2021 November Problems

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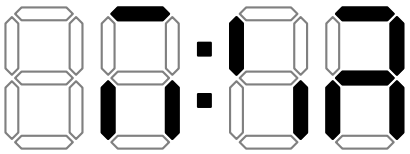
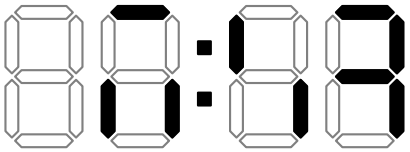
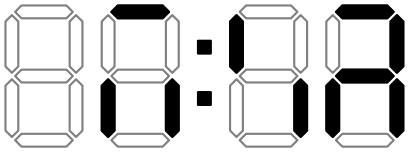
1. (3 points) After a failed escape attempt, your (12 hour) alarm clock has fallen off the table, and as a result some parts of the display are not working. You wake up startled to see your alarm clock looking like this:



Luckily for you, you have your trusty timer by your side. You set the timer for 115 minutes and doze off again. When the timer beeps, your alarm clock now looks like:



Interestingly, the clock has barely changed apperance except for one segment. Realizing that you still can't figure out what time it is, you set the timer again for 115 minutes and wake up to see the clock returned to its original position!



After some thinking, you figure out the time. What time is it now?

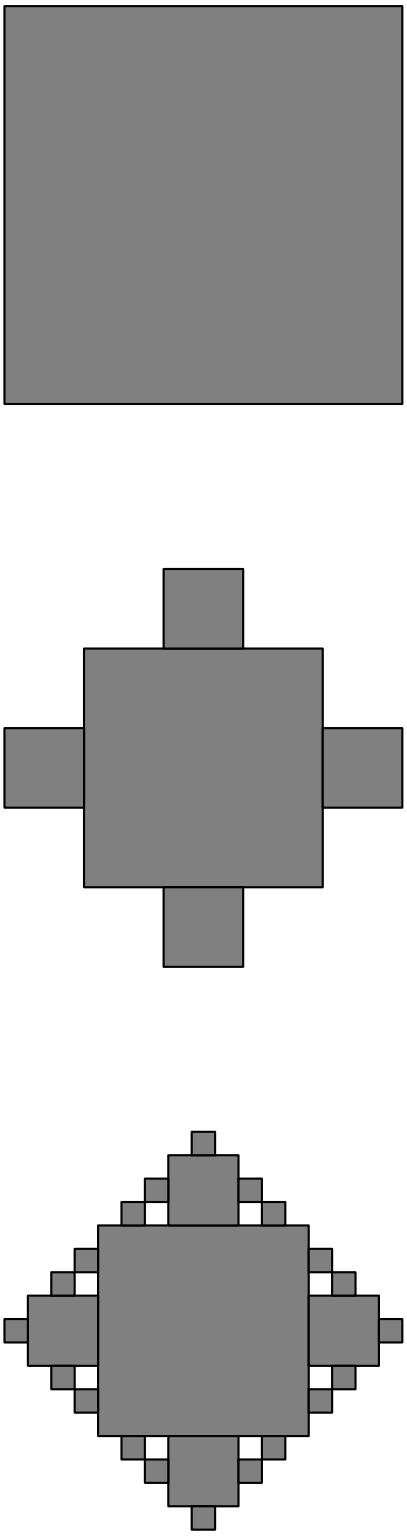
2. (3 points) On unit square $ABCD$, points M and N lie on AB and AD respectively such that $\frac{AM}{MB} = 2$ and $\frac{AN}{ND} = 2$. If I is the intersection of DM and BN, what is the area of the convex quadrilateral $DIBC$?

3. (2 points) Find all real solutions to $a^7 + 4a^4 + 4a = 0$. (No calculator)

4. (2 points) What two digits does 26^{47} end in? (No calculator)

5. (5 points) An ant walks on a cone with base radius 4 and height $2\sqrt{5}$. The cone is flat on the table, so the ant cannot walk on the interior of the base. It starts at some point on the perimeter of the base, walks along the cone to the point that is diameterically opposite its starting point on the base, then to the top of the cone. What is the smallest possible distance the ant could have walked?

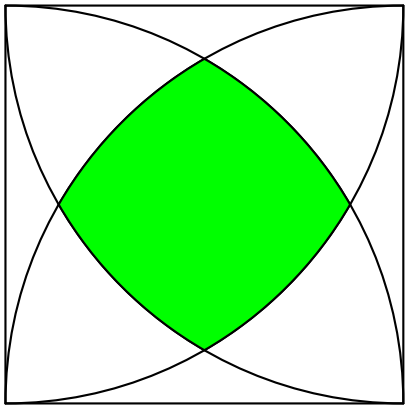
6. (4 points) A fractal is constructed using the following sequence, starting with a unit square and adding more squares on the edges:



After an infinite number of iterations, what is the total area of the fractal?

7. (2 points) There are 20 pokemon in a circle, each of which is a Squirtle, Pichu, or Charmander chosen uniformly at random. Call a pokemon "lonely" if neither of the pokemon next to it are of the same species. What is the expected number of lonely pokemon?

8. (5 points) The following arcs are placed inside a unit square, centered at each of its vertices. What is the green region's area?



9. (2 points) Suppose that x is a number such that $x^2 + x + 1 = 0$. Find

$$\sum_{i=0}^{99} x^i.$$

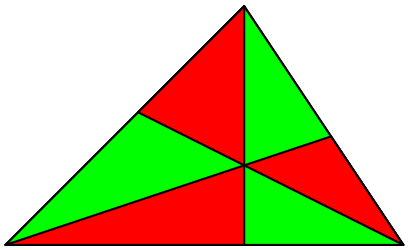
10. (10 points) Pichu and Pikachu are splitting a triangular shaped pizza in an interesting manner:

- a point P is chosen inside the triangle

- then, lines AP, BP, and CP are extended to hit BC, AC, and AB at D, E, and F respectively.

- The pizza is now split into 6 slices, and Pichu and Pikachu pick their slices one at a time counterclockwise.

In the example below, all red slices go to Pichu and all green slices go to Pikachu:



For what choices of point P will Pichu and Pikachu get an equal share of pizza, regardless of the shape of the pizza? Prove that all of your selection of points always works, and they are the only points that work.