Efficient Pricing of an Asian Put Option Using Stiff ODE Methods

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Contents

List of Tables

List of Figures

1 Abstract

Financial mathematics is a branch of mathematics that assesses the risk and value of various financial instruments. Banks, companies, and other institutions mitigate their risk through financial instruments known as derivatives, that derive their value from some underlying asset. The equations that arise from pricing and modeling can be very complex, leading to the necessity of numerical methods.

This project studied the use of certain numerical methods for the pricing of a particular type of option called an Asian option. Asian options can provide favorable risk profiles because the payout is determined based on the average value over a time period, rather than the final value. The price of an Asian option is governed by a partial differential equation in three variables: stock price, average price over the current time interval, and time.

The solution method was first to discretize the partial differential equation into a system of ordinary differential equations. Next, the ODE system was integrated using a stiff-ODE solver available in MATLAB. Enhancements to this solution method include specifying the sparsity pattern, implementing an iterative linear solver (GMRES [2]) in place of MATLAB's built-in direct linear solver, and using preconditioning to improve the solution characteristics of that solver.

2 Introduction

Asian options are a particular class of options that calculate payout on a time-interval average as opposed to the standard final value of an underlying instrument. As a result, Asian options have less volatility than their "vanilla" European counterparts and thus, a cheaper price.

Asian options can be classified by their method of averaging, such as arithmetic or geometric. In this project, the particular averaging equation can be found as Equation 1, where A is the average, T is the time until expiration, S is the price of the underlying instrument, and t is the time until maturity.

$$A = \frac{1}{T-t}\int_0^{T-t} S(\tau)\, d\tau \tag{1}$$

Financial mathematicians have developed theory to price Asian options, resulting in Equation 2, where P is the price, σ is the volatility, S is the underlying price, r is the risk-free rate, A is the time-interval average, T is the time until expiration, and t is the time until maturity [1].

$$\frac{\partial P}{\partial t} = \frac{\sigma^2 S^2}{2}\frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} + \frac{S-A}{T-t}\frac{\partial P}{\partial A} - rP \tag{2}$$

The equation is structured so that the starting time is at expiration and the equation is solved backwards in time until $t = T$ (today). The price of the option can then be found by looking at the diagonal $S = A$ and obtaining the current underlying price S from the marketplace. Ranges of S and A depend on the price properties of the underlying instrument. In this project, we set the maximum values of S and A to two-hundred.

Asian options come in two forms, puts and calls. Both forms have a payout based on the difference between the time-interval average price and the strike price. The strike price (K) is specified in the contract, and can be any value that the buyer and seller agree upon. The payout functions for puts and calls can be found in Equations 3 and 4. Puts are favorable to the option buyer if prices go down; calls are favorable to the option buyer if prices go up. This project addressed the problem of pricing an Asian put. Since the problem is solved backward in time, the initial condition for the integration was the put payout function, found in Equation 3.

$$P(S, A, 0) = max(K - A, 0) \tag{3}$$

$$P(S, A, 0) = max(A - K, 0) \tag{4}$$

3 Solution Methodology

3.1 Method of Lines

The method of lines is a technique for solving partial differential equations where all but one dimension is discretized, leading to a set of ordinary differential equations in the non-discretized dimension [3]. After the semi-discretization, methods and software that have been developed for ordinary differential equations can be used to integrate the system, yielding a solution of the partial differential equation.

This method is only suitable for certain classes of partial differential equations, namely initial value problems. The pricing of an Asian option meets this criteria because of its structure in time. An example of an unsuitable partial differential equation would be the standard Laplace equation which does not have any initial-value type conditions.

3.2 Stiffness

Since we are dealing with numerical methods, stability problems may arise. For the Asian option, the ordinary differential equations arising from the discretization are stiff. Stiffness is a property of an equation system that leads to instability when using explicit numerical methods, unless the time-steps are kept sufficiently small. As an example, consider the one-dimensional differential equation in Equation 5 with initial condition Equation 6.

$$\frac{du}{dt} = -20u \tag{5}$$

$$u(0) = 1 \tag{6}$$

Suppose we apply an Explicit Euler Integration, as found in Equation 7. The results for a number of h values (time step) are shown in Figure 1. A very small time step is required for a reasonable solution. However, if we use an Implicit Euler Integration, as found in Equation 8, the solution is reasonable even for the relatively large value of $h = .1$. This result can be found in Figure 2. A trade-off arises for stiff numerical differential equations. Explicit methods allow for direct calculation and computational simplicity. Implicit methods allow for improved stability with more complex computa-

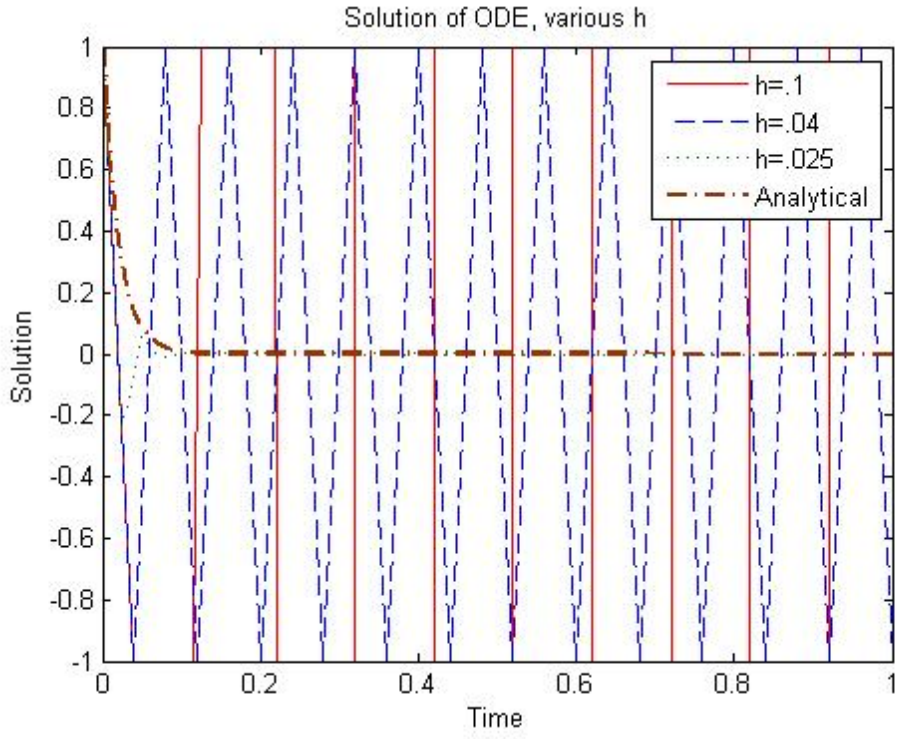

Figure 1: Explicit Euler Method

tions (solutions of linear or non-linear systems). In stiff problems, implicit methods are usually the preferred choice.

$$u_{n+1} = u_n + h(-20u_n) \tag{7}$$

$$u_{n+1} = u_n + h(-20u_{n+1}) \tag{8}$$

3.3 Linear Systems

Since we are using an implicit method in our solution, it is necessary to solve linear systems throughout the integration. There are many algorithms available. They are divided into two main types: direct and iterative solvers.

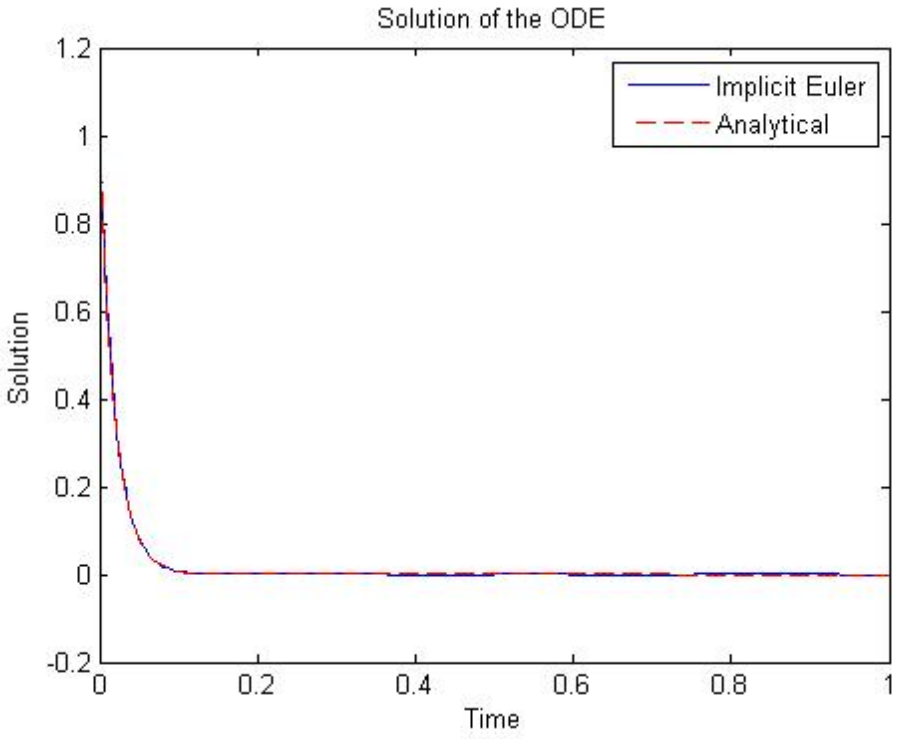

Figure 2: Implicit Euler Method, h=.1

Direct solvers attempt to solve the equation $Ax = b$ using an exact algorithm. Examples include Gaussian elimination and LU decomposition. There are many specialized algorithms within the set of direct solvers. For example, Cholesky decomposition is used for symmetric, positive-definite systems. For the Asian put pricing problem, many of the matrix elements are zero, meaning the systems that arise are extremely sparse. MATLAB's direct solvers can be chosen to exploit this sparsity.

Iterative solvers, as opposed to direct solvers, try to find successive approximations to the solution, starting from an initial guess. The stopping criteria can be a fixed number of iterations, a specified residual-norm reduction tolerance, or a combination of both. Examples include the Gauss-Seidel method and GMRES. Iterative solvers are often effective on large grid sizes where direct solvers can become inefficient or even impossible to use.

In this project, we implemented both direct and iterative solvers in order to find the best choice of method on a variety of grid sizes. The specific methods used are discussed in Chapter 4.

3.4 GMRES and Preconditioning

Other than the built-in MATLAB solvers, the linear systems solution method used was GMRES [2]. GMRES is an iterative linear systems method that is particularly suited for large, sparse systems. The method approximates the solution by the vector in a Krylov subspace with minimal residual.

In particular, if we define the nth Krylov subspace as K_n in Equation 9 and suppose we are trying to solve the matrix equation $Ax = b$, GMRES calculates $x_n = x_o + z_n, z_n \in K_n$ such that $\parallel Ax_n - b \parallel$ is minimized.

$$K_n = span(b, Ab, A^2b, ..., A^{n-1}b) \tag{9}$$

The performance of GMRES can be further enhanced through two methods. First, since GMRES is a Krylov method, matrix evaluations are not required; instead, a "matrix-vector product" routine can be supplied. Throughout the report, this method is known as the "matrix-free" method. Next, preconditioning can be implemented.

Preconditioning is the process of modifying a linear system in a specific way to improve its solvability. Suppose we are dealing with the generic linear equation $Ax = b$. The condition number, defined in Equation 10, is a crucial factor in the performance of linear solution methods. We consider

left-preconditioning here, in which the linear system is transformed with a left preconditioning matrix P as found in Equation 11. Improving the condition of the system is often stated as a nominal goal of preconditioning, since ill-conditioning of A usually implies poor performance of Krylov subspace methods. A good choice of P will improve the condition number of the modified system and speed up the convergence of the method sufficiently to outweigh the cost of implementing it.

$$\kappa(A) = ||A|| \cdot ||A^{-1}|| \tag{10}$$

$$P^{-1}Ax = P^{-1}b \tag{11}$$

4 Implementation Framework

4.1 Discretization

The partial differential equation was discretized on a square grid of S and A, with both variables ranging from zero to two hundred. There were $N+1$ nodes in each direction so that $h_A = h_S = \frac{1}{N}$. Using uniform spacing, $S_i = ih_S$ and $A_j = jh_A$. The derivatives were approximated using a finite difference scheme. For the S derivatives, a second order centered-difference scheme could be used. However, for the A derivatives, it was necessary to use upwinding since there is a lack of diffusion (second order derivative). In summary, we use the discretization found in [1] as shown below in Equations 12-16.

$$\frac{\partial P}{\partial t} = \frac{\sigma^2 S_i^2}{2} D_{S,S} + rS_i D_S + \frac{S_i - A_j}{T-t} D_A - rP \tag{12}$$

$$D_{S,S} = \frac{P(S_{i+1}, A_j, t) - 2P(S_i, A_j, t) + P(S_{i-1}, A_j, t)}{h_S^2} \tag{13}$$

$$D_S = \frac{P(S_{i+1}, A_j, t) - P(S_{i-1}, A_j, t)}{2h_S} \tag{14}$$

$$D_A = \frac{P(S_i, A_{j+1}, t) - P(S_i, A_j, t)}{h_A}, S_i > A_j \tag{15}$$

$$D_A = \frac{P(S_i, A_j, t) - P(S_i, A_{j-1}, t)}{h_A}, S_i \leq A_j \tag{16}$$

Equation 12 yields an ordinary differential equation in time for each node on the S-A grid. As a result, the partial differential equation has been discretized into a system of ordinary differential equations. A numerical solution can be obtained by using an ordinary differential equation solver such as those found in MATLAB.

4.2 Boundary Conditions

We need to consider four boundary conditions for the rectangular domain, $S = 0$, $S = S_c$, $A = 0$, and $A = A_c$. Along $S = 0$, the S derivatives vanish as they are multiplied by S in the equations. For $A = 0$, there is only

dependence on interior nodes since we are using upwinding. This is similar to the treatment of an outflow boundary in fluid mechanics.

For $S = S_c$, we impose the boundary condition $\frac{\partial P}{\partial S} = 0$ because the price of the option depends only on A for large values of S. For the boundary $A = A_c$, $lim_{A \to +\infty} P(t, S, A) = 0$ so we can set $P(t, S, A_c) = 0$ where $S > A_c$. For $S < A_c$, no condition has to be imposed due to upwinding and an outflow-like boundary.

4.3 MATLAB Implementation

The equation was solved using the MATLAB ODE Suite [4]. The suite has seven solvers that are used for various types of initial value problems: "ode23","ode45","ode113","ode15s","ode23s","ode23t", and "ode23tb". The solvers are distinguished from each other by their ability to solve stiff systems as well as their available order.

"ode23","ode45", and "ode113" are solvers designed for non-stiff problems. The others are used for stiff problems. The method we chose to use for this equation was "ode15s" due to its stiffness,. "ode15s" is the standard stiff solver, whereas "ode23s" and "ode23tb" are for crude error tolerances and "ode23t" is for a solution without numerical damping.

The MATLAB solvers themselves are relatively easy to implement. The minimum input includes a function handle for the time derivative, a time range, and an initial value. The solvers can accommodate systems of differential equations with ease: Rather than scalar arguments, vector arguments are used. For the implicit time-stepping solvers, performance can be enhanced by specifying a function that evaluates the Jacobian matrix, and the Jacobian's sparsity pattern.

In addition to using the standard "ode15s", several modified versions were created that implemented various versions of the GMRES algorithm for the solution of linear systems. Iterative methods will often perform better than standard direct algorithms on larger grid sizes. The standard MATLAB implementation of GMRES was used which unfortunately, is implemented as interpreted source code. This can cause the solution time to increase drastically relative to the compiled code used by MATLAB's direct solvers, as discussed in the results section.

5 Solution Methods

5.1 ode15s

In the "ode15s" case, we only provide the MATLAB stiff system solver the right-hand side of Equation 2, in addition to the time interval and initial data. The solver then proceeds to integrate the time derivatives and obtain the solution using full-matrix storage. The benefit of using this solver is that it is the most basic. However, that simplicity comes with a price because the method is very inefficient and slow. The main reason for the inefficiency is that a standard linear solution method is used that does not take advantage of the Jacobian sparsity.

5.2 ode15s with sparsity

For the "ode15s with sparsity" case, we provided MATLAB's ode15s solver with both the time derivative function and a matrix of zeros and ones indicating the sparsity pattern of the Jacobian. This allowed MATLAB's linear systems algorithms to exploit sparsity, enhancing speed and efficiency over ode15s on moderate and large grids. The sparsity pattern can be found in Figure 3. A close-up detailing the band structure can be found in Figure 4. The varying in lengths of the outer bands is a result of the upwinding scheme.

5.3 ode15s with GMRES

In the "ode15s with GMRES" case, we modified the ode15s solver code to use MATLAB's GMRES solver rather than its standard linear solver. The advantage of this is that GMRES is a solver suited especially for large sparse matrix systems. The disadvantages are that MATLAB's code had to be manually modified, and the GMRES implementation is interpreted.

5.4 ode15s with GMRES, static preconditioning

For "ode15s with GMRES, static preconditioning", a preconditioner is factored one time at the outset of the calculation and used throughout the integration. Preconditioning can have some desirable effects on the performance of GMRES, as discussed previously. Furthermore, static preconditioning has

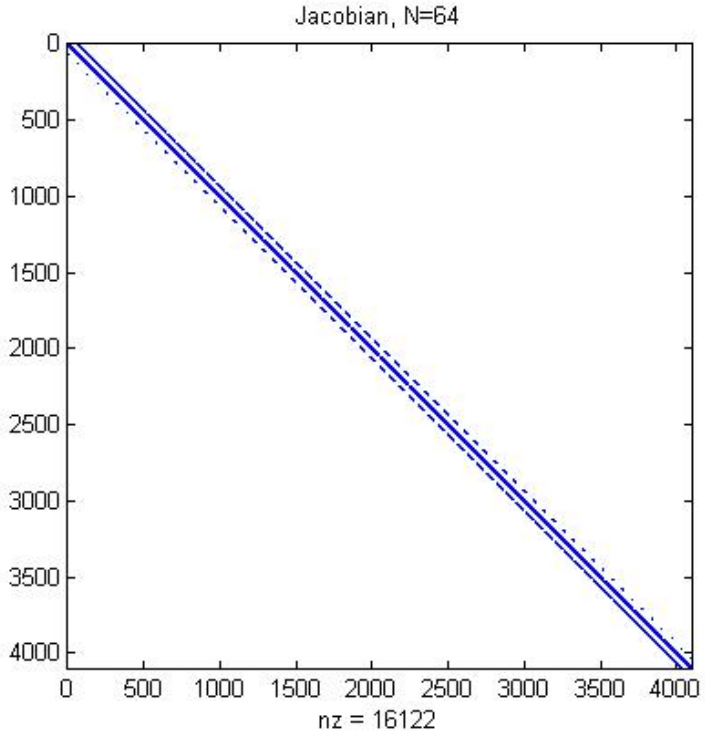

Figure 3: Jacobian Sparsity Pattern

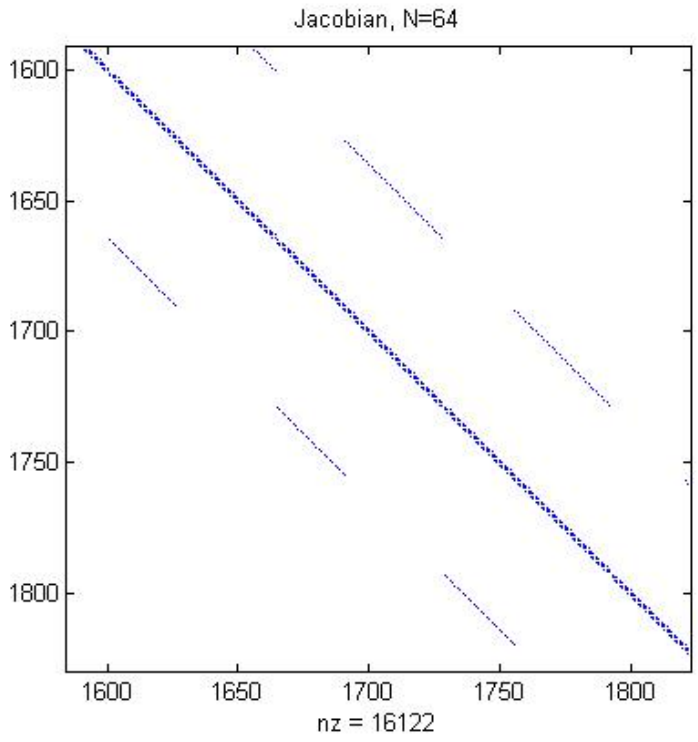

Figure 4: Jacobian Sparsity Pattern, Close-Up

a computational advantage over dynamic preconditioning as it requires only one factorization.

The static preconditioners used are based on the value of the Jacobian at the calculation outset. Various preconditioners include the tridiagonal part and main diagonal part of the Jacobian. It is difficult to find an effective static preconditioner because the implicit methods used in "ode15s" do not use the actual Jacobian matrix, but a matrix that includes method-specific constants which change at each step. An example of this "modified" matrix can be found as Equation 17 where "I" is the identity matrix of size N and α is a method constant.

$$M = I - \alpha \frac{\partial P}{\partial t} \tag{17}$$

5.5 ode15s with GMRES, dynamic preconditioning

In "ode15s with GMRES, dynamic preconditioning", we calculate a new preconditioner for each linear system to be solved. The advantage here is that a current preconditioner could be used at each time step. The disadvantage is that it requires more computation than either GMRES without preconditioning or GMRES with static preconditioning.

The dynamic preconditioner used is the tridiagonal part of the matrix found in Equation 17. This is calculated at each time step and is easily factored due to its tri-diagonal structure.

5.6 ode15s with GMRES, matrix-free version

Since GMRES only requires Jacobian-vector products, a "matrix-free" method can be introduced by providing GMRES with a function routine to evaluate those Jacobian-vector products. This routine is implemented by a finite-difference approximation. The advantage is that no Jacobian evaluations are required, although there is a substantial increase in the number of function evaluations.

5.7 ode15s with GMRES, matrix-free version with static preconditioning

The final method used is a modification of "ode15s with GMRES, matrix-free version", which is to introduce static preconditioning to help improve performance. There should only be a slight increase in the computation time required over the standard matrix-free version, because the preconditioner is evaluated and factored once at the computation start.

6 Results

The first result obtained was a base-case solution using the standard MATLAB solver "ode15s" on a moderate grid size. This was to check the discretization, initial condition, and other problem parameters against known results. After it was clear the solution method was working, the other methods were implemented, including the sparsity method and the various GMRES methods.

After some initial trials, it became clear that the best GMRES variation was the dynamic preconditioning method. A parametric study was done to find a reasonable set of GMRES parameters using this preconditioner. Once the set of parameters was found, a complete numerical study on a range of grid sizes for all methods was undertaken.

6.1 Base Case

Figures 5,6,and 7 show the solution plotted on the two-dimensional grid at various times. As mentioned before, time is integrated "backwards" so $t = 0$ actually corresponds to expiration. At $t = 1$ (today), the solution depends only on the current price (S), as expected [1]. This solution had $r = .05$, $K = 100$, $S_c = A_c = 200$, $T = 1$, and $\sigma = .2$. These values were used in all tests.

6.2 GMRES Parameters

After the base case, the next step was to find a reasonable set of parameters to use within GMRES. The three parameters varied were restart value, maximum iterations, and tolerance. A restart value of n tells GMRES to restart every n inner iterations. The maximum iterations is the maximum number of outer iterations. Finally, the tolerance is the maximum allowed relative residual norm defined as in Equation 18.

The grid size was $n = 128$, a mid-level grid. The default parameters were 10 maximum iterations, a restart value of 10, and a tolerance of 10^{-6}. The number of maximum outer iterations was varied from 10 to 100 in increments of 10. The restart value was varied from 1 to 50. Finally, the tolerance was varied from 10^{-1} to 10^{-10} , increasing the negative exponent.

The study indicated that a tolerance of 10^{-6}, a restart value of 15, and a maximum number of outer iterations 10 would be a good set of parameters to

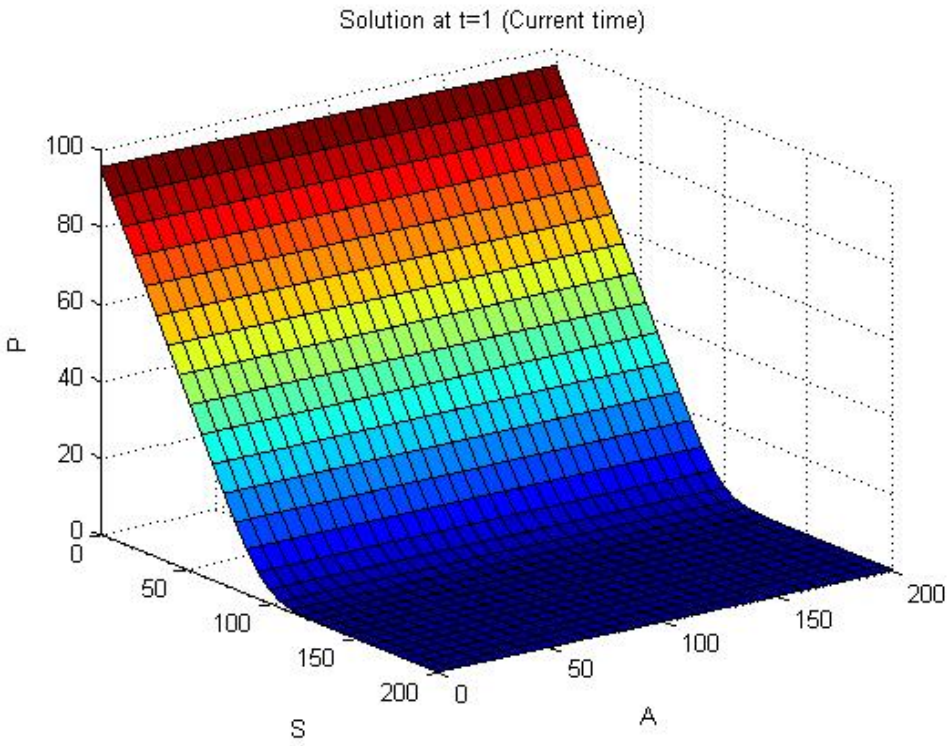

Figure 5: Solution at t=1 (Today)

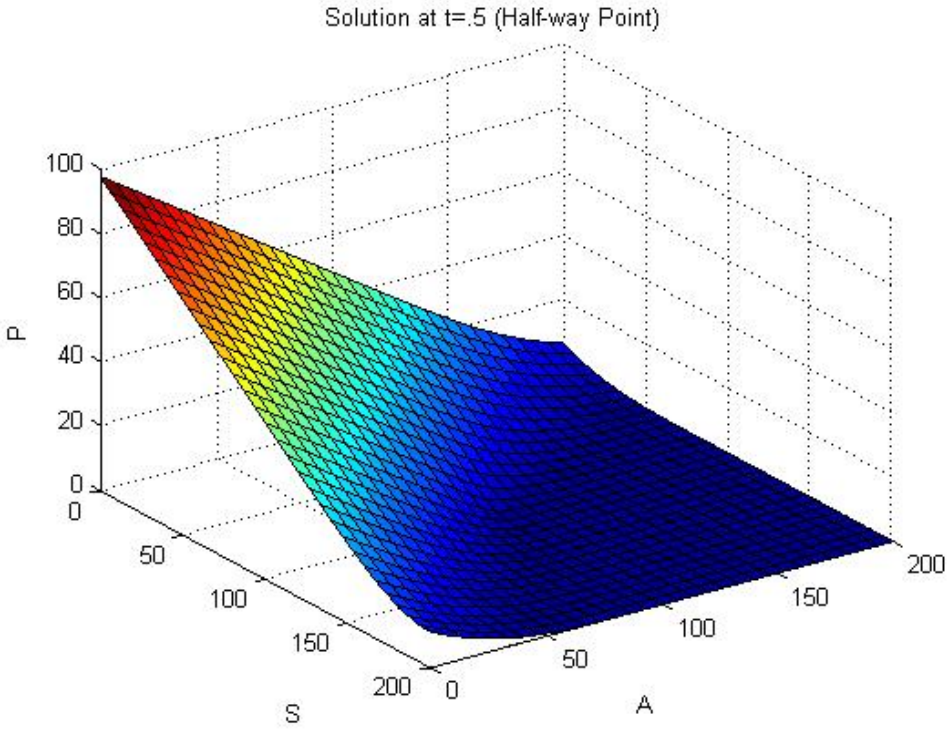

Figure 6: Solution at t=.5

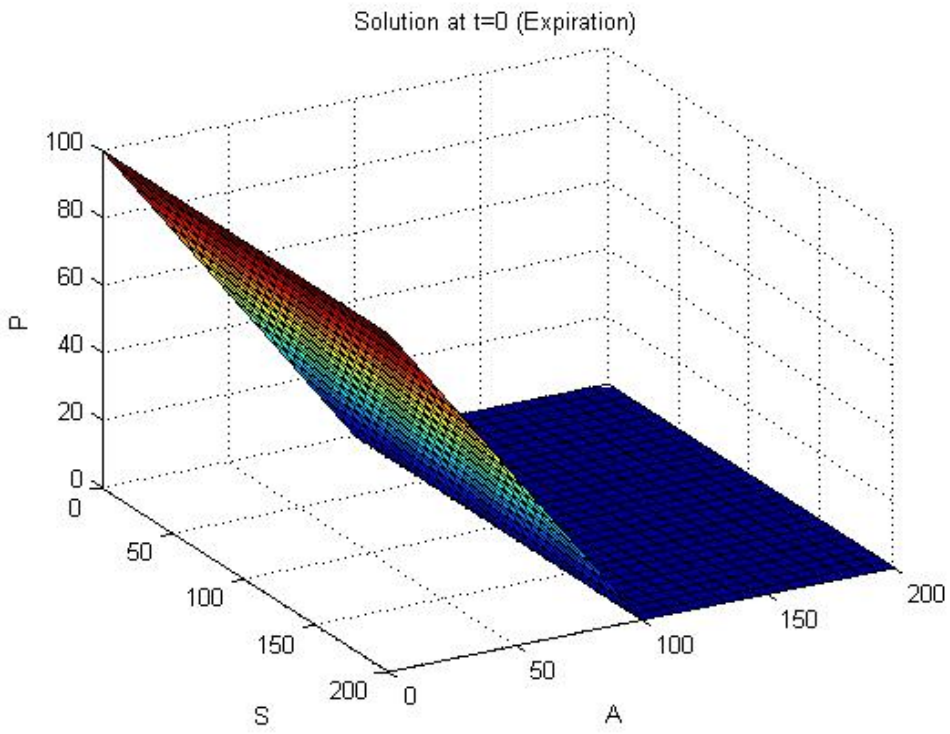

Figure 7: Solution at t=0 (Expiration)

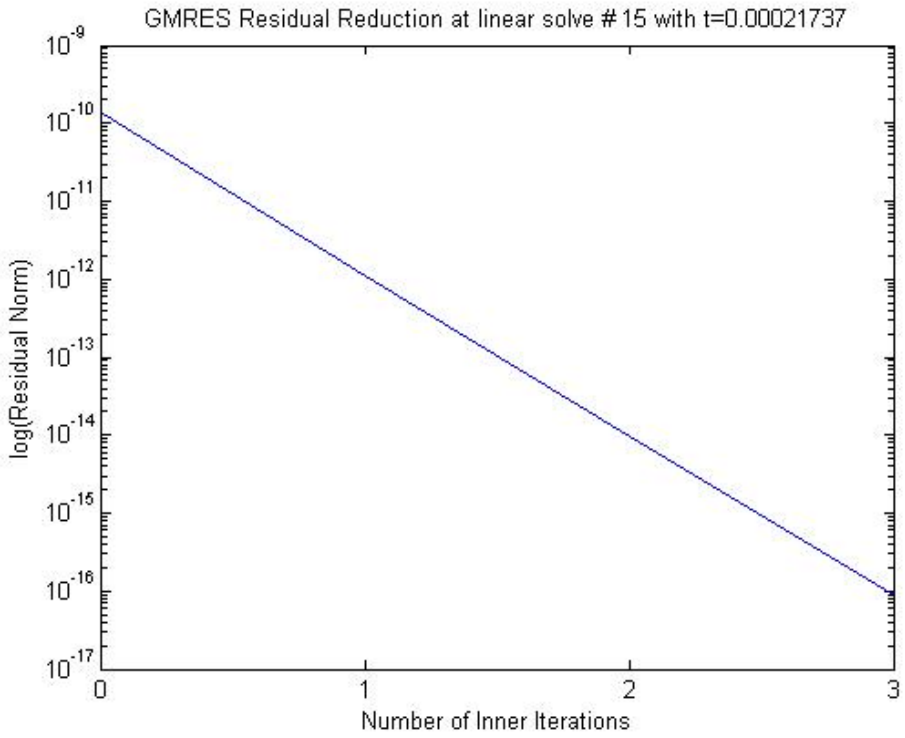

Figure 8: GMRES Performance, Preconditioned

use in the computations. Complete results for the GMRES parameter study are found in Appendix I.

$$||b - Ax||/||b|| \tag{18}$$

6.3 GMRES Performance

Residual reduction graphs of GMRES (log(residual norm) vs number of iterations) can show algorithm performance. A roughly linear behavior of residual norm reduction on a semi-log scale may show that effective preconditioning was used. The performance of GMRES was dependent on time. Figures 8 and 9 show GMRES performance at two times, a particularly easy time and one at which GMRES had trouble.

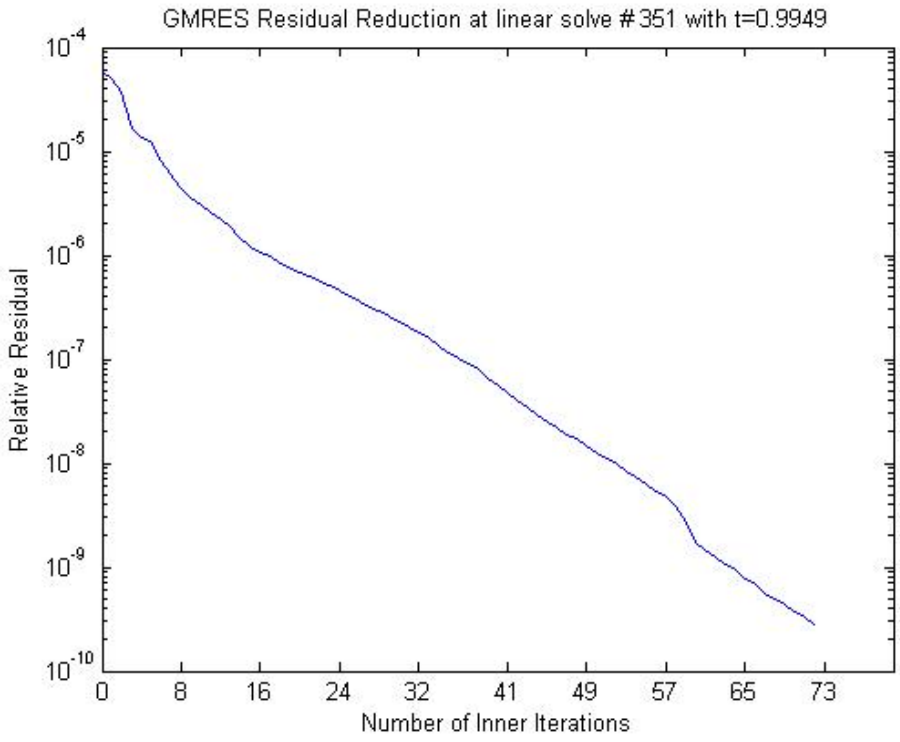

Figure 9: GMRES Performance, Preconditioned

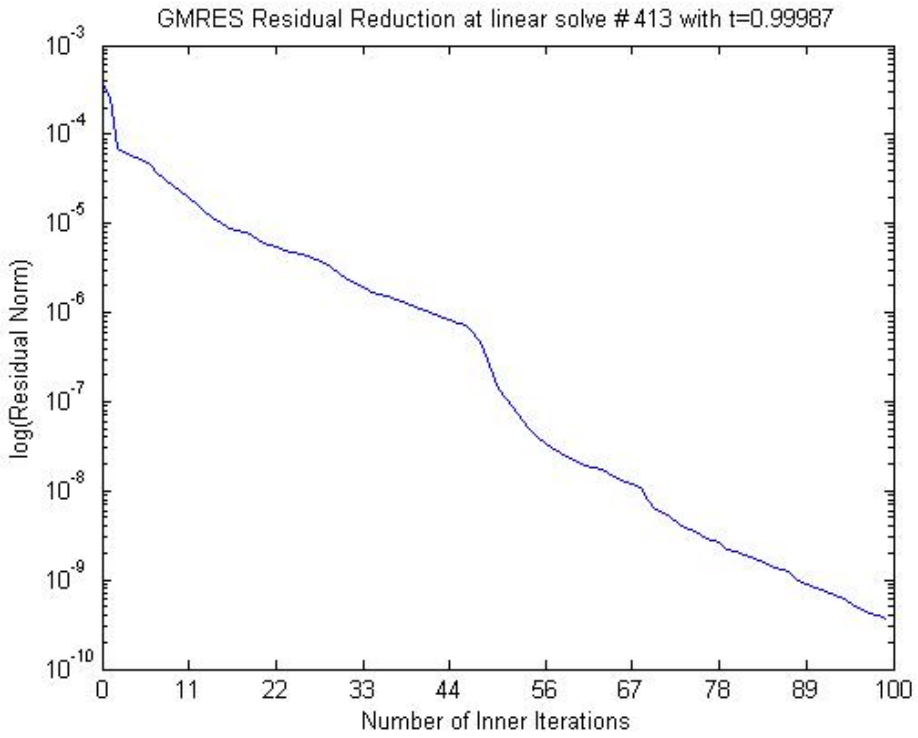

Figure 10: GMRES Performance, Not Preconditioned

Figure 10 shows performance of GMRES without preconditioning. The effect of preconditioning can be seen when comparing Figures 9 and 10.

6.4 Method Performance

Results were obtained for a variety of grid sizes ranging from $N = 16$ to $N = 512$. For the higher grid sizes, fewer methods were used because the calculation times became excessive for some methods. Tables 2-6 display the results for $N = 32, 64, 128, 256, 512$. Table 1 lists the various GMRES methods used in the cases, as well as other abbreviations used in Tables 2-5.

The "ode15s" case with full matrices was impractical to use for even moderate grid sizes, because the full matrix algorithms did not utilize the sparsity present in the jacobian. The use of a "static preconditioner" was not effective either; as the linear system changes over time, the optimal

Table 1: Abbreviations Key

Abbreviation	Full Name
GMRES-1	GMRES, No preconditioner
GMRES-2	GMRES, Updated preconditioner
GMRES-3	GMRES, Static preconditioner
GMRES-4	GMRES, Matrix-free
GMRES-5	GMRES, Matrix-free with preconditioner
NFE	Function Evaluations
NJE	Jacobian Evaluations
Jac. Fac.	Jacobian Factorizations
GMRES ITs.	GMRES Iterations
Lin. Solves	Linear System Solutions
NSS	Number of Successful Steps
NFA	Number of Failed Attempts

preconditioner to use also changes. With a static preconditioner, the user is hoping that the preconditioner calculated at the outset would demonstrate performance improvement for the method over the entire time interval, but this was not the case for moderate and large grid sizes.

The time required by each method was reported in each test case. However, on a time basis, comparing GMRES to MATLAB's default direct solver is not exactly a fair comparison. By default, within ode15s, MATLAB first calculates an L-U decomposition of the required matrix. Afterwards, it solves the system with back and forward substitution. A substantial reason for the time discrepancy is that while MATLAB's default solver is compiled, the GMRES solver used is interpreted. Compiled code will always provide a substantial time advantage. This is because compiled code is translated into machine code beforehand for fast and efficient usage, while interpreted code is translated into machine code line by line when the code is executed. However, other characteristics of the solution such as number of steps taken, number of linear solutions, and number of Jacobian evaluations can be compared fairly.

The two best methods were "ode15s with sparsity", and "GMRES with dynamic preconditioning". On a time basis, the default stiff solver had a substantial advantage. For the other performance measures, the numbers were comparable but "ode15s with sparsity" still had an advantage. This indicates that if a user were going to implement these methods within MATLAB, "ode15s with sparsity" would be the method of choice. However, if the user were going to compile his or her own code in another language (say C++), a viable alternative would be to implement "GMRES with dynamic preconditioning".

Complete results for all grid sizes can be found in Appendix II.

Table 2: $N = 32$

$N = 32$	NFE	NJE	Jac Fac.	GMRES ITs.	Lin. Solves	Time (secs.)	NSS	NFA
ode15s,Base	52799	51	106	x	515	76.21	193	80
ode15s,Sparsity	401	18	42	x	266	.75	134	24
GMRES-1	1198	77	0	5190	651	4.69	232	105
GMRES-2	1152	73	0	4532	633	7.18	230	102
GMRES-3	1216	80	0	5279	648	8.35	230	108
GMRES-4	4241	0	0	57292	442	40.08	214	55
GMRES-5	4319	0	0	58315	442	66.02	217	55

Table 3: $N = 64$

$N = 64$	NFE	NJE	Jac Fac.	GMRES ITs.	Lin. Solves	Time (secs.)	NSS	NFA
ode15s,Sparsity	1334	85	131	x	723	11.88	267	112
GMRES-1	1000	54	0	6489	607	66.46	245	86
GMRES-2	507	21	0	4186	345	26.86	174	28
GMRES-4	6498	0	0	57261	510	240.95	263	53
GMRES-5	6027	0	0	48577	462	362.83	248	45

Table 4: $N = 128$

$N = 128$	NFE	NJE	Jac Fac.	GMRES ITs.	Lin. Solves	Time (secs.)	NSS	NFA
ode15s,Sparsity	541	20	54	x	395	40.39	208	25
GMRES-1	980	47	0	17919	646	545.16	271	78
GMRES-2	601	23	0	6820	435	256.59	222	30
GMRES-4	7073	0	0	57970	527	1343.39	280	51

Table 5: $N = 256$

$N = 256$	NFE	NJE	Jac Fac.	GMRES ITs.	Lin. Solves	Time (secs.)	NSS	NFA
ode15s,Sparsity	568	15	53	x	462	313.21	254	18
GMRES-2	957	45	0	17238	642	3069.30	297	57

Table 6: $N = 512$

$N = 512$	NFE	NJE	Jac Fac.	GMRES ITs.	Lin. Solves	Time (secs.)	NSS	NFA
ode15s,Sparsity	774	23	69	x	612	3743.53	331	28
GMRES-2	1020	42	0	22247	726	15548.34	356	53

7 Conclusions and Future Work

7.1 Conclusions

An efficient solution of the pricing problem for an Asian put option was implemented on small, moderate, and large grid sizes using both MATLAB's stiff ordinary differential equations solver and several modified versions which incorporated various forms of GMRES.

Due to the unacceptable performance of "ode15s, base case" on even moderate grid sizes, it is clear that exploiting sparsity is necessary. The performances of the GMRES variations and "ode15s with sparsity" support this assertion.

In addition, the pricing problem showed strong sensitivity to both grid size and method selection. This is evident through the variations in performance numbers and time in all the test cases.

On most grid sizes, "ode15s with sparsity" had the best performance from both a time and efficiency perspective. However, as mentioned before, the time results are not a valid basis of comparison because the interpreted GMRES implementation is competing with the compiled internal solver. Aside from time, the performance numbers of "GMRES with dynamic preconditioning" were comparable on most grid sizes to those of "ode15s with sparsity", indicating that the GMRES method is viable although possibly inferior for these grid sizes.

7.2 Future Work

There are several improvements and new directions that were not implemented in the project. First, "ode15s" is an algorithm designed to solve non-linear differential equation systems. Thus, a Newton's Method apparatus is included in the code for new time steps. For a linear differential equation system, it is not necessary to resort to Newton's Method. As the Asian put pricing problem is linear, one modification could be to modify the "ode15s" code for all seven methods to solve for steps directly instead of using Newton's Method.

Another improvement would be to modify the GMRES code to accept right preconditioning. The MATLAB GMRES implementation only accepts left preconditioning. However, right preconditioning would be more compatible with the method by which "ode15s" evaluates tolerances and step

criteria.

Finally, throughout the methods, a finite-difference approximation of the Jacobian matrix was used to obtain both Jacobians and tri-diagonal preconditioners. If an analytic Jacobian were supplied, the methods may become more accurate and efficient.

8 References

References

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