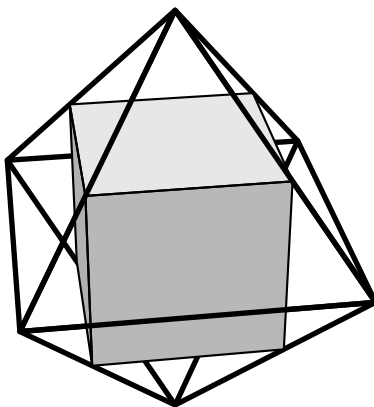


2021 December Problems

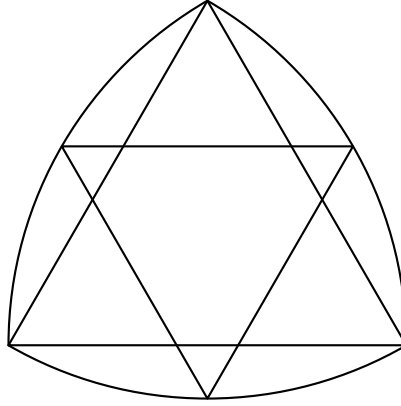
Ethan Lee

1. (4 points) A cube is placed in an octahedron of side length 1 as shown:



Find the volume of the cube.

2. (4 points) The graphs of $y = x^2$, $x + y = 42$, $x - 9y = -78$ form two closed regions, one which contains the intersection of the two lines. How many lattice points are inside this region, including those on the boundaries?
3. (3 points) Pichu and Pikachu are playing a card game. The deck of cards consists of two green cards and two red cards which are indistinguishable face down. All cards are face down at the start. They take turns with Pichu going first:
- The player picks two cards.
 - If the player picks two matching cards they win.
 - Otherwise, they shuffle only those two cards so that neither player knows which is which, and return them to the table. (Both players can tell which cards have just been returned to the table.)
- If both players are perfect logicians, play optimally, and wish for the game to end as soon as possible, what is the probability Pichu wins?
4. (5 points) Pichu is playing Snakes and Ladders, but has added the devastating rule that if you roll past the 100 square, you must restart from the beginning. Pichu will take all costs to avoid this! To make the game a bit easier, he has the choice of two dice, one with sides 1-2-3-4-5-6 and one with sides 1-2-2-3-3-10. He has an infinite number of turns, and is currently at square 97. He gets to choose which dice he uses at each turn. What is the probability of the dice carrying him to the end without overflowing if he plays perfectly optimally?
5. (2 points) Factor 7,999,992. Obviously, do not use a calculator!
6. (3 points) An equilateral triangle with side length 1 is drawn, and three circular arcs centered at each vertex that pass through the other vertices are drawn. There is another equilateral triangle that has its sides parallel to the first whose vertices all lie on the arcs. Find the area of this triangle.



7. (4 points) A square grid of dimensions 99 by 100 is drawn. There is one edge that contains the center of the entire grid; this edge is removed. Find the total number of squares such that all parts of their edges coincide with the edges of the grid.

8. (4 points) Find the smallest positive integer n such that

$$\frac{\sum_{i=0}^n 315i}{10!}$$

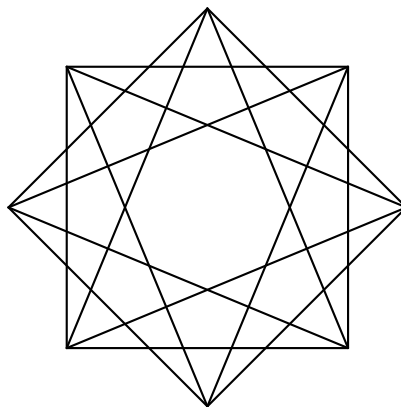
is an integer.

9. (6 points) Let $\omega = e^{\frac{\pi i}{6}}$. Find

$$\prod_{i=0}^{12} (\omega^i + \omega^{i+1}).$$

Your answer should be exact and expressed using radicals, exponents, and the 4 basic operations, but does not have to be fully simplified.

10. (4 points) Pichu makes the following star ornament for his Christmas tree:



However, there are even more stars, specifically pentagrams, hidden in this large star! How many pentagrams are there?