mathbf{R}(\mathbf{q})\mathbf{F}_{D,B}(\dot{\mathbf{x}}_B)$$

1.2 Translation

$$m\ddot{\mathbf{x}} = m\mathbf{g} + \mathbf{F}_R + \mathbf{F}_D$$

$$\ddot{\mathbf{x}} = \mathbf{g} + \frac{1}{m}\mathbf{F}_R + \frac{1}{m}\mathbf{F}_D$$

$$\ddot{\mathbf{x}} = \mathbf{g} + \frac{1}{m}\mathbf{R}\mathbf{F}_{R,B} + \frac{1}{m}\mathbf{R}\mathbf{F}_{D,B}$$

1.3 Rotation

$$\mathbf{I}\dot{\boldsymbol{\omega}} = \tau - \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega})$$

$$\dot{\boldsymbol{\omega}} = \mathbf{I}^{-1}\left(\boldsymbol{\tau} - \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega})\right)$$

$$I = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$

$$\boldsymbol{\tau}_B = \begin{bmatrix} Lk\left(\gamma_1 - \gamma_3\right) \\ Lk\left(\gamma_2 - \gamma_4\right) \\ b\left(\gamma_1 - \gamma_2 + \gamma_3 - \gamma_4\right) \end{bmatrix} = \begin{bmatrix} Lk & 0 & -Lk & 0 \\ 0 & Lk & 0 & -Lk \\ b & -b & b & -b \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{bmatrix}$$

1.3.1 Quaternion

$$\dot{\mathbf{q}} = \mathbf{A}_3\boldsymbol{\omega}$$

$$\ddot{\mathbf{q}} = \dot{\mathbf{A}}_3\boldsymbol{\omega} + \mathbf{A}_3\dot{\boldsymbol{\omega}}$$

$$\ddot{\mathbf{q}} = \dot{\mathbf{A}}_3\boldsymbol{\omega} + \mathbf{A}_3\mathbf{I}^{-1}\left(\boldsymbol{\tau} - \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega})\right)$$

1.3.2 Euler

$$\boldsymbol{\omega} = \begin{bmatrix} 1 & 0 & -s_{\boldsymbol{\theta}} \\ 0 & c_\phi & c_{\boldsymbol{\theta}} s_\phi \\ 0 & -s_\phi & c_{\boldsymbol{\theta}} s_\phi \end{bmatrix} \dot{\boldsymbol{\theta}} = \mathbf{A}_5^{-1}\dot{\boldsymbol{\theta}}$$

$$\dot{\boldsymbol{\theta}} = \mathbf{A}_5\boldsymbol{\omega}$$

$$\ddot{\boldsymbol{\theta}} = \dot{\mathbf{A}}_5\boldsymbol{\omega} + \mathbf{A}_5\dot{\boldsymbol{\omega}}$$

$$\ddot{\boldsymbol{\theta}} = \dot{\mathbf{A}}_5\boldsymbol{\omega} + \mathbf{A}_5\left(\mathbf{I}^{-1}\left(\boldsymbol{\tau} - \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega})\right)\right)$$

$$\ddot{\boldsymbol{\theta}} = \mathbf{S}^{-1}\left(\mathbf{I}^{-1}\left(\boldsymbol{\tau} - \boldsymbol{\omega} \times (I\boldsymbol{\omega})\right) - \dot{\mathbf{S}}\dot{\boldsymbol{\theta}}\right) \tag{1}$$

2 Control

2.1 Attitude

2.1.1 Quaternion

$$\mathbf{e}_1 = \mathbf{q}_{ref}\mathbf{q}^*$$

$$\mathbf{e}_2 = \boldsymbol{\omega}_{ref} - \boldsymbol{\omega}$$

2.2 Position

The position control law outputs net jerk in the inertial frame.

$$\begin{aligned}
\mathbf{j} = &+ \mathbf{C}_9\boldsymbol{\chi}_5 \\
&+ \mathbf{C}_{10}(\mathbf{x}_{ref} - \mathbf{x}) \\
&+ \mathbf{C}_{11}(\dot{\mathbf{x}}_{ref} - \mathbf{v}) \\
&+ \mathbf{C}_{12}(\ddot{\mathbf{x}}_{ref} - \mathbf{a}) \\
&+ \dddot{\mathbf{x}}_{ref}
\end{aligned}$$

From this jerk, x- and y-components of angular velcity and rate of change of net thrust are calculated.

Ignoring drag, the net force in the global frame is

$$\mathbf{f} = \mathbf{f}_g + \mathbf{f}_R$$

Differentiate in time

$$\dot{\mathbf{f}} = \dot{\mathbf{f}}_R = m\mathbf{j}$$

Rotor force in the inertial and body frame is related by

$$\mathbf{f}_R = \mathbf{q}^*\mathbf{f}_{RB}\mathbf{q}$$

Differentiate in time

$$\dot{\mathbf{f}}_R = \mathbf{q}^*\left(\dot{\mathbf{f}}_{RB} + \mathbf{f}_{RB} \times \boldsymbol{\omega}\right)\mathbf{q}$$

In our case

$$\mathbf{f}_{RB} = \begin{bmatrix} 0 \\ 0 \\ f_{RB} \end{bmatrix}$$

and the above becomes

$$\mathbf{q}\dot{\mathbf{f}}_R\mathbf{q}^* = \begin{bmatrix} -\omega_y f_{zRB} \\ \omega_x f_{zRB} \\ \dot{f}_{zRB} \end{bmatrix}$$

The left side is known. First, the derivative of rotor force from the z-component is integrated to get rotor force for the current time step.

$$f_{zRB}(t_i) = f_{zRB}(t_{i-1}) + \dot{f}_{zRB}$$

The angular velocity components can then be calculated. These are the reference values for the second controller with outputs torque.

2.2.1 Time Derivative of quaternion rotation

In general, if q rotates from y to x frame

$$\mathbf{y} = \mathbf{q}^*\mathbf{x}\mathbf{q}$$

$$\dot{\mathbf{y}} = \dot{\mathbf{q}}^*\mathbf{x}\mathbf{q} + \mathbf{q}^*\dot{\mathbf{x}}\mathbf{q} + \mathbf{q}^*\mathbf{x}\dot{\mathbf{q}}$$

$$\dot{\mathbf{y}} = \mathbf{q}^*\boldsymbol{\omega}^*\mathbf{x}\mathbf{q} + \mathbf{q}^*\dot{\mathbf{x}}\mathbf{q} + \mathbf{q}^*\mathbf{x}\boldsymbol{\omega}\mathbf{q}$$

$$\dot{\mathbf{y}} = \mathbf{q}^*\left(\boldsymbol{\omega}^*\mathbf{x} + \dot{\mathbf{x}} + \mathbf{x}\boldsymbol{\omega}\right)\mathbf{q}$$

$$\mathbf{q}\dot{\mathbf{y}}\mathbf{q}^* = \boldsymbol{\omega}^*\mathbf{x} + \dot{\mathbf{x}} + \mathbf{x}\boldsymbol{\omega}$$

which leads to the simple result

$$\boldsymbol{\omega}^*\mathbf{x} + \dot{\mathbf{x}} + \mathbf{x}\boldsymbol{\omega} = \begin{bmatrix} 0 \\ -\omega_y a_{RB} \\ \omega_x a_{RB} \\ \dot{a}_{RB} \end{bmatrix}$$

Notice that ω_z does not appear because it has no effect on the translation of the vehicle. In order to solve for ω_x, ω_y, and a_{RB}, we need a relationship between a_{RB} and its derivative. For example, we could use forward differencing.