Interpretation of Well-Block Pressures in Numerical Reservoir Simulation With Nonsquare Grid Blocks and Anisotropic Permeability

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Abstract

Previous work on the interpretation of well-block pressure (WBP) for a single isolated well is extended to the case of nonsquare grid blocks ($\Delta x \neq \Delta y$). Numerical solutions for the single-phase five-spot problem, involving various grid sizes, show that the effective well-block radius (where the actual flowing pressure equals the numerically calculated WBP) is given by

$$r_o = 0.14\ (\Delta x^2 + \Delta y^2)^{1/2}.$$

This relationship is verified by a mathematical derivation for a single well in an infinite grid. The exact value of the constant is shown to be $e^{-\gamma}/4$, where γ is Euler's constant.

Finally, the analysis is extended to include anisotropic permeability, and an expression for the effective well-block radius in terms of Δx, Δy, k_x, and k_y is derived.

Introduction

In the modeling of a reservoir by numerical methods, it is necessary to use grid blocks whose horizontal dimensions are much larger than the diameter of a well. As a result, the pressure calculated for a block containing a well, p_o, is greatly different from the flowing bottomhole pressure (BHP) of the well, p_{wf}. In a previous paper,[1] the equivalent radius of a well block, r_o, was defined as that radius at which the steady-state flowing pressure for the actual well is equal to the numerically calculated pressure for the well block. This definition for r_o gives

$$p_{wf} - p_o = \frac{q\mu}{2\pi kh} \ln \frac{r_w}{r_o}. \quad \text{(1)}$$

For a *square* grid ($\Delta x = \Delta y$), careful numerical experiments on a five-spot pattern[1] showed that the ratio of r_o to Δx ranges from 0.1936 (for a 3×3 grid) to a limit

$$\lim_{N\to\infty} \frac{r_o}{\Delta x} = 0.1982. \quad \text{(2)}$$

It was also shown that the pressures in the blocks adjacent to a well block approximately satisfy the steady-state radial flow equation

$$p = p_o + \frac{q\mu}{2\pi kh} \ln \frac{r}{r_o}. \quad \text{(3)}$$

By *assuming* that Eq. 3 is satisfied exactly, one can derive the relation

$$\frac{r_o}{\Delta x} = \exp(-\pi/2) = 0.208. \quad \text{(4)}$$

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Thus, for a square grid, we have the rule of thumb

$$r_o = 0.2\ \Delta x. \quad \text{(5)}$$

In this paper, we investigate the effects of a nonsquare grid ($\Delta x \neq \Delta y$), as well as anisotropic permeability ($k_x \neq k_y$), on the equivalent well-block radius.

Nonsquare Grid

In the literature[2-4] several equations are given for relating WBP to the BHP of the well. These may all be put in the form

$$p_{wf} - p_o = \frac{q\mu}{2\pi kh}\left[\ln\frac{r_w}{A(\Delta x \Delta y)^{1/2}} + B\right]. \quad \text{(6)}$$

This equation has been derived from the assumption that the WBP is an areal average pressure in a circle whose area equals that of the well block.[5] Comparison of Eq. 6 with Eq. 1 leads to the conclusion that r_o should satisfy the relation

$$r_o = C(\Delta x \Delta y)^{1/2}. \quad \text{(7)}$$

Since Eq. 7 should reduce to Eq. 5 for a square grid, we can write it as

$$\frac{r_o}{\Delta x} = 0.2\alpha^{1/2}, \quad \text{(8)}$$

where α is the aspect ratio, defined by

$$\alpha = \frac{\Delta y}{\Delta x}. \quad \text{(9)}$$

While the assumption that the WBP equals an areal average pressure has been shown to be false,[1] Eq. 7 has continued to be used, in the absence of evidence to the contrary.

An Analytic Derivation

Another approach to determine the effect of the aspect ratio, $\Delta y/\Delta x$, on r_o is to make the same assumption that led to Eq. 4. This assumption is that the pressures calculated for the blocks adjacent to the well block satisfy Eq. 3[1,6], so that (see Fig. 1)

$$p_1 - p_0 = p_3 - p_0 = \frac{q\mu}{2\pi kh}\ln(\Delta x/r_o), \quad \text{(10)}$$

and

$$p_2 - p_0 = p_4 - p_0 = \frac{q\mu}{2\pi kh}\ln(\Delta y/r_o). \quad \text{(11)}$$

The difference equation for the steady-state pressure distribution, written for Block 0, is

$$\frac{kh\Delta y}{\mu\Delta x}(p_3 - 2p_0 + p_1) + \frac{kh\Delta x}{\mu\Delta y}(p_2 - 2p_0 + p_4) = q. \quad \text{(12)}$$

Combining Eqs. 10 through 12 yields

$$\frac{\Delta y}{\Delta x}\ln\frac{\Delta x}{r_o} + \frac{\Delta x}{\Delta y}\ln\frac{\Delta y}{r_o} = \pi, \quad \text{(13)}$$

or

$$\frac{r_o}{\Delta x} = \exp\frac{\ln\alpha - \pi\alpha}{1+\alpha^2}. \quad \text{(14)}$$

Whether Eq. 8, Eq. 14, or some other equation correctly expresses the effect of the aspect ratio can be determined only by numerical experiments.

Numerical Calculation of Equivalent Radius for Various Aspect Ratios

Calculations very similar to those previously performed on a square grid[1] were carried out. The repeated five-spot pattern was solved for various aspect ratios, using different grid refinements. Fig. 2 shows a typical grid, for the case $\alpha=2$, $M=10$, and $N=5$. Details of the numerical calculation are given in Appendix A.

To calculate the equivalent well-block radius, we use Muskat's equation for the pressure drop between injection and producing wells in a repeated five-spot pattern,[7]

$$\Delta p = \frac{q\mu}{\pi kh}\left[\ln(d/r_w) - B\right], \quad \text{(15)}$$

where

$$B = 0.61738575. \quad \text{(16)}$$

Note that Muskat reported that $B=0.6190$. His derivation of Eq. 15 includes an infinite series, which he ignored, considering it negligible. If he had included only the first term, he would have obtained $B=0.617315$. Four or more terms give the value shown in Eq. 16, correct to eight digits. Use of the more accurate value of B in this study accounts for the slight difference (in the fourth significant digit) in the values of r_o calculated for a square grid compared with those reported previously.[1]

If we take Δp to be the difference in pressure between the injection and production well blocks, then r_w of Eq. 15 should be replaced by r_o. Further, we have

$$d = \sqrt{2}M\Delta x.$$

Then Eq. 15 may be rewritten as

$$\frac{\pi kh}{q\mu}\left(p_{M,N} - p_{o,o}\right) = \ln\left(\sqrt{2}M\ \Delta x/r_o\right) - B$$

or

$$\frac{r_o}{\Delta x} = \sqrt{2}M\exp\left[-B - \frac{\pi kh}{q\mu}\left(p_{M,N} - p_{o,o}\right)\right]. \quad \text{(17)}$$

TABLE 1—NUMERICAL CALCULATION OF PRESSURE DROP FOR REPEATED FIVE-SPOT PATTERN AND OF THE WELL-BLOCK EQUIVALENT RADIUS

α	M	N	$(kh/q\mu)(p_{M,N}-p_{o,o})$	$r_o/\Delta x$	$r_o/\Delta y$
1	2	2	0.666667	0.187860	0.187860
1	4	4	0.873950	0.195908	0.195908
1	8	8	1.091433	0.197858	0.197858
1	16	16	1.311287	0.198344	0.198344
1	32	32	1.531727	0.198466	0.198466
2	4	2	0.734921	0.303207	0.151604
2	8	4	0.947192	0.311280	0.155640
2	16	8	1.165846	0.313224	0.156612
2	32	16	1.385992	0.313706	0.156853
2	64	32	1.606506	0.313826	0.156913
4	8	2	0.759306	0.561693	0.140423
4	16	4	0.972676	0.574661	0.143665
4	32	8	1.191615	0.577732	0.144433
4	64	16	1.411834	0.578489	0.144622
4	128	32	1.632366	0.578676	0.144669
8	16	2	0.766128	1.099566	0.137446
8	32	4	0.979763	1.124016	0.140502
8	64	8	1.198772	1.129773	0.141222
8	128	16	1.419009	1.131189	0.141399
8	256	32	1.639545	1.131542	0.141443
16	32	2	0.767886	2.187021	0.136689
16	64	4	0.981586	2.235192	0.139700
16	128	8	1.200613	2.246517	0.140407
16	256	16	1.420855	2.249301	0.140581
16	512	32	1.641392	2.249994	0.140625
32	64	2	0.768328	4.367962	0.136499
32	128	4	0.982046	4.463939	0.139498
32	256	8	1.201077	4.486497	0.140203
32	512	16	1.421319	4.492043	0.140376
32	1,024	32	1.641857	4.493416	0.140419
64	128	2	0.768439	8.732872	0.136451
64	256	4	0.982161	8.924652	0.139448
64	512	8	1.201193	8.969717	0.140152
64	1,024	16	1.421435	8.980804	0.140325
128	256	2	0.768467	17.464233	0.136439
128	512	4	0.982189	17.847687	0.139435
128	1,024	8	1.201221	17.937820	0.140139
128	2,048	16	1.421465	17.959930	0.140312
256	512	2	0.768474	34.927704	0.136436
256	1,024	4	0.982197	35.694595	0.139432
256	2,048	8	1.201229	35.874802	0.140136

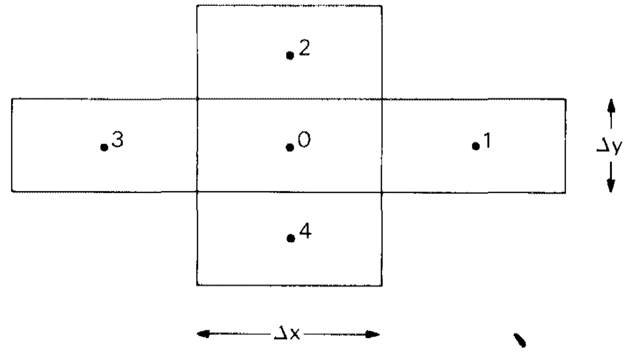

Fig. 1—Block 0, containing a well, and its four neighboring blocks.

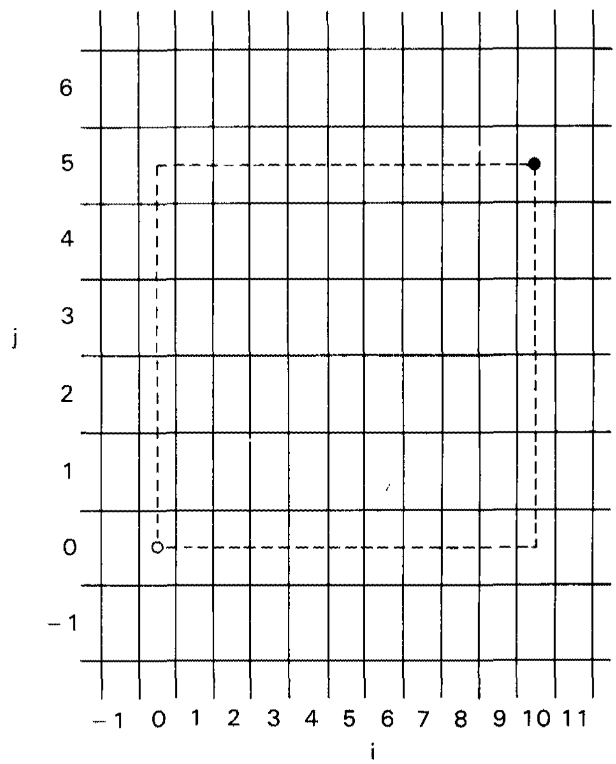

Fig. 2—Computing grid for repeated five-spot; $M=10$, $N=5$.

Similarly,

$$\frac{r_o}{\Delta y}=\sqrt{2}N\exp\left[-B-\frac{\pi kh}{q\mu}\left(p_{M,N}-p_{o,o}\right)\right]. \quad \ldots (18)$$

Calculations for grids with aspect ratios ranging from 1 to 256 are shown in Table 1. For each aspect ratio, the smallest grid was $N=2$, $M=2\alpha$; grids were doubled successively in each dimension until the storage required for solution exceeded the available memory. Col. 4 of this table lists the dimensionless pressure drop, $(kh/q\mu)\cdot(p_{M,N}-p_{o,o})$, obtained in the numerical calculation for each grid; Cols. 5 and 6 list the values of $r_o/\Delta x$ and $r_o/\Delta y$, respectively, calculated by using Eqs. 17 and 18.

For each aspect ratio, examination of $r_o/\Delta x$ and $r_o/\Delta y$ shows that they appear to be converging to a limit with an order of $1/M^2$. From this assumption, one can extrapolate the last two entries (for each aspect ratio) to infinite M and N by the equations

$$\left(r_o/\Delta x\right)_\infty \approx \left[4(r_o/\Delta x)_{2M}-(r_o/\Delta x)_M\right]/3$$

and

$$\left(r_o/\Delta y\right)_\infty \approx \left[4(r_o/\Delta y)_{2M}-(r_o/\Delta y)_M\right]/3.$$

These extrapolated values are listed in Cols. 2 and 3 of Table 2 for each aspect ratio.

For $\alpha=1$ (i.e., for a square grid), we have

$$r_o=0.198506\ \Delta x, \quad \ldots (19)$$

which is essentially the same as Eq. 2—which was ob-

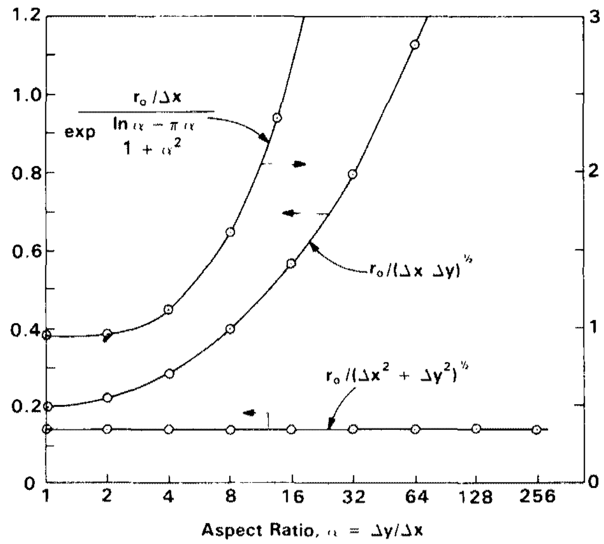

Fig. 3—Effect of aspect ratio on equivalent well-block radius.

TABLE 2—EXTRAPOLATED VALUES OF $r_o/\Delta x$ AND $r_o/\Delta y$, AND CERTAIN DERIVED QUANTITIES AS A FUNCTION OF ASPECT RATIO

α	$r_o/\Delta x$	$r_o/\Delta y$	$r_o/(\Delta x\Delta y)^{1/2}$	$r_o/(\Delta x^2+\Delta y^2)^{1/2}$
1	0.198506	0.198506	0.198506	0.140365
2	0.313866	0.156933	0.221936	0.140365
4	0.578739	0.144685	0.289369	0.140365
8	1.131660	0.141457	0.400102	0.140365
16	2.250225	0.140639	0.562556	0.140365
32	4.493874	0.140434	0.794412	0.140365
64	8.984497	0.140383	1.123062	0.140366
128	17.967300	0.140369	1.588099	0.140365
256	35.934860	0.140371	2.245929	0.140369

tained previously[1]—but with a more accurate value for the constant because of the use of the correct value for B in Eq. 15.

Effect of the Aspect Ratio

We first attempt to interpret these numerical results for $r_o/\Delta x$ and $r_o/\Delta y$ in terms of prior assumptions about WBP's. As discussed previously, the assumption that the WBP is an areal average pressure over a circle with the same area as that of the well block leads to the conclusion that the equivalent radius is proportional to the geometric mean of Δx and Δy (Eq. 7). To test this conclusion, the ratio $r_o/(\Delta x\Delta y)^{1/2}$ (which is equal to the geometric mean of $r_o/\Delta x$ and $r_o/\Delta y$) is listed in Col. 4 of Table 2 and is plotted in Fig. 3. It can be seen that Eq. 7 is far from valid.

Next, we test the assumption that the pressures calculated for the blocks adjacent to the well block satisfy the radial flow equation. This assumption is tested by dividing the value for $r_o/\Delta x$ listed in Table 2 by the value of $r_o/\Delta x$ calculated from Eq. 14, then plotting the quotient as a function of α, again in Fig. 3. The failure of this ratio to be constant for α outside the range 0.5 to 2 shows that Eq. 14 is not valid either.

The clue to finding the effect of α on r_o lies in the fact that $r_o/\Delta y$ approaches a constant for large α. It can easily be seen that $r_o/\Delta x$ and $r_o/\Delta y$ are interchanged in value when α is replaced by $1/\alpha$. It follows, then, that $r_o/\Delta x$ approaches the same constant as $\alpha \rightarrow 0$. Thus, we seek a length quantity that approaches Δx for small $\Delta y/\Delta x$ and that approaches Δy for small $\Delta x/\Delta y$. Such a quantity is the diagonal of the grid blocks $(\Delta x^2+\Delta y^2)^{1/2}$

The last column of Table 2 shows the ratio of r_o to this diagonal for each value of α. The constancy of this ratio for such a wide range of aspect ratios is truly startling. This ratio also is plotted in Fig. 3, where it appears, of course, as a horizontal line.

We conclude, therefore, that the well-block radius satisfies the equation

$$r_o = 0.140365(\Delta x^2+\Delta y^2)^{1/2}. \qquad (20)$$

The constant in Eq. 20 differs from that in Eq. 19 by the factor $1/\sqrt{2}$. The rule of thumb, Eq. 5, should now be replaced by the more general rule of thumb

$$r_o = 0.14(\Delta x^2+\Delta y^2)^{1/2}. \qquad (21)$$

Mathematical Derivation of Eq. 20

It cannot be fortuitous that Eq. 20 is valid over such a large range of the aspect ratio $\Delta y/\Delta x$. Although this equation was obtained solely from the numerical calculations, it suggests the possibility of deriving it mathematically. This has been done by deriving the pressure distribution for an infinite grid, as described in Appendix B. The constant of Eq. 20 is shown to be equal to $e^{-\gamma}/4$, where $\gamma = 0.5772157$ is Euler's constant.

Anisotropic Permeability

In discussing flow through an anisotropic medium, we assume that the principal axes of the permeability tensor are parallel to the x and y axes. Then the differential equation for steady-state pressure is

$$k_x \frac{\partial^2 p}{\partial x^2} + k_y \frac{\partial^2 p}{\partial x^2} = 0, \qquad (22)$$

with the boundary condition

$$p = p_{wf} \text{ at } r = (x^2+y^2)^{1/2} = r_w. \qquad (23)$$

By making the change of variables

$$u = (k_y/k_x)^{1/4} x \qquad (24a)$$

and

$$v = (k_x/k_y)^{1/4} y, \qquad (24b)$$

we can transform Eq. 22 into Laplace's equation

$$\frac{\partial^2 p}{\partial u^2} + \frac{\partial^2 p}{\partial v^2} = 0, \qquad (25)$$

with the boundary condition

$$p=p_{wf} \text{ at } (k_x/k_y)^{1/2}\ u^2+(k_y/k_x)^{1/2}\ v^2=r_w^2. \quad \ldots(26)$$

Because the boundary condition is specified on an ellipse rather than a circle, the solution to Eq. 25 in the u-v plane is not radial; rather, the isobars are a family of concentric ellipses. The exact solution to this problem is given in Appendix C. Because r_w is small relative to the size of the reservoir, the ellipse of Eq. 26 is correspondingly small. In Appendix C it is shown that, as a practical matter, the isobars in the u-v plane are essentially circular and that the pressure essentially satisfies the equation

$$p-p_{wf}=\frac{q\mu}{2\pi(k_xk_y)^{1/2}h}\ln\frac{r^{uv}}{\bar{r}_w}, \quad \ldots\ldots\ldots\ldots\ldots(27)$$

where

$$r^{uv}=(u^2+v^2)^{1/2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(28)$$

and

$$\bar{r}_w=\tfrac{1}{2}r_w\left[(k_y/k_x)^{1/4}+(k_x/k_y)^{1/4}\right]. \quad \ldots\ldots\ldots\ldots(29)$$

Now consider the difference equation for the steady-state pressure in an anisotropic medium. Instead of Eq. 12, we write

$$\frac{k_xh\Delta y}{\mu\Delta x}(p_3-2p_0+p_1)+\frac{k_yh\Delta x}{\mu\Delta y}(p_2-2p_0+p_4)=q.$$

With the change of variables of Eq. 24, this transforms to the difference equation on a grid in the u-v plane:

$$\frac{(k_xk_y)^{1/2}h\Delta v}{\mu\ \Delta u}(p_3-2p_0+p_1)+$$

$$\frac{(k_xk_y)^{1/2}h\Delta u}{\mu\ \Delta v}(p_2-2p_0+p_4)=q.$$

Thus, we have differential and difference problems in the u-v plane that are essentially identical to the isotropic problem that we have already solved in the x-y plane. Corresponding to Eq. 21 we then have

$$r_o^{uv}=0.14(\Delta u^2+\Delta v^2)^{1/2}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(30)$$

where r_o^{uv} is the radius of an almost circular isobar in the u-v plane that has the same pressure as the well block.

To complete the development, we extend the definition of the well-block equivalent radius in Eq. 1 to

$$p_o-p_{wf}=\frac{q\mu}{2\pi(k_xk_y)^{1/2}h}\ln(r_o/r_w).$$

But we also have, from Eq. 27,

$$p_o-p_{wf}=\frac{q\mu}{2\pi(k_xk_y)^{1/2}h}\ln(r_o^{uv}/\bar{r}_w),$$

so that

$$r_o=(r_w/\bar{r}_w)\ r_o^{uv}.$$

Substitution of Eqs. 24, 29, and 30 gives the final result,

$$r_o=0.28\frac{\left[(k_y/k_x)^{1/2}\Delta x^2+(k_x/k_y)^{1/2}\Delta y^2\right]^{1/2}}{(k_y/k_x)^{1/4}+(k_x/k_y)^{1/4}}. \quad \ldots(31)$$

Conclusions

1. For numerical reservoir simulations in which either square or nonsquare grid blocks are used, the pressure calculated for a well block is the same as the flowing pressure at an equivalent radius, r_o. The WBP, p_o, is related to the BHP by

$$p_o-p_{wf}=\frac{q\mu}{2\pi(k_xk_y)^{1/2}h}\ln\frac{r_o}{r_w}.$$

2. For any aspect ratio, $\Delta y/\Delta x$, the equivalent well-block radius for an isotropic system is

$$r_o=0.14(\Delta x^2+\Delta y^2)^{1/2}.$$

This relation was obtained by careful numerical calculations as well as by a mathematical derivation based on an infinite nonsquare grid.

3. For an anisotropic medium, the equivalent well-block radius is given by

$$r_o=0.28\frac{\left[(k_y/k_x)^{1/2}\Delta x^2+(k_x/k_y)^{1/2}\Delta y^2\right]^{1/2}}{(k_y/k_x)^{1/4}+(k_x/k_y)^{1/4}}.$$

Nomenclature

a_i^* = ½ if i=0 or M; unity otherwise
A_{pq} = coefficient in Fourier series
b = constant of conformal mapping
b_j^* = ½ if j=0 or N; unity otherwise
B = constant of Muskat's equation for pressure drop in a five-spot
C = constant of integration
d = diagonal distance between injection and production wells in five-spot pattern, m
E = relative error
h = reservoir thickness, m
k = isotropic permeability, m^2
k_x = permeability in x direction, m^2
k_y = permeability in y direction, m^2
$(2k)!!$ = (2)(4)(6) . . . ($2k$)
$(2k+1)!!$ = (1)(3)(5) . . . ($2k+1$)
M = number of blocks on side of computing grid in x direction

N = number of blocks on side of computing grid in y direction
p = pressure, Pa
p_D = dimensionless pressure $=(kh/q\mu)p$
p_o = simulator well-block pressure, Pa
p_{wf} = wellbore pressure, Pa
q = production rate of well, m^3/s
r = radius, m
r_o = equivalent radius of well block, m
r_w = wellbore radius, m
$\bar{r}$ = mean radius of elliptical isobar in u-v plane, m
r^{uv} = radius of circle in u-v plane, m
s = arc length along elliptical isobar in x-y plane, m
u = transformed distance coordinate in x direction, m
v = transformed distance coordinate in y direction, m
v_n = velocity normal to ellipse in x-y plane, m/s
v_x = velocity in x direction, m/s
v_y = velocity in y direction, m/s
α = aspect ratio of grid blocks, $\Delta y/\Delta x$
β = angle of line normal to ellipse in x-y plane
γ = Euler's constant, 0.5772157 . . .
δ_{ij} = 1 if i,j=0,0; −1 if $i,j=M,N$; zero otherwise
Δx = grid spacing in x direction, m
Δy = grid spacing in y direction, m
θ = variable of conformal mapping
μ = viscosity, Pa·s
ρ = variable of conformal mapping

Subscripts

i = grid index in x direction
j = grid index in y direction
p = index of Fourier component
q = index of Fourier component

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APPENDIX A
Numerical Solution for Repeated Five-Spot Pattern

Fig. 2 shows a portion of a repeated five-spot pattern that stretches to infinity in all directions. Because of symmetry, we need calculate only the quarter five-spot that is enclosed within the dashed lines. We divide that area into $M\times N$ blocks, using half-blocks on the boundaries. For all blocks $0\leq i\leq M$, $0\leq j\leq N$, the difference equation for the steady-state pressure distribution is

$$\frac{kh\Delta y}{\mu\Delta x}\left(p_{i+1,j}-2p_{ij}+p_{i-1,j}\right)+\frac{kh\Delta x}{\mu\Delta y}\left(p_{i,j+1}-2p_{ij}+p_{i,j-1}\right)=q_{ij}. \quad \text{.....(A-1)}$$

We assume production at rate q at the lower left corner and injection at rate q at the upper right corner. Thus,

$$q_{o,o}=q,$$

$$q_{M,N}=-q,$$

and

$$q_{ij}=0 \text{ for } i,j\neq 0,0 \text{ or } M,N.$$

If we define

$$p_D=(kh/q\mu)p,$$

then Eq. A-1 simplifies to

$$\alpha\left[(p_D)_{i-1,j}+(p_D)_{i+1,j}\right]+(1/\alpha)\cdot\left[(p_D)_{i,j-1}+(p_D)_{i,j+1}\right]-(2\alpha+2/\alpha)(p_D)_{ij}=\delta_{ij}, \quad \text{.....(A-2)}$$

where

$$\left.\begin{array}{l}\delta_{ij}=0 \text{ for } i,j\neq 0,0 \text{ or } M,N\\ \delta_{o,o}=1\\ \delta_{M,N}=-1\end{array}\right\}. \quad \text{.....(A-3)}$$

The following reflection conditions are used.

$$\left.\begin{array}{l}p_{-1,j}=p_{1,j}\\ p_{M+1,j}=p_{M-1,j}\end{array}\right\} \text{ for } 0\leq j\leq N, \quad \text{........(A-4)}$$

and

$$\left.\begin{array}{l} p_{i,-1}=p_{i,1} \\ p_{i,N+1}=p_{i,N-1} \end{array}\right\} \text{ for } 0\le i\le M. \quad \ldots\ldots\ldots\ldots (A\text{-}5)$$

Substitution of these reflection conditions yields a system of $(M+1)\times(N+1)$ equations. These equations were solved by direct solution in double precision on the IBM 370.™

APPENDIX B

Mathematical Derivation for Infinite Grid

Finite Fourier Series Representation

We seek the solution to the system of Eqs. A-2 through A-5 as the grid size (M and N) becomes infinite, keeping the ratio $M/N=\alpha$ constant. Assume a finite Fourier series solution of the form

$$\left(p_D\right)_{ij}=\sum_{p=0}^{M}\sum_{q=0}^{N} A_{pq}\cos a_p x_i \cos a_q y_j, \quad \ldots (B\text{-}1)$$

with

$$a_p x_i = p\pi i/M,$$

$$a_q y_j = q\pi j/N.$$

Each p,q component of this solution satisfies the reflection conditions, Eqs. A-4 and A-5.

Substitution of Eq. B-1 into the difference equation, Eq. A-2, yields

$$\sum_{p=0}^{M}\sum_{q=0}^{N} A_{pq}\cos(p\pi i/M)\cos(q\pi j/N)$$

$$\cdot[\alpha\sin^2(p\pi/2M)+(1/\alpha)\sin^2(q\pi/2N)]=-\delta_{ij}/4,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (B\text{-}2)$$

where δ_{ij} is defined by Eq. A-3.

To evaluate A_{pq}, multiply Eq. B-2 by

$$a_i^*\cos(r\pi i/M)\, b_j^*\cos(s\pi j/N) \quad \ldots\ldots\ldots\ldots (B\text{-}3)$$

and sum over i and j.

$$\sum_{i=0}^{M}\sum_{j=0}^{N} a_i^* b_j^*\cos(r\pi i/M)\cos(s\pi j/N)\sum_{p=0}^{M}\sum_{q=0}^{N}$$

$$\cdot A_{pq}\cos(p\pi i/M)\cos(q\pi j/N)[\alpha\sin^2(p\pi/2M)$$

$$+(1/\alpha)\sin^2(q\pi/2N)]$$

$$=-\sum_{i=0}^{M}\sum_{j=0}^{N} a_i^* b_j^*(\delta_{ij}/4)\cos(r\pi i/M)\cos(q\pi j/N).$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (B\text{-}4)$$

The functions of Eq. B-3 are orthogonal if we choose

$$a_i^*=½ \text{ for } i=0,M,$$

$$a_i^*=1 \text{ for } 1\le i\le M-1,$$

$$b_j^*=½ \text{ for } j=0,N,$$

$$b_j^*=1 \text{ for } 1\le j\le N-1.$$

Now, the summand in the right side of Eq. B-4 is zero except for $i,j=0,0$ or M,N. Thus, the right side is

$$(1/16)[\cos(r\pi)\cos(s\pi)-1]$$

or

$$(1/16)[(-1)^{r+s}-1].$$

Interchange the indices p,q with r,s and reorder the summations of Eq. B-4 to obtain

$$\sum_{r=0}^{M}\sum_{s=0}^{N} A_{rs}[\alpha\sin^2(r\pi/2M)+(1/\alpha)\sin^2(s\pi/2N)]$$

$$\cdot\sum_{i=0}^{M} a_i^*\cos(p\pi i/M)\cos(r\pi i/M)$$

$$\cdot\sum_{j=0}^{N} b_j^*\cos(q\pi j/N)\cos(s\pi j/N)$$

$$=(1/16)[(-1)^{p+q}-1]. \quad \ldots\ldots\ldots\ldots\ldots\ldots (B\text{-}5)$$

The one-dimensional sums can be evaluated. For $r\ne p$,

$$\sum_{i=0}^{M} a_i^*\cos(p\pi i/M)\cos(r\pi i/M)$$

$$=½+\sum_{i=1}^{M-1}\cos(p\pi i/M)\cos(r\pi i/M)$$

$$+½\cos(p\pi)\cos(r\pi).$$

Hildebrand[8] gives the formula (for $\alpha\ne\beta$)

$$\sum_{k=1}^{K}\cos k\alpha\cos k\beta=½[c_k(\alpha+\beta)+c_k(\alpha-\beta)],$$

where

$$c_k(\gamma)=\sum_{k=1}^{K}\cos k\gamma=\frac{\sin\dfrac{K}{2}\gamma\cos\dfrac{K+1}{2}\gamma}{\sin\dfrac{\gamma}{2}}.$$

Taking $\alpha=p\pi/M$, $\beta=r\pi/M$, $K=M-1$, we then can obtain

$$\sum_{i=1}^{M-1} \cos(p\pi i/M)\cos(r\pi i/M)$$

$$=-½-½\cos(p\pi)\cos(r\pi),$$

and, finally,

$$\sum_{i=0}^{M} a_i^* \cos(p\pi i/M)\cos(r\pi i/M)=0,\ r\neq p.$$

For the case $r=p$, we have

$$\sum_{i=0}^{M} a_i^*\cos^2(p\pi i/M)=½+\sum_{i=1}^{M-1}\cos^2(p\pi i/M)$$

$$+½\cos^2(p\pi)=1+\sum_{i=1}^{M-1}\cos^2(p\pi i/M).$$

Hildebrand[8] gives the formula

$$\sum_{k=1}^{K}\cos^2 k\alpha=\frac{K}{2}+\frac{\sin K\alpha\cos(K+1)\alpha}{2\sin\alpha}.$$

Again taking $\alpha=p\pi/M$, $K=M-1$, we obtain

$$\sum_{i=1}^{M-1}\cos^2 k\alpha=\frac{M-1}{2}-\frac{1}{2}\cos^2(Mp\pi/M)=\frac{M}{2}-1.$$

However, $p=0$ and $p=M$ are special cases. For $p=0$,

$$\sum_{i=1}^{M} a_i^*\cos^2(0)=½+(M-1)+½=M,$$

while for $p=M$,

$$\sum_{i=1}^{M} a_i^*\cos^2(\pi i)=½+(M-1)+½=M.$$

Thus, in general,

$$\sum_{i=0}^{M} a_i^*\cos^2(p\pi i/M)=\frac{M}{2a_p^*}.$$

Similarly,

$$\sum_{j=0}^{N} b_j^*\cos(q\pi j/N)\cos(s\pi j/N)=0,\ s\neq q,$$

and

$$\sum_{j=0}^{N} b_j^*\cos^2(q\pi j/N)=\frac{N}{2b_q^*}.$$

Substitution into Eq. B-5 gives

$$A_{pq}[\alpha\sin^2(p\pi/2M)+(1/\alpha)\sin^2(q\pi/2N)]\frac{M}{2a_p^*}\frac{N}{2b_q^*}$$

$$=(1/16)[(-1)^{p+q}-1]$$

or

$$A_{pq}=\frac{[(-1)^{p+q}-1]a_p^*b_q^*}{4MN[\alpha\sin^2(p\pi/2M)+(1/\alpha)\sin^2(q\pi/2N)]}. \qquad \text{(B-6)}$$

Integral Representation for Infinite Grid

In this form the finite Fourier series is not quite suitable for going to the limit of an infinite grid. For p even, A_{pq} is nonzero only for q odd, while for p odd, A_{pq} is nonzero only for q even. To cover all cases and leave out the zero terms, break up Eq. B-1 into two sums where

$$(p_D)_{ij}=S_1+S_2.$$

For the first sum, define indices m and n by

$p=2m$, $[m=0,\ 1\ .\ .\ .\ M/2]$, and

$q=2n+1$, $[n=0,\ 1\ .\ .\ .\ (N-2)/2]$.

Then

$$S_1=\sum_{m=0}^{M/2}\sum_{n=0}^{(N/2)-1}A'_{mn}\cos\frac{2m\pi i}{M}\cos\frac{(2n+1)\pi j}{N}$$

where

$$A'_{mn}=\frac{-a_{2m}^*b_{2n+1}^*}{2MN\left[\alpha\sin^2\frac{2m\pi}{2M}+(1/\alpha)\sin^2\frac{(2n+1)\pi}{2N}\right]}.$$

For the second sum, define m and n by

$p=2m+1$, $[m=0,\ 1\ .\ .\ .\ (M-2)/2]$, and

$q=2n$, $[n=0,\ 1\ .\ .\ .\ N/2]$.

Then

$$S_2=\sum_{m=0}^{(M/2)-1}\sum_{n=0}^{N/2}A''_{mn}\cos\frac{(2m+1)\pi i}{M}\cos\frac{2n\pi j}{N},$$

where

$$A''_{mn}=\frac{-a^*_{2m+1}b^*_{2n}}{2MN\left[\alpha\sin^2\frac{(2m+1)\pi}{2M}+(1/\alpha)\sin^2\frac{2n\pi}{2N}\right]}.$$

The dashed square in Fig. 2 has sides of length

$$L=M\Delta x=N\Delta y.$$

Let

$$\begin{aligned}u_m &= m\Delta x/L=m/M,\\ v_n &= n\Delta y/L=n/N,\\ \Delta u &= 1/M,\\ \Delta v &= 1/N.\end{aligned}$$

Then

$$S_1=-\sum_{u_m=0}^{½}\ \sum_{v_n=0}^{½-\Delta v} a^*_{2m}b^*_{2n+1}\Delta u\Delta v\cos(2u_m)\pi i$$

$$\cdot\cos(2v_n+\Delta v)\pi j/2[\alpha\sin^2(u_m\pi)$$

$$+(1/\alpha)\sin^2(v_n+\Delta v/2)\pi]$$

and

$$S_2=-\sum_{u_m=0}^{½-\Delta u}\ \sum_{v_n=0}^{½} a^*_{2m+1}b^*_{2n}\Delta u\Delta v$$

$$\cdot\cos(2u_m+\Delta u)\pi i\cos(2v_n)\pi j/2[\alpha\sin^2$$

$$\cdot(u_m+\Delta u/2)\pi+(1/\alpha)\sin^2(v_n\pi)].$$

Take the limit as Δu, $\Delta v\rightarrow 0$. This corresponds to letting M and N become infinite, keeping $M/N=\alpha$ constant. Both sums approach the same integral. The constants a^* and b^* are unity almost everywhere:

$$S_1=S_2=-\int_{u=0}^{½}\int_{v=0}^{½}\frac{\cos(2\pi iu)\cos(2\pi jv)du\,dv}{2[\alpha\sin^2(\pi u)+(1/\alpha)\sin^2(\pi v)]}.$$

Finally, with a slight change of dummy variables, we have

$$(p_D)_{ij}=\frac{-1}{\pi^2}\int_0^{\pi/2}\int_0^{\pi/2}\frac{\cos(2iu)\cos(2jv)du\,dv}{\alpha\sin^2u+(1/\alpha)\sin^2v}.\quad\text{(B-7)}$$

It is not difficult to verify that Eq. B-7 satisfies the difference equation, Eq. A-2, as well as the reflection conditions (Eqs. A-4 and A-5) at the lower and left boundaries.

Evaluation of Well-Block Equivalent Radius

It has not been possible to evaluate the double integral of Eq. B-7 for all i and j. However, to obtain the well-block equivalent radius, it is sufficient to evaluate the integral along some line through the origin. The most convenient such line is the horizontal axis.

For sufficiently large i, the solution on the infinite grid satisfies the exact radial solution. Along the horizontal axis ($j=0$), the exact radial solution is

$$(p_D)_{i,o}=(p_D)_{wf}+(1/2\pi)\ln(i\Delta x/r_w).$$

By the definition of the well-block equivalent radius,

$$(p_D)_{o,o}=(p_D)_{wf}+(1/2\pi)\ln(r_o/r_w).$$

Combining, we get

$$\ln(r_o/i\Delta x)=2\pi[(p_D)_{o,o}-(p_D)_{i,o}].$$

Substituting Eq. B-7 gives

$$\ln(r_o/i\Delta x)=$$

$$\frac{2\alpha}{\pi\cdot}\int_0^{\pi/2}\int_0^{\pi/2}\frac{\cos(2iu)-1}{\alpha^2\sin^2u+\sin^2v}dv\,du.\quad\text{........ (B-8)}$$

From several tables of definite integrals [Ref. 9, P. 76, Eq. 6 and Ref. 10, Eq. 3.653(2)],

$$\int_0^{\pi/2}\frac{dv}{1+a^2\sin^2v}=\frac{\pi}{2(1+a^2)^{½}},$$

from which it follows that

$$\int_0^{\pi/2}\frac{dv}{b^2+\sin^2v}=\frac{\pi}{2b(1+b^2)^{½}}.\quad\text{............ (B-9)}$$

Substituting Eq. B-9 into Eq. B-8 and taking $b=\alpha\sin u$ gives

$$\ln(r_o/i\Delta x)=\int_0^{\pi/2}\frac{\cos(2iu)-1}{\sin u(1+\alpha^2\sin^2u)^{½}}du.\quad\text{.. (B-10)}$$

Let

$$f=\alpha^2\sin^2u$$

and

$$g(f)=(1+f)^{-½}.$$

Expand $g(f)$ in a Taylor series in f:

$$g(f)=1+\sum_{k=1}^{\infty}\frac{(-1)^k(2k-1)!!}{(2k)!!}f^k.$$

Then Eq. B-10 becomes

$$\ln(r_o/\Delta x)=\ln i+\int_0^{\pi/2}\frac{\cos(2iu)-1}{\sin u}du$$

$$+\sum_{k=1}^{\infty}\frac{(-1)^k(2k-1)!!\alpha^{2k}}{(2k)!!}$$

$$\cdot\int_0^{\pi/2}\sin^{(2k-1)}u\,[\,\cos(2iu)-1]\,du. \quad \text{........(B-11)}$$

By Eq. 2.526(1) of Ref. 10,

$$\int\frac{du}{\sin u}=\ln\tan(u/2).$$

By Eq. 2.539(1) of Ref. 10,

$$\int\frac{\cos(2iu)du}{\sin u}=2\sum_{k=1}^{i}\frac{\cos(2k-1)u}{2k-1}+\ln\tan(u/2).$$

Then

$$\int\frac{\cos(2iu)-1}{\sin u}du=2\sum_{k=1}^{i}\frac{\cos(2k-1)u}{2k-1}$$

and

$$\int_0^{\pi/2}\frac{\cos(2iu)-1}{\sin u}du=-2\sum_{k=1}^{i}\frac{1}{2k-1}. \quad \text{.....(B-12)}$$

By Eq. 3.621(4) of Ref. 10,

$$\int_0^{\pi/2}\sin^{(2k-1)}u\,du=\frac{(2k-2)!!}{(2k-1)!!}. \quad \text{............(B-13)}$$

By Eq. 3.631(13) of Ref. 10,

$$\int_0^{\pi/2}\sin^{(2k-1)}u\cos(2iu)du$$

$$=\frac{2^{(k-1)}(k-1)!(2k-1)!!}{(2k+2i-1)!!}Y(i,k), \quad \text{...........(B-14)}$$

where

$$Y(i,k)=\frac{(-1)^i}{(2k-2i-1)!!}\ \text{for}\ k\geq i \quad \text{............(B-15)}$$

$$=(-1)^k(2i-2k+3)!!\ \text{for}\ k<i. \quad \text{......(B-16}')$$

Eq. B-16′ is incorrect, as can be seen by comparison with other evaluations of this integral for the case $k=1$. Ref. 9 (P. 69) gives the correct formula, which can be converted into the correct expression,

$$Y(i,k)=(-1)^k(2i-2k-1)!!\ \text{for}\ (k<i). \quad \text{.....(B-16)}$$

Substitution of Eqs. B-12 through B-16 into Eq. B-11 yields

$$\ln(r_o/\Delta x)=\ln i-2\sum_{k=1}^{i}\frac{1}{2k-1}-\sum_{k=1}^{\infty}\frac{(-1)^k\alpha^{2k}}{2k}$$

$$+\sum_{k=1}^{\infty}\alpha^{2k}Z(i,k), \quad \text{.............(B-17)}$$

where

$$Z(i,k)=\frac{(2k-1)!!(2k-1)!!(2i-2k-1)!!}{2k(2i+2k-1)!!}\quad(k<i),$$

and

$$Z(i,k)=\frac{(-1)^{i+k}(2k-1)!!(2k-1)!!}{2k(2k-2i-1)!!(2k+2i-1)!!}\quad(k\geq i).$$

$Z(i,k)$ approaches zero for large i, so that the last sum of Eq. B-17 also approaches zero for large i. Thus,

$$\lim_{i\to\infty}\ln(r_o/\Delta x)=\lim_{i\to\infty}\left[\ln i-2\sum_{k=1}^{i}1/(2k-1)\right]$$

$$-\tfrac{1}{2}\sum_{k=1}^{\infty}(-1)^k\alpha^{2k}/k. \quad \text{...............(B-18)}$$

Now

$$2\sum_{k=1}^{i}1/(2k-1)=2\sum_{k=1}^{2i}1/k-\sum_{k=1}^{i}1/k$$

and

$$\sum_{k=1}^{\infty}(-1)^k\alpha^{2k}/k=-\ln(1+\alpha^2),$$

so Eq. B-18 can be written

$$\lim_{i\to\infty}\ln(r_o/\Delta x)=$$

$$\lim_{i\to\infty}\left[2\ln 2i-2\sum_{k=1}^{2i}1/k-\ln i\right.$$

$$\left.+\sum_{k=1}^{i}1/k-2\ln 2\right]+\tfrac{1}{2}\ln(1+\alpha^2). \quad \text{........(B-19)}$$

One of the definitions of Euler's constant is

$$\gamma=\lim_{i\to\infty}\left[\sum_{k=1}^{i}1/k-\ln i\right],$$

so Eq. B-19 becomes

$$\lim_{i\to\infty} \ln(r_o/\Delta x) = -\gamma - 2\ln 2 + ½ \ln(1+\alpha^2),$$

$$\lim(r_o/\Delta x) = \frac{e^{-\gamma}}{4}(1+\alpha^2)^{½} = \frac{e^{-\gamma}}{4}\left(1+\frac{\Delta y^2}{\Delta x^2}\right)^{½},$$

and, finally,

$$\lim \frac{r_o}{(\Delta x^2+\Delta y^2)^{½}} = \frac{e^{-\gamma}}{4} = 0.1403649. \quad \text{(B-20)}$$

Note that, for a square grid,

$$\lim \frac{r_o}{\Delta x} = \frac{\sqrt{2}e^{-\gamma}}{4} = 0.198506. \quad \text{(B-21)}$$

APPENDIX C

Solution of Laplace's Equation With Elliptic Inner Boundary Condition

Transformation to Elliptic Coordinates

We seek the solution to Laplace's equation (Eq. 25) subject to a Dirichlet condition on an elliptical inner boundary (Eq. 26). To solve this problem, we introduce a conformal mapping to the elliptic coordinate system, ρ-θ, by the following change of variables.[7,11]

$$u = b \cosh\rho \cos\theta \quad \text{(C-1a)}$$

and

$$v = b \sinh\rho \sin\theta, \quad \text{(C-1b)}$$

where b is a constant to be determined later.

This transformation defines a family of concentric ellipses in the u-v plane, wherein ρ is a parameter identifying a particular ellipse. Let ρ_w be the parameter identifying the wellbore ellipse. Then, substitution of Eq. C-1 into

$$\cos^2\theta + \sin^2\theta = 1$$

yields

$$\frac{u^2}{b^2\cosh^2\rho_w} + \frac{v^2}{b^2\sinh^2\rho_w} = 1 \quad \text{(C-2)}$$

as an alternate equation for the wellbore ellipse. Comparison with Eq. 26 gives

$$\tanh^2\rho_w = k_x/k_y, \quad \text{(C-3)}$$

and

$$b^2 = r_w^2(k_y-k_x)/(k_x k_y)^{½}. \quad \text{(C-4)}$$

This derivation requires that k_y be greater than k_x. If k_y is less than k_x, it is necessary to interchange the roles of x and y in Eq. 24.

Solution in Elliptic Coordinates

Because Eq. C-1 is a conformal mapping, Laplace's equation transforms unchanged from the u-v plane to the ρ-θ plane:

$$\frac{\partial^2 p}{\partial \rho^2} + \frac{\partial^2 p}{\partial \theta^2} = 0. \quad \text{(C-5)}$$

The boundary condition, Eq. 26, transforms into the very simple condition

$$p = p_{wf} \text{ at } \rho = \rho_w.$$

Because the solution is independent of θ, Eq. C-5 is easily integrated:

$$p = p_{wf} + C(\rho - \rho_w), \quad \text{(C-6)}$$

where C is a constant of integration, to be determined from the flow rate.

Determination of the Constant of Integration

The flow rate, q, may be determined by integrating the normal velocity around any closed path. We choose for the closed path an elliptical isobar, for which ρ is a constant. Thus,

$$q = -h\int_\rho v_n ds = -h\int_0^{2\pi} v_n \left(\frac{\partial s}{\partial \theta}\right)_\rho d\theta, \quad \text{(C-7)}$$

where v_n is the velocity component normal to the ellipse in the x-y plane, and s is the arc length around the ellipse, also in the x-y plane. Now

$$\left(\frac{\partial s}{\partial \theta}\right)_\rho = \left[\left(\frac{\partial x}{\partial \theta}\right)_\rho^2 + \left(\frac{\partial y}{\partial \theta}\right)_\rho^2\right]^{½},$$

$$\left(\frac{\partial x}{\partial \theta}\right)_\rho = \frac{dx}{du}\left(\frac{\partial u}{\partial \theta}\right)_\rho$$

$$= -(k_x/k_y)^{¼}\, b\cosh\rho\sin\theta, \quad \text{(C-8a)}$$

$$\left(\frac{\partial y}{\partial \theta}\right)_\rho = \frac{dy}{dv}\left(\frac{\partial x}{\partial \theta}\right)_\rho$$

$$= (k_y/k_x)^{¼}\, b\sinh\rho\cos\theta, \quad \text{(C-8b)}$$

and

$$\left(\frac{\partial s}{\partial \theta}\right)_\rho = b[(k_x/k_y)^{½}\cosh^2\rho\sin^2\theta$$

$$+(k_y/k_x)^{½}\sinh^2\rho\cos^2\theta]^{½}. \quad \text{(C-9)}$$

To obtain v_n, we need to find v_x, v_y, and β, the angle of a line normal to the ellipse. This normal angle can be obtained from

$$\tan\beta=-\left(\frac{\partial x}{\partial y}\right)_\rho=-\left(\frac{\partial x}{\partial\theta}\right)_\rho\Big/\left(\frac{\partial y}{\partial\theta}\right)_\rho.$$

Substitution of Eqs. C-8a and C-8b gives

$$\tan\beta=(k_x/k_y)^{1/2}\tan\theta/\tanh\rho.\quad\text{.............(C-10)}$$

We now proceed to obtain v_x.

$$-v_x=\frac{k_x}{\mu}\left(\frac{\partial p}{\partial x}\right)_y=\frac{k_x}{\mu}\left[\left(\frac{\partial p}{\partial\rho}\right)_\theta\left(\frac{\partial\rho}{\partial x}\right)_y+\left(\frac{\partial p}{\partial\theta}\right)_\rho\left(\frac{\partial\theta}{\partial x}\right)_y\right].\quad\text{...............(C-11)}$$

But, from Eq. C-6, $(\partial p/\partial\theta)_\rho=0$, while $(\partial p/\partial\rho)_\theta=C$. To get $(\partial\rho/\partial x)_y$, we combine Eq. 24 with Eq. C-2 (which is valid for all ρ):

$$\frac{x^2(k_y/k_x)^{1/2}}{b^2\cosh^2\rho}+\frac{y^2(k_x/k_y)^{1/2}}{b^2\sinh^2\rho}=1.$$

Differentiation with respect to x at constant y and substitution of Eqs. 24, C-1, and C-11 yield

$$v_x=-\frac{k_xC}{\mu b}\frac{(k_y/k_x)^{1/4}\sinh\rho\cos\theta}{\sinh^2\rho+\sin^2\theta}.\quad\text{........(C-12)}$$

Similarly,

$$v_y=-\frac{k_yC}{\mu b}\frac{(k_x/k_y)^{1/4}\cosh\rho\sin\theta}{\cosh^2\rho-\cos^2\theta}.\quad\text{........(C-13)}$$

Now, the normal component of velocity is given by

$$v_n=v_x\cos\beta+v_y\sin\beta.$$

By making use of Eqs. C-10, C-12, and C-13, we obtain

$$v_n=\frac{-(k_xk_y)^{3/4}C/\mu b}{(k_y\sinh^2\rho\cos^2\theta+k_x\cosh^2\rho\sin^2\theta)^{1/2}}.$$

Multiplication by Eq. C-9 gives

$$v_n\left(\frac{\partial s}{\partial\theta}\right)_\rho=-\frac{(k_xk_y)^{1/2}C}{\mu},$$

and the integration indicated by Eq. C-7 yields

$$q=\frac{2\pi h(k_xk_y)^{1/2}C}{\mu}.$$

Substitution into Eq. C-6 then gives, for the exact solution,

$$p=p_{wf}+\frac{q\mu}{2\pi(k_xk_y)^{1/2}h}(\rho-\rho_w).\quad\text{...........(C-14)}$$

Solution in Terms of a Mean Radius, $\bar{r}$

For each isobar, the coefficients of $\sin\theta$ and $\cos\theta$ in Eq. C-1 are, respectively, $b\sinh\rho$ and $b\cosh\rho$. For large ρ, $\sinh\rho$ and $\cosh\rho$ are almost the same, leading to the conclusion that the isobars are essentially circular in the u-v plane. One possible definition of a mean radius for an isobar is the average of these coefficients, given by

$$\bar{r}=(b\sinh\rho+b\cosh\rho)/2=(b/2)\exp(\rho),\quad\text{......(C-15)}$$

or

$$\rho=\ln(2\bar{r}/b).$$

Substitution into Eq. C-14 gives

$$p=p_{wf}+\frac{q\mu}{2\pi(k_xk_y)^{1/2}h}\ln(\bar{r}/\bar{r}_w),\quad\text{............(C-16)}$$

where

$$\bar{r}_w=b(\sinh\rho_w+\cosh\rho_w)/2.$$

But, on the wellbore ellipse, substitution of $v=0$ and $u=(k_y/k_x)^{1/4}r_w$ into Eq. C-2 yields

$$b\cosh\rho_w=(k_y/k_x)^{1/4}r_w,$$

while substitution of $u=0$, $v=(k_x/k_y)^{1/4}r_w$ yields

$$b\sinh\rho_w=(k_x/k_y)^{1/4}r_w,$$

so that

$$\bar{r}_w=r_w[(k_x/k_y)^{1/4}+(k_y/k_x)^{1/4}]/2.\quad\text{..........(C-17)}$$

So far, no approximations have been introduced, and Eqs. C-16 and C-17 represent merely another form of the exact solution to Eqs. 25 and 26. The problem is that the mean radius $\bar{r}$, defined by Eq. C-15, is not the same as the more useful radius defined by Eq. 28—i.e.,

$$r^{uv}=(u^2+v^2)^{1/2}=b(\sinh^2\rho\sin^2\theta+\cosh^2\rho\cos^2\theta)^{1/2}.\quad\text{.....................(C-18)}$$

To evaluate the error of using r^{uv} instead of $\bar{r}$ in Eq. C-16, we compare $\ln(r^{uv}/\bar{r}_w)$ with $\ln(\bar{r}/\bar{r}_w)$ for various values of k_y/k_x and r/r_w. More specifically, we want the relative error, given by

$$E=1-\frac{\ln(r^{uv}/\bar{r}_w)}{\ln(\bar{r}/\bar{r}_w)}=\frac{\ln(\bar{r}/r^{uv})}{\rho-\rho_w}.\quad\text{............(C-19)}$$

Dividing Eq. C-15 by Eq. C-18 yields

$$\bar{r}/r^{uv} = \frac{1+\tanh\rho}{2(\cos^2\theta + \sin^2\theta\tanh^2\rho)^{1/2}}.$$

The error, E, takes on its maximum absolute value in the direction of maximum permeability (i.e., the y axis, where $\theta=\pi/2$). Then

$$\max_\theta |E| = \frac{\ln[(1+\tanh\rho)/(2\tanh\rho)]}{\rho-\rho_w}. \quad \text{.......(C-20)}$$

Also, on the y axis,

$$r/r_w = y/r_w = (k_y/k_x)^{1/4} b \sinh\rho/r_w$$

$$= [(k_y/k_x)-1]^{1/2} \sinh\rho. \quad \text{..............(C-21)}$$

From Eq. C-3 we have

$$\rho_w = \tanh^{-1}(k_x/k_y)^{1/2}$$

$$= \tfrac{1}{2}\ln\frac{1+(k_x/k_y)^{1/2}}{1-(k_x/k_y)^{1/2}}. \quad \text{................(C-22)}$$

Eqs. C-20, C-21, and C-22 can be used to estimate the relative error for any k_y/k_x and r/r_w. In particular, we want to know the radius beyond which the error is less than 0.1%. For $k_y/k_x=2$, $r/r_w>10.7$ suffices; for $k_y/k_x=10$, $r/r_w>29.2$. Thirty times the wellbore radius is small compared with any expected grid size; hence we conclude that the assumption of circularity of the isobars in the u-v plane is a good one, and that $\bar{r}$ can be replaced by r^{uv} safely in Eq. C-16 to yield Eq. 27.

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