Market Models for Inflation Derivatives

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Stylized facts

- Inflation-indexed (II) bonds have been issued since the 80's, but it is only in the very last years that these bonds, and inflation-indexed derivatives in general, have become quite popular.
- Inflation is defined as the percentage increment of a reference index, the Consumer Price Index (CPI), which is a basket of goods and services.
- Denoting by $I(t)$ the CPI's value at time t, the inflation rate over the time interval $[t, T]$ is therefore:
$$i(t, T) := \frac{I(T)}{I(t)} - 1.$$
- In theory, but also in practice, inflation can become negative.

Stylized facts

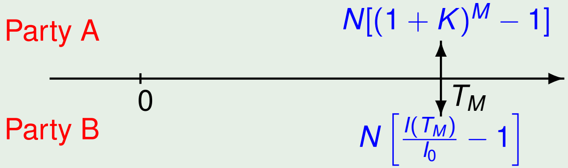
US CPI
Monthly closing values from Nov-00 to Oct-08.

Stylized facts

- Banks and governments are used to issue inflation-linked bonds, where an inflation-indexed floor is offered in conjunction with the “pure" bond.
- To grant positive coupons, the inflation rate (plus a spread) is floored at zero.
- Accordingly, floors with zero or negative strikes are traded in the inflation market.
- Besides caps and floors, other popular derivatives are inflation-indexed swaps.
- Two are the main inflation-indexed swaps traded in the market:
  - the zero coupon (ZC) swap
  - the year-on-year (YY) swap

Zero-coupon inflation-indexed swaps

Payment structure



In a ZCIIS, at time $T_M = M$ years, Party B pays Party A the fixed amount

$$N[(1+K)^M - 1],$$

where K and N are, respectively, the contract fixed rate and the contract nominal value.

Party A pays Party B, at the final time T_M, the floating amount

$$N\left[\frac{I(T_M)}{I_0} - 1\right].$$

Zero-coupon inflation-indexed swaps

Market quotes of USD ZC swaps as of December 2nd, 2008

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Term	Price Bid	Price Ask	Size (mio) Bid		Size (mio) Ask	Time
1) 2 Year	-3.250/	-2.950	25	X	25	10:04
2) 3 Year	-2.182/	-1.882	25	X	25	10:04
3) 4 Year	-1.028/	-0.778	25	X	25	10:04
4) 5 Year	-0.245/	0.005	25	X	25	10:04
5) 6 Year	-0.082/	0.168	25	X	25	10:04
6) 7 Year	0.365/	0.615	25	X	25	10:04
7) 8 Year	0.448/	0.698	25	X	25	10:04
8) 9 Year	0.732/	0.982	25	X	25	10:04
9) 10 Year	1.175/	1.425	25	X	25	10:04
10) 12 Year	1.235/	1.485	25	X	25	10:04
11) 15 Year	1.275/	1.525	25	X	25	10:04
12) 20 Year	1.310/	1.560	25	X	25	10:04
13) 25 Year	1.370/	1.620	25	X	25	10:04
14) 30 Year	1.425/	1.675	25	X	25	10:04

Zero Coupon Interp *Page F'wd for UK RPI / Back for EURO HIC
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Sales : +44 207 085 8209 |Euro Inflation ->{RBSE<GO>}
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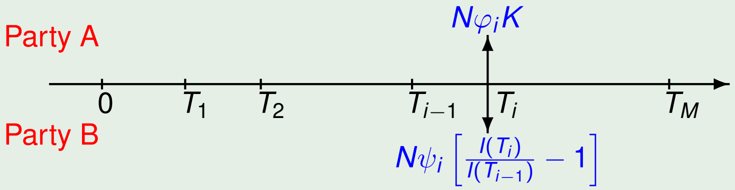
Year-on-year inflation-indexed swaps

Payment structure



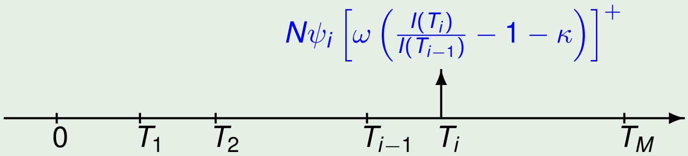
In a YYIIS, at each time T_i, Party B pays Party A the fixed amount $N\varphi_i K$, while Party A pays Party B the (floating) amount

$$N\psi_i \left[\frac{I(T_i)}{I(T_{i-1})} - 1\right],$$

where φ_i and ψ_i are, respectively, the fixed- and floating-leg year fractions for the interval $[T_{i-1}, T_i]$, $T_0 := 0$ and N is again the swap nominal value.

Inflation-Indexed Caps and Floors

Payment structure



II caps (floors) are strips of II caplets (floorlets).
At each time T_i, up to the contract maturity T_M, the payoff is

$$N\psi_i \left[\omega\left(\frac{I(T_i)}{I(T_{i-1})} - 1 - \kappa\right)\right]^+,$$

where κ is the strike, ψ_i is the year fraction for the interval $[T_{i-1}, T_i]$, N is the nominal value, and $\omega = 1$ for a caplet and $\omega = -1$ for a floorlet.

Inflation-Indexed Caps and Floors

USD market quotes: caps

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US CPI CPURNSA — Premium in BP upfront, USD 20m notional. Price for trade with inflation swap delta

USD Cap Term		Strike 2.0% Bid / Ask	Time		Strike 2.5% Bid / Ask	Time		Strike 3.0% Bid / Ask	Time
3 YEAR	1)	94 / 94	10:22	15)	76 / 76	10:22	29)	62 / 62	10:22
5 YEAR	2)	376 / 376	10:22	16)	309 / 309	10:22	30)	254 / 254	10:22
7 YEAR	3)	551 / 551	10:22	17)	453 / 453	10:22	31)	375 / 375	10:22
10 YEAR	4)	1069 /1069	10:22	18)	891 / 891	10:22	32)	743 / 743	10:22
15 YEAR	5)	1417 /1417	10:22	19)	1187 /1187	10:22	33)	1000 /1000	10:22
20 YEAR	6)	1746 /1746	10:22	20)	1470 /1470	10:22	34)	1248 /1248	10:22
30 YEAR	7)	2291 /2291	10:22	21)	1936 /1936	10:22	35)	1657 /1657	10:22

USD Cap Term		Strike 4.0% Bid / Ask	Time		Strike 5.0% Bid / Ask	Time		Strike 6.0% Bid / Ask	Time
3 YEAR	8)	43 / 43	10:22	22)	31 / 31	10:22	36)	23 / 23	10:22
5 YEAR	9)	176 / 176	10:22	23)	128 / 128	10:22	37)	97 / 97	10:22
7 YEAR	10)	266 / 266	10:22	24)	200 / 200	10:22	38)	158 / 158	10:22
10 YEAR	11)	531 / 531	10:22	25)	401 / 401	10:22	39)	320 / 320	10:22
15 YEAR	12)	739 / 739	10:22	26)	583 / 583	10:22	40)	484 / 484	10:22
20 YEAR	13)	947 / 947	10:22	27)	767 / 767	10:22	41)	651 / 651	10:22
30 YEAR	14)	1284 /1284	10:22	28)	1062 /1062	10:22	42)	919 / 919	10:22

Year on year Sep base * Page fwd for floors
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Inflation-Indexed Caps and Floors

USD market quotes: floors

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US CPI
CPURNSA

Premium in BP upfront, USD 20m notional
Price for trade with inflation swap delta

USD Floor		Strike -1.0%			Strike 0.0%			Strike 1.0%	
Term		Bid / Ask	Time		Bid / Ask	Time		Bid / Ask	Time
3 YEAR	1)	551 / 551	10:22	15)	736 / 736	10:22	29)	948 / 948	10:22
5 YEAR	2)	612 / 612	10:22	16)	817 / 817	10:22	30)	1064 /1064	10:22
7 YEAR	3)	693 / 693	10:22	17)	926 / 926	10:22	31)	1219 /1219	10:22
10 YEAR	4)	772 / 772	10:22	18)	1022 /1022	10:22	32)	1340 /1340	10:22
15 YEAR	5)	987 / 987	10:22	19)	1289 /1289	10:22	33)	1703 /1703	10:22
20 YEAR	6)	1183 /1183	10:22	20)	1528 /1528	10:22	34)	2021 /2021	10:22
30 YEAR	7)	1539 /1539	10:22	21)	1964 /1964	10:22	35)	2593 /2593	10:22

USD Floor		Strike 2.0%			Strike 2.5%			Strike 3.0%	
Term		Bid / Ask	Time		Bid / Ask	Time		Bid / Ask	Time
3 YEAR	8)	1184 /1184	10:22	22)	1310 /1310	10:22	36)	1440 /1440	10:22
5 YEAR	9)	1358 /1358	10:22	23)	1525 /1525	10:22	37)	1703 /1703	10:22
7 YEAR	10)	1592 /1592	10:22	24)	1811 /1811	10:22	38)	2050 /2050	10:22
10 YEAR	11)	1759 /1759	10:22	25)	2015 /2015	10:22	39)	2299 /2299	10:22
15 YEAR	12)	2294 /2294	10:22	26)	2667 /2667	10:22	40)	3083 /3083	10:22
20 YEAR	13)	2758 /2758	10:22	27)	3234 /3234	10:22	41)	3765 /3765	10:22
30 YEAR	14)	3571 /3571	10:22	28)	4212 /4212	10:22	42)	4930 /4930	10:22

Year on year Sep base * Page fwd for caps
Trading: Tel: 44 207 085 0682 | Inflation swaps -> {RILS<GO>}

Some general considerations

- Similarly to interest rates, inflation is not a tradable asset.
- The no-arbitrage requirements to impose on its dynamics may be not so straightforward, as they depend on
  - The nature of the modeled rate.
  - The reference probability measure.
- Contrary to (nominal) interest rates, which are forward looking, future inflation rates may refer to time intervals whose initial point is in the past.
- Historically, researchers preferred to model the dynamics of the CPI, obtaining inflation rates as percentage increments of the CPI's values at the relevant times.
- Two are the main approaches followed by practitioners:
  - Models based on the foreign-currency analogy.
  - Market models.

The two main approaches

The Foreign-Currency Analogy (FCA)

- There are two economies: the nominal ad the real ones;
- The CPI can be viewed as the exchange rate between them;
- Pricing an inflation-indexed contract is equivalent to pricing a multi-currency interest rate derivative.

Market Models

- One models the evolution of suitable market quantities under some reference measure;
- Two main objectives:
  - To model more realistic interest and inflation rates;
  - To derive simpler formulas for plain vanilla instruments.

The Jarrow and Yildirim (JY) (2003) model

- Using the FCA, Jarrow and Yildirim assume that (instantaneous) nominal and real rates evolve according to correlated one-factor Hull-White models.
- The (nominal) risk-neutral dynamics of nominal and real rates and the CPI are, respectively,

$$
\begin{aligned}
dn(t) &= [\vartheta_n(t) - a_n n(t)]\, dt + \sigma_n\, dW_n(t) \\
dr(t) &= [\vartheta_r(t) - \rho_{r,I}\sigma_I\sigma_r - a_r r(t)]\, dt + \sigma_r\, dW_r(t) \\
dI(t) &= I(t)[n(t) - r(t)]\, dt + \sigma_I I(t)\, dW_I(t)
\end{aligned}
$$

  where W_n, W_r, W_I are correlated Brownian motions.
- Knowing the risk-neutral dynamics, we can price any inflation-indexed derivative (and interest rate hybrid).
- In particular, we can derive closed-form formulas for YYII swaps and caps.

The market model of forward CPIs

- Kazziha (1999), Belgrade, Benhamou and Koehler (2004) and Mercurio (2005) choose to model the evolution of the forward CPIs.
- Kazziha (1999) defines the T_i-forward CPI at time t, $\mathcal{I}_i(t)$, as the fixed amount X to be exchanged at time T_i for the CPI $I(T_i)$, so that the swap has zero value at time t.
- Using no-arbitrage pricing theory, we have:

$$\mathcal{I}_i(t) = E^{T_i}\left[I(T_i)|\mathcal{F}_t\right]$$

- The value at time zero of a T_i-forward CPI can be immediately obtained from the market quote $K(T_i)$. In fact

$$\mathcal{I}_i(0) = I(0)(1 + K(T_i))^i$$

Consistency with the foreign-currency analogy

- The previous definition of forward CPI is consistent with the FCA.
- In the FCA, the CPI is equivalent to an exchange rate, so it makes sense to define the T_i-forward CPI as

$$\mathcal{I}_i^{\mathrm{FCA}}(t) := I(t)\frac{P_r(t,T_i)}{P_n(t,T_i)}$$

  where we denote by $P_r(t,T_i)$ and $P_n(t,T_i)$ the real and nominal discount factors for maturity T_i, respectively.
- Since $I(t)P_r(t,T_i)$ is a tradable asset in the nominal economy, when we divide it by $P_n(t,T_i)$ we get a martingale under the nominal T_i-forward measure.
- Therefore, since $P_r(T_i,T_i) = P_n(T_i,T_i) = 1$,

$$\mathcal{I}_i^{\mathrm{FCA}}(t) = E^{T_i}\left[\mathcal{I}_i^{\mathrm{FCA}}(T_i)|\mathcal{F}_t\right] = E^{T_i}\left[I(T_i)|\mathcal{F}_t\right] = \mathcal{I}_i(t)$$

The lognormal market model (LMM) for forward CPIs

- In the LMM one assumes that, under the T_i-forward measure Q^{T_i},

$$d\mathcal{I}_i(t) = \sigma_{I,i}(t)\mathcal{I}_i(t)\,dZ_i^I(t)$$

- An analogous evolution is assumed to hold for $\mathcal{I}_{i-1}$ under $Q^{T_{i-1}}$.
- Using classic drift-freezing techniques, we obtain:

$$E^{T_i}\left\{\frac{\mathcal{I}_i(T_{i-1})}{\mathcal{I}_{i-1}(T_{i-1})}\Big|\mathcal{F}_t\right\} = \frac{\mathcal{I}_i(t)}{\mathcal{I}_{i-1}(t)}\,e^{D_i(t)}$$

  where D_i is a deterministic function.
- As a consequence, we can price in closed form:
  - YYII swaps.
  - YYII caplets (with a Black-Scholes type formula).

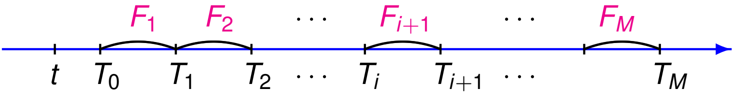
What model should one use?

- The JY model and the LMM for forward CPIs are almost equivalent:
  - Forward CPIs are lognormally distributed.
  - One can derive explicit formulas for YYII swaps and options (both on YY and ZC rates).
  - One can calibrate exactly only one strike for each maturity.

- What are the desiderata for a good inflation model?
  - Can price in closed form:
    - The YY convexity adjustment, and hence YYII swaps.
    - YYII caplets: $\left[\frac{I(T_i)}{I(T_{i-1})} - 1 - \kappa\right]^+$.
    - ZC options: $\left[\frac{I(T_M)}{I_0} - 1 - \kappa\right]^+$.
  - Can well accommodate market quotes.

Market models for forward inflation rates

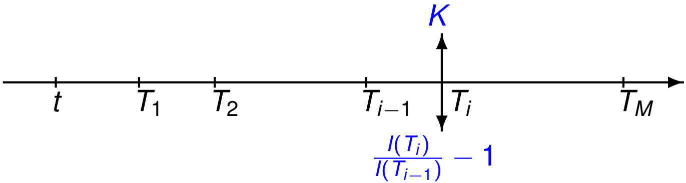
- An inflation market model is not necessarily based on forward CPIs.
- For instance, in the main market model for interest rates, one models the joint evolution of consecutive forward LIBOR rates:



  where $T_0, T_1, \ldots, T_M$ is a given time structure.
- We may follow a similar approach in modeling inflation and jointly model consecutive forward inflation rates.

Market models for forward inflation rates

- To this end, let us define the time-t forward inflation rate for the future interval $[T_{i-1}, T_i]$ as the rate K that, at time t, gives zero value to the swaplet where, at time T_i, K is exchanged for $I(T_i)/I(T_{i-1}) - 1$:



- We get:

$$K = \mathcal{Y}_i(t) := E^{T_i}\left\{ \frac{I(T_i)}{I(T_{i-1})} - 1 \middle| \mathcal{F}_t \right\}$$

A market model for the forward inflation rates $\mathcal{Y}_i(t)$

- A market model for consecutive inflation rates $\mathcal{Y}_1(t), \mathcal{Y}_2(t), \dots$ is defined by specifying their (instantaneous) covariance structure:

$$d\mathcal{Y}_i(t) = \varsigma_i(t)\, dZ_i(t), \quad \text{under } Q^{T_i}$$
$$\rho_{i,j} dt = dZ_i(t) dZ_j(t)$$

- To derive joint dynamics under a common measure, we use the change-of-measure technique and remember that each $\mathcal{Y}_i(t)$ is a martingale under the corresponding Q^{T_i}.
- The pricing of caplets is straightforward (if we smartly choose the volatility coefficient ς_i), since

$$E_t^{T_i}\left\{\left[\frac{I(T_i)}{I(T_{i-1})} - 1 - \kappa\right]^+\right\} = E_t^{T_i}\left\{\left[\mathcal{Y}_i(T_i) - \kappa\right]^+\right\}$$

A market model for the forward inflation rates $\mathcal{Y}_i(t)$

Caplet pricing

- A popular choice is to assume that $\mathcal{Y}_i(t)$ follows a Gaussian SABR model:

$$
\begin{aligned}
d\mathcal{Y}_i(t) &= V_i(t)\, dZ_i(t) \\
V_i(t) &= \epsilon_i V_i(t)\, dW_i(t)
\end{aligned}
$$

  where $V_i(0) = \alpha_i$ and $dZ_i(t) dW_i(t) = \rho_i dt$.
- As is well known, this model leads to a closed-form formula that is extremely easy to implement.
- The calibration to market data is fast and accurate.
- However, the model needs in input the value $\mathcal{Y}_i(0)$, which may be not provided by the market, unless we trivially assume a zero convexity correction.

A market model for the forward inflation rates $\mathcal{Y}_i(t)$

- One may also define a forward inflation rate as the corresponding forward CPI ratio minus one:

$$Y_i(t) := \frac{\mathcal{I}_i(t)}{\mathcal{I}_{i-1}(t)} - 1, \quad t \leq T_{i-1}$$

- Extending the definition of Y_i after T_{i-1}, we have that $Y_i(t) = \mathcal{Y}_i(t)$ for any $t \geq T_{i-1}$.
- The difference between the two rates is the following:
  - $Y_i(t)$ is known as soon as the corresponding ZCIIS rates are known;
  - $\mathcal{Y}_i(t)$ is known as soon as the corresponding YYIIS rates are known;
- From a mathematical point of view, another difference is that $\mathcal{Y}_i(t)$ is a martingale under the T_i-forward measure, whereas $Y_i(t)$ is not.

A market model for the forward inflation rates $Y_i(t)$

- Introducing a market model for consecutive inflation rates $Y_1(t), Y_2(t), \ldots$ is less straightforward.
- In fact, since $Y_i(t)$ is a not martingale under the T_i-forward measure, the no-arbitrage conditions to impose on the rates dynamics are not so obvious as in the previous case.
- However, given that Y_i is defined by a (shifted) forward CPI ratio, we can model $Y_1(t), Y_2(t), \ldots$ by suitably modeling $\mathcal{I}_1(t), \mathcal{I}_2(t), \ldots$.
- For instance, the LMM of forward CPIs

$$d\mathcal{I}_k(t) = \sigma_{I,k}\mathcal{I}_k(t)\, dZ_k^I(t), \quad k = i-1, 1$$

leads to the following dynamics

$$dY_i(t) = [1 + Y_i(t)]\Big[c_i(t)\, dt + \sigma_{I,i}\, dZ_i^I(t) - \sigma_{I,i-1}\, d\bar{Z}_{i-1}^I(t)\Big]$$

A market model for the forward inflation rates $Y_i(t)$

- Our idea is to suitably model forward CPIs so as to imply as simple as possible dynamics for the forward CPI ratios Y_i.
- In fact, we start from some dynamics for the forward CPI ratios Y_i and derive the forward CPI model that implies them by using the relation:

$$\mathcal{I}_M(T_M) = I_0 \prod_{j=1}^{M} [1 + Y_j(T_j)]$$

- An obvious choice to start with is to assume a shifted-lognormal volatility for the forward CPI ratios Y_i:

$$dY_i(t) = \cdots dt + [1 + Y_i(t)]\sigma_i 1_{\{t \le T_j\}} dZ_i(t),$$

which leads to the Black formula for a shifted-lognormal process.

A LMM for the forward inflation rates $Y_i(t)$

- Let us assume that each $\mathcal{I}_i(t)$ evolves, under the corresponding forward measure Q^{T_i}, according to

$$d\mathcal{I}_i(t) = \mathcal{I}_i(t) \sum_{l=\beta(t)}^{i} \sigma_l dZ_l^i(t)$$

where $dZ_h^i(t) dZ_k^i(t) = \rho_{h,k} dt$, and $\beta(t)$ is the index of the first time T_j that is strictly larger than t.

- After a measure change for $\mathcal{I}_{i-1}(t)$, Ito's lemma leads to:

$$dY_i(t) = (1+Y_i(t)) \left[\sum_{l=\beta(t)}^{i-1} \sigma_l \left(\rho_{i,l}^{F,Y} \frac{\sigma_{F,i} \tau_i F_i(t)}{1+\tau_i F_i(t)} - \sigma_i \rho_{i,l} \right) dt + \sigma_i dZ_i^i(t) \right]$$

where $\sigma_{F,i}$ is the (assumed constant) volatility of $F_i(t)$ and $\rho_{i,l}^{F,Y}$ is the (instantaneous) correlation between F_i and Y_l.

A LMM for the forward inflation rates $Y_i(t)$

- Freezing $F_i(t)$ at its time-0 value $F_i(0)$, we have that $\bar{Y}(t) := 1 + Y_i(t)$ evolves according to the geometric Brownian motion:

$$d\bar{Y}_i(t) = \bar{Y}_i(t)\left[\sum_{l=\beta(t)}^{i-1} \sigma_l\left(\rho_{i,l}^{F,Y}\frac{\sigma_{F,i}\tau_i F_i(0)}{1+\tau_i F_i(0)} - \sigma_i\rho_{i,l}\right)dt + \sigma_i dZ_i^I(t)\right]$$

- Denoting by $\gamma(t)$ the drift of $\bar{Y}(t)$, the YYIIS and IICpl prices can be obtained as in the LMM for forward CPIs:

$$\mathcal{Y}_i(t) = E_t^{T_i}\left\{\bar{Y}_i(T_i)\right\} - 1 = \bar{Y}_i(t)e^{\int_0^t \gamma(u)du} - 1$$

$$E_t^{T_i}\left\{\left[Y_i(T_i) - \kappa\right]^+\right\} = \mathrm{Bl}\left(1 + \mathcal{Y}_i(t), 1 + \kappa, \sigma_i, T_i - t\right)$$

- The calibration to a given strike κ is almost automatic (provided we know $\mathcal{Y}_i(0)$).

A LMM for the forward inflation rates $Y_i(t)$

- Modeling forward inflation rates $Y_i(t)$ has the following advantages:
  - Almost automatic calibration to a given strike.
  - Possibility of assuming an exogenous correlation structure for forward inflation rates (in the market model for forward CPIs this correlation is endogenous).
- However, we still have the problem of calibrating cap smiles.
- This issue can be addressed by introducing stochastic volatility:

$$d\mathcal{I}_i(t) = \mathcal{I}_i(t) \sum_{j=\beta(t)}^{i} V_j(t) dZ_j^i(t)$$

where V_j, $j = 1, \dots, M$, are stochastic processes.

A LMM for the forward inflation rates $Y_i(t)$

- One can assume Heston's dynamics:

$$dV_i(t) = k[\theta - V_i(t)]dt + \nu_i \sqrt{V_i(t)} dW_i^I(t)$$

or SABR dynamics:

$$dV_i(t) = \nu_i V_i(t) dW_i^I(t)$$

- In both cases one gets:

$$\frac{d\bar{Y}_i(t)}{\bar{Y}_i(t)} = \sum_{l=\beta(t)}^{i-1} V_l(t) \left(\rho_{i,l}^{F,Y} \frac{\sigma_{F,i}\tau_i F_i(0)}{1+\tau_i F_i(0)} - V_i(t)\rho_{i,l} \right) dt + V_i(t) dZ_i^I(t)$$

- Freezing the drift at its time-0 value, we can write

$$d\bar{Y}_i(t) = \bar{Y}_i(t) \left[D_i(t)\, dt + V_i(t)\, dW_i^I(t) \right]$$

A LMM for the forward inflation rates $Y_i(t)$

- To derive an explicit option price (especially in the SABR case), we notice that

$$\bar{Y}_i(T_i) = \tilde{Y}_i(T_i),$$

where the process $\tilde{Y}_i$ is defined by

$$d\tilde{Y}_i(t) = \tilde{Y}_i(t) V_i(t)\, dZ_i^I(t), \quad \tilde{Y}_i(0) = \bar{Y}_i(0)\, e^{\int_0^{T_i} D_i(t)dt}$$

Therefore, setting $K := 1 + \kappa$, we have:

$$\begin{aligned} \textbf{IICplt}(t, T_{i-1}, T_i, K, \omega) &= P(t, T_i) E^{T_i} \left\{ \left[\omega Y_i(T_i) - \omega \kappa \right]^+ \middle| \mathcal{F}_t \right\} \\ &= P(t, T_i) E^{T_i} \left\{ \left[\omega \bar{Y}_i(T_i) - \omega K \right]^+ \middle| \mathcal{F}_t \right\} \\ &= P(t, T_i) E^{T_i} \left\{ \left[\omega \tilde{Y}_i(T_i) - \omega K \right]^+ \middle| \mathcal{F}_t \right\}, \end{aligned}$$

A LMM for the forward inflation rates $Y_i(t)$

- Assuming, for instance, SABR dynamics for the volatility, the caplet price can be valued by

$$\mathbf{IICplt}(t, T_{i-1}, T_i, K, \omega) = \omega P(t, T_i)\big[\tilde{Y}_i(t)\Phi(\omega d_+) - K\Phi(\omega d_-)\big]$$

where ρ_i is the instantaneous correlation between $\tilde{Y}_i$ and V_i,

$$d_\pm = \frac{\ln \frac{\tilde{Y}_i(t)}{K} \pm \frac{1}{2}\sigma^2(K)(T_i - t)}{\sigma(K)\sqrt{T_i - t}}$$

$$\sigma(K) = \alpha_i \frac{z}{x(z)} \left\{1 + \left[\frac{\rho_i \nu_i \alpha_i}{4} + \nu_i^2 \frac{2 - 3\rho_i^2}{24}\right](T_i - t)\right\}$$

$$z := \frac{\nu_i}{\alpha_i} \ln\left(\frac{\tilde{Y}_i(t)}{K}\right), \quad x(z) := \ln\left\{\frac{\sqrt{1 - 2\rho_i z + z^2} + z - \rho_i}{1 - \rho_i}\right\}.$$

Conclusions

- We have started by analyzing stylized facts of the inflation market.
- We have described the foreign currency analogy and shown how to price under a lognormal market model.
- We have finally introduced two possible definitions of forward inflation rates and considered the associated market models.
- The definition based on the CPI ratio, though less natural, turns out to be more effective.
- In fact, we can derive closed form formulas for YY caps and, with similar approximation techniques also for ZC options (in the SABR case).
- This is the first model in the financial literature that incorporates YY and ZC rates in a single consistent framework.