Homework 3

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1 Problem 1:

Show that ⟨ x, y ⟩ is an inner product:

$$\langle x, y \rangle = x_1 y_1 - (x_1 y_2 + x_2 y_1) + 2 x_2 y_2$$

Note that:

$$\langle y, x \rangle = y_1 x_1 - (y_1 x_2 + y_2 x_1) + 2 y_2 x_2$$

Therefore $\langle x, y \rangle = \langle y, x \rangle$ and satisfy the symmetric property Also:

$$\langle x, x \rangle = x_1^2 - (x_1 x_2 + x_2 x_1) + 2 x_2^2$$

$$\langle x, x \rangle = x_1^2 - 2 x_1 x_2 + 2 x_2^2$$

$$\langle x, x \rangle = (x_1 - x_2)^2 + x_2^2$$

Note that: $\langle x, x \rangle \geq 0$ and only equals to 0 if and only if $x_1 = x_2 = 0$

2 Problem 2:

Use x = $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and y = $\begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$ to compute the distance:

(a) $\langle x, y \rangle = x^T y$

$$x - y = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$$

$$dist(x, y) = \sqrt{\langle x - y, x - y \rangle} = \sqrt{\begin{bmatrix} 2 & 3 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}} = \sqrt{22}$$

(b) $\langle x, y\rangle = x^T By$ with $B = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

$$dist(x, y) = \sqrt{\langle x - y, x - y\rangle} = \sqrt{\begin{bmatrix} 2 & 3 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}} = \sqrt{\begin{bmatrix} 7 & 8 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}} = \sqrt{39}$$

3 Problem 3:

Use x = $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and y = $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$ to compute the angle:

(a) $\langle x, y\rangle = x^T y$

With $\langle x, y\rangle = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ = -3, $\|x\| = \sqrt{\langle x, x\rangle} = \sqrt{5}$, and $\|y\| = \sqrt{\langle y, y\rangle}$ $= \sqrt{2}$. Therefore:

$$\cos w = \frac{\langle x, y\rangle}{\|x\|\|y\|} = \frac{-3}{\sqrt{10}}$$

(b) $\langle x, y\rangle = x^T By$ with $B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$

With $\langle x, y\rangle = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 & 7 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ = -11, $\|x\| = \sqrt{\langle x, x\rangle}$ $= \sqrt{\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}} = \sqrt{\begin{bmatrix} 4 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}} = \sqrt{18}$, and $\|y\| = \sqrt{\langle y, y\rangle} =$ $\sqrt{\begin{bmatrix} -1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix}} = \sqrt{\begin{bmatrix} -3 & -4 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix}} = \sqrt{7}$.

$$\cos w = \frac{\langle x, y\rangle}{\|x\|\|y\|} = \frac{-11}{\sqrt{126}}$$

4 Problem 4:

U = Span($\begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 5 \\ 0 \\ 7 \end{bmatrix}$) and x = $\begin{bmatrix} -1 \\ -9 \\ -1 \\ 4 \\ 1 \end{bmatrix}$

(a) Determine orthogonal project $\pi_U(x)$ of x onto U

Note that:

$$B = \begin{bmatrix} 0 & 1 & -3 & -1 \\ -1 & -3 & 4 & -3 \\ 2 & 1 & 1 & 5 \\ 0 & -1 & 2 & 0 \\ 2 & 2 & 1 & 7 \end{bmatrix} \leftrightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the last column of the matrix B contains free variable, the last vector is linear dependent from the first 3 vectors. Therefore, the basis of B consists the first 3 vectors

$$\pi_U(x) = B(B^T B)^{-1} B^T x$$

$$\pi_U(x) = \begin{bmatrix} 1 \\ -5 \\ -1 \\ -2 \\ 3 \end{bmatrix}$$

(b) Determine the distance d(x, U)

Since the dot product is defined, $x - \pi_U(x) = \begin{bmatrix} -1 \\ -9 \\ -1 \\ 4 \\ 1 \end{bmatrix}$ - $\begin{bmatrix} 1 \\ -5 \\ -1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \\ 0 \\ 6 \\ -2 \end{bmatrix}$

$$d(x, U) = \sqrt{\langle x - \pi_U(x), x - \pi_U(x) \rangle} = \sqrt{4 + 16 + 36 + 4} = 2\sqrt{15}$$

5 Problem 5:

$$\langle x, y \rangle = x^T \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} y$$

(a) Determine the orthogonal project $\pi_U(e_2)$ of e_2 onto U = span(e_1, e_3)
Note that since e_1 and e_3 are 2 vectors in standard basis of R^3, therefore, U = span(e_1, e_3) is also a basis.

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$B^T E B \alpha = B^T E x$$

$$\alpha = (B^T EB)^{-1} B^T Ex$$

$$\pi_U(x) = B(B^T EB)^{-1} B^T Ex = \begin{bmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix}$$

(b) Compute the distance d(e_2, U)

$$x - \pi_U(x) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ \frac{1}{2} \end{bmatrix}$$

$$d(e_2, U) = \sqrt{\langle x - \pi_U(x), x - \pi_U(x) \rangle} = 1$$

6 Problem 6:

Let V be a vector space and π an endormorphism of V

(a) Prove that π is a projection if and only if id_V - π is a projection, where id_V is an identity endormorphism on V
Suppose x is a vector and x $\in$ V:

$$(id_V - \pi) \circ (id_V - \pi)(x) = (id_V - \pi)((id_V - \pi)(x)) = (id_V - \pi)(id_V(x) - \pi(x))$$

Note that $\pi^2(x) = \pi(x)$, since π is a projection function. Also, $id_V^2(x) = id_V(id_V(x)) = id_V(x) =$ x, since $id_V(x)$ is an identity function in vector space V.

$$(id_V - \pi)(id_V(x) - \pi(x)) = id_V(x) - id_V(\pi(x)) - \pi(x) + \pi^2(x)$$

$$(id_V - \pi)(id_V(x) - \pi(x)) = id_V(x) - 2\pi(x) + \pi^2(X)$$

$$(id_V - \pi)(id_V(x) - \pi(x)) = id_V(x) - \pi(x)$$

Therefore, $(id_V - \pi) \circ (id_V - \pi)$, or $(id_V - \pi)^2$, equals to $id_V - \pi$. By definition, $id_V - \pi$ is a projection

(b) Assuming π is a projection, calculate Image(id_V - π) and Ker(id_V - π)
Suppose, x $\in$ V, we note that to find Ker($id_V - \pi$), we need to find that $(id_V - \pi)$(x) = 0. From the above, $(id_V - \pi)$(x) = $(id_V - \pi)^2$(x). In other word:

$$(id_V - \pi)^2(x) = (id_V - \pi)(x) = 0$$

Note that"

$$(id_V - \pi)^2 = (id_V - \pi) \circ (id_V - \pi)(x) = (id_V - \pi)(0) = 0$$

Therefore, Ker($id_V - \pi$(x)) consists only vector 0.