Homework 2

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1 Problem 1:

(a) Find transformation matrix A_ϕ
Solution:

$$\phi(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}) = \begin{bmatrix} 3x_1 + 2x_2 + x_3 \\ x_1 + x_2 + x_3 \\ x_1 - 3x_2 \\ 2x_1 + 3x_2 + x_3 \end{bmatrix}$$

$$\phi(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}) = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & -3 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Therefore, the transformation matrix A_ϕ is $\begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & -3 & 0 \\ 2 & 3 & 1 \end{bmatrix}$

(b) Determine rank(A_ϕ)
Solution:

$$A_\phi = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & -3 & 0 \\ 2 & 3 & 1 \end{bmatrix} \leftrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 1 & -3 & 0 \\ 2 & 3 & 1 \end{bmatrix} \leftrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 4 & 1 \\ 0 & 1 & -1 \end{bmatrix} \leftrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 7 \\ 0 & 0 & 3 \end{bmatrix}$$

Note that every column has pivots. Therefore, the transformation matrix A_ϕ has rank 3.

(c) Kernel and image of A_ϕ and dim(ker(A_ϕ)) and dim(image(A_ϕ)
Solution: From above, the transformation matrix A_ϕ is:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 7 \\ 0 & 0 & 3 \end{bmatrix} \leftrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

To find the ker(ϕ), compute $\phi(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}) = 0$. In other word, $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 7 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $= 0$. Therefore, $x_1 = x_2 = x_3 = 0$ and $\ker(\phi) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and $\dim(\ker) = 0$.

To find the image(ϕ), we need to find the column space of the transformation matrix. The col(ϕ) $= (\begin{bmatrix} 3 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1. \end{bmatrix})$. Therefore, the image($\phi$) $= (\begin{bmatrix} 3 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1. \end{bmatrix})$ and the dim(image(ϕ)) $= 3$.

2 Problem 2:

Find the matrix to rotate the vector $x_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $x_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ by $\pi/6$.

Solution: To determine the rotation matrix, we have:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\pi/6 & -\sin\pi/6 \\ \sin\pi/6 & \cos\pi/6 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

Apply the rotation matrix, we have a new vector v_1 and v_2:

$$x_1 = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{2\sqrt{3}-3}{3} \\ \frac{2+3\sqrt{3}}{3} \end{bmatrix}$$

$$x_2 = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{bmatrix}$$

3 Problem 3:

Which of the following mappings are linear

(a) ϕ: $\mathbb{R} \to \mathbb{R}$
x $\to \cos x$
Solution: Not a linear transformation since $\sin 3x \neq \sin x + \sin 2x$ for $\pi/2$

(b) $\phi : \mathbb{R}^{\nVDash} \to \mathbb{R}^2$
x $\to \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 3 \end{bmatrix}$x

Solution: A linear transformation for general x_1 and x_2. Note that $\phi(x_1) = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 3 \end{bmatrix} x_1$ and $\phi(x_2) = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 3 \end{bmatrix} x_2$.

$$\phi(\alpha x_1 + \beta x_2) = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 3 \end{bmatrix} (\alpha x_1 + \beta x_2) = \alpha\phi(x_1) + \beta\phi(x_2)$$

4 Problem 4:

$$A_\phi = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

(a) Determine Ker(ϕ) and Image(ϕ)
Solution:

$$A_\phi = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \leftrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \leftrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that the matrix A_ϕ has pivots in every row and column. Therefore, the Ker(ϕ) $= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and Image(ϕ) $= (\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix})$

(b) Determine transformation matrix C_θ with respect to new basis ($\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$) and perform change of basis toward new basis with

Solution: $B^{-1} = \begin{bmatrix} 0 & -1 & 2 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$

$$C_\phi = B^{-1} A_\phi B = \begin{bmatrix} 0 & -1 & 2 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$C_\phi = \begin{bmatrix} 6 & 9 & 1 \\ -3 & -5 & 0 \\ -1 & -1 & 0 \end{bmatrix}$$

5 Problem 5:

Given $b_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $b_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$, $b_1' = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$, $b_2' = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and bases B $= (b_1, b_2)$ and B' $= (b_1', b_2')$

(a) Prove that B and B' are two bases of $\mathbb{R}^2$ and draw those basis

Solution: Put two bases in the 2×2 matrices and apply row reduced operation to have

$$B = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix} \leftrightarrow \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} \leftrightarrow \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$B' = \begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix} \leftrightarrow \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \leftrightarrow \begin{bmatrix} 1 & 1 \\ 0 & 4 \end{bmatrix}$$

Note that both B and B matrices have pivots in every row and column. Also B and B' spans over $\mathbb{R}^2$. Therefore, B and B' are 2 bases of $\mathbb{R}^2$. Draw a graph

(b) Compute P_1 that performs basis change from B' to B

Solution: To perform the basis change from B' to B or compute $P_1 = P_{B' \leftarrow B}$, we simply puts 2 matrices B' and B side by side and do the row reduced echelon form. In other word, we have:

$$[BB'] = \left[\begin{array}{cc|cc} 2 & -1 & 2 & 1 \\ 1 & -1 & -2 & 1 \end{array}\right] \leftrightarrow \left[\begin{array}{cc|cc} 1 & -1 & -2 & 1 \\ 2 & -1 & 2 & 1 \end{array}\right] \leftrightarrow \left[\begin{array}{cc|cc} 1 & 0 & 4 & 0 \\ 0 & 1 & 6 & -1 \end{array}\right]$$

Therefore, the P_1 matrix is $\begin{bmatrix} 4 & 0 \\ 6 & -1 \end{bmatrix}$

(c) Given C = ($\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$), show that C = $(c_1,\, c_2,\, c_3)$ is a basis by using determinants

Solution: Putting the 3 vectors in C into 3×3 matrix to determine the determinant of the matrix.

$$C = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ -1 & 2 & -1 \end{bmatrix}$$

$$det(C) = 1 \begin{bmatrix} -1 & 0 \\ 2 & -1 \end{bmatrix} + 1 \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = (-1)(-1) - (2)(0) + (2)(2) - (-1)(-1) = 4$$

Since the det(C) is $\neq 0$, C is a basis of R^3

(d) Given C' is a standard basis of $\mathbb{R}^3$. Determine P_2 that performs the basis change from C to C'

Solution: Apply the similar process like above, we have [C'C] matrix to find P_2 or $P_{C' \leftarrow C}$

$$[C'C] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right]$$

Notice that the P_2 matrix is actually the C' matrix. Therefore, $P_2 = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ -1 & 2 & -1 \end{bmatrix}$

(e) Consider the following linear transformation ϕ: $R^2 \to R^3$ such that

$$\phi(b_1 + b_2) = c_2 + c_3$$

$$\phi(b_1 - b_2) = 2c_1 - c_2 + 3c_3$$

Determine transformation matrix A_ϕ of ϕ with respect to bases B and C
Solution: Note that ϕ is a linear transformation, therefore, we can write $\phi(b_1 + b_2) = \phi(b_1) + \phi(b_2)$. Similiarly, $\phi(b_1 - b_2) = \phi(b_1)$ - $\phi(b_2)$. Rewrite the given system of equation, we have:

$$\phi(b_1) + \phi(b_2) = c_2 + c_3$$

$$\phi(b_1) - \phi(b_2) = 2c_1 - c_2 + 3c_3$$

Add 2 equations together, we have

$$2\phi(b_1) = 2c_1 + 4c_3 \leftrightarrow \phi(b_1) = c_1 + 2c_3$$

Subtract 2 equations together, we have

$$2\phi(b_2) = -2c_1 + 2c_2 - 2c_3 \leftrightarrow \phi(b_2) = -c_1 + c_2 - c_3$$

Therefore, the transformation matrix $A_\phi = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & -1 \end{bmatrix}$

(f) Determine transformation matrix A'_ϕ of ϕ with respect to bases B' and C'
Solution: Note that $P_{C' \leftarrow B'} = P_{C' \leftarrow C}\ P_{C \leftarrow B}\ P_{B \leftarrow B'}$

$$P_{C' \leftarrow B'} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 6 & -1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 2 & -3 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 6 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -10 & 3 \\ 12 & -4 \end{bmatrix}$$