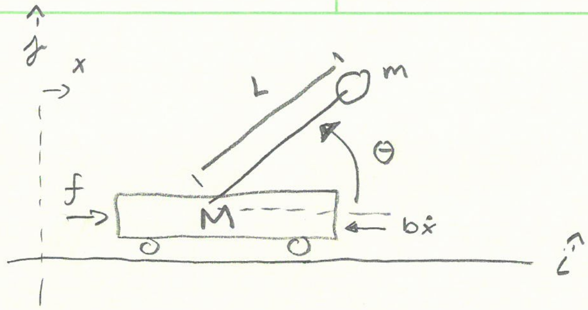
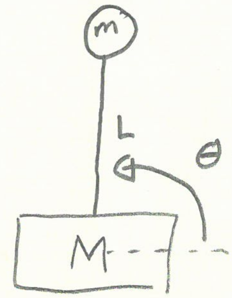
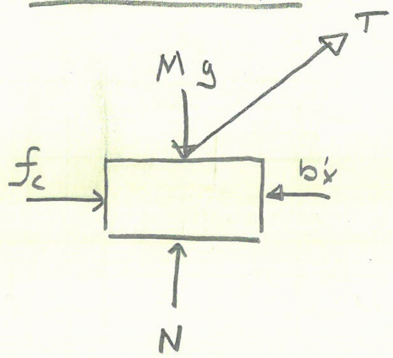




FBD OF CART



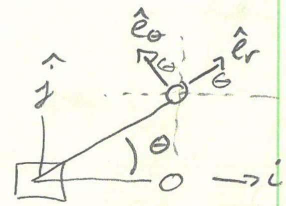
WEIGHT OF CART $= -Mg\,\hat{j}$

FORCE OF ROAD $= N\hat{j}$

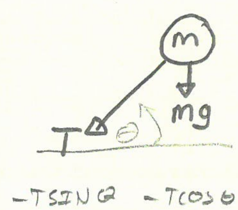
TENSION IN ROD $= T\cos\theta\,\hat{i} + T\sin\theta\,\hat{j}$

$= T(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$

CONTROL FORCE $= f_c(t)\,\hat{i}$



FBD OF PENDULUM MASS



WEIGHT OF PENDULUM $= -mg\,\hat{j}$

TENSION IN ROD $= -T(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$

↳ EQUAL AND OPPOSITE FROM TENSION IN CART

CART DYNAMICS

$$\sum F_{\hat{i}} = f_c(t) + T\cos\theta = M\ddot{x} \quad (1)$$

$$\sum F_{\hat{j}} = N - Mg - T\sin\theta = M\ddot{y} = \cancel{0}$$

$$\underline{X} = \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix}$$

PENDULUM DYNAMICS

$$\sum F_{\hat{i}} = -T\cos\theta = m a_{p_x}$$

$$\sum F_{\hat{j}} = -T\sin\theta - mg = m a_{p_y}$$

$$a_p = a_c + a_{p/c}$$

$$a_c = \ddot{x}$$

$$a_{p/c} = L\ddot{\theta}\,\hat{e}_\theta - L\dot{\theta}^2\,\hat{e}_r$$

$$\hat{e}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

$$\hat{e}_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

$$a_p = \ddot{x} + L\ddot{\theta}\left[-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}\right] - L\dot{\theta}^2\left[\cos\theta\,\hat{i} + \sin\theta\,\hat{j}\right]$$

CART DYNAMICS

$$\sum F_{\hat{i}} = f_c(t) = M_c \ddot{x} + b\dot{x} + T\cos\theta \qquad (1)$$

$$\sum F_{\hat{j}} = N - M_c g - T\sin\theta = M_c \ddot{y} = \emptyset$$

PENDULUM DYNAMICS

$$\sum F_{\hat{i}} = -T\cos\theta = m_p a_{p_x}$$

$$\sum F_{\hat{j}} = -T\sin\theta - mg = m_p a_{p_y}$$

$$a_p = a_c + a_{p/c}$$

$$a_c = \ddot{x}\,\hat{i}$$

$$a_{p/c} = L\ddot{\theta}\,\hat{e}_\theta - L\dot{\theta}^2\,\hat{e}_r$$

$$a_{p/c} = L\ddot{\theta}\left[-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}\right] - L\dot{\theta}^2\left[\cos\theta\,\hat{i} + \sin\theta\,\hat{j}\right]$$

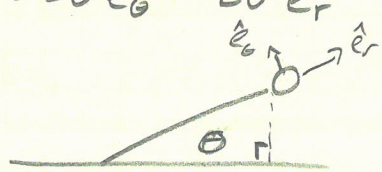
$$r = L\hat{e}_r$$

$$\dot{r} = \dot{\theta}\,\hat{e}_3 \times L e_r$$

$$\dot{r} = L\dot{\theta}\,\hat{e}_\theta$$

$$\ddot{r} = L\ddot{\theta}\,\hat{e}_\theta + \dot{\theta}\,\hat{e}_3 \times L\dot{\theta}\,\hat{e}_\theta$$

$$\ddot{r} = L\ddot{\theta}\,\hat{e}_\theta - L\dot{\theta}^2\,\hat{e}_r$$



ROTATING TO INERTIAL

$$\hat{e}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

$$\hat{e}_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

$$a_p = \left[\ddot{x} - L\ddot{\theta}\sin\theta - L\dot{\theta}^2\cos\theta\right]\hat{i} + \left[L\ddot{\theta}\cos\theta - L\dot{\theta}^2\sin\theta\right]\hat{j}$$

$$\sum F_{\hat{i}} = -T\cos\theta = m_p\left[\ddot{x} - L\ddot{\theta}\sin\theta - L\dot{\theta}^2\cos\theta\right]$$

$$-T\cos\theta = m_p\ddot{x} - m_p L\ddot{\theta}\sin\theta - m_p L\dot{\theta}^2\cos\theta \qquad (2)$$

$$\sum F_{\hat{j}} = -T\sin\theta - mg = m_p\left[L\ddot{\theta}\cos\theta - L\dot{\theta}^2\sin\theta\right]$$

$$-T\sin\theta - m_p g = m_p L\ddot{\theta}\cos\theta - m_p L\dot{\theta}^2\sin\theta \qquad (3)$$

REWRITING EQN (2) + (3)

$$-T\cos\theta = m_p\ddot{x} - m_pL\ddot{\theta}\sin\theta - m_pL\dot{\theta}^2\cos\theta \qquad (2)$$

$$-T\sin\theta - m_pg = m_pL\ddot{\theta}\cos\theta - m_pL\dot{\theta}^2\sin\theta \qquad (3)$$

TO SOLVE FOR T (2) SIN θ - (3) COS θ

$$-T\cos\theta\sin\theta = m_p\ddot{x}\sin\theta - m_pL\ddot{\theta}\sin^2\theta - m_pL\dot{\theta}^2\cos\theta\sin\theta$$

$$+T\cos\theta\sin\theta + m_pg\cos\theta = -m_pL\ddot{\theta}\cos^2\theta + m_pL\dot{\theta}^2\cos\theta\sin\theta$$

YIELDS:

$$\boxed{m_pg\cos\theta = m_p\ddot{x}\sin\theta - m_pL\ddot{\theta} \qquad (4)}$$

USE EQ (2) FOR A SUB OF $T\cos\theta$

$$T\cos\theta = m_pL\ddot{\theta}\sin\theta + m_pL\dot{\theta}^2\cos\theta - m_p\ddot{x} \qquad (5)$$

SUB (5) INTO (1)

$$(m_p + M_c)\ddot{x} = f_c(t) + mL\ddot{\theta}\sin\theta + mL\dot{\theta}^2\cos\theta - b\dot{x}$$

$$\boxed{f_c(t) = [m_p + M_c]\ddot{x} - mL\ddot{\theta}\sin\theta - mL\dot{\theta}^2\cos\theta + b\dot{x} \qquad (6)}$$

GOVERNING DYNAMICS EQUATIONS ARE EQ (4) + EQ (6)

EQUATIONS ARE COUPLED NON-LINEAR DIFFERENTIAL EOM.

LINEARIZE ABOUT $\theta = \pi/2$

$$\theta = \pi/2 + \Delta\theta$$

$$\cos\theta = \cos(\pi/2 + \Delta\theta) \approx \Delta\theta$$

$$\sin\theta = \sin(\pi/2 + \Delta\theta) \simeq 1$$

$$\dot{\theta}^2 \simeq \emptyset$$

- SUB LINEAR APPROXIMATIONS INTO NON-LIN DYNAMICS

$$m_p g \cancel{\cos\theta} \xrightarrow{} \theta = m_p \ddot{x} \cancel{\sin\theta} \xrightarrow{} 1 - m_p L \ddot{\theta}$$

$$m_p g \theta = m_p \ddot{x} - m_p L \ddot{\theta} \qquad (7)$$

$$f_c(t) = [m_p + M_c]\ddot{x} - m_p L \ddot{\theta} \cancel{\sin\theta} \xrightarrow{} 1 - mL \cancel{\dot{\theta}^2} \xrightarrow{} 0 \; \cancel{\cos\theta} \xrightarrow{} \theta + b\dot{x}$$

$$f_c(t) = [m_p + M_c]\ddot{x} - m_p L \ddot{\theta} + b\dot{x} \qquad (8)$$

EQN (7) + (8) ARE LINEAR EOM

$$m_p L \ddot{\theta} = m_p g \theta - m_p \ddot{x}$$

$$\ddot{\theta} = \frac{g\theta}{L} - \frac{\ddot{x}}{L}$$

$$[m_p + M_c]\ddot{x} = f_c(t) + m_p L \ddot{\theta} + b\dot{x}$$

$$[m_p + M_c]\ddot{x} - m_p L\left[\frac{g\theta}{L} - \frac{\ddot{x}}{L}\right] = f_c(t) + b\dot{x}$$

$$[m_p + M_c]\ddot{x} + m_p \ddot{x} = f_c(t) + b\dot{x} + m_p g \theta$$

$$[2m_p + M_c]\ddot{x} = f_c(t) + b\dot{x} + m_p g\theta$$

$$\boxed{\ddot{x} = \frac{b}{2m_p + M_c}\dot{x} + \frac{m_p g}{2m_p + M_c}\theta + \frac{1}{2m_p + M_c} f_c(t)}$$

$$\ddot{\theta} = \frac{g\theta}{L} - \frac{1}{L}\left[\frac{b}{2m_p + M_c}\dot{x} + \frac{m_p g}{2m_p + M_c}\theta + \frac{1}{2m_p + M_c} f_c(t)\right]$$

$$\boxed{\ddot{\theta} = -\frac{b}{L(2m_p + M_c)}\dot{x} + \left[\frac{g}{L} - \frac{m_p g}{2m_p + M_c}\right]\theta - \frac{1}{L(2m_p + M_c)} f_c(t)}$$

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \emptyset & 1 & \emptyset & \emptyset \\ \emptyset & \frac{b}{2m_p + M_c} & \frac{m_p g}{2m_p + M_c} & \emptyset \\ \emptyset & \emptyset & \emptyset & 1 \\ \emptyset & \frac{-b}{L(2m_p + M_c)} & \left(\frac{g}{L} - \frac{m_p g}{2m_p + M_c}\right) & \emptyset \end{bmatrix} + \begin{bmatrix} \emptyset \\ \frac{1}{2m_p + M_c} \\ \emptyset \\ \frac{-1}{L(2m_p + M_c)} \end{bmatrix} f_c(t)$$

$$y(t) = \begin{bmatrix} 1 & \emptyset & \emptyset & \emptyset \\ \emptyset & 1 & \emptyset & \emptyset \\ \emptyset & \emptyset & 1 & \emptyset \\ \emptyset & \emptyset & \emptyset & 1 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix}$$