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1 Equations of Motion

1.1 Reference

Gravity force in inertial frame

$$\mathbf{g} = -g\mathbf{k}$$

Rotor force in body frame

$$\mathbf{F}_{R,B} = k(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4)\mathbf{k}$$

Rotor force in inertial frame

$$\mathbf{F}_R = \mathbf{R}(\mathbf{q})\mathbf{F}_{R,B}(\gamma)$$

Drag force in body frame

$$\mathbf{F}_{D,B}(\dot{\mathbf{x}}_B)$$

$$\dot{\mathbf{x}}_B = \mathbf{R}^{-1}\dot{\mathbf{x}}$$

Drag force in inertial frame

$$\mathbf{F}_D = \mathbf{R}(\mathbf{q})\mathbf{F}_{D,B}(\dot{\mathbf{x}}_B)$$

1.2 Translation

$$m\ddot{\mathbf{x}} = m\mathbf{g} + \mathbf{F}_R + \mathbf{F}_D$$

$$\ddot{\mathbf{x}} = \mathbf{g} + \frac{1}{m}\mathbf{F}_R + \frac{1}{m}\mathbf{F}_D$$

$$\ddot{\mathbf{x}} = \mathbf{g} + \frac{1}{m}\mathbf{R}\mathbf{F}_{R,B} + \frac{1}{m}\mathbf{R}\mathbf{F}_{D,B}$$

1.3 Rotation

$$\mathbf{I}\dot{\boldsymbol{\omega}} = \boldsymbol{\tau} - \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega})$$

$$\dot{\boldsymbol{\omega}} = \mathbf{I}^{-1} (\boldsymbol{\tau} - \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega}))$$

$$I = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$

$$\boldsymbol{\tau}_B = \begin{bmatrix} Lk(\gamma_1 - \gamma_3) \\ Lk(\gamma_2 - \gamma_4) \\ b(\gamma_1 - \gamma_2 + \gamma_3 - \gamma_4) \end{bmatrix} = \begin{bmatrix} Lk & 0 & -Lk & 0 \\ 0 & Lk & 0 & -Lk \\ b & -b & b & -b \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{bmatrix}$$

1.3.1 Quaternion

$$\dot{\mathbf{q}} = \mathbf{A}_3\boldsymbol{\omega}$$

$$\ddot{\mathbf{q}} = \dot{\mathbf{A}}_3\boldsymbol{\omega} + \mathbf{A}_3\dot{\boldsymbol{\omega}}$$

$$\ddot{\mathbf{q}} = \dot{\mathbf{A}}_3\boldsymbol{\omega} + \mathbf{A}_3\mathbf{I}^{-1} (\boldsymbol{\tau} - \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega}))$$

1.3.2 Euler

$$\boldsymbol{\omega} = \begin{bmatrix} 1 & 0 & -s_\theta \\ 0 & c_\phi & c_\theta s_\phi \\ 0 & -s_\phi & c_\theta s_\phi \end{bmatrix} \dot{\boldsymbol{\theta}} = \mathbf{A}_5^{-1} \dot{\boldsymbol{\theta}}$$

$$\dot{\boldsymbol{\theta}} = \mathbf{A}_5\boldsymbol{\omega}$$

$$\ddot{\boldsymbol{\theta}} = \dot{\mathbf{A}}_5\boldsymbol{\omega} + \mathbf{A}_5\dot{\boldsymbol{\omega}}$$

$$\ddot{\boldsymbol{\theta}} = \dot{\mathbf{A}}_5\boldsymbol{\omega} + \mathbf{A}_5 (\mathbf{I}^{-1} (\boldsymbol{\tau} - \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega})))$$

$$\ddot{\boldsymbol{\theta}} = \mathbf{S}^{-1} \left(\mathbf{I}^{-1} (\boldsymbol{\tau} - \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega})) - \dot{\mathbf{S}}\dot{\boldsymbol{\theta}} \right) \quad (1)$$

2 Control

2.1 Attitude

2.1.1 Quaternion

$$\mathbf{e}_1 = \mathbf{q}_{ref} \mathbf{q}^*$$

$$\mathbf{e}_2 = \boldsymbol{\omega}_{ref} - \boldsymbol{\omega}$$

2.2 Position

$$\mathbf{e}_5 = \mathbf{x}_{ref} - \mathbf{x}$$

$$\dot{\mathbf{e}}_5 = \dot{\mathbf{x}}_{ref} - \dot{\mathbf{x}}$$

$$\dot{\mathbf{e}}_5 = \dot{\mathbf{x}}_{ref} - \mathbf{v}$$

$$\mathbf{v}_{ref} = \mathbf{C}_5 \mathbf{e}_5 + \dot{\mathbf{x}}_{ref} + \boldsymbol{\Lambda}_5 \boldsymbol{\chi}_5$$

$$\dot{\mathbf{v}}_{ref} = \mathbf{C}_5 \dot{\mathbf{e}}_5 + \ddot{\mathbf{x}}_{ref} + \boldsymbol{\Lambda}_5 \mathbf{e}_5$$

$$\mathbf{e}_6 = \mathbf{v}_{ref} - \mathbf{v}$$

$$\mathbf{e}_6 = \mathbf{C}_5 \mathbf{e}_5 + \dot{\mathbf{x}}_{ref} + \boldsymbol{\Lambda}_5 \boldsymbol{\chi}_5 - \mathbf{v}$$

$$\dot{\mathbf{e}}_6 = \dot{\mathbf{v}}_{ref} - \dot{\mathbf{v}} = \dot{\mathbf{v}}_{ref} - \mathbf{a}$$

$$\dot{\mathbf{e}}_6 = \mathbf{C}_5 \dot{\mathbf{e}}_5 + \ddot{\mathbf{x}}_{ref} + \boldsymbol{\Lambda}_5 \mathbf{e}_5 - \ddot{\mathbf{x}}$$

$$\dot{\mathbf{x}}_{ref} = \mathbf{v}_{ref} - \mathbf{C}_5 \mathbf{e}_5 - \boldsymbol{\Lambda}_5 \boldsymbol{\chi}_5$$

$$\dot{\mathbf{x}}_{ref} = \mathbf{e}_6 + \mathbf{v} - \mathbf{C}_5 \mathbf{e}_5 - \boldsymbol{\Lambda}_5 \boldsymbol{\chi}_5$$

$$\dot{\mathbf{e}}_5 = \mathbf{e}_6 - \mathbf{C}_5 \mathbf{e}_5 - \boldsymbol{\Lambda}_5 \boldsymbol{\chi}_5$$

$$\dot{\mathbf{e}}_6 = \mathbf{f}_6(\mathbf{e}_5, \mathbf{e}_6)$$

$$\mathbf{f}_6(\mathbf{e}_5, \mathbf{e}_6) = \mathbf{C}_5 (\mathbf{e}_6 - \mathbf{C}_5 \mathbf{e}_5 - \boldsymbol{\Lambda}_5 \boldsymbol{\chi}_5) + \ddot{\mathbf{x}}_{ref} + \boldsymbol{\Lambda}_5 \mathbf{e}_5 - \ddot{\mathbf{x}}$$

$$\ddot{\mathbf{x}} = -\mathbf{f}_6(\mathbf{e}_5, \mathbf{e}_6) + \mathbf{C}_5 (\mathbf{e}_6 - \mathbf{C}_5 \mathbf{e}_5 - \mathbf{\Lambda}_5 \chi_5) + \ddot{\mathbf{x}}_{ref} + \mathbf{\Lambda}_5 \mathbf{e}_5$$

$$\mathbf{F}_R = m \left(-\mathbf{C}_7 \mathbf{e}_5 - \mathbf{C}_8 \mathbf{e}_6 + \mathbf{C}_5 (\mathbf{e}_6 - \mathbf{C}_5 \mathbf{e}_5 - \mathbf{\Lambda}_5 \chi_5) + \ddot{\mathbf{x}}_{ref} + \mathbf{\Lambda}_5 \mathbf{e}_5 - \mathbf{g} - \frac{1}{m} \mathbf{F}_D \right)$$

$$\mathbf{F}_R = m \left((-\mathbf{C}_7 - \mathbf{C}_5^2 + \mathbf{\Lambda}_5) \mathbf{e}_5 + (\mathbf{C}_8 + \mathbf{C}_5) \mathbf{e}_6 - \mathbf{C}_5 \mathbf{\Lambda}_5 \chi_5 + \ddot{\mathbf{x}}_{ref} - \mathbf{g} - \frac{1}{m} \mathbf{F}_D \right)$$

$$\begin{aligned} \mathbf{F}_R = m \big(& (-\mathbf{C}_7 - \mathbf{C}_5^2 + \mathbf{\Lambda}_5) \mathbf{e}_5 \\ & + (-\mathbf{C}_8 + \mathbf{C}_5) (\mathbf{C}_5 \mathbf{e}_5 + \dot{\mathbf{x}}_{ref} + \mathbf{\Lambda}_5 \chi_5 - \mathbf{v}) \\ & - \mathbf{C}_5 \mathbf{\Lambda}_5 \chi_5 + \ddot{\mathbf{x}}_{ref} - \mathbf{g} - \frac{1}{m} \mathbf{F}_D \big) \end{aligned}$$

$$\begin{aligned} \frac{1}{m} \mathbf{F}_R = & (-\mathbf{C}_7 - \mathbf{C}_5^2 + \mathbf{\Lambda}_5 + (-\mathbf{C}_8 + \mathbf{C}_5) \mathbf{C}_5) \mathbf{x}_{ref} \\ & - (-\mathbf{C}_7 - \mathbf{C}_5^2 + \mathbf{\Lambda}_5 + (-\mathbf{C}_8 + \mathbf{C}_5) \mathbf{C}_5) \mathbf{x} \\ & + (-\mathbf{C}_8 + \mathbf{C}_5) \dot{\mathbf{x}}_{ref} \\ & + (-\mathbf{C}_8) \mathbf{\Lambda}_5 \chi_5 \\ & - (-\mathbf{C}_8 + \mathbf{C}_5) \mathbf{v} \\ & + \ddot{\mathbf{x}}_{ref} - \mathbf{g} - \frac{1}{m} \mathbf{F}_D \end{aligned}$$

$$\begin{aligned} 0 = & + \mathbf{C}_{11} \chi_5 \\ & + \mathbf{C}_9 (\mathbf{x}_{ref} - \mathbf{x}) \\ & + \mathbf{C}_{10} (\dot{\mathbf{x}}_{ref} - \mathbf{v}) \\ & + \ddot{\mathbf{x}}_{ref} - \ddot{\mathbf{x}} \end{aligned}$$

2.3 Dynamics of Acceleration

Instead of assuming $\ddot{\mathbf{x}}$ is our control input, we can continue the above process. We guess that the result will be

$$\begin{aligned} 0 = & + \mathbf{C}_9 \chi_5 \\ & + \mathbf{C}_{10} (\mathbf{x}_{ref} - \mathbf{x}) \\ & + \mathbf{C}_{11} (\dot{\mathbf{x}}_{ref} - \mathbf{v}) \\ & + \mathbf{C}_{12} (\ddot{\mathbf{x}}_{ref} - \mathbf{a}) \\ & + \ddot{\mathbf{x}}_{ref} - \ddot{\mathbf{a}} \end{aligned}$$

$$\mathbf{a} = \mathbf{a}_g + \mathbf{a}_R$$

$$\dot{\mathbf{a}} = \dot{\mathbf{a}}_R$$

$$\mathbf{a}_R = \mathbf{A}_{\mathbf{q}^*} \mathbf{B}_{\mathbf{q}^*} \mathbf{a}_{RB}$$

$$\dot{\mathbf{a}}_R = \mathbf{A}_{\dot{\mathbf{q}}^*} \mathbf{B}_{\mathbf{q}^*} \mathbf{a}_{RB} + \mathbf{A}_{\mathbf{q}^*} \mathbf{B}_{\dot{\mathbf{q}}^*} \mathbf{a}_{RB} + \mathbf{A}_{\mathbf{q}^*} \mathbf{B}_{\mathbf{q}^*} \dot{\mathbf{a}}_{RB}$$

2.3.1 Transfer Function

$$\begin{aligned} \ddot{x} = & (c_9 y - c_9 x + c_{10} \dot{y} - c_{10} \dot{x} \\ & + c_{11} \int y - c_{11} \int x \\ & + \ddot{y} - g - \frac{F_D}{m}) \end{aligned}$$

$$\begin{aligned} s^2 X - sx(0) - \dot{x}(0) = & (c_9 Y - c_9 X \\ & + c_{10} s Y - c_{10} y(0) - c_{10} s X + c_{10} x(0) \\ & + c_{11} \frac{Y}{s} - c_{11} \int x \\ & + \ddot{y} - g - \frac{F_D}{m}) \end{aligned}$$