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## 1 Equations of Motion

### 1.1 Reference

Gravity force in inertial frame

$$\mathbf{g} = -g\mathbf{k}$$

Rotor force in body frame

$$\mathbf{F}_{R,B} = k(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4)\mathbf{k}$$

Rotor force in inertial frame

$$\mathbf{F}_R = \mathbf{R}(\mathbf{q})\mathbf{F}_{R,B}(\gamma)$$

Drag force in body frame

$$\mathbf{F}_{D,B}(\dot{\mathbf{x}}_B)$$

$$\dot{\mathbf{x}}_B = \mathbf{R}^{-1}\dot{\mathbf{x}}$$

Drag force in inertial frame

$$\mathbf{F}_D = \mathbf{R}(\mathbf{q})\mathbf{F}_{D,B}(\dot{\mathbf{x}}_B)$$

## 1.2 Translation

$$m\ddot{\mathbf{x}} = m\mathbf{g} + \mathbf{F}_R + \mathbf{F}_D$$

$$\ddot{\mathbf{x}} = \mathbf{g} + \frac{1}{m}\mathbf{F}_R + \frac{1}{m}\mathbf{F}_D$$

$$\ddot{\mathbf{x}} = \mathbf{g} + \frac{1}{m}\mathbf{R}\mathbf{F}_{R,B} + \frac{1}{m}\mathbf{R}\mathbf{F}_{D,B}$$

## 1.3 Rotation

$$\mathbf{I}\dot{\boldsymbol{\omega}} = \boldsymbol{\tau} - \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega})$$

$$\dot{\boldsymbol{\omega}} = \mathbf{I}^{-1} (\boldsymbol{\tau} - \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega}))$$

$$I = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$

$$\boldsymbol{\tau}_B = \begin{bmatrix} Lk(\gamma_1 - \gamma_3) \\ Lk(\gamma_2 - \gamma_4) \\ b(\gamma_1 - \gamma_2 + \gamma_3 - \gamma_4) \end{bmatrix} = \begin{bmatrix} Lk & 0 & -Lk & 0 \\ 0 & Lk & 0 & -Lk \\ b & -b & b & -b \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{bmatrix}$$

### 1.3.1 Quaternion

$$\dot{\mathbf{q}} = \mathbf{A}_3\boldsymbol{\omega}$$

$$\ddot{\mathbf{q}} = \dot{\mathbf{A}}_3\boldsymbol{\omega} + \mathbf{A}_3\dot{\boldsymbol{\omega}}$$

$$\ddot{\mathbf{q}} = \dot{\mathbf{A}}_3\boldsymbol{\omega} + \mathbf{A}_3\mathbf{I}^{-1} (\boldsymbol{\tau} - \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega}))$$

### 1.3.2 Euler

$$\boldsymbol{\omega} = \begin{bmatrix} 1 & 0 & -s_\theta \\ 0 & c_\phi & c_\theta s_\phi \\ 0 & -s_\phi & c_\theta s_\phi \end{bmatrix} \dot{\boldsymbol{\theta}} = \mathbf{A}_5^{-1} \dot{\boldsymbol{\theta}}$$

$$\dot{\boldsymbol{\theta}} = \mathbf{A}_5\boldsymbol{\omega}$$

$$\ddot{\boldsymbol{\theta}} = \dot{\mathbf{A}}_5\boldsymbol{\omega} + \mathbf{A}_5\dot{\boldsymbol{\omega}}$$

$$\ddot{\boldsymbol{\theta}} = \dot{\mathbf{A}}_5\boldsymbol{\omega} + \mathbf{A}_5 (\mathbf{I}^{-1} (\boldsymbol{\tau} - \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega})))$$

$$\ddot{\boldsymbol{\theta}} = \mathbf{S}^{-1} \left( \mathbf{I}^{-1} (\boldsymbol{\tau} - \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega})) - \dot{\mathbf{S}}\dot{\boldsymbol{\theta}} \right) \quad (1)$$

## 2 Control

### 2.1 Attitude

#### 2.1.1 Euler

Tracking error

$$\mathbf{e}_1 = \boldsymbol{\theta}_{ref} - \boldsymbol{\theta}$$

$$\dot{\mathbf{e}}_1 = \dot{\boldsymbol{\theta}}_{ref} - \dot{\boldsymbol{\theta}}$$

$$\dot{\mathbf{e}}_1 = \dot{\boldsymbol{\theta}}_{ref} - \mathbf{A}_5 \boldsymbol{\omega} \quad (2)$$

The  $\boldsymbol{\omega}$  is not our control input and has its own dynamic. So we set for it a desired behavior and consider it as our virtual control

$$\boldsymbol{\omega}_{ref} = c_1 \mathbf{e}_1 + \mathbf{A}_5^{-1} \dot{\boldsymbol{\theta}}_{ref} + \lambda_1 \boldsymbol{\chi}_1 \quad (3)$$

$$\mathbf{e}_2 = \boldsymbol{\omega}_{ref} - \boldsymbol{\omega} \quad (4)$$

$$\dot{\mathbf{e}}_2 = \dot{\boldsymbol{\omega}}_{ref} - \dot{\boldsymbol{\omega}}$$

$$\dot{\mathbf{e}}_2 = c_1 \dot{\mathbf{e}}_1 + \ddot{\boldsymbol{\theta}}_{ref} + \lambda_1 \mathbf{e}_1 - \ddot{\boldsymbol{\theta}} \quad (5)$$

Solve (3) for  $\dot{\boldsymbol{\theta}}_{ref}$

$$\dot{\boldsymbol{\theta}}_{ref} = \mathbf{A}_5 (\boldsymbol{\omega}_{ref} - c_1 \mathbf{e}_1 - \lambda_1 \boldsymbol{\chi}_1) \quad (6)$$

Solve (4) for  $\boldsymbol{\omega}_{ref}$

$$\boldsymbol{\omega}_{ref} = \mathbf{e}_2 + \boldsymbol{\omega} \quad (7)$$

Combine (6) and (7)

$$\dot{\boldsymbol{\theta}}_{ref} = \mathbf{A}_5 (\mathbf{e}_2 + \boldsymbol{\omega} - c_1 \mathbf{e}_1 - \lambda_1 \boldsymbol{\chi}_1) \quad (8)$$

Rewrite (2) using (8)

$$\dot{\mathbf{e}}_1 = \mathbf{A}_5 (\mathbf{e}_2 + \boldsymbol{\omega} - c_1 \mathbf{e}_1 - \lambda_1 \boldsymbol{\chi}_1) - \mathbf{A}_5 \boldsymbol{\omega}$$

$$\dot{\mathbf{e}}_1 = \mathbf{A}_5 (\mathbf{e}_2 - c_1 \mathbf{e}_1 - \lambda_1 \boldsymbol{\chi}_1)$$

Replace  $\ddot{\boldsymbol{\theta}}$  in (5) with its model in (1)

$$\dot{\mathbf{e}}_2 = c_1 \dot{\mathbf{e}}_1 + \ddot{\boldsymbol{\theta}}_{ref} + \lambda_1 \mathbf{e}_1 - \left( \dot{\mathbf{A}}_3 \boldsymbol{\omega} + \mathbf{A}_5 \mathbf{I}^{-1} (\boldsymbol{\tau} - \boldsymbol{\omega} \times (\mathbf{I} \boldsymbol{\omega})) \right) \quad (9)$$

$$\begin{aligned}\dot{\mathbf{e}}_2 = & \mathbf{C}_1 \mathbf{A}_5 (\mathbf{e}_2 - \mathbf{C}_1 \mathbf{e}_1 - \mathbf{\Lambda}_1 \chi_1) + \ddot{\boldsymbol{\theta}}_{ref} + \mathbf{\Lambda}_1 \mathbf{e}_1 \\ & - \left( \dot{\mathbf{A}}_3 \boldsymbol{\omega} + \mathbf{A}_5 \mathbf{I}^{-1} (\boldsymbol{\tau} - \boldsymbol{\omega} \times (\mathbf{I} \boldsymbol{\omega})) \right)\end{aligned}\quad (10)$$

We now choose a desirable form for the dynamics of the angular speed tracking error. An example would be

$$\dot{\mathbf{e}}_2 = -c_2 \mathbf{e}_2 - \mathbf{e}_1$$

but for now we will generalize this as

$$\dot{\mathbf{e}}_2 = \mathbf{f}_2 (\mathbf{e}_1, \mathbf{e}_2)$$

$$\begin{aligned}\mathbf{f}_2 = & \mathbf{C}_1 \mathbf{A}_5 (\mathbf{e}_2 - \mathbf{C}_1 \mathbf{e}_1 - \mathbf{\Lambda}_1 \chi_1) + \ddot{\boldsymbol{\theta}}_{ref} + \mathbf{\Lambda}_1 \mathbf{e}_1 \\ & - \left( \dot{\mathbf{A}}_3 \boldsymbol{\omega} + \mathbf{A}_5 \mathbf{I}^{-1} (\boldsymbol{\tau} - \boldsymbol{\omega} \times (\mathbf{I} \boldsymbol{\omega})) \right)\end{aligned}\quad (11)$$

$$\boldsymbol{\tau} = \mathbf{I} \mathbf{A}_5^{-1} \left( -\mathbf{f}_2 + \mathbf{C}_1 \mathbf{A}_5 (\mathbf{e}_2 - \mathbf{C}_1 \mathbf{e}_1 - \mathbf{\Lambda}_1 \chi_1) + \ddot{\boldsymbol{\theta}}_{ref} + \mathbf{\Lambda}_1 \mathbf{e}_1 - \dot{\mathbf{A}}_3 \boldsymbol{\omega} \right) + \boldsymbol{\omega} \times (\mathbf{I} \boldsymbol{\omega})$$

## 2.2 Altitude

$$m\ddot{z} = c_\phi c_\theta F_R - mg$$

$$\ddot{z} = \frac{c_\phi c_\theta}{m} F_R - g$$

$$e_3 = z_{ref} - z$$

$$\dot{e}_3 = \dot{z}_{ref} - \dot{z}$$

$$\dot{e}_3 = \dot{z}_{ref} - v_z$$

$$v_{z,ref} = c_3 e_3 + \dot{z}_{ref} + \lambda_3 \chi_3$$

$$e_4 = v_{z,ref} - v_z$$

$$\dot{e}_4 = \dot{v}_{z,ref} - \dot{v}_z$$

$$\dot{e}_4 = c_3 \dot{e}_3 + \ddot{z}_{ref} + \lambda_3 e_3 - \dot{v}_z$$

$$\dot{z}_{ref} = v_{z,ref} - c_3 e_3 - \lambda_3 \chi_3$$

$$\dot{z}_{ref} = e_4 + v_z - c_3 e_3 - \lambda_3 \chi_3$$

$$\dot{e}_3 = e_4 - c_3 e_3 - \lambda_3 \chi_3$$

$$\dot{e}_4 = c_3 (e_4 - c_3 e_3 - \lambda_3 \chi_3) + \ddot{z}_{ref} + \lambda_3 e_3 - \left( \frac{c_\phi c_\theta}{m} F_R - g \right)$$

$$\dot{e}_4 = f_4(e_3, e_4)$$

$$F_R = \frac{m}{c_\phi c_\theta} (-f_4(e_3, e_4) + c_3 (e_4 - c_3 e_3 - \lambda_3 \chi_3) + \ddot{z}_{ref} + \lambda_3 e_3 + g)$$

## 2.3 Position

$$\mathbf{e}_5 = \mathbf{x}_{ref} - \mathbf{x}$$

$$\dot{\mathbf{e}}_5 = \dot{\mathbf{x}}_{ref} - \dot{\mathbf{x}}$$

$$\dot{\mathbf{e}}_5 = \dot{\mathbf{x}}_{ref} - \mathbf{v}$$

$$\mathbf{v}_{ref} = \mathbf{C}_5 \mathbf{e}_5 + \dot{\mathbf{x}}_{ref} + \mathbf{\Lambda}_5 \chi_5$$

$$\dot{\mathbf{v}}_{ref} = \mathbf{C}_5 \dot{\mathbf{e}}_5 + \ddot{\mathbf{x}}_{ref} + \mathbf{\Lambda}_5 \mathbf{e}_5$$

$$\mathbf{e}_6 = \mathbf{v}_{ref} - \mathbf{v}$$

$$\mathbf{e}_6 = \mathbf{C}_5 \mathbf{e}_5 + \dot{\mathbf{x}}_{ref} + \mathbf{\Lambda}_5 \chi_5 - \mathbf{v}$$

$$\dot{\mathbf{e}}_6 = \dot{\mathbf{v}}_{ref} - \dot{\mathbf{v}}$$

$$\dot{\mathbf{e}}_6 = \mathbf{C}_5 \dot{\mathbf{e}}_5 + \ddot{\mathbf{x}}_{ref} + \mathbf{\Lambda}_5 \mathbf{e}_5 - \ddot{\mathbf{x}}$$

$$\dot{\mathbf{x}}_{ref} = \mathbf{v}_{ref} - \mathbf{C}_5 \mathbf{e}_5 - \mathbf{\Lambda}_5 \chi_5$$

$$\dot{\mathbf{x}}_{ref} = \mathbf{e}_6 + \mathbf{v} - \mathbf{C}_5 \mathbf{e}_5 - \mathbf{\Lambda}_5 \chi_5$$

$$\dot{\mathbf{e}}_5 = \mathbf{e}_6 - \mathbf{C}_5 \mathbf{e}_5 - \mathbf{\Lambda}_5 \chi_5$$

$$\dot{\mathbf{e}}_6 = \mathbf{f}_6(\mathbf{e}_5, \mathbf{e}_6)$$

$$\mathbf{f}_6(\mathbf{e}_5, \mathbf{e}_6) = \mathbf{C}_5 (\mathbf{e}_6 - \mathbf{C}_5 \mathbf{e}_5 - \mathbf{\Lambda}_5 \chi_5) + \ddot{\mathbf{x}}_{ref} + \mathbf{\Lambda}_5 \mathbf{e}_5 - \left( \mathbf{g} + \frac{1}{m} \mathbf{F}_R + \frac{1}{m} \mathbf{F}_D \right)$$

$$\mathbf{F}_R = m \left( -\mathbf{f}_6(\mathbf{e}_5, \mathbf{e}_6) + \mathbf{C}_5 (\mathbf{e}_6 - \mathbf{C}_5 \mathbf{e}_5 - \mathbf{\Lambda}_5 \chi_5) + \ddot{\mathbf{x}}_{ref} + \mathbf{\Lambda}_5 \mathbf{e}_5 - \mathbf{g} - \frac{1}{m} \mathbf{F}_D \right)$$

$$\mathbf{F}_R = m \left( -\mathbf{C}_7 \mathbf{e}_5 - \mathbf{C}_8 \mathbf{e}_6 + \mathbf{C}_5 (\mathbf{e}_6 - \mathbf{C}_5 \mathbf{e}_5 - \mathbf{\Lambda}_5 \chi_5) + \ddot{\mathbf{x}}_{ref} + \mathbf{\Lambda}_5 \mathbf{e}_5 - \mathbf{g} - \frac{1}{m} \mathbf{F}_D \right)$$

$$\mathbf{F}_R = m \left( (-\mathbf{C}_7 - \mathbf{C}_5^2 + \mathbf{\Lambda}_5) \mathbf{e}_5 + (\mathbf{C}_8 + \mathbf{C}_5) \mathbf{e}_6 - \mathbf{C}_5 \mathbf{\Lambda}_5 \chi_5 + \ddot{\mathbf{x}}_{ref} - \mathbf{g} - \frac{1}{m} \mathbf{F}_D \right)$$

$$\begin{aligned} \mathbf{F}_R = m \big( & (-\mathbf{C}_7 - \mathbf{C}_5^2 + \mathbf{\Lambda}_5) \mathbf{e}_5 \\ & + (-\mathbf{C}_8 + \mathbf{C}_5) (\mathbf{C}_5 \mathbf{e}_5 + \dot{\mathbf{x}}_{ref} + \mathbf{\Lambda}_5 \chi_5 - \mathbf{v}) \\ & - \mathbf{C}_5 \mathbf{\Lambda}_5 \chi_5 + \ddot{\mathbf{x}}_{ref} - \mathbf{g} - \frac{1}{m} \mathbf{F}_D \big) \end{aligned}$$

$$\begin{aligned} \mathbf{F}_R = m \big( & (-\mathbf{C}_7 - \mathbf{C}_5^2 + \mathbf{\Lambda}_5 + (-\mathbf{C}_8 + \mathbf{C}_5) \mathbf{C}_5) \mathbf{x}_{ref} \\ & - (-\mathbf{C}_7 - \mathbf{C}_5^2 + \mathbf{\Lambda}_5 + (-\mathbf{C}_8 + \mathbf{C}_5) \mathbf{C}_5) \mathbf{x} \\ & + (-\mathbf{C}_8 + \mathbf{C}_5) \dot{\mathbf{x}}_{ref} \\ & + (-\mathbf{C}_8) \mathbf{\Lambda}_5 \chi_5 \\ & - (-\mathbf{C}_8 + \mathbf{C}_5) \mathbf{v} \\ & + \ddot{\mathbf{x}}_{ref} - \mathbf{g} - \frac{1}{m} \mathbf{F}_D \big) \end{aligned}$$

$$\begin{aligned} \mathbf{F}_R = m \big( & \mathbf{C}_9 \mathbf{x}_{ref} - \mathbf{C}_9 \mathbf{x} + \mathbf{C}_{10} \dot{\mathbf{x}}_{ref} - \mathbf{C}_{10} \mathbf{v} \\ & + \mathbf{C}_{11} \chi_5 + \ddot{\mathbf{x}}_{ref} - \mathbf{g} - \frac{1}{m} \mathbf{F}_D \big) \end{aligned}$$

### 2.3.1 Transfer Function

$$\begin{aligned}\ddot{x} = & \left( c_9 y - c_9 x + c_{10} \dot{y} - c_{10} \dot{x} \right. \\ & + c_{11} \int y - c_{11} \int x \\ & \left. + \ddot{y} - g - \frac{F_D}{m} \right)\end{aligned}$$

$$\begin{aligned}s^2 X - sx(0) - \dot{x}(0) = & \left( c_9 Y - c_9 X \right. \\ & + c_{10} sY - c_{10} y(0) - c_{10} sX + c_{10} x(0) \\ & + c_{11} \frac{Y}{s} - c_{11} \int x \\ & \left. + \ddot{y} - g - \frac{F_D}{m} \right)\end{aligned}$$

### 2.4 Attitude and Altitude

$$\begin{aligned}\begin{bmatrix} \tau_B \\ F_R \end{bmatrix} &= \begin{bmatrix} Lk & 0 & -Lk & 0 \\ 0 & Lk & 0 & -Lk \\ b & -b & b & -b \\ k & k & k & k \end{bmatrix} \gamma = \mathbf{A}_1 \gamma \\ \gamma &= \mathbf{A}_1^{-1} \begin{bmatrix} \tau_B \\ F_R \end{bmatrix}\end{aligned}$$