Homework #6 - PCTL

ComS 412/512 - Due: Tuesday, April 23, 2024

We discussed in class an informal definition for Probabilistic CTL (PCTL). The formal semantics of PCTL is defined below. Let $\Pi_{(s_0,\ldots,s_n)}$ be the set of paths with prefix $(s_0, \ldots, s_n)$. Given path $\pi = (s_0, s_1, \ldots)$, let $\pi_i = s_i$. We define function $\mu(\Pi)$ as follows:

$$\mu(\Pi) = \sum_{\Pi_{(s_0,\ldots,s_n)} \in \Pi} (\mathbf{P}[s_0, s_1] \times \ldots \times \mathbf{P}[s_{n-1}, s_n])$$

Informally, $\mu(\Pi)$ is the probability of taking one of the paths in Π.

We define state formulas $\Phi \in \{\ a \mid \neg\Phi \mid \Phi_1 \wedge \Phi_2 \mid \mathtt{P}_{\bowtie\,\alpha}(\Psi)\ \}$ as follows (where $\bowtie \in \{<, \leq, =, \geq, >\}$):

- $s \models a$ iff $a \in \mathcal{L}(s)$
- $s \models \neg\Phi$ iff $s \not\models \Phi$
- $s \models \Phi_1 \wedge \Phi_2$ iff $s \models \Phi_1 \wedge s \models \Phi_2$
- $s \models \mathtt{P}_{\bowtie\,\alpha}(\Psi)$ iff $\mu(\{\ \pi \mid \pi \models \Psi\ \wedge \pi_0 = s\ \}) \bowtie \alpha$

We define path formulas $\Psi \in \{\neg\Psi \mid \Psi_1 \wedge \Psi_2 \mid \mathtt{X}\Phi \mid \mathtt{F}\Phi \mid \mathtt{G}\Phi \mid \Phi_1\ \mathtt{U}\ \Phi_2 \mid \Phi_1\ \mathtt{U}^{\leq t}\ \Phi_2\ \}$, with $\pi = \{s_0, s_1, \ldots\}$:

- $\pi \models \mathtt{X}\Phi$ iff $s_1 \models \Phi$
- $\pi \models \mathtt{F}\Phi$ iff $\exists i \geq 0, s_i \models \Phi$
- $\pi \models \mathtt{G}\Phi$ iff $\forall i \geq 0, s_i \models \Phi$
- $\pi \models \Phi_1\ \mathtt{U}\ \Phi_2$ iff $\exists j \geq 0, (s[j] \models \Phi_2) \wedge (\forall i < j, s[i] \models \Phi_1)$
- $\pi \models \Phi_1\ \mathtt{U}^{\leq t}\ \Phi_2$ iff $\exists j \in [0, t], (s[j] \models \Phi_2) \wedge (\forall i < j, s[i] \models \Phi_1)$

1. In section 16.2 of the lecture notes, it is shown that not all the PCTL formulas defined above are needed to fully express standard PCTL logic. **Provide a minimal set of PCTL path formulas and relational operators that can express all PCTL formulas, and formally prove any reductions on the set using PCTL semantics**. Note: the reduced set provided in Section 16.2 of the class notes may not be minimal! 20pt

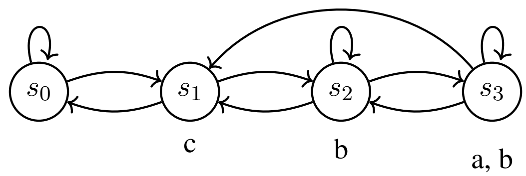
2. Let us expand our PCTL formula definitions to now include the two path formulas $\Phi_1\ \mathtt{U}^{[t_1,t_2]}\ \Phi_2$ and $\Phi_1\ \mathtt{U}^{[t,\infty)}\ \Phi_2$ defined by: 30pt

   - $\pi \models \Phi_1\ \mathtt{U}^{[t_1,t_2]}\ \Phi_2$ iff $\exists j \in [t_1, t_2], (s[j] \models \Phi_2) \wedge (\forall i < j, s[i] \models \Phi_1)$
   - $\pi \models \Phi_1\ \mathtt{U}^{[t,\infty)}\ \Phi_2$ iff $\exists j \geq t, (s[j] \models \Phi_2) \wedge (\forall i < j, s[i] \models \Phi_1)$

   Provide a minimal set of PCTL operators for this new logic, and formally prove any reductions on the set.

This page can be removed in your submissions

All homeworks should be typed using LaTeX. **Homeworks not typeset in LaTeX will receive 0 credit**. You may use this document as a starting point, and www.overleaf.com/learn/latex/Tutorials provides a number of good tutorials on basic Latex and TikZ Graphics tutorials. Additionally, an example TikZ diagram (recognize it?) is provided to help get you started - we recommend utilizing it!



Students in the class are allowed to discuss the homework problems in the public online forum for the course, or privately with me or with the TA, **but not with anybody else!** Unless stated otherwise, sharing of homework solutions (such as viewing or accessing someone else's files, hardcopy outputs, or handwritten notes), will be considered cheating. If you have even the slightest doubt about whether a certain activity is admissible, *ask before you do it!*