Contents

1 Equations of Motion

1.1 Reference

Gravity force in inertial frame

$$\mathbf{g} = -g\mathbf{k}$$

Rotor force in body frame

$$\mathbf{F}_{R,B} = k\left(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4\right)\mathbf{k}$$

Rotor force in inertial frame

$$\mathbf{F}_R = \mathbf{R}(\mathbf{q})\mathbf{F}_{R,B}(\gamma)$$

Drag force in body frame

$$\mathbf{F}_{D,B}(\dot{\mathbf{x}}_B)$$

$$\dot{\mathbf{x}}_B = \mathbf{R}^{-1}\dot{\mathbf{x}}$$

Drag force in inertial frame

$$\mathbf{F}_D = \mathbf{R}(\mathbf{q})\mathbf{F}_{D,B}(\dot{\mathbf{x}}_B)$$

1.2 Translation

$$m\ddot{\mathbf{x}} = m\mathbf{g} + \mathbf{F}_R + \mathbf{F}_D$$

$$\ddot{\mathbf{x}} = \mathbf{g} + \frac{1}{m}\mathbf{F}_R + \frac{1}{m}\mathbf{F}_D$$

$$\ddot{\mathbf{x}} = \mathbf{g} + \frac{1}{m}\mathbf{R}\mathbf{F}_{R,B} + \frac{1}{m}\mathbf{R}\mathbf{F}_{D,B}$$

1.3 Rotation

$$\mathbf{I}\dot{\boldsymbol{\omega}} = \tau - \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega})$$

$$\dot{\boldsymbol{\omega}} = \mathbf{I}^{-1}\left(\boldsymbol{\tau} - \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega})\right)$$

$$I = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$

$$\boldsymbol{\tau}_B = \begin{bmatrix} Lk\left(\gamma_1 - \gamma_3\right) \\ Lk\left(\gamma_2 - \gamma_4\right) \\ b\left(\gamma_1 - \gamma_2 + \gamma_3 - \gamma_4\right) \end{bmatrix} = \begin{bmatrix} Lk & 0 & -Lk & 0 \\ 0 & Lk & 0 & -Lk \\ b & -b & b & -b \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{bmatrix}$$

1.3.1 Quaternion

$$\dot{\mathbf{q}} = \mathbf{A}_3\boldsymbol{\omega}$$

$$\ddot{\mathbf{q}} = \dot{\mathbf{A}}_3\boldsymbol{\omega} + \mathbf{A}_3\dot{\boldsymbol{\omega}}$$

$$\ddot{\mathbf{q}} = \dot{\mathbf{A}}_3\boldsymbol{\omega} + \mathbf{A}_3\mathbf{I}^{-1}\left(\boldsymbol{\tau} - \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega})\right)$$

1.3.2 Euler

$$\boldsymbol{\omega} = \begin{bmatrix} 1 & 0 & -s_{\boldsymbol{\theta}} \\ 0 & c_\phi & c_{\boldsymbol{\theta}} s_\phi \\ 0 & -s_\phi & c_{\boldsymbol{\theta}} s_\phi \end{bmatrix} \dot{\boldsymbol{\theta}} = \mathbf{A}_5^{-1}\dot{\boldsymbol{\theta}}$$

$$\dot{\boldsymbol{\theta}} = \mathbf{A}_5\boldsymbol{\omega}$$

$$\ddot{\boldsymbol{\theta}} = \dot{\mathbf{A}}_5\boldsymbol{\omega} + \mathbf{A}_5\dot{\boldsymbol{\omega}}$$

$$\ddot{\boldsymbol{\theta}} = \dot{\mathbf{A}}_5\boldsymbol{\omega} + \mathbf{A}_5\left(\mathbf{I}^{-1}\left(\boldsymbol{\tau} - \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega})\right)\right)$$

$$\ddot{\boldsymbol{\theta}} = \mathbf{S}^{-1}\left(\mathbf{I}^{-1}\left(\boldsymbol{\tau} - \boldsymbol{\omega} \times (I\boldsymbol{\omega})\right) - \dot{\mathbf{S}}\dot{\boldsymbol{\theta}}\right) \tag{1}$$

2 Control

2.1 Attitude

2.1.1 Quaternion

Tracking error

$$\mathbf{e}_1 = \mathbf{q}_{ref} - \mathbf{q}$$

$$\dot{\mathbf{e}}_1 = \dot{\mathbf{q}}_{ref} - \dot{\mathbf{q}}$$

$$\dot{\mathbf{e}}_1 = \dot{\mathbf{q}}_{ref} - \mathbf{A}_3\boldsymbol{\omega} \tag{2}$$

The $\boldsymbol{\omega}$ is not our control input and has its own dynamic. So we set fot it a desired behavior and consider it as our virtual control

$$\boldsymbol{\omega}_{ref} = c_1\mathbf{e}_1 + \mathbf{A}_3^{-1}\dot{\mathbf{q}}_{ref} + \lambda_1\chi_1 \tag{3}$$

$$\mathbf{e}_2 = \boldsymbol{\omega}_{ref} - \boldsymbol{\omega} \tag{4}$$

$$\dot{\mathbf{e}}_2 = \dot{\boldsymbol{\omega}}_{ref} - \dot{\boldsymbol{\omega}}$$

$$\dot{\mathbf{e}}_2 = c_1\dot{\mathbf{e}}_1 + \ddot{\mathbf{q}}_{ref} + \lambda_1\mathbf{e}_1 - \ddot{\mathbf{q}} \tag{5}$$

Solve (12) for $\dot{\mathbf{q}}_{ref}$

$$\dot{\mathbf{q}}_{ref} = \mathbf{A}_3\left(\boldsymbol{\omega}_{ref} - c_1\mathbf{e}_1 - \lambda_1\boldsymbol{\chi}_1\right) \tag{6}$$

Solve (13) for $\boldsymbol{\omega}_{ref}$

$$\boldsymbol{\omega}_{ref} = \mathbf{e}_2 + \boldsymbol{\omega} \tag{7}$$

Combine (15) and (16)

$$\dot{\mathbf{q}}_{ref} = \mathbf{A}_3\left(\mathbf{e}_2 + \boldsymbol{\omega} - c_1\mathbf{e}_1 - \lambda_1\boldsymbol{\chi}_1\right) \tag{8}$$

Rewrite (11) using (17)

$$\dot{\mathbf{e}}_1 = \mathbf{A}_3\left(\mathbf{e}_2 + \boldsymbol{\omega} - c_1\mathbf{e}_1 - \lambda_1\boldsymbol{\chi}_1\right) - \mathbf{A}_3\boldsymbol{\omega}$$

$$\dot{\mathbf{e}}_1 = \mathbf{A}_3\left(\mathbf{e}_2 - c_1\mathbf{e}_1 - \lambda_1\boldsymbol{\chi}_1\right)$$

Replace $\ddot{\mathbf{q}}$ in (14) with its model in (1)

$$\dot{\mathbf{e}}_2 = c_1\dot{\mathbf{e}}_1 + \ddot{\mathbf{q}}_{ref} + \boldsymbol{\Lambda}_1\mathbf{e}_1 - \left(\dot{\mathbf{A}}_3\boldsymbol{\omega} + \mathbf{A}_3\mathbf{I}^{-1}\left(\boldsymbol{\tau} - \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega})\right)\right) \tag{9}$$

$$
\begin{aligned}
\dot{\mathbf{e}}_2 =& \mathbf{C}_1\mathbf{A}_3\left(\mathbf{e}_2 - \mathbf{C}_1\mathbf{e}_1 - \boldsymbol{\Lambda}_1\boldsymbol{\chi}_1\right) + \ddot{\mathbf{q}}_{ref} + \boldsymbol{\Lambda}_1\mathbf{e}_1 \\
&- \left(\dot{\mathbf{A}}_3\boldsymbol{\omega} + \mathbf{A}_3\mathbf{I}^{-1}\left(\boldsymbol{\tau} - \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega})\right)\right)
\end{aligned} \tag{10}
$$

We now choose a desirable form for the dynamics of the angular speed tracking error. An example would be

$$\dot{\mathbf{e}}_2 = -c_2\mathbf{e}_2 - \mathbf{e}_1$$

but for now we will generalize this as

$$\dot{\mathbf{e}}_2 = \mathbf{f}_2\left(\mathbf{e}_1, \mathbf{e}_2\right)$$

$$\boldsymbol{\tau} = \mathbf{I}\mathbf{A}_3^{-1}\left(-\mathbf{f}_2\left(\mathbf{e}_1, \mathbf{e}_2\right) + \mathbf{C}_1\mathbf{A}_3\left(\mathbf{e}_2 - \mathbf{C}_1\mathbf{e}_1 - \boldsymbol{\Lambda}_1\boldsymbol{\chi}_1\right) + \ddot{\mathbf{q}}_{ref} + \boldsymbol{\Lambda}_1\mathbf{e}_1 - \dot{\mathbf{A}}_3\boldsymbol{\omega}\right) + \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega})$$

2.1.2 Euler

Tracking error

$$\mathbf{e}_1 = \boldsymbol{\theta}_{ref} - \boldsymbol{\theta}$$

$$\dot{\mathbf{e}}_1 = \dot{\boldsymbol{\theta}}_{ref} - \dot{\boldsymbol{\theta}}$$

$$\dot{\mathbf{e}}_1 = \dot{\boldsymbol{\theta}}_{ref} - \mathbf{A}_5\boldsymbol{\omega} \tag{11}$$

The $\boldsymbol{\omega}$ is not our control input and has its own dynamic. So we set fot it a desired behavior and consider it as our virtual control

$$\boldsymbol{\omega}_{ref} = c_1\mathbf{e}_1 + \mathbf{A}_5^{-1}\dot{\boldsymbol{\theta}}_{ref} + \lambda_1\chi_1 \tag{12}$$

$$\mathbf{e}_2 = \boldsymbol{\omega}_{ref} - \boldsymbol{\omega} \tag{13}$$

$$\dot{\mathbf{e}}_2 = \dot{\boldsymbol{\omega}}_{ref} - \dot{\boldsymbol{\omega}}$$

$$\dot{\mathbf{e}}_2 = c_1\dot{\mathbf{e}}_1 + \ddot{\boldsymbol{\theta}}_{ref} + \lambda_1\mathbf{e}_1 - \ddot{\boldsymbol{\theta}} \tag{14}$$

Solve (12) for $\dot{\boldsymbol{\theta}}_{ref}$

$$\dot{\boldsymbol{\theta}}_{ref} = \mathbf{A}_5\left(\boldsymbol{\omega}_{ref} - c_1\mathbf{e}_1 - \lambda_1\boldsymbol{\chi}_1\right) \tag{15}$$

Solve (13) for $\boldsymbol{\omega}_{ref}$

$$\boldsymbol{\omega}_{ref} = \mathbf{e}_2 + \boldsymbol{\omega} \tag{16}$$

Combine (15) and (16)

$$\dot{\boldsymbol{\theta}}_{ref} = \mathbf{A}_5 \left(\mathbf{e}_2 + \boldsymbol{\omega} - c_1 \mathbf{e}_1 - \lambda_1 \boldsymbol{\chi}_1 \right) \tag{17}$$

Rewrite (11) using (17)

$$\dot{\mathbf{e}}_1 = \mathbf{A}_5 \left(\mathbf{e}_2 + \boldsymbol{\omega} - c_1 \mathbf{e}_1 - \lambda_1 \boldsymbol{\chi}_1 \right) - \mathbf{A}_5 \boldsymbol{\omega}$$

$$\dot{\mathbf{e}}_1 = \mathbf{A}_5 \left(\mathbf{e}_2 - c_1 \mathbf{e}_1 - \lambda_1 \boldsymbol{\chi}_1 \right)$$

Replace $\ddot{\boldsymbol{\theta}}$ in (14) with its model in (1)

$$\dot{\mathbf{e}}_2 = c_1 \dot{\mathbf{e}}_1 + \ddot{\boldsymbol{\theta}}_{ref} + \boldsymbol{\Lambda}_1 \mathbf{e}_1 - \left(\dot{\mathbf{A}}_3 \boldsymbol{\omega} + \mathbf{A}_5 \mathbf{I}^{-1} \left(\boldsymbol{\tau} - \boldsymbol{\omega} \times (\mathbf{I} \boldsymbol{\omega}) \right) \right) \tag{18}$$

$$\begin{aligned} \dot{\mathbf{e}}_2 =& \mathbf{C}_1 \mathbf{A}_5 \left(\mathbf{e}_2 - \mathbf{C}_1 \mathbf{e}_1 - \boldsymbol{\Lambda}_1 \boldsymbol{\chi}_1 \right) + \ddot{\boldsymbol{\theta}}_{ref} + \boldsymbol{\Lambda}_1 \mathbf{e}_1 \\ & - \left(\dot{\mathbf{A}}_3 \boldsymbol{\omega} + \mathbf{A}_5 \mathbf{I}^{-1} \left(\boldsymbol{\tau} - \boldsymbol{\omega} \times (\mathbf{I} \boldsymbol{\omega}) \right) \right) \end{aligned} \tag{19}$$

We now choose a desirable form for the dynamics of the angular speed tracking error. An example would be

$$\dot{\mathbf{e}}_2 = -c_2 \mathbf{e}_2 - \mathbf{e}_1$$

but for now we will generalize this as

$$\dot{\mathbf{e}}_2 = \mathbf{f}_2 \left(\mathbf{e}_1, \mathbf{e}_2 \right)$$

$$\begin{aligned} \mathbf{f}_2 =& \mathbf{C}_1 \mathbf{A}_5 \left(\mathbf{e}_2 - \mathbf{C}_1 \mathbf{e}_1 - \boldsymbol{\Lambda}_1 \boldsymbol{\chi}_1 \right) + \ddot{\boldsymbol{\theta}}_{ref} + \boldsymbol{\Lambda}_1 \mathbf{e}_1 \\ & - \left(\dot{\mathbf{A}}_3 \boldsymbol{\omega} + \mathbf{A}_5 \mathbf{I}^{-1} \left(\boldsymbol{\tau} - \boldsymbol{\omega} \times (\mathbf{I} \boldsymbol{\omega}) \right) \right) \end{aligned} \tag{20}$$

2.2 Altitude

$$m \ddot{z} = c_\phi c_\theta F_R - mg$$

$$\ddot{z} = \frac{c_\phi c_\theta}{m} F_R - g$$

$$e_3 = z_{ref} - z$$

$$\dot{e}_3 = \dot{z}_{ref} - \dot{z}$$

$$\dot{e}_3 = \dot{z}_{ref} - v_z$$

$$v_{z,ref} = c_3 e_3 + \dot{z}_{ref} + \lambda_3 \chi_3$$

$$e_4 = v_{z,ref} - v_z$$

$$\dot{e}_4 = \dot{v}_{z,ref} - \dot{v}_z$$

$$\dot{e}_4 = c_3 \dot{e}_3 + \ddot{z}_{ref} + \lambda_3 e_3 - \dot{v}_z$$

$$\dot{z}_{ref} = v_{z,ref} - c_3 e_3 - \lambda_3 \chi_3$$

$$\dot{z}_{ref} = e_4 + v_z - c_3 e_3 - \lambda_3 \chi_3$$

$$\dot{e}_3 = e_4 - c_3 e_3 - \lambda_3 \chi_3$$

$$\dot{e}_4 = c_3 \left(e_4 - c_3 e_3 - \lambda_3 \chi_3\right) + \ddot{z}_{ref} + \lambda_3 e_3 - \left(\frac{c_\phi c_\theta}{m} F_R - g\right)$$

$$\dot{e}_4 = f_4(e_3, e_4)$$

$$F_R = \frac{m}{c_\phi c_\theta} \left(-f_4(e_3, e_4) + c_3 \left(e_4 - c_3 e_3 - \lambda_3 \chi_3\right) + \ddot{z}_{ref} + \lambda_3 e_3 + g\right)$$

2.3 Position

$$\mathbf{e}_5 = \mathbf{x}_{ref} - \mathbf{x}$$

$$\dot{\mathbf{e}}_5 = \dot{\mathbf{x}}_{ref} - \dot{\mathbf{x}}$$

$$\dot{\mathbf{e}}_5 = \dot{\mathbf{x}}_{ref} - \mathbf{v}$$

$$\mathbf{v}_{ref} = \mathbf{C}_5 \mathbf{e}_5 + \dot{\mathbf{x}}_{ref} + \mathbf{\Lambda}_5 \boldsymbol{\chi}_5$$

$$\dot{\mathbf{v}}_{ref} = \mathbf{C}_5 \dot{\mathbf{e}}_5 + \ddot{\mathbf{x}}_{ref} + \mathbf{\Lambda}_5 \mathbf{e}_5$$

$$\mathbf{e}_6 = \mathbf{v}_{ref} - \mathbf{v}$$

$$\dot{\mathbf{e}}_6 = \dot{\mathbf{v}}_{ref} - \dot{\mathbf{v}}$$

$$\dot{\mathbf{e}}_6 = \mathbf{C}_5\dot{\mathbf{e}}_5 + \ddot{\mathbf{x}}_{ref} + \boldsymbol{\Lambda}_5\mathbf{e}_5 - \ddot{\mathbf{x}}$$

$$\dot{\mathbf{x}}_{ref} = \mathbf{v}_{ref} - \mathbf{C}_5\mathbf{e}_5 - \boldsymbol{\Lambda}_5\boldsymbol{\chi}_5$$

$$\dot{\mathbf{x}}_{ref} = \mathbf{e}_6 + \mathbf{v} - \mathbf{C}_5\mathbf{e}_5 - \boldsymbol{\Lambda}_5\boldsymbol{\chi}_5$$

$$\dot{\mathbf{e}}_5 = \mathbf{e}_6 - \mathbf{C}_5\mathbf{e}_5 - \boldsymbol{\Lambda}_5\boldsymbol{\chi}_5$$

$$\dot{\mathbf{e}}_6 = \mathbf{f}_6(\mathbf{e}_5, \mathbf{e}_6)$$

$$\mathbf{f}_6(\mathbf{e}_5, \mathbf{e}_6) = \mathbf{C}_5\left(\mathbf{e}_6 - \mathbf{C}_5\mathbf{e}_5 - \boldsymbol{\Lambda}_5\boldsymbol{\chi}_5\right) + \ddot{\mathbf{x}}_{ref} + \boldsymbol{\Lambda}_5\mathbf{e}_5 - \left(\mathbf{g} + \frac{1}{m}\mathbf{F}_R + \frac{1}{m}\mathbf{F}_D\right)$$

$$\mathbf{F}_R = m\left(-\mathbf{f}_6(\mathbf{e}_5, \mathbf{e}_6) + \mathbf{C}_5\left(\mathbf{e}_6 - \mathbf{C}_5\mathbf{e}_5 - \boldsymbol{\Lambda}_5\boldsymbol{\chi}_5\right) + \ddot{\mathbf{x}}_{ref} + \boldsymbol{\Lambda}_5\mathbf{e}_5 - \mathbf{g} - \frac{1}{m}\mathbf{F}_D\right)$$

2.4 Attitude and Altitude

$$\begin{bmatrix} \boldsymbol{\tau}_B \\ F_R \end{bmatrix} = \begin{bmatrix} Lk & 0 & -Lk & 0 \\ 0 & Lk & 0 & -Lk \\ b & -b & b & -b \\ k & k & k & k \end{bmatrix} \boldsymbol{\gamma} = \mathbf{A}_1\boldsymbol{\gamma}$$

$$\boldsymbol{\gamma} = \mathbf{A}_1^{-1}\begin{bmatrix} \boldsymbol{\tau}_B \\ F_R \end{bmatrix}$$