Finite Difference Method

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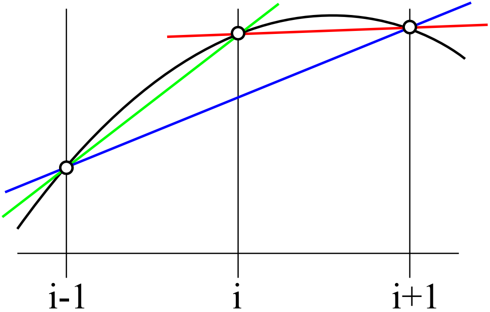
Introduction to Finite Difference Methods

1. analytical solution diffucult/impossible for real problems
2. finite number of compute elements
3. approximation: accuracy and stability
4. only $u(x_i)$ known $\rightarrow$ derivatives unknown

Introduction to Finite Difference Methods

- one-sided approximation
- centered approximation



Examples

- one-sided:

$$D_+u(x_i) = \frac{u(x_i + h) - u(x_i)}{h} \tag{1}$$

$$D_-u(x_i) = \frac{u(x_i) - u(x_i - h)}{h} \tag{2}$$

- centered approximation:

$$D_0u(x_i) = \frac{u(x_i + h) - u(x_i - h)}{2h} = \frac{1}{2}\left(D_+u(x_i) - D_-u(x_i)\right) \tag{3}$$

- third order approximation:

$$D_3u(x_i) = \frac{1}{6h}\left(2u(x_i + h) + 3u(x_i) - 6u(x_i - h) + u(x_i - 2h)\right) \tag{4}$$

Truncation Error

- expand $u(x_i)$ in a Taylor series:

$$u(x_i + h) = u(x_i) + hu'(x_i) + \frac{1}{2}h^2u''(x_i) + \frac{1}{6}h^3u'''(x_i) + O(h^4) \tag{5}$$

$$u(x_i - h) = u(x_i) - hu'(x_i) + \frac{1}{2}h^2u''(x_i) - \frac{1}{6}h^3u'''(x_i) + O(h^4) \tag{6}$$

- example: one-sided approx:

$$D_+u(x_i) = \frac{u(x_i + h) - u(x_i)}{h} = u'(x_i) + \frac{1}{2}hu''(x_i) + \frac{1}{6}h^2u'''(x_i) + O(h^3) \tag{7}$$

$$E(h) = D_+u(x_i) - u'(x_i) = \frac{1}{2}hu''(x_i) + \dots \tag{8}$$

Truncation Error

- example: centered approx:

$$u(x_i + h) - u(x_i - h) = 2hu'(x_i) + \frac{1}{3}h^3 u'''(x_i) + O(h^5) \quad (9)$$

$$\frac{u(x_i + h) - u(x_i - h)}{2h} - u'(x_i) = \frac{1}{6}h^2 u'''(x_i) + O(h^4) \quad (10)$$

Deriving Finite Difference Approximations

- example: FD approx for $u'(x_i)$ based on $u(x_i)$, $u(x_i - h)$, $u(x_i - 2h)$

$$D_2 u(x_i) = au(x_i) + bu(x_i - h) + cu(x_i - 2h) \tag{11}$$

- Taylor:

$$\begin{aligned} D_2 u(x_i) &= (a+b+c)u(x_i) - (b+2c)hu'(x_i) + \frac{1}{2}(b+4c)h^2 u''(x_i) \dots \\ &\dots - \frac{1}{6}(b+8c)h^3 u'''(x_i) + \dots \end{aligned} \tag{12}$$

- solve system:

$$a + b + c = 0 \qquad b + 2c = -1/n \qquad b + 4c = 0 \tag{13}$$

- resulting stencil:

$$D_2 u(x_i) = \frac{3u(x_i) - 4u(x_i - h) + u(x_i - 2h)}{2h} \tag{14}$$

Deriving Finite Difference Approximations

- second order derivative (example):

$$D^2 u(x_i) = D_+ D_- u(x_i) = \frac{1}{h^2}\left[u(x_i - h) - 2u(x_i) + u(x_i + h)\right] \qquad (15)$$

$$\cdots = u''(x) + \frac{1}{12} h^2 u'''(x_i) + O(h^4) \qquad (16)$$

1D Elliptic Equation

- class of differential operators
- $\sum a_{ij}(x)\partial^2_{x_i x_j} u(x) + \sum b_i(x)\partial x_i u(x) + c(x)u(x) = f(x)$
- stationary problems
- minimal energy
- example: diffusion equation (isotropic)

$$u_t(x,t) = Du_{xx}(x,t) + \Psi(x,t) \tag{17}$$

- IC: $u(x,0) = u_0(x)$, BC: $u(a,t) = \alpha(t)$, $u(b,t) = \beta(t)$
- steady state with $f(x) = -\Psi(x)/D$

$$u''(x) = f(x) \qquad 0 < x < 1 \qquad u(0) = \alpha \quad u(1) = \beta \tag{18}$$

1D Elliptic Equation

- discretization: $\{x_0, x_1, \ldots, x_{m-1}, x_m, x_{m+1}\}$, $h = \frac{1}{m+1}$
- unknowns: $\{x1, x2, \ldots, x_m\}$
- apply second order centerd difference:

$$\frac{1}{h^2}\left[u_{i-1} - 2u_i + u_{i+1}\right] = f(x_i) \qquad\qquad i = 1 \ldots m \tag{19}$$

- write as $AU = F$:

$$\begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ \ldots \\ \\ \\ u_m \end{bmatrix} = \begin{bmatrix} f(x_1) - \alpha/h^2 \\ f(x_2) \\ \ldots \\ f(x_{m-1}) \\ f(x_m) - \beta/h^2 \end{bmatrix} \tag{20}$$

Neumann Boundary Condition

- e.g. at left boundary: $u'(0) = \sigma$
- one-sided approx: $\frac{u_1 - u_0}{h} = \sigma$
- centered approx:

$$\frac{1}{h^2}\left[u_{-1} - 2u_0 + u_1\right] = f(x_0) \tag{21}$$

$$\frac{1}{2h}\left[u_1 - u_{-1}\right] = \sigma \tag{22}$$

$$\frac{1}{h}\left[-u_0 + u_1\right] = \sigma + \frac{h}{2} f(x_0) \qquad O(h) \tag{23}$$

- we want to use u_1, u_2, u_3 (see deriving FD approx):

$$\frac{1}{h}\left[\frac{3}{2}u_0 - 2u_1 + \frac{1}{2}u_2\right] = \sigma \qquad O(h^2) \tag{24}$$

Neumann Boundary Condition

$$\frac{1}{h^2}\begin{bmatrix} \frac{3}{2}h & -2h & \frac{1}{2}h & & \\ 1 & -2 & 1 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{bmatrix}\begin{bmatrix} u_0 \\ u_1 \\ \cdots \\ \\ u_m \end{bmatrix} = \begin{bmatrix} \sigma \\ f(x_1) \\ \cdots \\ \\ f(x_m) - \beta/h^2 \end{bmatrix} \tag{25}$$

2D Elliptic Equations

- $a_1 u_{xx} + a_2 u_{xy} + a_3 u_{yy} + a_4 u_x + a_5 u_y + a_6 u = f$
- $a_2^2 - 4 a_1 a_3 < 0$
- example: Poisson equation (diffusion)

$$u_{xx} + u_{yy} = f \tag{26}$$

- IC: $u(x, y, 0) = u_0(x, y, 0)$, BC: $u(x, y, t)\ \{x, y\} \in \partial\Omega$
- centerd differences:

$$\frac{1}{h_x^2}\left[u_{i-1,j} - 2u_{ij} + u_{i+1,j}\right] + \frac{1}{h_y^2}\left[u_{i,j-1} - 2u_{ij} + u_{i,j+1}\right] = f_{ij} \tag{27}$$

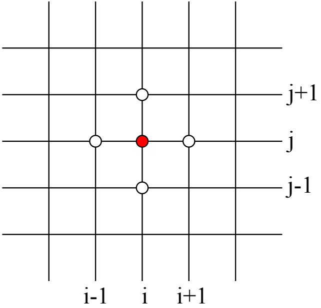
$$\frac{1}{h^2}\left[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{ij}\right] = f_{ij} \tag{28}$$

- lexicographic ordering: $u = [u_{11} \dots u_{m1}, u_{12} \dots u_{m2}, \dots, u_{1m} \dots u_{mm}]$

2D Elliptic Equations

- centered differences:

$$\nabla_h^2 u(x) = \frac{1}{h^2}\left[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + j_{i,j+1} - 4u_{ij}\right] \tag{29}$$



2D Elliptic Equations

- the system now looks like:

$$A = \begin{bmatrix} T & I & & & \\ I & T & I & & \\ & I & T & I & \\ & & & \dots & \dots \end{bmatrix} \qquad \in \mathbb{R}^{m^2 \times m^2} \tag{30}$$

- with

$$T = \begin{bmatrix} -4 & 1 & & & \\ 1 & -4 & 1 & & \\ & 1 & -4 & 1 & \\ & & & \dots & \dots \end{bmatrix} \qquad \in \mathbb{R}^{m \times m} \tag{31}$$

Initial Value Problem for ODE

- example:

$$u'(t) = f(u(t), t) \qquad \text{IC: } u(t_0) = \eta \tag{32}$$

- very simple: forward Euler

$$\frac{u^{n+1} - u^n}{\Delta t} = f(u^n) \qquad O(\Delta t) \tag{33}$$

- less simple: backward Euler

$$\frac{u^{n+1} - u^n}{\Delta t} = f(u^{n+1}) \qquad O(\Delta t) \tag{34}$$

- less simple: trapezoidal method

$$\frac{u^{n+1} - u^n}{\Delta t} = \frac{1}{2}\left[f(u^n) + f(u^{n+1})\right] \qquad O(\Delta t^2) \tag{35}$$

Initial Value Problem for ODE

- multistep methods: e.g.

$$\frac{u(t+\Delta t) - u(t-\Delta t)}{2\Delta t} = u'(t) + \frac{1}{6}\Delta t^2 u'''(t) + O(\Delta t^3) \qquad (36)$$

- example: leapfrog/midpoint method

$$\frac{u(t+\Delta t) - u(t-\Delta t)}{2\Delta t} = f(u^n) \qquad O(\Delta t^2) \qquad \text{explicit 2-step} \qquad (37)$$

- multistep methods for higher accuracy (but high memory consumption)
- accuracy can be achieved by one-step methods (e.g. Runge-Kutta)

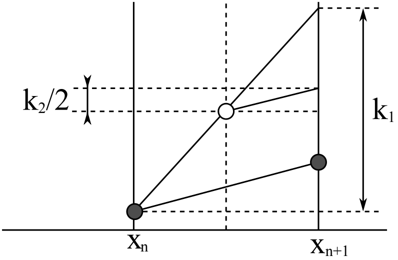
Runge-Kutta Method

- Forward Euler: $u^{n+1} = u^n + \Delta t \cdot f(x_n, u_n)\ O(h)$
- second order RK:
  - trial step to the midpoint of the interval
  - use info there

$$k_1 = \Delta t \cdot f(x_n, u_n) \tag{38}$$

$$k_2 = \Delta t \cdot f(x_n + \frac{1}{2}\Delta t, u_n + \frac{1}{2}k_1) \tag{39}$$

$$u^{n+1} = u^n + k_2 + O(\Delta t^3) \tag{40}$$



Diffusion Equation (parabolic)

$$u_t = u_{xx} \tag{41}$$

- IC: $u(x,0) = \eta(x)$, BC: $u(0,t) = g_0(t)$, $u(1,t) = g_1(t)$
- discretization: $x_i = i \cdot \Delta x$, $t_n = n \cdot \Delta t$
- very simple: centered difference in space, forware Euler in time.

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{1}{h^2}\left[u_{i-1}^n - 2u_i^n + u_{i+1}^n\right] \qquad \frac{\Delta t}{h^2} < \frac{1}{2} \tag{42}$$

- Crank-Nicholson:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{1}{2}\left[D^2 u_i^n + D^2 u_i^{n+1}\right] \tag{43}$$

$$\cdots = \frac{1}{2h^2}\left[u_{i-1}^n - 2u_i^n + u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i+1}^{n+1}\right] \tag{44}$$

Crank-Nicholson

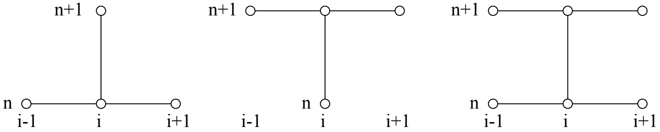
$$-ru_{i-1}^{n+1}+(1+2r)u_i^{n+1}-ru_{i+1}^{n+1}=ru_{i-1}^{n}+(1-2r)u_i^{n}+ru_{i+1}^{n} \qquad r=\frac{\Delta t}{2h^2} \tag{45}$$

$$\begin{bmatrix} (1+2r) & -r & & & \\ -r & (1+2r) & r & & \\ & -r & (1+2r) & -r & \\ & & & & \end{bmatrix}\begin{bmatrix} u_1^{n+1} \\ u_2^{n+1} \\ u_3^{n+1} \\ \cdots \end{bmatrix}=\begin{bmatrix} r(g_0^n+g_0^{n+1})+(1+2r)u_1^n+ru_2^n \\ ru_1^n+(1+2r)u_2^n+ru_3^n \\ \cdots \\ \end{bmatrix} \tag{46}$$

1. solve this system in $O(m)$

Stencils

- stencil of forward Euler, backward Euler, Crank-Nicholson:



Order of Accuracy

- lets get back to forward Euler:

$$u_i^{n+1} = u_i^n + \frac{\Delta t}{h^2}\left[u_{i-1}^n - 2u_i^n + u_{i+1}^n\right] \tag{47}$$

- local truncation error:

$$\tau(x,t) = \frac{u(x,t+\Delta t) - u(x,t)}{\Delta t} - \frac{1}{h^2}\left[u(x-h,t) - 2u(x,t) + u(x+h,t)\right] \tag{48}$$

- Taylor expansion in (48):

$$\tau(x,t) = \left(u_t + \frac{1}{2}\Delta t u_{tt} + \frac{1}{6}\Delta t^2 u_{ttt} + \dots\right) - \left(u_{xx} + \frac{1}{12}h^2 u_{xxxx} + \dots\right) \tag{49}$$

- recall: $u_t = u_{xx}$, $u_{tt} = u_{txx} = u_{xxxx}$

$$\tau(x,t) = \left(\frac{1}{2}\Delta t - \frac{1}{12}h^2\right) u_{xxxx} + O(\Delta t^2 + h^4) \tag{50}$$

- conclusion: $O(\Delta t)$ in time, $O(\Delta h^2)$ in space
- Crank-Nicholson: $O(\Delta t^2)$ in time, $O(\Delta h^2)$ in space

Multidimensional Problem

- example: 2D diffusion

$$u_t = \Delta u = u_{xx} + u_{yy} \qquad u(x, y, 0) = \eta(x, y) \quad u(x, y, t) = u_\partial(x, y, ts) \tag{51}$$

- discretized Laplacian:

$$\nabla_h^2 u_{ij} = \frac{1}{h^2}\left[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} 1 u_{i,j+1} - 4u_{ij}\right] \tag{52}$$

- discretize in time: e.g. CN

$$\frac{u_{ij}^{n+1} - u_{ij}^{n}}{\Delta t} = \frac{1}{2}\left[\nabla_h^2 u_{ij}^{n} + \nabla_h^2 u_{ij}^{n+1}\right] \tag{53}$$

- implicit: solve linear system in every time step

Advection Equation (hyperbolic)

- waves
- advective transport
- example:

$$u_t + au_x = 0 \qquad \text{IC: } u(x,0) = \eta(x) \tag{54}$$

- analytical solution: $u(x,t) = \eta(x - at)$
- centered difference in space, forward Euler in time (not very stable):

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = -\frac{a}{2h}\left(u_{j+1}^n - u_{j-1}^n\right) \tag{55}$$

- replace u_j^n by $\frac{1}{2}\left(u_{j-1}^n + u_{j+1}^n\right)$: Lax-Friedrichs method

$$u_j^{n+1} = \frac{1}{2}\left(u_{j-1}^n + u_{j+1}^n\right) - \frac{a\Delta t}{2h}\left(u_{j+1}^n - u_{j-1}^n\right) \qquad \left|\frac{a\Delta t}{h}\right| \leq 1 \tag{56}$$

Literature

- Press, Teukolsky Vetterling, Flannery, *Numerical Recipes*, Cambridge University Press, 2007
- LeVeque, *Finite Differece Methods for Ordinary and Partial Differential Equations*
- Ralf Hiptmayr, *Numerical Methods for Partial Differential Equations*, lecture notes, 2010
- Gerald W. Recktenwald, *Finite-Difference Approximations to the Heat Equation*, 2011

Thanks!!

Thanks for your attention!

Slides for this talk will be available at:

`http://www.bsse.ethz.ch/cobi/education`

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