Variational Equations and Variational Theorems

Notes for CE231: Version 1.0

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Contents

Chapter 1

Introduction

In these notes, Chapters 2 and 3 provide an introduction to formulation of problems in both a *strong* and a *weak* form. The strong form of a problem is given as a set of partial differential equations; whereas, the weak form of a problem is associated with either a variational equation or a variational theorem. Vainberg's theorem is introduced to indicate when a variational theorem exists for a given variational equation. A variational statement provides a convenient basis for constructing approximate solutions (e.g., finite element methods). Initially, the linear heat equation is used as an example problem to describe some of the details concerning use of strong and weak forms.

Chapter 4 provides a summary of the strong form for a linear elasticity problem. In Chapter 5 variational theorems are presented in three forms. The first form, known as the Hu-Washizu form, presents a variational theorem which includes all the equations for elastostatics: Balance of linear momentum, strain-displacement equations, constitutive equations, and boundary conditions for both traction and displacement. The second form, known as the Hellinger-Reissner form, includes all the equations except the strain-displacement equations which must therefore be provided in a strong form. Finally, the last form discussed is the principle of minimum potential energy which includes only the momentum equations (in terms of displacement) and the traction boundary conditions. All other equations must be provided in strong form.

Chapter 2

Introduction to Strong and Weak Forms

2.1 Strong form for problems in engineering

Many problems in engineering are modeled using partial differential equations (PDE). The set of partial differential equations describing such problems is often referred to as the *strong form* of the problem. The differential equations may be either linear or non-linear. Linear equations are characterized by the appearance of the dependent variable(s) in linear form only, whereas, non-linear equations include nonlinear terms also. Very few partial differential equations may be solved in closed form - one case being the linear wave equation in one space dimension and time. Some equations admit use of solutions written as series of products of one dimensional functions for which exact solutions may be constructed for each function. Again, in general it is not possible to treat general boundary conditions or problem shapes using this approach. As an example consider the Poisson equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = q(x, y) \tag{2.1}$$

defined on the region $0 \leq x \leq a$, $0 \leq y \leq b$ with the boundary condition $u = 0$ on all edges. This differential equation may be solved by writing u as a product form

$$u = \sum_m \sum_n \sin(\frac{m\pi x}{a}) \sin(\frac{n\pi y}{b}) u_{mn} \tag{2.2}$$

which when substituted into the equation yields

$$\sum_m \sum_n \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \right] \sin(\frac{m\pi x}{a}) \sin(\frac{n\pi y}{b}) u_{mn} = q(x, y) \tag{2.3}$$

The solution may now be completed by expanding the right hand side as a double sine series (i.e., Fourier series) and matching terms between the left and right sides. Evaluation of the

solution requires the summation of the series for each point (x, y) of interest. Consequently, it is not possible to get an *exact* solution in closed form. Indeed, use of a finite set of terms leads to an *approximate* solution with the accuracy depending on the number of terms used.

More general solutions may be constructed using separable solution; however, again, the solutions are obtained only in series form. In the sequel, we will be concerned with the construction of approximate solutions to problems in elasticity – one such approximation being a reduced theory for beams, plates, or shells. Accordingly, we now direct our attention to rewriting the set of equations in a form we call the *weak* form of the problem. The weak form will be the basis for constructing our approximate solutions.

2.2 Construction of a weak form

A weak form of a set of differential equations is constructed by considering 4 steps:

1. Multiply the differential equation by an arbitrary function which contracts the equations to a scalar.
2. Integrate the result of 1. over the domain of consideration, Ω.
3. Integrate by parts using Green's theorem to reduce derivatives to their minimum order.
4. Replace the boundary conditions by an appropriate construction.

2.3 Heat conduction problem: Strong form

The above steps are made more concrete by considering an example. The governing partial differential equation set for the transient heat conduction equation is given by

$$-\sum_{i=1}^{d} \frac{\partial q_i}{\partial x_i} \; + \; Q \; = \; \rho\, c\, \frac{\partial T}{\partial t} \tag{2.4}$$

where: d is the spatial dimension of the problem; q_i is the component of the heat flux in the x_i direction; Q is the volumetric heat generation per unit volume per unit time, T is temperature; ρ is density; c is specific heat; and t is time. The equations hold for all points x_i in the domain of interest, Ω.

The following notation is introduced for use throughout this report. Partial derivatives in space will be denoted by

$$(\;\cdot\;)_{,i} \; = \; \frac{\partial(\;\cdot\;)}{\partial x_i} \tag{2.5}$$

and in time by

$$\dot{T} \;=\; \frac{\partial T}{\partial t} \tag{2.6}$$

In addition, summation convention is used where

$$a_i b_i \;=\; \sum_{i=1}^{d} a_i b_i \tag{2.7}$$

With this notation, the divergence of the flux may be written as

$$q_{i,i} \;=\; \sum_{i=1}^{d} \frac{\partial q_i}{\partial x_i} \tag{2.8}$$

Boundary conditions are given by

$$T(x_j, t) \;=\; \bar{T} \tag{2.9}$$

where $\bar{T}$ is a specified temperature for points x_j on the boundary, Γ_T,; and

$$q_n \;=\; q_i n_i \;=\; \bar{q}_n \tag{2.10}$$

where $\bar{q}n_n$ is a specified flux for points x_j on the flux boundary, Γ_q, and n_i are direction cosines of the unit outward pointing normal to the boundary. Initial conditions are given by

$$T(x_i, 0) \;=\; \bar{T}_0(x_i) \tag{2.11}$$

for points in the domain, Ω, at time zero. The equations are completed by giving a relationship between the gradient of temperature and the heat flux (called the thermal constitutive equation). The Fourier law is a linear relationship given as

$$q_i \;=\; -\,k_{ij} T_{,j} \tag{2.12}$$

where k_{ij} is a symmetric, second rank thermal conductivity tensor. For an isotropic material

$$k_{ij} \;=\; k\delta_{ij} \tag{2.13}$$

in which δ_{ij} is the Kronecker delta function ($\delta_{ij} = 1$ for i = j; $= 0$ for i $\neq$ j). Hence for an isotropic material the Fourier law becomes

$$q_i \;=\; -\,kT_{,i} \tag{2.14}$$

The differential equation may be expressed in terms of temperature by substituting Eq. 2.14 into Eq. 2.4. The result is

$$(kT_{,i})_{,i} \; + \; Q \; = \; \rho c \dot{T} \tag{2.15}$$

The equation is a second order differential equation and for isotropic materials with constant k is expanded for two dimensional plane bodies as

$$k \left(\frac{\partial^2 T}{\partial x_1^2} \; + \; \frac{\partial^2 T}{\partial x_2^2} \right) \; + \; Q \; = \; \rho c \frac{\partial T}{\partial t} \tag{2.16}$$

We note that it is necessary to compute second derivatives of the temperature to compute a solution to the differential equation. In the following, we show that, expressed as a weak form, it is only necessary to approximate first derivatives of functions to obtain a solution. Thus, the solution process is simplified by considering weak (variational) forms. The partial differential equation together with the boundary and initial conditions is called the *strong form* of the problem.

2.4 Heat conduction problem: Weak form

In step 1, we multiply Eq. 2.4 by an arbitrary function $W(x_i)$, which transforms the set of differential equations onto a scalar function. The equation is first written on one side of an equal sign. Thus

$$g(W, q_i, T) \; = \; W(x_i) \left(\rho c \dot{T} \; - \; Q \; + \; q_{i,i} \right) \; = \; 0 \tag{2.17}$$

In step 2 we integrate over the domain, Ω. Thus,

$$G(W, q_i, T) \; = \; \int_\Omega W(x_i) \left(\rho c \dot{T} \; - \; Q \; + \; q_{i,i} \right) d\Omega \; = \; 0 \tag{2.18}$$

In step 3 we integrate by parts the terms involving the spatial derivatives (i.e., the thermal flux vector in our case). Green's theorem is given by

$$\int_\Omega \phi_{,i} \, d\Omega \; = \; \int_\Gamma \phi \, n_i d \, \Gamma \tag{2.19}$$

Normally, ϕ is the product of two functions. Thus for

$$\phi \; = \; V \, U \tag{2.20}$$

we have

$$\int_\Omega (U\,V)_{,i} d\Omega \;=\; \int_\Gamma (U\,V) n_i d\Gamma \tag{2.21}$$

The left hand side expands to give

$$\int_\Omega \left[U\,V_{,i} \;+\; U_{,i}\,V\right] d\Omega \;=\; \int_\Gamma (U\,V) n_i d\Gamma \tag{2.22}$$

which may be rearranged as

$$\int_\Omega U\,V_{,i} d\Omega \;=\; -\int_\Omega U_{,i}\,V d\Omega \;+\; \int_\Gamma (U\,V) n_i d\Gamma \tag{2.23}$$

which we observe is an integration by parts.

Applying the integration by parts to the heat equation gives

$$\begin{aligned} G(W, q_i, T) \;&=\; \int_\Omega W(x_i) \left(\rho c \dot{T} \;-\; Q\right) d\Omega \;-\; \int_\Omega W_{,i}\, q_i d\,\Omega \\ &+\; \int_\Gamma W q_i n_i d\Gamma \;=\; 0 \end{aligned} \tag{2.24}$$

Introducing q_n, the boundary term may be split into two parts and expressed as

$$\int_\Gamma W q_n d\Gamma \;=\; \int_{\Gamma_T} W q_n d\Gamma \;+\; \int_{\Gamma_q} W q_n d\Gamma \tag{2.25}$$

Now the boundary condition Eq. 2.10 may be used for the part on Γ_q and (without any loss in what we need to do) we can set W to zero on Γ_u (Note that W is arbitrary, hence our equation must be valid even if W is zero for some parts of the domain). Substituting all the above into Eq. 2.24 completes step 4 and we obtain the final expression

$$\begin{aligned} G(W, q_i, T) \;&=\; \int_\Omega W(x_i) \left(\rho c \dot{T} \;-\; Q\right) d\Omega \;-\; \int_\Omega W_{,i}\, q_i \, d\Omega \\ &+\; \int_{\Gamma_q} W \bar{q}_n d\Gamma \;=\; 0 \end{aligned} \tag{2.26}$$

If in addition to the use of the boundary condition we assume that the Fourier law is satisfied at each point in Ω the above integral becomes

$$\begin{aligned} G \;&=\; \int_\Omega W \left(\rho\, c\, \dot{T} \;-\; Q\right) d\Omega \;+\; \int_\Omega W_{,i}\, k\, T_{,i}\, d\Omega \\ &+\; \int_{\Gamma_q} W\, \bar{q}_n\, d\Gamma \;=\; 0 \end{aligned} \tag{2.27}$$

We note that the above form only involves first derivatives of quantities instead of the second derivatives in the original differential equation. This leads to weaker conditions to define solutions of the problem and thus the notion of a *weak form* is established. Furthermore, there are no additional equations that can be used to give any additional reductions; thus, Eq. 2.27 is sometimes said to be *irreducible* [4, Chapter 9].

Chapter 3

Introduction to Variational Theorems

3.1 Derivatives of functionals: The variation

The weak form of a differential equation is also called a *variational equation*. The notion of a variation is associated with the concept of a derivative of a functional (i.e., a function of functions). In order to construct a derivative of a functional, it is necessary to introduce a *scalar parameter* which may be used as the limiting parameter in the derivative [1]. This may be done by introducing a parameter η and defining a family of functions given by

$$T^\eta(\mathbf{x}) \;=\; T(\mathbf{x}) \;+\; \eta\,\tau(\mathbf{x}) \tag{3.1}$$

The function τ is an arbitrary function and is related to the arbitrary function W introduced in the construction of the weak form. The function $\eta\tau$ is called the variation of the function T and often written as δT ($\tau(\mathbf{x})$ alone also may be called the variation of the function) [1].

Introducing the family of functions T^η into the functional we obtain, using the steady state heat equation as an example, the result

$$\begin{aligned} G^\eta \;&=\; G(W,T^\eta) \;=\; \int_\Omega W_{,i}\,k\,T^\eta_{,i}\,d\Omega \;-\; \int_\Omega W\,Q\,d\Omega \\ &+\; \int_{\Gamma_q} W\,\bar{q}_n\,d\Gamma \end{aligned} \tag{3.2}$$

The derivative of the functional with respect to η now may be constructed using conventional methods of calculus. Thus,

$$\frac{dG}{d\eta} \;=\; \lim_{\eta\to 0}\frac{G^\eta \;-\; G^0}{\eta} \tag{3.3}$$

where G^0 is the value of G^η for η equal to 0. The construction of the derivative of the functional requires the computation of variations of derivatives of T. Using the above definition we obtain

$$\frac{d(T^\eta)_{,i}}{d\eta} \;=\; \frac{d}{d\eta}\left(T_{,i} \;+\; \eta\tau_{,i}\right) \;=\; \tau_{,i} \tag{3.4}$$

With this result in hand, the derivative of the functional with respect to η is given by

$$\frac{dG}{d\eta} = \int_\Omega W_{,i}\, k\, \tau_{,i}\, d\Omega \tag{3.5}$$

The limit of the *derivative* as η goes to zero is called the variation of the functional. For the linear steady state heat equation the derivative with respect to η is constant, hence the derivative is a variation of G. We shall define the derivative of the functional representing the weak form of a differential equation as

$$\frac{dG}{d\eta} = A(W,\tau) \tag{3.6}$$

This is a notation commonly used to define inner products.

3.2 Symmetry of inner products

Symmetry of inner product relations is fundamental to the derivation of variational theorems. To investigate symmetry of a functional we consider only terms which include both the dependent variable and the arbitrary function. An inner product is symmetric if

$$A(W,\tau) = A(\tau,W) \tag{3.7}$$

Symmetry of the inner product resulting from the variation of a weak form is a sufficient condition for the existence of a variational theorem which may also be used to generate a weak form. Symmetry of the functional A also implies that the tangent matrix (computed from the second variation of the theorem or the first variation of the weak form) of a Galerkin method will be symmetric.

A variational theorem, given by a functional $I(T)$, has a first variation which is identical to the weak form. Thus, given a functional $I(T)$ we can construct $G(W,T)$ as

$$\lim_{\eta\to 0} \frac{dI(T^\eta)}{d\eta} = G(\tau,T) \tag{3.8}$$

Note that use of Eq. 3.1 leads to a result where τ replaces W in the weak form. Thus, for the variational equation to be equivalent to the weak form τ must be an arbitrary function with the same restrictions as we established in defining W. Variational theorems are quite common for several problem classes; however, often we may only have a functional G and desire to know if a variational theorem exists. In practice we seldom need to have the variational theorem, but knowledge that it exists is helpful since it implies properties of the discrete problem which are beneficial (e.g., symmetry of the tangent matrices, minimum or stationary value, etc.). Also, existence of a variational theorem yields a weak form directly by using Eq. 3.8.

3.3 Vainberg's method: Construction of variational theorems

The construction of a variational theorem from a weak form is performed as follows [2]:

1. Check symmetry of the functional $A(W,\tau)$. If symmetric then to 2; otherwise, stop: no variational theorem exists.

2. Perform the following substitutions in $G(W,T)$

$$W(\mathbf{x}) \;\rightarrow\; T(\mathbf{x},t) \tag{3.9}$$

$$T(\mathbf{x},t) \;\rightarrow\; \eta T(\mathbf{x},t) \tag{3.10}$$

   to define $G(T,\eta T)$

3. Integrate the functional result from (b) with respect to η over the interval 0 to 1.

The result of the above process gives

$$I(T) \;=\; \int_0^1 G(T,\eta T)d\eta \tag{3.11}$$

Performing the variation of I and setting to zero gives

$$\lim_{\eta\rightarrow 0}\frac{dI(T^\eta)}{d\eta} \;=\; G(\tau,T) \;=\; 0 \tag{3.12}$$

and a problem commonly referred to as a *variational theorem*. A variational theorem is a functional whose first variation, when set to zero, yields the governing differential equations and boundary conditions associated with some problem.

For the steady state heat equation we have

$$G(T,\eta T) \;=\; \int_\Omega T_{,i}\,k\,\eta\,T_{,i}\,d\Omega \;-\; \int_\Omega T\,Q\,d\Omega \;+\; \int_{\Gamma_q} T\,\bar{q}_n\,d\Gamma \tag{3.13}$$

The integral is trivial and gives

$$I(T) \;=\; \frac{1}{2}\int_\Omega T_{,i}kT_{,i}d\Omega \;-\; \int_\Omega TQd\Omega \;+\; \int_{\Gamma_q} T\bar{q}_n d\Gamma \tag{3.14}$$

Reversing the process, the first variation of the variational theorem generates a variational equation which is the weak form of the partial differential equation. The first variation is defined by replacing T by

$$T^\eta \;=\; T \;+\; \eta\tau \tag{3.15}$$

and performing the derivative defined by Eq. 3.12. The second variation of the theorem generates the inner product

$$A(\tau, \tau) \tag{3.16}$$

If the second variation is strictly positive (i.e., A is positive for all τ), the variational theorem is called a *minimum principle* and the discrete tangent matrix is positive definite. If the second variation can have either positive or negative values the variational theorem is a stationary principle and the discrete tangent matrix is indefinite.

The transient heat equation with weak form given by

$$\begin{aligned} G &= \int_\Omega W \left(\rho\, c\, \dot{T} - Q \right) d\Omega + \int_\Omega W_{,i}\, k\, T_{,i}\, d\Omega \\ &+ \int_{\Gamma_q} W\, \bar{q}_n\, d\Gamma = 0 \end{aligned} \tag{3.17}$$

does not lead to a variational theorem due to the lack of the symmetry condition for the transient term

$$A = \left(\dot{T}, \eta\tau \right) \neq (\eta\dot{\tau}, T) \tag{3.18}$$

If however, we first discretize the transient term using some time integration method, we can often restore symmetry to the functional and then deduce a variational theorem for the discrete problem. For example if at each time t_n we have

$$T(t_n) \approx T_n \tag{3.19}$$

then we can approximate the time derivative by the finite difference

$$\dot{T}(t_n) \approx \frac{T_{n+1} - T_n}{t_{n+1} - t_n} \tag{3.20}$$

Letting $t_{n+1} - t_n = \Delta t$ and omitting the subscripts for quantities evaluated at t_{n+1}, the rate term which includes both T and τ becomes

$$A = \left(\frac{T}{\Delta t}, \eta\tau \right) = \left(\eta \frac{\tau}{\Delta t}, T \right) \tag{3.21}$$

since scalars can be moved from either term without affecting the value of the term. That is,

$$A = (T, \eta\,\tau) = (\eta\, T, \tau) \tag{3.22}$$

3.4 Variational notation

A formalism for constructing a variation of a functional may be identified and is similar to constructing the differential of a function. The differential of a function $f(x_i)$ may be written as

$$df = \frac{\partial f}{\partial x_i} dx_i \tag{3.23}$$

where x_i are the set of independent variables. Similarly, we may formally write a first variation as

$$\delta I = \frac{\partial I}{\partial u}\delta u + \frac{\partial I}{\partial u_{,i}}\delta u_{,i} + \cdots \tag{3.24}$$

where u, $u_{,i}$ are the dependent variables of the functional, δu is the variation of the variable (i.e., it is formally the $\eta\tau(x)$), and δI is called the first variation of the functional. This construction is a formal process as the indicated partial derivatives have no direct definition (indeed the result of the derivative is obtained from Eq. 3.3). However, applying this construction can be formally performed using usual constructions for a derivative of a function. For the functional Eq. 3.14, we obtain the result

$$\begin{aligned} \delta I &= \frac{1}{2}\int_\Omega \frac{\partial}{\partial T_{,i}}(T_{,i}\, k\, T_{,i})\, \delta T_{,i}\, d\Omega - \int_\Omega \frac{\partial}{\partial T}(T\, Q)\, \delta T\, d\Omega \\ &+ \int_{\Gamma_q} \frac{\partial}{\partial T}(T\, \bar{q}_n)\, \delta T\, d\Gamma \end{aligned} \tag{3.25}$$

Performing the derivatives leads to

$$\delta I = \frac{1}{2}\int_\Omega (k\, T_{,i} + T_{,i}\, k)\, \delta T_{,i}\, d\Omega - \int_\Omega Q\, \delta T\, d\Omega + \int_{\Gamma_q} \bar{q}_n\, \delta T\, d\Gamma \tag{3.26}$$

Collecting terms we have

$$\delta I = \int_\Omega \delta T_{,i}\, k\, T_{,i}\, d\Omega - \int_\Omega Q\, \delta T\, d\Omega + \int_{\Gamma_q} \bar{q}_n\, \delta T\, d\Gamma \tag{3.27}$$

which is identical to Eq. 3.2 with δT replacing W, etc.

This formal construction is easy to apply but masks the meaning of a variation. We may also use the above process to perform a linearization of variational equations in order to construct solution processes based on Newton's method.

Chapter 4

Small Deformation: Linear Elasticity

A summary of the governing equations for linear elasticity is given below. The equations are presented using *direct* notation. For a presentation using *indicial* notation see [4, Chapter 6]. The presentation below assumes small (infinitesimal) deformations and general three dimensional behavior in a Cartesian coordinate system, $\mathbf{x}$, where the domain of analysis is Ω with boundary Γ. The dependent variables are given in terms of the displacement vector, $\mathbf{u}$, the stress tensor, $\boldsymbol{\sigma}$, and the strain tensor, $\boldsymbol{\epsilon}$. The basic governing equations are:

1. Balance of linear momentum expressed as

$$\nabla \cdot \boldsymbol{\sigma} \; + \; \rho \, \mathbf{b}_m \; = \; \rho \, \ddot{\mathbf{u}} \tag{4.1}$$

   where ρ is the mass density, $\mathbf{b}_m$ is the body force per unit mass, ∇ is the gradient operator, and $\ddot{\mathbf{u}}$ is the acceleration.

2. Balance of angular momentum, which leads to symmetry of the stress tensor

$$\boldsymbol{\sigma} \; = \; \boldsymbol{\sigma}^T \tag{4.2}$$

3. Deformation measures based upon the gradient of the displacement vector, $\nabla \mathbf{u}$, which may be split as follows

$$\nabla \mathbf{u} \; = \; \nabla^{(s)} \mathbf{u} \; + \; \nabla^{(a)} \mathbf{u} \tag{4.3}$$

   where the symmetric part is

$$\nabla^{(s)} \mathbf{u} \; = \; \frac{1}{2} \left[\nabla \mathbf{u} \; + \; (\nabla \mathbf{u})^T \right] \tag{4.4}$$

   and the skew symmetric part is

$$\nabla^{(a)} \mathbf{u} \; = \; \frac{1}{2} \left[\nabla \mathbf{u} \; - \; (\nabla \mathbf{u})^T \right] \tag{4.5}$$

   Based upon this split, the symmetric part defines the strain

$$\boldsymbol{\epsilon} \; = \; \nabla^{(s)} \mathbf{u} \tag{4.6}$$

and the skew symmetric part defines the spin, or small rotation,

$$\boldsymbol{\omega} \;=\; \nabla^{(a)}\mathbf{u} \tag{4.7}$$

In a three dimensional setting the above tensors have 9 components. However, if the tensor is symmetric only 6 are independent and if the tensor is skew symmetric only 3 are independent. The component ordering for each of the tensors is given by

$$\boldsymbol{\sigma} \;\rightarrow\; \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \tag{4.8}$$

which from the balance of angular momentum must be symmetric, hence

$$\sigma_{ij} \;=\; \sigma_{ji} \tag{4.9}$$

The gradient of the displacement has the components ordered as (with no symmetries)

$$\nabla\mathbf{u} \;\rightarrow\; \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} \\ u_{2,1} & u_{2,2} & u_{2,3} \\ u_{3,1} & u_{3,2} & u_{3,3} \end{bmatrix} \tag{4.10}$$

The strain tensor is the symmetric part with components

$$\boldsymbol{\epsilon} \;\rightarrow\; \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix} \tag{4.11}$$

and the symmetry condition

$$\epsilon_{ij} \;=\; \epsilon_{ji} \tag{4.12}$$

The spin tensor is skew symmetric,thus,

$$\omega_{ij} \;=\; \omega_{ji} \tag{4.13}$$

which implies $\omega_{11} \;=\; \omega_{22} \;=\; \omega_{33} \;=\; 0$. Accordingly,

$$\boldsymbol{\omega} \;\rightarrow\; \begin{bmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & \omega_{23} \\ \omega_{31} & \omega_{32} & \omega_{33} \end{bmatrix} \;=\; \begin{bmatrix} 0 & \omega_{12} & \omega_{13} \\ -\omega_{12} & 0 & \omega_{23} \\ -\omega_{13} & -\omega_{23} & 0 \end{bmatrix} \tag{4.14}$$

The basic equations which are independent of material constitution are completed by specifying the boundary conditions. For this purpose the boundary, Γ, is split into two parts:

- Specified displacements on the part Γ_u, given as:

$$\mathbf{u} \;=\; \bar{\mathbf{u}} \tag{4.15}$$

where $\bar{\mathbf{u}}$ is a specified quantity; and

- specified tractions on the part Γ_t, given as:

$$\mathbf{t} \;=\; \boldsymbol{\sigma}^{\mathbf{n}} \;=\; \bar{\mathbf{t}} \tag{4.16}$$

  where $\bar{\mathbf{t}}$ is a specified quantity.

In the balance of momentum, the body force was specified per unit of mass. This may be converted to a body force per unit volume (i.e., unit weight/volume) using

$$\rho\, \mathbf{b}_m \;=\; \mathbf{b}_v \tag{4.17}$$

Static or quasi-static problems are considered by omitting the acceleration term from the momentum equation (Eq. 4.1). Inclusion of inertial forces requires the specification of the initial conditions

$$\mathbf{u}(\mathbf{x}, 0) \;=\; \mathbf{d}_0(\mathbf{x}) \tag{4.18}$$

$$\dot{\mathbf{u}}(\mathbf{x}, 0) \;=\; \mathbf{v}_0(\mathbf{x}) \tag{4.19}$$

where $\mathbf{d}_0$ is the initial displacement field, and $\mathbf{v}_0$ is the initial velocity field.

4.1 Constitutive Equations for Linear Elasticity

The linear theory is completed by specifying the constitutive behavior for the material. In small deformation analysis the strain is expressed as an *additive* sum of parts. We shall consider several alternatives for splits during the course; however, we begin by considering a linear elastic material with an additional known strain. Accordingly,

$$\boldsymbol{\epsilon} \;=\; \boldsymbol{\epsilon}^m \;+\; \boldsymbol{\epsilon}^0 \tag{4.20}$$

where $\boldsymbol{\epsilon}^m$ is the strain caused by stresses and is called the *mechanical* part, $\boldsymbol{\epsilon}^0$ is a second part which we assume is a specified strain. For example, $\boldsymbol{\epsilon}^0$ as a thermal strain is given by

$$\boldsymbol{\epsilon}^0 \;=\; \boldsymbol{\epsilon}^{th} \;=\; \boldsymbol{\alpha}(T \;-\; T_0) \tag{4.21}$$

.LP where T is temperature and T_0 is a stress free temperature. The constitutive equations relating stress to mechanical strain may be written (in matrix notation, which is also called Voigt notation) as

$$\boldsymbol{\sigma} \;=\; \mathbf{D}\boldsymbol{\epsilon}^m \;=\; \mathbf{D}(\boldsymbol{\epsilon} \;-\; \boldsymbol{\epsilon}^0) \tag{4.22}$$

where the matrix of stresses is ordered as the vector

$$\boldsymbol{\sigma} \;=\; \begin{bmatrix}\sigma_{11} & \sigma_{22} & \sigma_{33} & \sigma_{12} & \sigma_{23} & \sigma_{31}\end{bmatrix}^T \tag{4.23}$$

the matrix of strains is ordered as the vector (note factors of 2 are used to make shearing components the engineering strains, γ_{ij})

$$\boldsymbol{\epsilon} \;=\; \begin{bmatrix}\epsilon_{11} & \epsilon_{22} & \epsilon_{33} & 2\,\epsilon_{12} & 2\,\epsilon_{23} & 2\,\epsilon_{31}\end{bmatrix}^T \tag{4.24}$$

Tensor Index	Matrix Index					
	1	2	3	4	5	6
ab	11	22	33	12	23	31
				21	32	13

Table 4.1: Transformation of indices from tensor to matrix form

and $\mathbf{D}$ is the matrix of elastic constants given by

$$\mathbf{D} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} \\ D_{21} & D_{22} & D_{23} & D_{24} & D_{25} & D_{26} \\ D_{31} & D_{32} & D_{33} & D_{34} & D_{35} & D_{36} \\ D_{41} & D_{42} & D_{43} & D_{44} & D_{45} & D_{46} \\ D_{51} & D_{52} & D_{53} & D_{54} & D_{55} & D_{56} \\ D_{61} & D_{62} & D_{63} & D_{64} & D_{65} & D_{66} \end{bmatrix} \tag{4.25}$$

Assuming the existence of a *strain energy density*, $W(\boldsymbol{\epsilon}^m)$, from which stresses are computed as

$$\sigma_{ab} = \frac{\partial W}{\partial \epsilon^m_{ab}} \tag{4.26}$$

the elastic modulus matrix is symmetric and satisfies

$$D_{ij} = D_{ji} \tag{4.27}$$

Using tensor quantities, the constitutive equation for linear elasticity is written in indicial notation as:

$$\sigma_{ab} = C_{abcd}(\epsilon_{cd} - \epsilon^0_{cd}) \tag{4.28}$$

The transformation from the tensor to the matrix (Voigt) form is accomplished by the index transformations shown in Table 4.1

Thus, using this table, we have

$$C_{1111} \rightarrow D_{11} \;\; ; \;\; C_{1233} \rightarrow D_{43} \;\; ; \;\; \text{etc.} \tag{4.29}$$

The above set of equations defines the governing equations for use in solving linear elastic boundary value problems in which the inertial forces may be ignored. We next discuss some variational theorems which include the elasticity equations in a form amenable for development of approximate solutions.

The inclusion of inertial forces precludes the development of variational theorems in a simple form as noted in the previous chapter. We can add the inertial effects to the variational equations which result from any variational theorem using d'Alemberts principle.

Chapter 5

Variational Theorems: Linear Elasticity

5.1 Hu-Washizu Variational Theorem

Instead of constructing the weak form of the equations and then deducing the existence of a variational theorem, as done for the thermal problem, a variational theorem which includes *all* the equations for the linear theory of elasticity (without inertial forces) will be stated. The variational theorem is a result of the work of the Chinese scholar, Hu, and the Japanese scholar, K. Washizu [3], and, thus, is known as the *Hu-Washizu variational theorem*. The theorem may be written as

$$\begin{aligned} I(\mathbf{u},\boldsymbol{\sigma},\boldsymbol{\epsilon}) \;=\; & \frac{1}{2}\int_\Omega \boldsymbol{\epsilon}^T\,\mathbf{D}\,\boldsymbol{\epsilon}\,d\Omega \;-\; \int_\Omega \boldsymbol{\epsilon}^T\,\mathbf{D}\,\boldsymbol{\epsilon}^0\,d\Omega \\ & + \int_\Omega \boldsymbol{\sigma}^T(\nabla^{(s)}\mathbf{u} \;-\; \boldsymbol{\epsilon})\,d\Omega \;-\; \int_\Omega \mathbf{u}^T\,\mathbf{b}_v\,{}^d\Omega \\ & - \int_{\Gamma_t} Bu^T\,\bar{\mathbf{t}}\,d\Gamma \;-\; \int_{\Gamma_u} \mathbf{t}^T(\mathbf{u} \;-\; \bar{\mathbf{u}})d\Gamma \;=\; \textit{Stationary} \end{aligned} \tag{5.1}$$

Note that the integral defining the variational theorem is a *scalar*; hence, a transpose may be introduced into each term without changing the meaning. For example,

$$I \;=\; \int_\Omega \mathbf{a}^T\,\mathbf{b}\,d\Omega \;=\; \int_\Omega (\mathbf{a}^T\,\mathbf{b})^T\,d\Omega \;=\; \int_\Omega \mathbf{b}^T\,\mathbf{a}\,d\Omega \tag{5.2}$$

A variational theorem is stationary when the arguments (e.g., $\mathbf{u}$, $\boldsymbol{\sigma}$, $\boldsymbol{\epsilon}$) satisfy the conditions where the first variation vanishes. To construct the first variation, we proceed as in the previous chapter. Accordingly, we introduce the variations to the displacement, $\mathbf{U}$, the stress, $\mathbf{S}$, and the strain, $\mathbf{E}$, as

$$\mathbf{u}^\eta \;=\; \mathbf{u} \;+\; \eta\,\mathbf{U} \tag{5.3}$$

$$\boldsymbol{\sigma}^\eta \;=\; \boldsymbol{\sigma} \;+\; \eta\,\mathbf{S} \tag{5.4}$$

$$\boldsymbol{\epsilon}^{\eta} \;=\; \boldsymbol{\epsilon} \;+\; \eta\,\mathbf{E} \tag{5.5}$$

and define the single parameter functional

$$I^{\eta} \;=\; I(\mathbf{u}^{\eta}, \boldsymbol{\sigma}^{\eta}, \boldsymbol{\epsilon}^{\eta}) \tag{5.6}$$

The first variation is then defined as the derivative of I^{η} with respect to η and evaluated at $\eta \;=\; 0$. For the Hu-Washizu theorem the first variation defining the stationary condition is given by

$$\begin{aligned}
\left.\frac{dI^{\eta}}{d\eta}\right|_{\eta=0} \;&=\; \int_{\Omega} \mathbf{E}^T\mathbf{D}\boldsymbol{\epsilon} d\Omega \;-\; \int_{\Omega} \mathbf{E}^T\mathbf{D}\boldsymbol{\epsilon}^0 d\Omega \\
&+\; \int_{\Omega} \mathbf{S}^T(\nabla^{(s)}\mathbf{u} \;-\; \boldsymbol{\epsilon}) d\Omega \;+\; \int_{\Omega} \boldsymbol{\sigma}^T(\nabla^{(s)}\mathbf{U} \;-\; \mathbf{E}) d\Omega \\
&-\; \int_{\Omega} \mathbf{U}^T\mathbf{b}_v d\Omega \;-\; \int_{\Gamma_t} \mathbf{U}^T\bar{\mathbf{t}} d\Gamma \\
&-\; \int_{\Gamma_u} \mathbf{n}^T\mathbf{S}(\mathbf{u} \;-\; \bar{\mathbf{u}}) d\Gamma \;-\; \int_{\Gamma_u} \mathbf{t}^T\mathbf{U} d\Gamma \;=\; 0
\end{aligned} \tag{5.7}$$

The first variation may also be constructed using 3.23 for each of the variables. The result is

$$\begin{aligned}
\delta I \;&=\; \int_{\Omega} \delta\boldsymbol{\epsilon}^T\mathbf{D}\boldsymbol{\epsilon} d\Omega \;-\; \int_{\Omega} \delta\boldsymbol{\epsilon}^T\mathbf{D}\boldsymbol{\epsilon}^0 d\Omega \\
&+\; \int_{\Omega} \delta\boldsymbol{\sigma}^T(\nabla^{(s)}\mathbf{u} \;-\; \boldsymbol{\epsilon}) d\Omega \;+\; \int_{\Omega} \boldsymbol{\sigma}^T(\nabla^{(s)}\delta\mathbf{u} \;-\; \delta\boldsymbol{\epsilon}) d\Omega \\
&-\; \int_{\Omega} \delta\mathbf{u}^T\mathbf{b}_v d\Omega \;-\; \int_{\Gamma_t} \delta\mathbf{u}^T\bar{\mathbf{t}} d\Gamma \\
&-\; \int_{\Gamma_u} \mathbf{n}^T\delta\boldsymbol{\sigma}(\mathbf{u} \;-\; \bar{\mathbf{u}}) d\Gamma \;-\; \int_{\Gamma_u} \mathbf{t}^T\delta\mathbf{u} d\Gamma \;=\; 0
\end{aligned} \tag{5.8}$$

and the two forms lead to identical results.

In order to show that the theorem in form 5.7 is equivalent to the equations for linear elasticity, we need to group all the terms together which multiply each variation function (e.g., the $\mathbf{U}$, $\mathbf{S}$, $\mathbf{E}$). To accomplish the grouping it is necessary to integrate by parts the term involving $\nabla^{(s)}\mathbf{U}$. Accordingly,

$$\int_{\Omega} \boldsymbol{\sigma}^T\nabla^{(s)}\mathbf{U} d\Omega \;=\; -\int_{\Omega} \mathbf{U}^T\nabla\cdot\boldsymbol{\sigma} d\Omega \;+\; \int_{\Gamma_t} \mathbf{t}^T\mathbf{U} d\Gamma \;+\; \int_{\Gamma_u} \mathbf{t}^T\mathbf{U} d\Gamma \tag{5.9}$$

Grouping all the terms we obtain

$$\begin{aligned}
\left.\frac{dI^{\eta}}{d\eta}\right|_{\eta=0} \;&=\; \int_{\Omega} \mathbf{E}^T[\mathbf{D}(\boldsymbol{\epsilon} \;-\; \boldsymbol{\epsilon}^0) \;-\; \boldsymbol{\sigma}] d\Omega \\
&+\; \int_{\Omega} \mathbf{S}^T(\nabla^{(s)}\mathbf{u} \;-\; \boldsymbol{\epsilon}) d\Omega \;-\; \int_{\Omega} \mathbf{U}^T(\nabla\cdot\boldsymbol{\sigma} \;+\; \mathbf{b}_v) d\Omega \\
&+\; \int_{\Gamma_t} \mathbf{U}^T(\mathbf{t} \;-\; \bar{\mathbf{t}}) d\Gamma \;-\; \int_{\Gamma_u} \mathbf{n}^T\mathbf{S}(\mathbf{u} \;-\; \bar{\mathbf{u}}) d\Gamma \;=\; 0
\end{aligned} \tag{5.10}$$

The fundamental lemma of the calculus of variations states that each expression multiplying an arbitrary function in each integral type must vanish at each point in the domain of the integral. The lemma is easy to prove. Suppose that an expression does not vanish at a point, then, since the variation is arbitrary, we can assume that it is equal to the value of the non-vanishing expression. This results in the integral of the *square* of a function, which must then be positive, and hence the integral will not be zero. This leads to a contradiction, and thus the only possibility is that the assumption of a non-vanishing expression is false.

The expression which multiplies each variation function is called an *Euler equation* of the variational theorem. For the Hu-Washizu theorem, the variations multiply the constitutive equation, the strain-displacement equation, the balance of linear momentum, the traction boundary condition, and the displacement boundary condition. Indeed, the only equation not contained is the balance of angular momentum.

The Hu-Washizu variational principle serves as a convenient basis for constructing approximate solutions to problems in linear elasticity. There are other variational principles which can be deduced directly from the principle. Two of these, the *Hellinger-Reissner principle* and the *principle of minimum potential energy* are presented below since they are also often used in constructing approximate solutions in linear elasticity.

5.2 Hellinger-Reissner Variational Theorem

The Hellinger-Reissner principle eliminates the strain as a primary dependent variable; consequently, only the displacement, $\mathbf{u}$, and the stress, $\boldsymbol{\sigma}$, remain as arguments in the functional for which variations are constructed. The strains are eliminated by developing an expression in terms of the stresses. For linear elasticity this leads to

$$\boldsymbol{\epsilon} \;=\; \boldsymbol{\epsilon}^0 \;+\; \mathbf{D}^{-1}\boldsymbol{\sigma} \tag{5.11}$$

The need to develop an expression for strains in terms of stresses limits the application of the Hellinger-Reissner principle. For example, in finite deformation elasticity the development of a relation similar to 5.11 is not possible in general. On the other hand, the Hellinger-Reissner principle is an important limiting case when considering problems with constraints (e.g., linear elastic incompressible problems, thin plates as a limit case of the thick Mindlin-Reissner theory). Thus, we shall on occasion use the principle in our studies. Introducing 5.11 into the Hu-Washizu principle leads to the result

$$\begin{aligned} I(\mathbf{u},\boldsymbol{\sigma}) \;=\; & -\frac{1}{2}\int_\Omega \boldsymbol{\epsilon}^{0T}\mathbf{D}\boldsymbol{\epsilon}^0 d\Omega \;-\; \frac{1}{2}\int_\Omega \boldsymbol{\sigma}^T\mathbf{D}^{-1}\boldsymbol{\sigma} d\Omega \\ & -\; \int_\Omega \boldsymbol{\sigma}^T\boldsymbol{\epsilon}^0 d\Omega \;+\; \int_\Omega \boldsymbol{\sigma}^T\nabla^{(s)}\mathbf{u} d\Omega \;-\; \int_\Omega \mathbf{u}^T\mathbf{b}_v d\Omega \\ & -\; \int_{\Gamma_t} \mathbf{u}^T\bar{\mathbf{t}} d\Gamma \;-\; \int_{\Gamma_u} \mathbf{t}^T(\mathbf{u} \;-\; \bar{\mathbf{u}}) d\Gamma \end{aligned} \tag{5.12}$$

The Euler equations for this principle are

$$\nabla^{(s)}\mathbf{u} \;=\; \boldsymbol{\epsilon}^0 \;+\; \mathbf{D}^{-1}\boldsymbol{\sigma} \tag{5.13}$$

together with 4.1, 4.15 and 4.16. The strain-displacement equations are deduced by either directly stating 4.6 or comparing 5.11 to 5.13. The first term in 5.12 may be omitted since its first variation is zero.

5.3 Minimum Potential Energy Theorem

The principle of minimum potential energy eliminates both the stress, $\boldsymbol{\sigma}$, and the strain, $\boldsymbol{\epsilon}$, as arguments of the functional. In addition, the displacement boundary conditions are assumed to be imposed as a constraint on the principle. The MPE theorem may be deduced by assuming

$$\boldsymbol{\epsilon} \;=\; \nabla^{(s)}\mathbf{u} \tag{5.14}$$

and

$$\mathbf{u} \;=\; \bar{\mathbf{u}} \tag{5.15}$$

are satisfied at each point of Ω and Γ, respectively. Thus, the variational theorem is given by the integral functional

$$\begin{aligned} I(\mathbf{u}) \;&=\; \frac{1}{2}\int_{\Omega}(\nabla^{(s)}\mathbf{u})^T\mathbf{D}(\nabla^{(s)}\mathbf{u})d\Omega \;-\; \int_{\Omega}(\nabla^{(s)}\mathbf{u})^T\mathbf{D}\boldsymbol{\epsilon}^0 d\Omega \\ &-\; \int_{\Omega}\mathbf{u}^T\mathbf{b}_v d\Omega \;-\; \int_{\Gamma_t}\mathbf{u}^T\bar{\mathbf{t}}d\Gamma \end{aligned} \tag{5.16}$$

Since stress does not appear explicitly in the theorem, the constitutive equation must be given. Accordingly, in addition to 5.14 and 5.15 the relation

$$\boldsymbol{\sigma} \;=\; \mathbf{D}(\boldsymbol{\epsilon} \;-\; \boldsymbol{\epsilon}^0) \tag{5.17}$$

is given.

The principle of minimum potential energy is often used as the basis for developing approximate solutions in which only *displacement* approximations are needed.

Bibliography

[1] F.B. Hildebrand. *Methods of Applied Mathematics*. Prentice-Hall, Englewood Cliffs NJ, 2 edition, 1965 (also Dover Press, 1992).

[2] M.M. Vainberg. *Variational Methods for the Study of Nonlinear Operators*. Holden-Day Inc., San Francisco, CA, 1964.

[3] K. Washizu. *Variational Methods in Elasticity and Plasticity*. Pergamon Press, New York, 3 edition, 1982.

[4] O.C. Zienkiewicz and R.L. Taylor. *The Finite Element Method*, volume 1. McGraw-Hill, London, 4 edition, 1989.