Nonlinear Control Systems - Lecture 25

- **Backstepping**

Consider the system

$$\begin{aligned}\dot z &= f(z) + g(z)\xi \\ \dot \xi &= u\end{aligned}$$

We want to design a state feedback control law to stabilize the origin. We view this as a cascade connection of two components of which the first is an integrator (see figure 1).

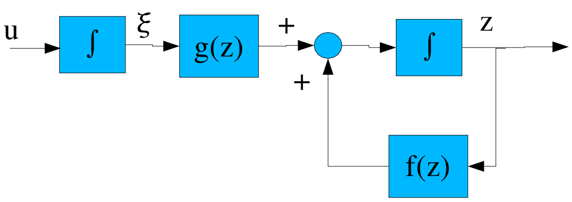

Fig. 1. Backstepping - initial system

Suppose we can asympotically stabilize the first system,

$$\dot z = f(z) + g(z)\xi$$

with a control law $\xi = \phi(z)$ in which $\phi(0) = 0$. This implies that the origin of

$$\dot z = f(z) + g(z)\phi(z)$$

is asymptotically stable.

Suppose also that we know a Lyapunov function $V(z)$ that satisfies the inequality,

$$\frac{\partial V}{\partial z}[f + g\phi] \le -W(z)$$

where $W(z)$ is positive definite. Add and subtract $g(z)\phi(z)$ to obtain

$$\begin{aligned}\dot z &= [f(z) + g(z)\phi(z)] + g(z)[\xi - \phi(z)] \\ \dot \xi &= u\end{aligned}$$

This resulting system is shown in figure 2.

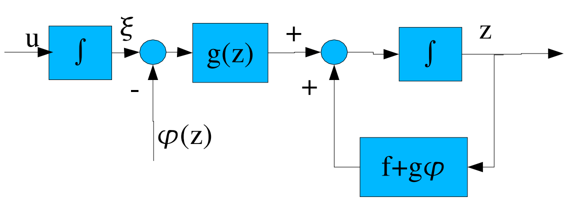

Fig. 2. Introducing Control

We now introduce the change of variables,

$$y = \xi - \phi(z)$$

which generates the following system equations

$$\begin{aligned}\dot z &= [f(z) + g(z)\phi(z)] + g(z)y \\ \dot y &= u - \dot\phi(z)\end{aligned}$$

which is shown in figure 3. This change of variables is often called **backstepping** since it "backsteps" the control $-\phi(z)$ through the integrator.

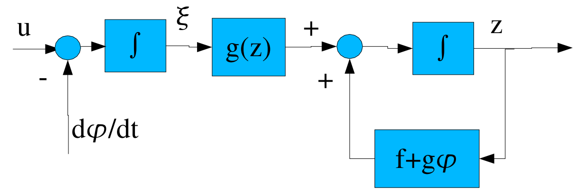

Fig. 3. Backstepping Control Through Integrator

Since f, g, and ϕ are known, the derivative can be written as

$$\dot\phi(z) = \frac{\partial \phi}{\partial z}[f(z) + g(z)\xi]$$

Letting $v = u - \dot\phi$ reduces our system to

$$\begin{aligned}\dot z &= [f(z) + g(z)\phi(z)] + g(z)y \\ \dot y &= v\end{aligned}$$

which has the same "form" as the system we started with the exception that we now know the first system is asymptotically stable at the origin. This modular property of backstepping will be exploited to stabilize the overall system.

This is done by considering a candidate Lyapunov function

$$\begin{aligned}V_c(z,\xi) &= V(z) + \frac{1}{2}y^2 \\ &= V(z) + \frac{1}{2}(\xi - \phi(z))^2\end{aligned}$$

The derivative of V_c is

$$\begin{aligned}\dot V_c &= \frac{\partial V}{\partial z}[f(z) + g(z)\phi(z)] + \frac{\partial V}{\partial z} g(z) y + yv \\ &\le -W(z) + \frac{\partial V}{\partial z} g(z) y + yv\end{aligned}$$

Choose

$$v = -\frac{\partial V}{\partial z} g(z) - ky$$

where $k > 0$. This implies that

$$\dot V_c \le -W(z) = ky^2$$

which implies the origin is asympoitcally stable ($z = 0, y = 0$). Since $\phi(0) = 0$, this implies that $z = 0$ and $\xi = 0$ is also asymptotically stable.

Backstepping Lemma: Consider the system

$$\dot z = f(z) + g(z)\xi \tag{1}$$

$$\dot \xi = u \tag{2}$$

Let $\phi(z)$ be a stabilizing state feedback law for the system in equation 1 where $\phi(0) = 0$. Let $V(z)$ be a Lyapunov function such that

$$\frac{\partial V}{\partial z}\left[f(z) + g(z)\phi(z)\right] \leq -W(z)$$

for some positive definite W. Then the feedback law

$$u = \frac{\partial \phi}{\partial z}\left[f + g\xi\right] - \frac{\partial V}{\partial z} g(z) - k\left[\xi - \phi(z)\right]$$

for $k > 0$ stabilizes the origin with the Lyapunov function

$$V(z) = \frac{1}{2}\left[\xi - \phi(z)\right]^2$$

Example: Consider the system

$$\begin{aligned} \dot{z} &= z^2 - z^3 + \xi \\ \dot{\xi} &= u \end{aligned}$$

Consider the scalar system,

$$\dot{z} = z^2 - z^3 + \xi$$

and design a feedback control law which might be

$$\xi = \phi(z) = -z^2 - z$$

(cancels quadratic term and adds some damping). This implies that

$$\dot{z} = -z - z^3$$

with candidate Lyapunov function $V(z) = z^2/2$. Note that

$$\dot{V} = -z^2 - z^4 \leq -z^2$$

which implies that $z = 0$ is asymptotically stable equilibrium point.

Now use the backstepping change of variables,

$$y = \xi - \phi(z) = \xi + z + z^2$$

to transform our system to

$$\begin{aligned} \dot{z} &= -z - z^3 + y \\ \dot{y} &= u + (1 + 2z)(-z - z^3 + y) \end{aligned}$$

and let

$$V_c = \frac{1}{2}z^2 + \frac{1}{2}y^2$$

Then

$$\begin{aligned} \dot{V}_c &= z(-z - z^3 + y) + y(u + (1 + 2z)(-z - z^3 + y)) \\ &= -z^2 - z^4 + y(z + (1 + 2z)(-z - z^3 + y) + u) \end{aligned}$$

and choose

$$u = -z - (1 + 2z)(-z - z^3 + y) - y$$

to force

$$\dot{V}_c = -z^2 - z^4 - y^2$$

This is clearly negative definite so $(z, y) = (0, 0)$ is asymptotically stable.

The actual control in our original coordinates is

$$\begin{aligned} u &= -z - (1 + 2z)(-z - z^3 + \xi + z + z^2) \\ &\quad -\xi - z - z^2 \\ &= -z - (1 + 2z)(z^2 - z^3 + \xi) - \xi - z - z^2 \end{aligned}$$

The real value of backstepping is seen in higher order system where we can exploit the modularity of the approach. Consider this example,

$$\begin{aligned} \dot{z}_1 &= z_1^2 - z_1^3 + z_2 \\ \dot{z}_2 &= \xi \\ \dot{\xi} &= u \end{aligned}$$

This system conatins the second order system of the preceding example with an additional integrator as the input.

After the first backstep, we know the second order system

$$\begin{aligned} \dot{z}_1 &= z_1^2 - z_1^3 + z_2 \\ \dot{z}_2 &= \xi \end{aligned}$$

can be globally stabilized by the control

$$\begin{aligned} \xi &= -z_1 - (1 + 2z_1)(z_1^2 - z_1^3 + z_2) - (z_2 + z_1 + z_1^2) \\ &= \phi(z_1, z_2) \end{aligned}$$

with Lyapunov function

$$V(z_1, z_2) = \frac{1}{2}z_1^2 + \frac{1}{2}(z_2 + z_1 + z_1^2)^2$$

To do the second backstep, we apply the change of variables

$$y = \xi - \phi(z_1, z_2)$$

to obtain

$$\begin{aligned} \dot{z}_1 &= z_1^2 - z_1^3 + z_2 \\ \dot{z}_2 &= \phi(z_1, z_2) + y \\ \dot{y} &= u - \frac{\partial \phi}{\partial z_1}(z_1^2 - z_1^3 + z_2) - \frac{\partial \phi}{\partial z_2}(\phi + y) \end{aligned}$$

using $V_c = V + \frac{1}{2}z_3^2$ as the candidate Lyapunov function , we obtain

$$\begin{aligned} \dot{V}_c &= \frac{\partial V}{\partial z_1}(z_1^2 - z_1^3 + z_2) + \frac{\partial V}{\partial z_2}(y + \phi) \\ &\quad + y\left[u - \frac{\partial \phi}{\partial z_1}(z_1^2 - z_1^3 + z_2) - \frac{\partial \phi}{\partial z_2}(y + \phi)\right] \\ &= -z_1^2 - z_1^4 - (z_2 + z_1 + z_1^2)^2 \\ &\quad + y\left[\frac{\partial V}{\partial z_2} - \frac{\partial \phi}{\partial z_1}(z_1^2 - z_1^3 + z_2)\right. \\ &\quad \left. - \frac{\partial \phi}{\partial z_2}(y + \phi) + u\right] \end{aligned}$$

So we take

$$u = -\frac{\partial V}{\partial z_2} + \frac{\partial \phi}{\partial z_1}(z_1^2 - z_1^3 + z_2) + \frac{\partial \phi}{\partial z_2}(y + \phi) - y$$

to force $\dot{V}_c < 0$ and thereby stabilize the origin.

We now consider the more general system

$$\begin{aligned} \dot{z} &= f(z) + g(z)\xi \\ \dot{\xi} &= f_a(z,\xi) + g_a(z,\xi)u \end{aligned}$$

This can be reduced to our origina form using the control input

$$u = \frac{1}{g_a(z,\xi)}\left[v - f_a(z,\xi)\right]$$

so that $\dot{\xi} = v$.

Recursive application of backstepping allows us to stabilze systems that are in **strict feedback form**

$$\begin{aligned} \dot{x} &= f_0(x) + g_0(x)z_1 \\ \dot{z}_1 &= f_1(x,z_1) + g_1(x,z_1)z_2 \\ \dot{z}_2 &= f_2(x,z_1,z_2) + g_2(x,z_1,z_3)z_3 \\ \vdots &= \vdots \\ \dot{z}_{k-1} &= f_{k-1}(x,z_1,\cdots,z_{k-1}) + g_{k-1}(x,z_1,\cdots,z_{k-1})z_k \\ \dot{z}_k &= f_k(x,z_1,\cdots,z_k) + g(x,z_1,\cdots,z_k)u \end{aligned}$$

Note that in this form $\dot{z}_i$ only depends on states x, z_1, through z_i.

The recursive application of backstepping to this system with

$$\dot{x} = f_0(x) + g_0(x)z_1$$

(viewing z_1 as the input). It is important that we find a control law (somehow) that stabilizes this SCALAR system. But once this is done, we can simply apply the backstepping recursions to find a control that stabilizes the entire system.