

# 2021 October Problems

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1. (5 points) Let  $O$  be the circumcircle of acute  $\triangle ABC$ . Line  $\ell$  passes through  $A$ , is perpendicular to  $BC$ , and intersects  $O$  at another point than  $A$  which is  $D$ .  $D$  is reflected over  $BC$ , and its image is  $D'$ . The line  $BD'$  intersects  $AC$  at  $E$ . Find  $\angle BEA$ .

2. (3 points) Find all functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$xf(y + f(x)) = yf(x)^2$$

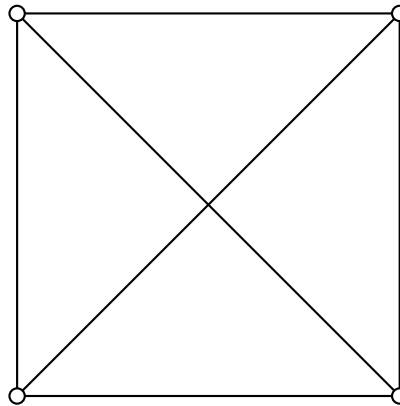
for all real  $x$  and  $y$ .

3. (2 points) Let  $P(x) = x^4 + ax^3 + bx^2 + cx + d$ . Given that  $P(2) = P(3) = P(5) = P(7) = 0$ , find  $a + b + c + d$ .

4. (3 points) Consider the sphere with radius 1 centered at  $(0, 1, 0)$  and the cone with base centered at  $(0, 0, 0)$  parallel to the  $xz$ -plane, and apex at  $(0, 1, 0)$  with a radius of 1. Find the volume of the union of the two solids.

5. (3 points) Let the subscript  $b$  denote that a number is represented in base  $b$ . For example,  $185 = 1310$ . Prove there are no integers  $a, b \geq 16$  such that  $14AC9_a = 6392_b$ , where  $A$  and  $C$  are digits with value 10 and 12 respectively.

6. (4 points) lethan3 wants to make a new language out of special characters. All of them will be made by tracing along the lines of the diagram below:



When writing a character, all segments must be made from one white point to another white point (no stopping in the center intersection and turning). Also, since the language must be efficient, the entire character must be able to be written using one stroke of the pen, and no line may be traced over more than once (points can be visited twice or more). Characters cannot be empty; if one character is a translation of another treat them as distinct characters. How many characters are possible in this language?

Examples of valid characters:

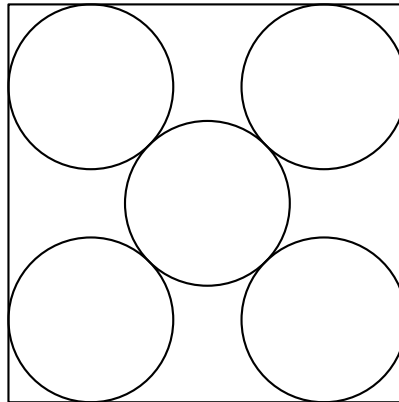




Examples of invalid characters:



7. (1 point) Given that the radius of each circle is 1, find the side length of the square.



8. (4 points) Find the side length of the largest equilateral triangle able to fit in a regular dodecagon of side length 1. Express your answer in exact, fully simplified, form.

9. (4 points) Pikachu dreams of being in a 6 by 6 grid of ketchup-filled squares! Each square has some number of ketchup bottles, shown in the figure below. He is currently in the top-left square, and needs to exit through the bottom right. However he is in a rush and wants to get out of the grid in 10 steps while collecting all the bottles on the way. Find the optimal path which yields the maximum number of ketchup bottles (1 point) and prove it is optimal or else Pikachu won't believe you. (3 points)

0	5	8	10	2	1
10	5	7	5	9	8
7	8	3	1	2	7
8	1	5	4	6	1
6	6	8	7	9	6
10	7	5	4	9	6

10. (7 points) Pikachu has made up a crazy new dance! (No, it's literally crazy.) He starts at  $(1, 1)$  in the coordinate plane, and has some mental variables  $a$  and  $b$  which are both set to 1 at the beginning of his dance.

He defines a *rep* as follows: 1. Move  $a + b$  steps (units) to the right. 2. Move  $2a + b$  steps (units) upwards. 3. Set  $a$  to his current  $x$  position and  $b$  to his current  $y$  position.

Pikachu wants to know where he will end up after 2021 reps. What are the exact coordinates of his final location? Express your answer in exact numerical form. (Your answer does not have to be fully simplified, but the answer should be made up of solely numerical calculations such as exponents, radicals, basic operations, constants etc.)