CS 284 Problem Set 1 Solutions (v2)

February 15, 2016

1. **Install Drake [10 points]**.

   Solution: Please consult the FAQs and open/closed Drake issues on Github if you run into installation errors. On Mac and Linux systems, teaching staff may be able to help. On Windows systems, installing the precompiled binary is sufficient for the course. □

2. **Fully-actuated definition [10 points]**. True or false. A sufficient condition for a system to be fully-actuated is that the number of degrees of freedom is equal to the number of actuators (i.e. $|q| = |u|$). Explain your answer.

   Solution: False. A demonstrating counterexample is a double pendulum with two joints and two actuators. If the two actuators are acting on the same joint (leaving the other joint freely moving) then the system is not fully-actuated. □

3. **Underactuated systems [25 points]**. For the following dynamical systems, determine if they are underactuated. Justify your responses.

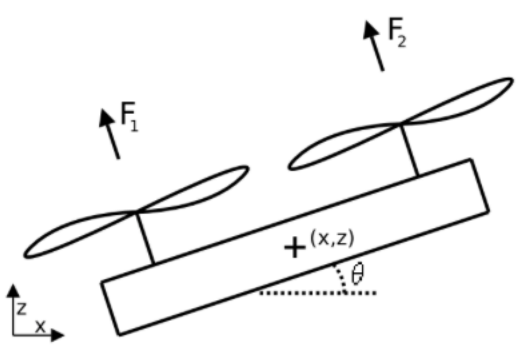


Figure 1: Planar quadrotor

   (a) *Planar quadrotor [5 points].* A helicopter with two rotors is constrained to move in the x-z plane. Assume gravity acting on the

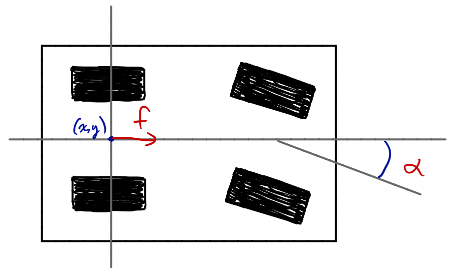


Figure 2: Simple car

helicopter. The task is to control the position, x, z and pitch, θ, by varying the thrust produced by the two rotors. Decide whether this system is fully-actuated (in all states), or if there are any states in which the problem is underactuated. Use the definition of underactuated provided in lecture. Explain your answers.

Solution: This system is trivially underactuated since it has three degrees of freedom and only two control inputs. The dynamics of the quadrotor can be written as:

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{z} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ g \\ 0 \end{bmatrix} = \begin{bmatrix} -\sin(\theta) & -\sin(\theta) \\ \cos(\theta) & \cos(\theta) \\ -L & L \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

Note that the matrix multiplying the control inputs on the right hand side of the equation (labeled B in the manipulator equations) has rank two. □

(b) *Simple car [5 points].* Figure 2 shows a simple car system with two inputs: the throttle f and steering angle α. Is this system underactuated?

Solution: This system is also trivially underactuated since it has three degrees of freedom (x, y, and θ) and only two control inputs. Therefore, the B matrix can have at most rank two. □

(c) *Holonomic constraints [5 points].* Holonomic constraints are equality constraints that can be expressed purely in terms of the configuration of the system, q. These include joint limits on a robot arm, for example.

Nonholonomic constraints are nonintegrable constraints on velocity (e.g., the system can get to any configuration, but cannot get there by any arbitrary path). The simple car is a classical example of a nonholonomic system. If nonholonomic systems have constraints on velocity, then are all nonholonomic systems also underactuated? Explain your answer.

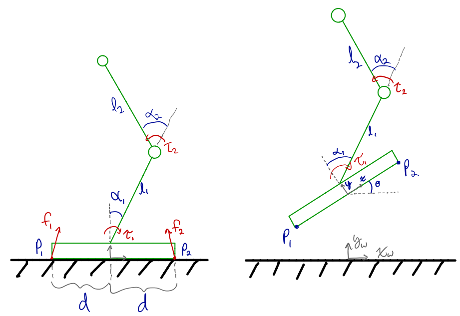


Figure 3: Robot with a big foot

Solution: Nonholonomic velocity constraints imply underactuation. To see this, assume we have a constraint of the following form:

$$g(q, \dot{q}) = 0$$

Now use the chain rule to differentiate both sides of the equation with respect to time

$$\frac{\partial g}{\partial q}\dot{q} + \frac{\partial g}{\partial \dot{q}}\ddot{q} = 0$$

This equation tells us that arbitrary accelerations cannot be achieved in all degrees of freedom. Therefore the system is underactuated. □

(d) *Robot with a big foot [10 points].* Figure 3 shows a double pendulum attached to a foot of length $2d$. Conceptually we can think of this as a single robot leg. The robot can apply torques, τ_1, τ_2, directly to the "ankle" and "knee" joints. When in contact with the ground, it can experience contact forces $f_1, f_2 \in \mathbb{R}^2$ at two contact points at the points p_1, p_2 at the heel and toe. The configuration of the robot is described by $q = [x, y, \theta, \alpha_1, \alpha_2]$.

i. When the foot is flat on the ground (i.e. $y = 0, \theta = 0$) does the system satisfy the rank condition for underactuation? Hint: the contact forces f_1, f_2 can be thought of as inputs that are mapped to generalized forces by the transpose of their *Jacobian* matrices $J_1^T = \frac{\partial p_1}{\partial q}^T$ and $J_2^T = \frac{\partial p_2}{\partial q}^T$.
Solution: The dynamics when the foot is flat on the ground can be modeled as follows:

$$H(q)\begin{bmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{\theta}\\ \ddot{\alpha}_1\\ \ddot{\alpha}_2\end{bmatrix} + C(q,\dot{q}) + G(q) = \begin{bmatrix}0 & 0 & 1 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 & 0 & 1\\ 0 & 0 & 0 & -d & 0 & d\\ 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\tau_1\\ \tau_2\\ f_{1x}\\ f_{1y}\\ f_{2x}\\ f_{2y}\end{bmatrix}$$

Note that the B matrix for the system has rank five, which satisfies the condition for full actuation. $\square$

ii. Is the system underactuated? Explain your answer.
Solution: The system is underactuated since there is are additional constraints $f_{1y} \geq 0$ and $f_{2y} \geq 0$. In physical terms, the contact forces cannot push the foot down. $\square$

4. **Feedback Linearization [5 points]** True or False. Consider a robot whose dynamics are given by the manipulator equations, $\ddot{q} = H(q)^{-1}(-C(q,\dot{q}) + Bu)$. For $q \in \mathbb{R}^n$, suppose B is rank n. Using feedback linearization, you can build a controller to instantaneously achieve any $\dot{q}$ and $\ddot{q}$. Explain your answer.

   Solution: False. In a second order system (e.g. $Force = (mass)(\ddot{q})$), we can achieve any instantaneous acceleration $\ddot{q}$ if the system is fully actuated. However, we cannot achieve any $\dot{q}$ instantaneously. In other words, by applying appropriate input u, $\ddot{q}$ can be discontinuously changed to a desired value where as $\dot{q}$ must be continuous (but need not be smooth) and therefore cannot suddenly change to some value. $\square$

5. **Pendulum dynamics [10 points]**. Recall the dynamics of the simple pendulum from lecture,

$$ml^2\ddot{\theta} + mgl\sin(\theta) = u - b\dot{\theta}. \tag{1}$$

   Let $m = 1, l = 1, g = 10, b = 1$.

   (a) Compute the fixed points in the interval $[-\pi, \pi]$ when $u = -5$. Identify them as stable or unstable.
   Solution: Substituting $m = 1, l = 1, g = 10, b = 1$, we have

$$\ddot{\theta} + 10\sin(\theta) = u - \dot{\theta}$$

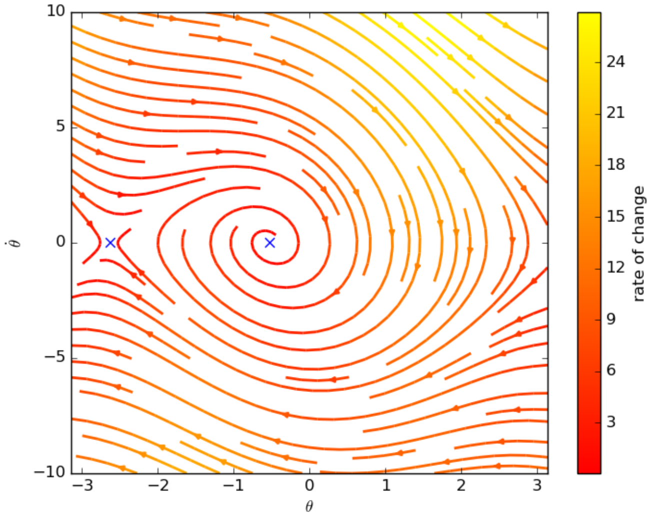
   and for $u = -5$, we have

$$\ddot{\theta} + 10\sin(\theta) = -5 - \dot{\theta}.$$

   As we are interested in finding fixed points, we set $\ddot{\theta} = 0$ and $\dot{\theta} = 0$. Thus,

$$10\sin(\theta) = -5 \Rightarrow \sin(\theta) = -\frac{1}{2} \Rightarrow \theta = -\frac{\pi}{6} \; or \; -\frac{5\pi}{6}.$$

   We make the phase portrait below. The fixed points are marked with 'x' in blue. Judging from the directions of the surrounding arrows, we conclude that $\theta = -\frac{\pi}{6}$ is a stable fixed point and $-\frac{5\pi}{6}$ an unstable fixed point.

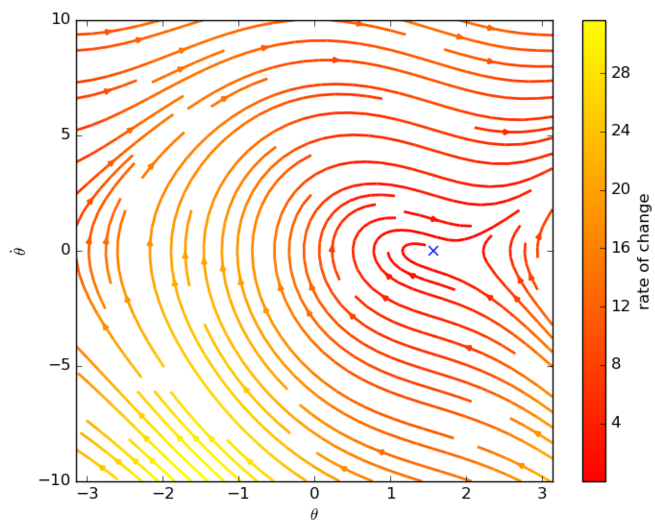


$\square$

(b) Do the same for $u = 10$. What happens when $u > 10$?

Solution: For $u = 10$, we have $\theta = \frac{\pi}{2}$ as the fixed point after doing the same analysis as above. From the directions of the surrounding arrows, we conclude that the fixed point is unstable: although moving away from the fixed point to the left will return to the fixed point, moving away to the right will diverge (unstable).

There is no fixed point for $u > 10$.



[Note: We only covered the graphical method in class so we present it as the solution above. However, if a student's solution applied eigenvalue analysis or perturbation analysis correctly, it was accepted as well.]□

6. **Graphical analysis [5 points]**. Consider a system with the dynamics given by

$$\dot{x} = 5x^3 - 7x^2 - 13x + 17. \tag{2}$$

Draw a flow diagram for this system and determine the three equilibrium points. Identify them as stable or unstable.

Solution: The system has two unstable equilibria located approximately at $x \approx -1.5873$ and $x \approx 1.7922$ and one stable equilibrium located at $x \approx 1.1952$. These are marked in red and green, respectively, in Figure 4. □

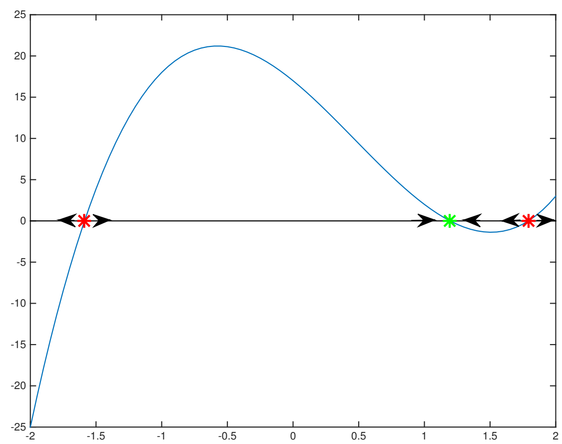


Figure 4: Flow diagram.

7. **Lyapunov analysis [10 points]**. For the following system, use the quadratic Lyapunov function candidate $V = \frac{1}{2}x^T x$ to show that the origin is asymptotically stable (meaning $\dot{V}(x) < 0$). State whether the system is locally or globally asymptotically stable and justify your response.

$$\dot{x}_1 = -x_2 - x_1(1 - x_1^2 - x_2^2), \quad \dot{x}_2 = x_1 - x_2(1 - x_1^2 - x_2^2)$$

Solution: We start by writing out the candidate function,

$$\begin{aligned} V &= \frac{1}{2}x^T x \\ &= \frac{1}{2}(x_1^2 + x_2^2). \end{aligned}$$

So, for $\frac{dV}{dt}$, we have

$$\begin{aligned}\frac{dV}{dt} &= \frac{1}{2}(2x_1\dot{x_1} + 2x_2\dot{x_2}) \\ &= x_1\dot{x_1} + x_2\dot{x_2}\end{aligned}$$

and substituting for $\dot{x_1}$ and $\dot{x_2}$, we have

$$\begin{aligned}\frac{dV}{dt} &= x_1(-x_2 - x_1(1 - x_1^2 - x_2^2)) + x_2(x_1 - x_2(1 - x_1^2 - x_2^2)) \\ &= -x_1^2(1 - x_1^2 - x_2^2)) - x_2^2(1 - x_1^2 - x_2^2)) \\ &= -(x_1^2 + x_2^2)(1 - x_1^2 - x_2^2).\end{aligned}$$

Thus, $\dot{V} := \frac{dV}{dt}$ is negative if and only if $1 - x_1^2 - x_2^2 > 0$. In other words, $x_1^2 + x_2^2 < 1$ leads to $\dot{V} < 0$, and the system is asymptotically stable in the unit circle. $\dot{V} > 0$ outside the unit circle. Thus, it is not globally stable.□

8. **Discrete time systems [15 points]**. We have seen that for a univariate dynamical system, $\dot{x} = f(x)$, the point x^* is a locally stable equilibrium if the following conditions hold

$$f(x^*) = 0 \tag{3}$$

$$\frac{\partial f}{\partial x}(x^*) < 0. \tag{4}$$

Now consider a simple time discretization of this system using forward Euler integration:

$$x[k+1] = x[k] + hf(x[k]), \tag{5}$$

where $h > 0$ is a time step parameter. For arbitrary h, the two conditions above are not sufficient for stability of the discrete time system.

(a) Provide a counterexample demonstrating this by giving values for x^*, $f(x)$, and h.

Solution: There are many possible correct answers to this question. One example is:

$$f(x) = -x$$

$$x^* = 0$$

$$h = 3.$$

□

(b) Find an upper bound, h^*, such that all $h < h^*$ result in a stable discrete system. Give your answer in units of $g = \|\frac{\partial f}{\partial x}(x^*)\|$.

Solution: For the values given in part (a), $h^* = 2$. For a general first-order linear system $f(x) = -ax$, the following discrete-time update rule is obtained by applying forward Euler integration:

$$x[k+1] = (1 - ha)x[k]$$

Stability of this system can be assessed by taking the limit of x as $k \to \infty$:

$$\lim_{k\to\infty} x[k] = (1 - ha)^k x[0] = 0 \implies \text{stability}$$

The system is stable if $\|(1 - ha)\| < 1$, or equivalently if $h < \frac{2}{g}$. $\square$