

Musical Mathematics

ON THE ART AND SCIENCE OF ACOUSTIC INSTRUMENTS

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CHAPTER 8: *SIMPLE FLUTES*

Part I: Equations for the Placement of Tone Holes on Concert Flutes and Simple Flutes

Flutes, harps, and drums are the oldest musical instruments created by man. Wind instruments are unique, however, because they alone embody the physical dimensions of scales and tunings. In a work entitled *The Greek Aulos*, Kathleen Schlesinger (1862–1953) attempted to reconstruct Greek music theory by analyzing the remains of ancient reed flutes.^[1] It is possible to approach the subject of flute tunings from two different perspectives. We may predict the tuning of an existing instrument by first measuring various flute bore, embouchure hole, and tone hole dimensions, and then substituting these data into a sequence of equations. This method provides convenient solutions when a flute is extremely fragile and cannot be played, or is simply not available for playing. On the other hand, we may realize a given tuning by making a flute according to another sequence of equations. From a mathematical perspective, these two approaches are distinctly different and require separate discussions.

As all experienced flute players know, the intonation of a transverse flute — with either a very simple or a very complex embouchure hole — depends not only on the precision of instrument construction, but also on the performer. The mathematics of flute tubes, embouchure holes, and tone holes does not necessarily produce an accurate sounding instrument. The intonation of a flute is also governed by the strength of the airstream, and by the amount the lips cover the embouchure hole. Because these two variables exist beyond the realm of mathematical predictability, they depend exclusively on the skill of the performer.

Due to the overall complexity of flutes, this chapter is divided into three parts. Part I investigates equations for the placement of tone holes, and Part II, mathematical procedures required to analyze existing flutes. Since Part II is unintelligible without a thorough understanding of Part I, the reader should study this chapter from beginning to end. Finally, Part III gives some suggestions on how to make very inexpensive yet highly accurate simple flutes.

Section 8.1

In writing this chapter, I am indebted to Cornelis J. Nederveen. In his book entitled *Acoustical Aspects of Woodwind Instruments*,^[2] Nederveen carefully defines all the mathematical variables needed for a thorough investigation into woodwind acoustics. Numerous tables of woodwind instrument dimensions are included at the end of the book. Because a full description of flute acoustics requires many different variables, this discussion begins with a list of symbols originally defined by Nederveen. Since most of these symbols appear only in this chapter, they are not included in the List of Symbols at the beginning of this book. Furthermore, the List of Flute Symbols below gives seven symbols that do not appear in Nederveen’s book: effective length $L_{B(h)}$ replaces λ_H ; effective length $L_{B(e)}$ replaces λ_E ; correction $\Delta \ell_E$

replaces $L_{B(e)}$ in the context of flute length calculations; $\Delta\ell_H$ and $\Delta\ell_T$, and length ℓ_T , represent simplifications. Finally, L_A replaces L_S when, in predicting the frequencies of an existing flute, we cannot determine L_S , which represents an *exact* acoustic half-wavelength; in this context, we must calculate L_A , which represents an *approximate* acoustic half-wavelength. In preparation for Chapter 8, the reader should read, study, and absorb Chapter 7. Knowledge of longitudinal pressure waves and end correction terminology is essential for an understanding of flute acoustics.

LIST OF FLUTE SYMBOLS

d_1	Bore diameter at tone hole.
d_0	Bore diameter at embouchure hole.
d_H	Tone hole diameter.
d_E	Embouchure hole diameter.
S_E	Surface area of embouchure hole.
S_0	Surface area of bore at embouchure hole.
ℓ_H	Geometric or measured length (or height) of tone hole; that is, (1) the shortest distance of a tone hole chimney on a concert flute with key pads, or (2) the wall thickness of a simple flute without key pads.
L_H	Acoustic or effective length of tone hole. (See Note 13.)
$L_{B(h)}$	Acoustic or effective length of bore at tone hole. (Nederveen: λ_H , p. 64.)
ℓ_E	Geometric or measured length of embouchure hole.
L_E	Acoustic or effective length of embouchure hole. (See Note 13.)
$L_{B(e)}$	Acoustic or effective length of bore at embouchure hole. (Nederveen: λ_E , p. 26.)
Δ	Correction coefficient.
$\Delta\ell_E$	Approximate correction at embouchure hole. In principle, the same as $L_{B(e)}$. Although $\Delta\ell_E$ does not have an exact mathematical value, $\Delta\ell_E$ is always greater than $L_{B(e)}$.
$\Delta\ell_H$	Correction at tone hole. (Nederveen: L_X , p. 13; and zL_S , p. 48.)
$\Delta\ell_T$	End correction at open tube end: $0.3d_1$. (Nederveen: ξa , p. 27.)
$\Delta\ell_K$	Correction at key pad. (Nederveen: $\Delta\ell_d$, p. 64.)
h	Geometric or measured distance of key pad in the open position above the center of a tone hole.

ℓ_T	Geometric or measured length of flute tube from the embouchure hole center to the open tube end.
ℓ_L	Geometric or measured length of the flute tube from the embouchure hole center to a tone hole center.
g	Interval ratio minus one between two adjacent tones on the flute. In 12-tone equal temperament, $g = \sqrt[12]{2} - 1$. In just intonation, $g = x/y - 1$, where x and y are positive integers whose quotients are greater than one and less than two.
z	Dimensionless tone hole parameter.
L_S	<i>Exact</i> acoustic half-wavelength of Nederveen's <i>substitution tube</i> , calculated according to a frequency variable: $L_S = c/(2F)$.
L_A	<i>Approximate</i> acoustic half-wavelength, calculated according to flute variables: $L_A = \ell_T + \Delta\ell_E + \Delta\ell_T$, or $L_A = \ell_L + \Delta\ell_E + \Delta\ell_H$. (Unfortunately, on p. 47, Nederveen also uses L_S in this context of approximate half-wavelengths.)
L_T	$L_T = \ell_T + \Delta\ell_E + \Delta\ell_T$, or $L_T = \ell_L + \Delta\ell_E + \Delta\ell_H$. (This apparent redundancy with respect to the two previous equations for L_A is explained in Section 8.10.)
L_L	Approximate length of the flute tube from a tone hole center to the correction at embouchure hole: $L_L = \ell_L + \Delta\ell_E$.
L_R	Tube-piece that extends past an open tone hole: $L_R = L_T - L_L$.

Recall that in Chapter 7, Section 12, we discussed the great convenience of centimeter measurements for cutting resonator lengths. A similar advantage also applies to flute construction. However, because many flute dimensions are much smaller than resonator dimensions, we will use millimeter measurements throughout this chapter. To convert inches to millimeters, multiply inches by 25.40; and to convert millimeters to inches, multiply millimeters by 0.039370.[\[3\]](#)

Finally, we must also define a dimensionally consistent speed of sound (c) *inside* a flute. Heat, water vapor, and carbon dioxide from the player's breath, as well as mechanical boundary layer effects that are frequency sensitive, all affect the value of c .[\[4\]](#) For simplicity, assume an average temperature of 25°C (or 77°F) inside the flute, and not accounting for changes due to frequency,[\[5\]](#) we will therefore use $c = 346,000$ mm/s in all future calculations.[\[6\]](#)

Section 8.2

We begin this discussion on the mathematics of flute construction by carefully distinguishing between a tube's *invisible* acoustic length and its *visible* geometric length. Figure 8.1 shows that for the fundamental mode of vibration, the acoustic length of a tube open at both ends spans the distance of a

half-wavelength. Similarly, Chapter 7, Figures 12(a) and 12(b), show that the acoustic length of a resonator closed at one end spans the distance of a quarter-wavelength. The latter three figures illustrate that a tube's *acoustic or effective length* (L_{eff}) is slightly longer than its *geometric or measured length* (L). In this context, the acoustic length is of paramount importance because it alone determines a tube's fundamental wavelength (λ_1). For a tube open at both ends, $\lambda_1 = 2L_{\text{eff}}$. (See Equations 7.22 and 7.24.) And for a tube closed at one end, $\lambda_1 = 4L_{\text{eff}}$. (See Equations 7.27 and 7.29.)

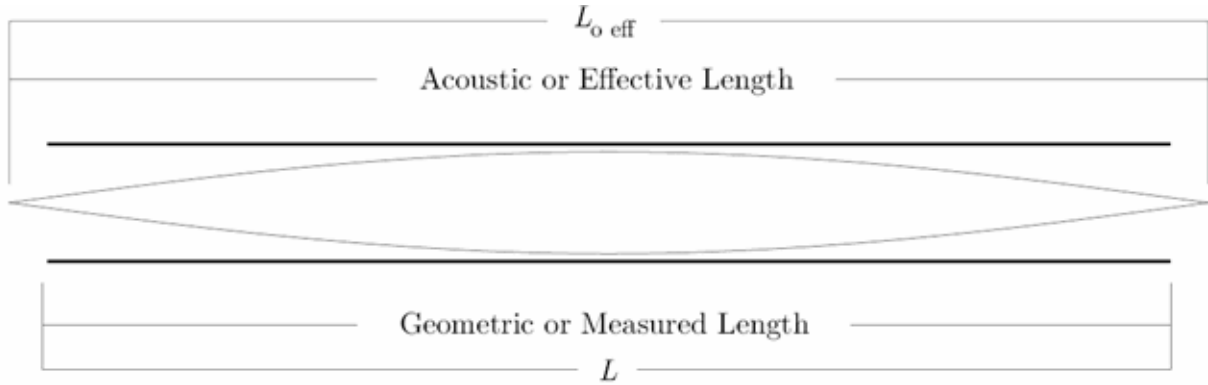


Figure 8.1 The acoustic or effective length, and the geometric or measured length of a tube open at both ends. The reflection of pressure waves occurs a small distance beyond the physical length of the tube. Consequently, the tube's acoustic length is longer than its geometric length. (Not to scale.)

Until now, we have not considered tubes closed at both ends because they serve no practical musical purpose. However, such tubes are fundamentally important because they provide a theoretical basis for flute length calculations. Figure 8.2 shows a tube closed at both ends, also called a closed-closed tube. For a closed-closed tube, the distinction between an acoustic/effective length and a geometric/measured length does *not* exist. Since there are no openings, pressure waves or displacement waves inside the tube do not reflect a small distance beyond the tube's physical length. Instead, all waves reflect precisely at the tube's closed ends. Consequently, end corrections are not a factor,^[7] which means that for a tube closed at both ends, $\lambda_1 = 2L_{\text{closed-closed}}$. The equation for the theoretical and actual fundamental frequency (F_1) of a closed-closed tube states

$$F_{1 \text{ closed-closed: theoretical and actual}} = \frac{c}{\lambda_1} = \frac{c}{2L} \quad (8.1)$$

where L is the tube's inner measured length.

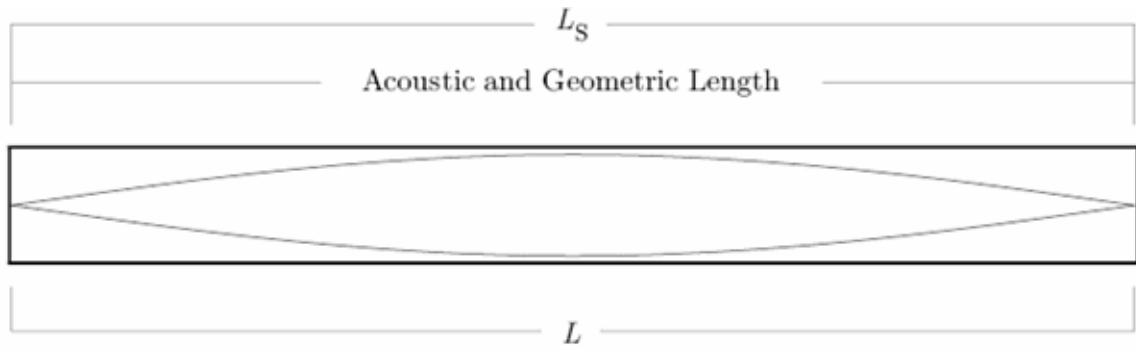


Figure 8.2 For a tube closed at both ends, the acoustic and geometric lengths are identical. Nederveen refers to this tube as a *substitution tube*, and notates its exact acoustic half-wavelength as L_S .

Nederveen refers to a closed-closed tube as a *substitution tube*, and notates its inner geometric length, L or its exact acoustic half-wavelength, as L_S .^[8] Since the acoustic and geometric lengths are identical, we now set $L_S = L$. Substitute L_S into Equation 8.1 and solve for the acoustic half-wavelength in the following manner:^[9]

$$F = \frac{c}{\lambda}$$

$$F = \frac{c}{2L_S} \quad (8.2A)$$

$$L_S = \frac{c}{2F} \quad (8.2B)$$

A transverse flute consists of a tube open at both ends. The embouchure hole provides the first opening, and the far end of the flute, or an open tone hole, provides the second opening. Turn to Figures 8.3(a), 8.3(b), and 8.3(c), and observe that each illustration consists of two sections. The top portion shows a substitution tube closed at both ends, and the bottom portion shows a flute tube open at both ends. Nederveen's substitution tube exemplifies a tube *without* any openings that generates exactly the same frequency as a flute tube *with* two openings.^[10] Stated another way, L_S represents the acoustic half-wavelength that a flute tube produces when it sounds a given frequency. Therefore, we now write the following condition for the purposes of flute length calculations:

$$L_S \equiv L_{o \text{ eff}}$$

This expression states that the acoustic/geometric length of a substitution tube, or of a tube closed at both ends, is by definition identical to the acoustic/effective length of a tube open at both ends.

Due to the reflection of waves a small distance beyond the flute openings, the geometric/measured length from the embouchure hole center to the open tube end (ℓ_T), and the geometric/measured length

from the embouchure hole center to a tone hole center (ℓ_L), are always shorter than L_S . Figure 8.3(a) shows that ℓ_T is shorter than L_S because of an added embouchure correction ($\Delta\ell_E$) at the *top or high end*, and an added end correction ($\Delta\ell_T$) at the *bottom or low end*. Similarly, Figures 8.3(b) and 8.3(c) show that ℓ_L is also shorter than L_S because of $\Delta\ell_E$ at the top end, and a tone hole correction ($\Delta\ell_H$) past the tone hole center at the bottom end.

are closed, evaluate Equation 8.2b and then replace L_S with $L_{S(t)}$; and for all higher frequencies when one or more tone holes are open, again evaluate Equation 8.2b and then replace L_S with $L_{S(h)}$. (2) Next, calculate the corrections $\Delta\ell_E$, $\Delta\ell_T$, and $\Delta\ell_H$. (3) Finally, determine ℓ_T by subtracting $\Delta\ell_E$ and $\Delta\ell_T$ from $L_{S(t)}$,

$$\ell_T = L_{S(t)} - \Delta\ell_E - \Delta\ell_T \quad (8.3)$$

and (4) determine ℓ_L by subtracting $\Delta\ell_E$ and $\Delta\ell_H$ from $L_{S(h)}$:

$$\ell_L = L_{S(h)} - \Delta\ell_E - \Delta\ell_H \quad (8.4)$$

In Section 8.3, a solution to Equation 8.3 for a concert flute with keys predicts ℓ_T within 3.5 mm (0.138 in.) of the measured length. And in Section 8.6, twelve solutions to Equation 8.4 for the same concert flute predict ℓ_L within 1 mm (0.039 in.) of the measured lengths.

Before we continue, it is important to first clear up a semantic problem. In Section 8.1, the List of Symbols defines four corrections: $\Delta\ell_E$, $\Delta\ell_H$, $\Delta\ell_T$, and $\Delta\ell_K$. The last two corrections have simple, straightforward solutions. However, as explained in Sections 8.3 and 8.5, the first two corrections are more complicated because they are derived from proportions. The List of Symbols carefully defines $\Delta\ell_E$ as a “correction at the embouchure hole.” It does *not* define $\Delta\ell_E$ as an “embouchure hole correction.” The latter definition suggests that $\Delta\ell_E$ is a sole function of the embouchure hole. This is not the case. Section 8.3 explains that $\Delta\ell_E$ has its origins in the following proportion:

$$\frac{S_E}{L_E} = \frac{S_0}{L_{B(e)}}$$

where S_0 is the *surface area of the bore* at the location of the embouchure hole, and $L_{B(e)}$ is the *effective length of the bore* at the location of the embouchure hole. At the end of Section 8.3, we will replace $L_{B(e)}$ with $\Delta\ell_E$. This example demonstrates why it is technically more accurate to call $\Delta\ell_E$ a “correction at the embouchure hole” than to give the impression that $\Delta\ell_E$ is an “embouchure hole correction.” However, phrases like “the correction at the embouchure hole” and “the correction at a tone hole” are awkward in the context of a difficult discussion. Therefore, we will continue to use the phrases “embouchure hole correction” and “tone hole correction” even though such expressions do not give exactly correct descriptions.

Section 8.3

We turn our attention now to solutions for $\Delta\ell_T$ and $\Delta\ell_E$. Chapter 7, Section 11, discussed at length the end correction phenomenon that occurs at the open ends of cylindrical resonators. Since the end

correction for a flute tube is the same as for a resonator tube, refer to Equation 7.20, set $\Delta\ell_T = \Delta\ell$, and use this equation for the end correction:

$$\Delta\ell_T = 0.3d_1 \quad (8.5)$$

Unfortunately, an exact equation for the embouchure correction does not exist. $\Delta\ell_E$ depends on the position of the player's lips over the embouchure hole. Increased lip coverage increases $\Delta\ell_E$.^[11] This causes an increase in L_S (see Equation 8.20), which in turn decreases the frequencies of the flute. However, Nederveen does give a solution for the minimum theoretical value^[12] for the effective length of the bore at the embouchure hole,^[13]

$$L_{B(e)} = (\ell_E + d_E) \left(\frac{d_0}{d_E} \right)^2 - 0.45d_0 \quad (8.6)$$

where ℓ_E is the measured length or height of the embouchure hole, in millimeters; d_E is the diameter of the embouchure hole, in millimeters; and d_0 is the diameter of the flute bore at the location of the embouchure hole, also in millimeters.

To understand the origins of Equation 8.6, we must first consider the acoustic admittance (Y_A) of a flute bore or flute hole. This quantity measures the ease with which air flows through a duct. Acoustic admittance is the reciprocal (or opposite) of acoustic impedance:^[14]

$$Y_A = \frac{1}{Z_A}$$

As the impedance of a duct increases, the admittance decreases; *vice versa*. A non-complex definition^[15] of acoustic impedance is

$$Z_A = \frac{\rho c}{S} = \frac{\rho c k}{\kappa} \quad (8.7)$$

Therefore, a non-complex definition of acoustic admittance is

$$Y_A = \frac{S}{\rho c} = \frac{\kappa}{\rho c k} \quad (8.8)$$

where S is the surface area of the duct, in square millimeters; ρ is the density of air, in kilograms per cubic millimeters; c is the speed of sound, in millimeters per second; k (Lat.) is the angular wave number,^[16] in radians per millimeter; and κ (Gk.) is the conductivity of the hole, in millimeters. Conductivity is the reciprocal (or opposite) of resistance. We define conductivity as the ratio of a duct's surface area divided by its effective length^[17]

$$\kappa = \frac{S}{L_{\text{eff}}} = \frac{S}{\ell + \Delta(d)} \quad (8.9)$$

where S is the surface area of a bore or hole, in square millimeters; ℓ is the measured length of a bore or hole, in millimeters; d is the diameter of a bore or hole, also in millimeters; and Δ is the correction coefficient of a bore or hole, [18] dimensionless. Equation 8.9 states that the airflow through a hole is directly proportional to surface area, and inversely proportional to effective length. A wide shallow hole conducts air more efficiently than a narrow deep hole. Note, however, when κ is too large or too small for a given bore, the flute tube is either too long or too short and will not work. Similarly, when κ is too small for a given tone hole, the hole is too narrow and will not work. Equations 8.7 and 8.8 show that Z_A and Y_A are functions of conductivity. For a given bore or hole, the ratio of S to $\ell + \Delta(d)$ must be correct, otherwise the terminating admittance (or terminating impedance) will not cause a reflection of pressure waves from the openings of a bore, nor from the opening of a hole. [19] Without the reflection of pressure waves, there can be no standing waves, and without standing waves, there is no sound.

Consider now the locations on a flute where two admittances (of conductivities κ) are simultaneously present. Such a place exists at the top end of the flute. Here the conductivity of the embouchure hole interacts with the conductivity of the flute bore at the location of the embouchure hole. The flute provides us with the dimensions of the diameter and the length of the embouchure hole, and the diameter of the bore. However, because the effective length of the bore at the location of the embouchure hole remains unknown, we may solve for $L_{B(e)}$ by considering the following proportion. Equation 8.10 states that the surface area of the embouchure hole (S_E) is to the effective length of the embouchure hole (L_E), as the surface area of the main bore at the embouchure hole (S_0) is to $L_{B(e)}$:

$$\left[\frac{S_E}{L_E} = \frac{S_0}{L_{B(e)}} \right] = \left[\frac{\pi \left(\frac{d_E}{2} \right)^2}{\ell_E + \Delta(d_E)} = \frac{\pi \left(\frac{d_0}{2} \right)^2}{L_{B(e)}} \right] \quad (8.10)$$

To solve for $L_{B(e)}$, cross-multiply, divide, cancel, and simplify:

$$\begin{aligned}
L_{B(e)} \times \pi \left(\frac{d_E}{2} \right)^2 &= \pi \left(\frac{d_0}{2} \right)^2 \times [\ell_E + \Delta(d_E)] \\
L_{B(e)} &= \frac{\pi \left(\frac{d_0}{2} \right)^2 [\ell_E + \Delta(d_E)]}{\pi \left(\frac{d_E}{2} \right)^2} \\
L_{B(e)} &= \frac{\pi \left(\frac{d_0}{2} \right)^2 [\ell_E + \Delta(d_E)]}{\pi \left(\frac{d_E}{2} \right)^2} \\
L_{B(e)} &= \left(\frac{d_0}{d_E} \right)^2 [\ell_E + \Delta(d_E)] \tag{8.11}
\end{aligned}$$

Equation 8.10 defines the conductivity of one duct in terms of another duct. The difference between these two ducts is that the left side of Equation 8.10 represents the actual duct of the embouchure hole, whereas the right side of Equation 8.10 represents a fictitious duct associated with the bore. This fictitious duct has the same diameter as the flute tube, and by definition, the same conductivity as the embouchure hole. Figures 8.3(a), 8.3(b), and 8.3(c) show that $L_{B(e)}$ (or $\Delta \ell_E$) of the fictitious duct has the effect of extending the bore beyond the top end of the flute. Equation 8.10 states that the actual and the fictitious ducts must have the same conductivity, otherwise two different admittances would result. If two different admittances existed, then the pressure waves would not reflect from the same location to the left of the bore. We conclude, therefore, that for a given frequency, the combined terminating admittances of the embouchure hole and the bore cause a reflection of pressure waves a moderate distance $L_{B(e)}$ beyond the physical location of the embouchure hole center. Similarly, recall that in Chapter 7, Section 11, the impedance of a resonator causes a reflection of pressure waves a small distance $\Delta \ell$ beyond the physical length of the tube.

According to Arthur H. Benade,[\[20\]](#) a good estimate for the coefficient of both the inside and outside corrections[\[21\]](#) of an embouchure hole or tone hole is $\Delta = 0.75$. To ensure that the results of Equations 8.6 and 8.11 are very close, we will use a slightly smaller correction coefficient, $\Delta = 0.71$. Consider now the following embouchure and bore dimensions of a typical concert metal flute:

$\ell_E = 4.3$ mm. Due to the presence of a lip plate, the length of the embouchure hole is much greater than the thickness of the flute tube; the wall of a typical concert flute is only 0.5 mm thick.

$d_E = 11.2$ mm. On most concert flutes the embouchure hole is in the shape of an oval and has a diameter of 12 mm; here 11.2 mm represents a mean value.[\[22\]](#)

$d_0 = 17.4$ mm. The embouchure end of a concert flute — also known as the head joint — has a slight taper that decreases the tube’s diameter from 19.0 mm at the joint to approximately 16.8 mm at the far upper end of the tube.

A substitution of these values into Equation 8.6 gives $L_{B(e)} = 29.58$ mm, and into Equation 8.11 gives $L_{B(e)} = 29.57$ mm.

Four years after the publication of his book, Nederveen concluded that empirical observations over a wide frequency range indicate $L_{B(e)} \approx 50$ mm.[\[23\]](#) Theobald Boehm[\[24\]](#) (1794–1881), the inventor of the modern concert flute, and Benade give the same approximation. Since the effective length of the bore at the embouchure hole acts as a correction, express the following condition for the purposes of flute length calculations:

$$\Delta \ell_E \equiv L_{B(e)}$$

and assign the value $\Delta \ell_E = 50$ mm. Now, to calculate ℓ_T for a concert flute tuned to middle C (or C_4 at 261.63 cps), first use Equation 8.2b to compute the acoustic length of the substitution tube:

$$L_{S(t)} = \frac{346,000 \text{ mm/s}}{2(261.63 \text{ cps})} = 661.2 \text{ mm}$$

Next, substitute a 19.0 mm diameter bore into Equation 8.5 to obtain the end correction:

$$\Delta \ell_T = 0.3(19.0 \text{ mm}) = 5.7 \text{ mm}$$

Finally, substitute the values for $L_{S(t)}$, $\Delta \ell_E$, and $\Delta \ell_T$ into Equation 8.3 to obtain the tube length:

$$\ell_T = 661.2 \text{ mm} - 50 \text{ mm} - 5.7 \text{ mm} = 605.5 \text{ mm}$$

Section 8.4

On the Armstrong Concert Flute (Serial Number 104-25-29992), $\ell_T = 602$ mm, which is 3.5 mm shorter than given by the previous calculation. To account for this shorter length, we must examine the profile of a flute bore in greater detail. While playing the lowest frequency, a flute bore does not have a perfectly smooth surface. Instead, a series of closed tone holes gives the bore an interrupted and irregular profile. Closed tone hole cavities have the effect of enlarging the volume of the bore and perturbing the flow of air through the bore.[\[25\]](#) This causes the flute to sound flat. Furthermore, the viscous and thermal losses associated with a thin boundary layer against the flute’s inner surface, and the shape of the tapered head joint at the top end of the flute, also effectively decrease the frequencies of a flute throughout the first “octave” of its three “octave” range. For a Gemeinhardt Concert Flute, [\[26\]](#) which is very similar to the Armstrong Concert Flute, Nederveen shows that these combined

corrections cause the flute to sound C_4 between 15–30 ¢ flat. To help offset these effects, flute makers intentionally shorten ℓ_T by a small amount. According to Equation 8.1, as ℓ_T or L decreases, λ_1 decreases, and so F_1 increases. Although a 3.5 mm shorter length for this flute sounds F_1 only 9.2 ¢ higher, the performer’s blowing technique at the embouchure hole corrects any remaining intonational discrepancies.

During the early stages of making a simple flute, mark the location of the embouchure hole 32 mm (1 1/4 in.) from the upper end of the tube; now cut the tube 10 mm longer than given by Equation 8.3. Drill the embouchure hole, and place a cork into the tube so that the cork’s inner surface sits 17 mm to the left of the embouchure hole center. (In concert flutes, 17.0 mm is the standard distance between the embouchure hole center and cork.) Test the frequency of the tube *without* tone holes. Gradually shorten the tube until it gives a frequency about 30 ¢ flat of the fundamental frequency. Stop for now. For the embouchure hole correction use $\Delta\ell_E = 52$ mm (see Section 8.7), and calculate the tone hole locations as specified in Section 8.5. Drill the tone holes with a small diameter drill at first: this causes all the holes to sound flat. Fine-tune the flute by gradually enlarging the tone hole diameters: this raises the frequencies. Intermittently test the lowest frequency of the flute with all tone holes closed. Finally, fine-tune the flute bore by gradually decreasing the length of the tube. As ℓ_T decreases, F_1 increases and comes into tune.

Section 8.5

Our next task is to calculate the tone hole correction. On first impression, Figure 8.3(b) seems to suggest that $\Delta\ell_E$ and $\Delta\ell_H$ represent equivalent quantities. This is not the case. Equation 8.17 and Equation 8.18 show that $\Delta\ell_H$ is a function of the effective length of the bore at the location of a tone hole ($L_{B(h)}$). Unlike $L_{B(e)}$, $L_{B(h)}$ has a predictable value because it does not depend on the technique of the flute player. Analogous to $L_{B(e)}$, $L_{B(h)}$ represents the length of a fictitious duct that has the same diameter as the main tube, and the same conductivity as the tone hole. However, $L_{B(h)}$ alone does not define $\Delta\ell_H$. With respect to $L_{B(e)}$, the flute tube ends near the embouchure hole. But with respect to $L_{B(h)}$, the flute tube does not end near the open tone hole. Instead, pressure waves continue to travel inside the bore beyond the physical location of the open tone hole. For this reason, we must also account for the terminating admittance of the short tube-piece that extends past the open hole. For the first tone hole in Figure 8.3(b), the effective length of this tube-piece spans the distance from the tone hole center to the dashed line. This line marks the end correction at the open end of the tube. Similarly, in the case of two adjacent open holes, the effective length of the tube-piece in Figure 8.3(c) spans the distance from the second tone hole center to the dashed line. Here the line marks the tone hole correction of the first tone hole. We conclude, therefore, that $\Delta\ell_H$ is a function of three admittances. For a given frequency, the combined terminating admittances of the tone hole, the flute bore, and the tube-piece cause the reflection of pressure waves a moderate distance $\Delta\ell_H$ beyond the physical location of the tone hole center.[\[27\]](#)

Before proceeding with detailed calculations, let us first distinguish between flutes with key pads, and those without key pads. A key pad has the effect of decreasing the frequency of a tone hole. Consequently, a tone hole with a key pad sits higher on the bore (or nearer the embouchure hole) than a tone hole without a key pad, which sits lower on the bore (or nearer the flute's open end). For a tone hole with a key pad, the higher position compensates for this flattening effect and, thereby, increases the frequency of the hole. Since most modern concert flutes come equipped with key pads, we continue our analysis of the previously mentioned concert flute.

For a flute with key pads, $\Delta\ell_H$ requires preliminary solutions to Equations 8.2B, 8.12, 8.13, 8.16, and 8.17; and for a flute without key pads, $\Delta\ell_H$ requires solutions to Equations 8.2B, 8.14, 8.16, and 8.17. Nederveen gives the following equation for the effective length of the bore at the location of a tone hole:[28]

$$L_{B(h)} = (\ell_H + d_H + \Delta\ell_K) \left(\frac{d_1}{d_H} \right)^2 - 0.45d_1 \quad (8.12)$$

where ℓ_H is the shortest distance of a tone hole chimney,[29] in millimeters; d_H is the diameter of the tone hole, in millimeters; $\Delta\ell_K$ is the key pad correction, in millimeters; and d_1 is the diameter of the bore, also in millimeters. With the exception of the $\Delta\ell_K$ variable, notice the similarities between Equations 8.6 and 8.12. Nederveen and Benade give[30]

$$\Delta\ell_K = 0.325d_H \left[\left(\frac{d_H}{2h} \right)^{0.39} - 1 \right] \quad (8.13)$$

where h is the key pad distance in an open position above the center of the tone hole, in millimeters. Remember, however, that for a simple flute without key pads, we must rewrite Equation 8.12 without the $\Delta\ell_K$ variable:

$$L_{B(h)} = (\ell_H + d_H) \left(\frac{d_1}{d_H} \right)^2 - 0.45d_1 \quad (8.14)$$

Equation 8.14 suggests that the proportion expressed by Equation 8.10 applies not only to embouchure holes, but to tone holes as well. Consequently, Equation 8.15 also gives good results:

$$L_{B(h)} = \left(\frac{d_1}{d_H} \right)^2 (\ell_H + 0.71d_H) + \Delta\ell_K \quad (8.15)$$

Consider now the following C#₄ tone hole dimensions, key pad and bore dimensions, and key pad correction of the same concert flute:

$$\begin{aligned}
\ell_H &= 2.0 \text{ mm} \\
d_H &= 15.3 \text{ mm} \\
h &= 3.0 \text{ mm} \\
d_1 &= 19.0 \text{ mm} \\
\Delta\ell_K &= 2.2 \text{ mm}
\end{aligned}$$

A substitution of these values into Equation 8.12 gives $L_{B(h)} = 21.5 \text{ mm}$, and into Equation 8.15 gives 22.0 mm.

To determine $\Delta\ell_H$, we must also consider a musical interval or the interval ratio between two adjacent tone holes on the flute. In this context, Nederveen gives this variable,[\[31\]](#)

$$g = x/y - 1 \quad (8.16)$$

where x/y is either a rational or irrational interval ratio. For a scale in 12-tone equal temperament,[\[32\]](#) where all the intervals between adjacent tones are constant, g equals the 12th root of 2 minus one:

$$g = \sqrt[12]{2} - 1 = 1.05946 - 1 = 0.05946$$

However, for a scale in just intonation,[\[33\]](#) g is not necessarily a constant. For example, if the first tone hole is a 9/8, or a “just major second,” then the interval ratio between the lowest frequency of the flute tube and the first hole is a 9/8,

$$\frac{9}{8} \div \frac{1}{1} = \frac{9}{8} \times \frac{1}{1} = \frac{9}{8}$$

and so,

$$g = \frac{9}{8} - 1 = 1.125 - 1 = 0.125$$

But if the second tone hole is a 5/4, or a “just major third,” then the interval ratio between the first hole and the second hole is a 10/9,

$$\frac{5}{4} \div \frac{9}{8} = \frac{5}{4} \times \frac{8^2}{9} = \frac{10}{9}$$

and so,

$$g = \frac{10}{9} - 1 = 1.111 - 1 = 0.111$$

Nederveen combines variables $L_{B(h)}$, g , and L_S into a dimensionless tone hole parameter:[\[34\]](#)

$$z = \frac{1}{2}g\sqrt{1 + 4\frac{L_{B(h)}}{gL_S}} - \frac{1}{2}g \quad (8.17)$$

According to Equation 8.2b, for the first C#₄ tone hole

$$L_{S(h)} = \frac{346,000 \text{ mm/s}}{2(277.18 \text{ cps})} = 624.1 \text{ mm}$$

so that,

$$z = \frac{0.05946}{2}\sqrt{1 + 4\frac{21.5 \text{ mm}}{0.05946(624.1 \text{ mm})}} - \frac{0.05946}{2} = 0.0244$$

Finally, Nederveen gives the tone hole correction[35]

$$\Delta\ell_H = zL_S \quad (8.18)$$

so that,

$$\Delta\ell_H \text{ for C}\sharp_4 = 0.0244(624.1 \text{ mm}) = 15.2 \text{ mm}$$

For the location of the C#₄ tone hole, substitute the values for $L_{S(h)}$, $\Delta\ell_E$, and $\Delta\ell_H$ into Equation 8.4 to obtain the tube length:

$$\ell_L = 624.1 \text{ mm} - 50 \text{ mm} - 15.2 \text{ mm} = 559 \text{ mm}$$

This corresponds to the exact location of the first hole on the concert flute.

Section 8.6

Table 8.1, Columns 3–5, list tone hole and key pad dimensions of the Armstrong Concert Flute, and Columns 6–10 list the results of five equations required to compute ℓ_L . Finally, Column 11 gives the calculated tone hole locations, and Column 12 gives the measured tone hole locations, rounded here to the nearest millimeter. With the exception of the C#₅ tone hole, there is very good agreement between calculated and measured locations. The C#₅ hole also serves as a vent or register hole for D₅ and D#₅. [36] Its location is designed to correct the lengthening (or flattening) effect of the tapered head joint. [37] For this reason, the C#₅ hole has a comparatively small diameter and sits 3.5 millimeters higher on the bore than predicted by theory.

Table 8.1
ARMSTRONG CONCERT FLUTE TONE HOLE
DIMENSIONS AND LOCATIONS

Scale	F	ℓ_H	d_H	h	$\Delta\ell_K$	$L_{B(h)}$	L_S	z	$\Delta\ell_H$	Calc. ℓ_L	Meas. ℓ_L
C \sharp_4	277.18	2.0	15.3	3.0	2.2	21.5	624.1	0.0244	15.2	559	559
D $_4$	293.66	2.0	15.3	3.0	2.2	21.5	589.1	0.0255	15.0	524	524
D \sharp_4	311.13	2.0	15.3	3.0	2.2	21.5	556.0	0.0267	14.8	491	491
E $_4$	329.63	2.0	13.8	2.0	2.8	26.7	524.8	0.0328	17.2	458	457
F $_4$	349.23	2.0	13.8	2.0	2.8	26.7	495.4	0.0342	16.9	428	427
F \sharp_4	369.99	2.0	13.8	2.0	2.8	26.7	467.6	0.0357	16.7	401	400
G $_4$	392.00	2.0	13.8	2.0	2.8	26.7	441.3	0.0372	16.4	375	375
G \sharp_4	415.30	2.0	12.8	2.0	2.4	29.3	416.6	0.0415	17.3	349	350
A $_4$	440.00	2.0	12.8	2.0	2.4	29.3	393.2	0.0432	17.0	326	327
A \sharp_4	466.16	2.0	12.8	2.0	2.4	29.3	371.1	0.0450	16.7	304	304
B $_4$	493.88	2.0	12.7	2.0	2.3	29.6	350.3	0.0471	16.5	284	283
C $_5$	523.25	2.0	12.7	2.0	2.3	29.6	330.6	0.0491	16.2	264	264
C \sharp_5	554.37	2.0	7.3	2.0	0.6	58.7	312.1	0.0801	25.0	237	233
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)

Section 8.7

One of the most important relations to emerge from Table 8.1 is that ℓ_L is directly proportional to d_H :

$$\ell_L \propto d_H$$

As the diameter of a tone hole decreases, the distance from the embouchure hole center to a tone hole center decreases as well; *vice versa*. For a given frequency, a small diameter hole sits higher on the bore than a large diameter hole. For example, if we decrease the diameter of the C \sharp_4 hole in Table 8.1 to 14.3 mm, then $\ell_L = 558$ mm, which means that this smaller hole would have a position 1 mm higher on the bore than the existing hole. This is an important consideration for makers of simple keyless flutes. The usefulness of a simple flute depends in large part on the spaces between tone holes, because if the tone holes are too difficult to reach, the instrument is unplayable.

Most simple flutes consist of an embouchure hole and six tone holes. One plays the fundamental by closing all the holes, then six more tones by opening each hole, and finally the eighth tone, or the “octave,” by closing all the holes again and overblowing at the embouchure hole.[38] Consider such a flute tuned to G $_4$ at 392.0 cps. To play a *just intoned* diatonic scale on this instrument requires six tones holes that produce the following six frequency ratios and six interval ratios:[39]

	①	②	③	④	⑤	⑥	
G ₄	A ₄	B ₄	C ₅	D ₅	E ₅	F ₅ [#]	G ₅
1/1	9/8	5/4	4/3	3/2	5/3	15/8	2/1
	9/8	10/9	16/15	9/8	10/9	9/8	

Between 15/8 and 2/1, we do not include the last interval, ratio 16/15, because it occurs naturally while overblowing the flute.

Table 8.2(a) lists tone hole dimensions and solutions for such a flute tuned to 392.0 cps., where $d_1 = 19.0$ mm (3/4 in.), $\ell_H = 3.2$ mm (1/8 in.), and $\ell_T = 384$ mm (15.1 in.). The latter value results when we substitute $\Delta\ell_E = 52.0$ mm into Equation 8.3:

$$\ell_T = \frac{346,000 \text{ mm/s}}{2(392.0 \text{ cps})} - 52 \text{ mm} - 5.7 \text{ mm} = 384 \text{ mm}$$

The embouchure hole has the following dimensions: $d_0 = 19.0$ mm, $\ell_E = 3.2$ mm, and $d_E = 9.5$ mm (3/8 in.). Substitute these values into Equation 8.6 and calculate $L_{B(e)} = 42.3$ mm. Tests show that for such a cylindrical flute with a simple circular embouchure hole, and without a tapered head joint, [40] $\Delta\ell_E = 52.0$ mm. This actual correction is typically between 9–10 mm greater than the theoretical correction, and 2.0 mm greater than the empirical correction of a standard concert flute. Such conclusions are based not only on experimental observation, but also on personal preference. Equation 8.4 shows that $\Delta\ell_E$ plays a critical role in determining the locations of all tone holes. That is, as $\Delta\ell_E$ increases, ℓ_L decreases, which means that a higher value for $\Delta\ell_E$ places all the tone holes closer to the top end of the flute. Since ℓ_L is inversely proportional to F (see Section 8.8), as ℓ_L decreases, F increases. Note, however, that the intonation of a tone hole depends not only on location, but also on the strength of a player's airstream, and on the lip coverage of the embouchure hole. A weak airstream and a large lip coverage tend to flatten (or decrease), and a strong airstream and a small lip coverage tend to sharpen (or increase) all the tones of a flute. Consequently, when a series of tone holes sits 2.0 mm higher on the bore, the player is more relaxed because high tone hole locations require less air pressure and allow more lip coverage without the risk of sounding flat. On the other hand, if the player chooses to play loud, or play in the second or third registers, the flute requires greater air pressure and less lip coverage, two conditions that tend to sharpen all the tones. These potentially conflicting relations indicate that there are no hard rules governing $\Delta\ell_E$. The final value of the embouchure correction is (within limits) subject to the discretion of the flute maker.

We return now to Table 8.2(a) where Column 7 gives the frequencies of the tone holes in just intonation. To calculate these frequencies, multiply 392.0 cps by a given frequency ratio. For example, frequency of F₅[#] is $392.0 \text{ cps} \times 15/8 = 735.0 \text{ cps}$. (As before, use $c = 346,000$ mm/s to calculate L_S .)

For all holes $d_H = 9.5$ mm (3/8 in.). Notice that the space between holes ① and ② is 38 mm, and between holes ② and ③ is 17 mm, which constitutes an irregular spacing, but between holes ④ and ⑤ it is 28 mm, and between ⑤ and ⑥ it is 29 mm, which constitutes a regular spacing. The irregular spacing of the first three holes is due in large part to the semitone between B_4 and C_5 . If we decrease the diameters of holes ① and ③ to $d_H = 6.4$ mm (1/4 in.), then ℓ_L for ① = 299 mm, and for ③ = 248 mm, so that both holes would occupy higher locations on the bore. Now the spacing between holes ① and ② would be 24 mm, and between holes ② and ③ would be 27 mm. This is a much more comfortable design. The only drawback for such small diameter holes is that the surface areas are also smaller. According to Equation 8.7, as S decreases, Z_A increases, which means that smaller holes have a higher wave impedance and, therefore, radiate acoustic energy less efficiently than large holes.^[41] Consequently, for a given blowing pressure, small holes are not as loud as large holes. This discrepancy between 9.5 mm holes and 6.4 mm holes in the lower part of the bore is rather noticeable. However, Table 8.2(b) shows that if we decrease the diameters of holes ① and ③ from the original 9.5 mm to 7.2 mm (9/32 in.), then the space between holes ① and ② is 29 mm, and between holes ② and ③ is 24 mm. Although a 9/32 in. hole is only 1/32 in. larger than a 1/4 in. hole, the difference in amplitude is strikingly perceptible and very similar to the amplitude of a large 3/8 in. hole. Note that the new spacing between the lower three holes also represents a well-balanced design. In Plate 11, the flute on the left — or Flute 1 — has the dimensions and tuning of this flute described in Table 8.2(b). (See [Instruments and Music > Simple Flutes.](#))

Table 8.2(a)
SIMPLE FLUTE WITH CONSTANT
TONE HOLE DIAMETERS

Scale	Ratio	g	ℓ_H	d_H	$L_{B(h)}$	F	L_S	z	$\Delta\ell_H$	ℓ_L
A_4	$9/8$	0.125	3.2	9.5	42.3	441.0	392.3	0.0693	27.2	① = 313
B_4	$5/4$	0.111	3.2	9.5	42.3	490.0	353.1	0.0724	25.6	② = 275
C_5	$4/3$	0.067	3.2	9.5	42.3	522.7	331.0	0.0648	21.4	③ = 258
D_5	$3/2$	0.125	3.2	9.5	42.3	588.0	294.2	0.0853	25.1	④ = 217
E_5	$5/3$	0.111	3.2	9.5	42.3	653.3	264.8	0.0887	23.5	⑤ = 189
$F\sharp_5$	$15/8$	0.125	3.2	9.5	42.3	735.0	235.4	0.0998	23.5	⑥ = 160
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)

Table 8.2(b)
SIMPLE FLUTE WITH TWO VARIED
TONE HOLE DIAMETERS

Scale	Ratio	g	ℓ_H	d_H	$L_{B(h)}$	F	L_S	z	$\Delta\ell_H$	ℓ_L
A ₄	$\frac{9}{8}$	0.125	3.2	7.2	63.9	441.0	392.3	0.0933	36.6	① = 304
B ₄	$\frac{5}{4}$	0.111	3.2	9.5	42.3	490.0	353.1	0.0724	25.6	② = 275
C ₅	$\frac{4}{3}$	0.067	3.2	7.2	63.9	522.7	331.0	0.0849	28.1	③ = 251
D ₅	$\frac{3}{2}$	0.125	3.2	9.5	42.3	588.0	294.2	0.0853	25.1	④ = 217
E ₅	$\frac{5}{3}$	0.111	3.2	9.5	42.3	653.3	264.8	0.0887	23.5	⑤ = 189
F ₅ [#]	$\frac{15}{8}$	0.125	3.2	9.5	42.3	735.0	235.4	0.0998	23.5	⑥ = 160
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)

Section 8.8

In conclusion to Part I, let us continue the discussion begun in Section 8.7 and examine the interaction between flute variables in greater detail. A careful examination of the various flute equations indicates the following proportionalities with respect to the ℓ_L :

$$\ell_L \propto \frac{1}{F}$$

$$\ell_L \propto \frac{1}{d_1}$$

$$\ell_L \propto \frac{1}{\ell_H}$$

$$\ell_L \propto d_H$$

In words, ℓ_L is inversely proportional to F , d_1 , and ℓ_H . An increase in F , d_1 , or ℓ_H produces a decrease in ℓ_L , and so moves a hole to a higher location on the bore. However, an increase in d_H also increases ℓ_L and, therefore, moves a hole to a lower location on the bore. A similar analysis indicates the following proportionalities with respect to F :

$$F \propto \frac{1}{\ell_L}$$

$$F \propto \frac{1}{d_1}$$

$$F \propto \frac{1}{\ell_H}$$

$$F \propto d_H$$

Again in words, F is inversely proportional to ℓ_L , d_1 , and ℓ_H , so that increases in these variables produce decreases in F . However, an increase in d_H also increases F , which means that after a flute maker has drilled all the tone holes, subtle increases in hole diameter constitute the most important technique for fine-tuning a flute.[\[42\]](#)

[\[1\]](#)Schlesinger, K. (1939). *The Greek Aulos*, Plate 12 between pp. 74–75, and Plate 17 between pp. 420–421. Methuen & Co. Ltd., London, England.

[\[2\]](#)Nederveen, C.J. (1969). *Acoustical Aspects of Woodwind Instruments*. Frits Knuf, Amsterdam, Netherlands.

[\[3\]](#)See Appendix B for a complete list of conversion factors.

[\[4\]](#)*Acoustical Aspects of Woodwind Instruments*, pp. 15–17.

[\[5\]](#)Coltman, J.W. (1979). Acoustical analysis of the Boehm flute. *Journal of the Acoustical Society of America* **65**, No. 2, pp. 499–506.

[\[6\]](#)See Chapter 7, Section 2.

[\[7\]](#)See Chapter 7, Section 11.

[\[8\]\(a\)](#) *Acoustical Aspects of Woodwind Instruments*, p. 13, pp. 47–48.

[\(b\)](#) Nederveen, C.J. (1973). Blown, passive and calculated resonance frequencies of the flute. *Acustica* **28**, pp. 13–14.

[\[9\]](#)Appendix A lists the frequencies of eight “octaves” in 12-tone equal temperament.

[\[10\]](#)In Figures 8.3(a), 8.3(b), and 8.3(c), the *substitution tubes* show *displacement standing waves* with nodes at the ends and an antinode in the center, while the flute tubes show *pressure standing waves* with nodes at the ends and an antinode in the center. Although such inconsistencies may lead to

confusion, here the emphasis is on wavelength and visual imagery and not on wave physics. For a given mode of vibration, a displacement standing wave and a pressure standing wave have exactly the same length.

[11] Benade, A.H., and French, J.W. (1965). Analysis of the flute head joint. *Journal of the Acoustical Society of America* **37**, No. 4, pp. 679–691.

[12] This value assumes no lip coverage at the embouchure hole.

[13] *Acoustical Aspects of Woodwind Instruments*, p. 26, 47.

With the exception of the key pad correction ($\Delta\ell_K$), Equation 38.8 on p. 64 of Nederveen's book (or Equation 7 on p. 14 of Nederveen's article in *Acustica* **28**, 1973) functions as a variable in determining (1) the effective length of the bore at the embouchure hole (as in Equation 8.6 with d_E , ℓ_E , and d_0 variables), and (2) the effective length of the bore at a tone hole (as in Equation 8.14 with d_H , ℓ_H , and d_1 variables). For example, Equation 8.14 is derived in the following manner. On p. 63 of his book, Nederveen gives the outside correction of a tone hole as

$$\Delta\ell_{\text{out}} = \frac{1}{2} 0.7 d_H$$

Also on p. 63, Nederveen's Equation 38.3 is Benade's solution (see Note 30) for the inside correction of a tone hole. This equation states

$$\Delta\ell_{\text{in}} = \frac{1}{2} \left(1.3 - 0.9 \frac{d_H}{d_1} \right) d_H$$

The effective length of a tone hole (L_H) is then given by

$$L_H = \ell_H + \Delta\ell_{\text{in}} + \Delta\ell_{\text{out}}$$

To use this equation, first simplify $\Delta\ell_{\text{in}}$ and $\Delta\ell_{\text{out}}$:

$$L_H = \ell_H + \left(0.65 d_H - 0.45 \frac{d_H^2}{d_1} \right) + 0.35 d_H$$

Now add these two corrections and simplify again:

$$L_H = \ell_H + d_H - 0.45 \frac{d_H^2}{d_1}$$

Therefore, the equivalent effective length of an embouchure hole (L_E) equals

$$L_E = \ell_E + d_E - 0.45 \frac{d_E^2}{d_0}$$

This latter equation is more accurate than the denominator of Equation 8.9. Refer to Equation 8.11, and note that a substitution of L_H and two equivalent tone hole variables into this equation gives the effective length of the bore at a tone hole location:

$$L_{B(h)} = \left(\frac{d_1}{d_H} \right)^2 L_H$$

$$L_{B(h)} = \left(\frac{d_1}{d_H} \right)^2 \left(\ell_H + d_H - 0.45 \frac{d_H^2}{d_1} \right)$$

A simplification of this equation gives Equation 8.14 in Section 8.5:

$$L_{B(h)} = (\ell_H + d_H) \left(\frac{d_1}{d_H} \right)^2 - 0.45 d_1$$

And a similar substitution of L_E into Equation 8.11 gives Equation 8.6 in Section 8.3:

$$L_{B(e)} = (\ell_E + d_E) \left(\frac{d_0}{d_E} \right)^2 - 0.45 d_0$$

[14] See Chapter 7.

[15] See Chapter 7, Section 5.

[16] The angular wave number is given by

$$k = \frac{\omega}{c} = \frac{2\pi F}{c}$$

[17] Richardson, E.G. (1929). *The Acoustics of Orchestral Instruments and of the Organ*, p. 49, 148. Edward Arnold & Co., London, England.

[18] See Chapter 7, Section 12.

[19] A flute tube three centimeters long has a terminating impedance that is too small for the production of pressure standing waves. In contrast, a tube three meters long has a terminating impedance that is too large for human beings to produce pressure standing waves. Most musicians are not able to deliver the air pressure required for sound production from very long tubes. Furthermore, with respect to tone holes, a very small diameter hole drilled into a flute tube has an extremely large terminating impedance. Pressure waves inside the tube propagate past the tiny hole as though it was not there. Consequently, the presence of such a hole does not change the frequency of the tube.

[20] Benade, A.H. (1976). *Fundamentals of Musical Acoustics*, p. 449, 495. Dover Publications, Inc., New York, 1990.

Also, see Equation 8.26.

[21] *Acustica* **28**, pp. 12–23.

Nederveen states, “Regarding the outside correction for a hole . . . we observe that the main tube acts as a partial flange to the hole-end, so that we have an end correction in between those for an unflanged end and a fully flanged end.”

[22] Steinkopf, O. (1983). *Zur Akustik der Blasinstrumente*, p. 19. Moeck Verlag, Celle, Germany.

[23] *Acustica* **28**, 1973, p. 16.

[24](a) Boehm, T. (1847). *On the Construction of Flutes, Über den Flötenbau*, p. 55. Frits Knuf Buren, Amsterdam, Netherlands, 1982.

The English version of this essay does not have page numbers. A reference to the 50 mm correction in the English translation appears four pages from the end of the book.

(b) Boehm, T. (1871). *The Flute and Flute-Playing*, p. 42. Dover Publications, Inc., New York, 1964.

Here Boehm gives a slightly longer correction at 51.5 mm.

(c) *Fundamentals of Musical Acoustics*, p. 495.

[25] This cavity effect is especially significant on concert metal flutes. Here most tone holes have a small chimney that sits above the curved outer surface of the tube. These chimneys provide a straight airtight closing surface for the key pads.

[26] *Acoustical Aspects of Woodwind Instruments*, p. 76.

[27] *Ibid.*, p. 48.

[28] *Ibid.*, p. 64.

[29] See Note 25.

[30](a) *Acoustical Aspects of Woodwind Instruments*, p. 64.

(b) Benade, A.H. (1967). Measured end corrections for woodwind toneholes. *Journal of the Acoustical Society of America* **41**, No. 6, p. 1609.

[31] *Ibid.*, p. 48.

[32] See Chapter 9, Section 13.

[33] See Chapter 9, Section 14.

[34] *Acoustical Aspects of Woodwind Instruments*, p. 48.

[35] *Ibid.*, p. 48, in Figure 32.3.

[36] Rossing, T.D. (1989). *The Science of Sound*, 2nd ed., pp. 246–247. Addison-Wesley Publishing Co., Inc., Reading, Massachusetts, 1990.

[37] *JASA* **37**, 1965, p. 689.

[38] According to Equation 7.12, a tube open at both ends generates a complete harmonic series of both even-number and odd-number harmonics; and according to Equation 7.17, a tube closed at one end generates an incomplete harmonic series of only odd-number harmonics. The act of overblowing a flute — or increasing the blowing pressure — causes these faint harmonics to sound as plainly audible tones. Therefore, the first overblown tone of a flute open at both ends produces the second harmonic, ratio 2/1; and the first overblown tone of a flute closed at the far end produces the third harmonic, ratio 3/1.

[39] See Chapter 10, Figure 15.

[40] *JASA* 37, 1965, p. 688.

[41] See Chapter 7, Section 5 through Section 8.

[42] *Fundamentals of Musical Acoustics*, pp. 473–480.

Another important fine-tuning technique consists of changing the bore diameter of woodwind instruments at the location of a pressure antinode, or at the location of a displacement antinode for a given mode of vibration. If we expand the bore by lightly sanding the inner surface of the flute tube at the location of a pressure antinode, such an increase in diameter causes a local increase in volume (V), which in turn produces a local decrease in pressure. Refer to Equation 7.36, and note that as V increases, the springiness of the air decreases. We may interpret such a local decrease in the air spring constant as causing a decrease in the potential energy of the vibrating system. When the potential energy of a pressure standing wave decreases, the frequency decreases as well. In contrast, if we constrict the bore by applying several coats of lacquer at the location of a pressure antinode, the frequency of a given mode of vibration increases.

On the other hand, a contraction at the location of a displacement antinode decreases the frequency of a given mode of vibration. A local constriction causes an increase in the inertia — or in the resistance to motion — of the air as it flows through the constriction. However, for a given amplitude of vibration, the same volume of air must flow through the constricted portion of the bore per second of time as through the regular bore. This requires an increase in the particle velocity of the air (u) as it passes through the constriction. We may interpret such an increase in particle velocity as causing an increase in the kinetic energy of the vibrating system. When the kinetic energy of a displacement standing wave increases, the frequency decreases. In contrast, if we expand the bore at the location of a displacement antinode, the frequency of a given mode of vibration increases.

Reminiscent of the discussion in Chapter 6, Section 10, a pattern of overlapping pressure antinodes and displacement antinodes means that a constriction or expansion anywhere along the length of the bore affects numerous modes simultaneously. Moreover, the physical length of a given constriction or expansion also determines which mode frequencies are affected. For example, on the Armstrong flute the constriction of the head joint taper extends 114 mm to the right of the embouchure hole center. Since the embouchure hole marks the location of a displacement antinode of the first mode of vibration, the constriction flattens the frequencies of the first “octave.” Furthermore, according to Chapter 7, Figure 9, a pressure antinode of the second mode of vibration is located a quarter-wavelength ($0.25 \times \lambda_2$) from either end of an open tube. On the Armstrong flute, the C_5 pressure antinode exists approximately 115.3 mm to the right of the embouchure hole center. That is, $\lambda_2 = c / F_2$

$= 346,000 \text{ mm/s} \div 523.25 \text{ cps} = 661.3 \text{ mm}$, so that $0.25(661.3 \text{ mm}) = 165.3 \text{ mm}$, and $165.3 \text{ mm} - 50 \text{ mm} = 115.3 \text{ mm}$. (To locate the position of the second mode pressure antinode with respect to the embouchure hole center, we must account for the embouchure hole correction and, therefore, subtract 50 mm.) Since the 114 mm constriction extends into the vicinity of this pressure antinode, it also sharpens the frequencies of the second “octave.” This sharpening effect on the second register produces tones that are *less flat* than in the first register. According to Nederveen’s graph in *Acustica* **28**, 1973, p. 17, the head joint taper causes G_4 at 392.0 cps to sound 51 cents flat, which gives a frequency of 380.6 cps, and G_5 at 784.0 cps to sound only 10 cents flat, which gives a frequency of 779.5 cps. Note, however, that several factors such as increases in the speed of sound due to the warmth and chemical composition of the player’s breath (see Coltman in *JASA* **40**, 1966, pp. 99–107), and increases in the player’s blowing pressure at the embouchure hole, act to sharpen and correct the tendencies of these tones to sound flat. The acoustic need for a tapered head joint is discussed at length by Benade and French in *JASA* **37**, 1965, pp. 679–691, and by Benade in *Fundamentals of Musical Acoustics*, pp. 495–497.