Homework 1

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December 24, 2020

Problem 1: $\mathbb{G} = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{3x3} \; |x, y, z \in \mathbb{R}$

Solution:
$\mathbb{G}$ is in the G because it satisfies the first 4 properties. For the first condition, the matrix multiplication will always produce a real 3×3 matrix with real entries. For the second condition, associative will be always true for every matrix multiplication. For the third condition, the e matrix is the invertible matrix of x. Since $\forall$ x, y, z $\in \mathbb{R}$, the matrix has a shape of a upper triangular with all of diagonal entries are different from 0, then the matrix is invertible. For the same reason, the last condition is also true. However, the matrix is not in Abelian group because not every x, y, z will give us the same result of the matrix multiplication. For instance, 2 matrices $A_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ will not be the same if we multiply $A_1 A_2$ and $A_2 A_1$

Problem 2: Which of the following sets are subspace of $\mathbb{R}^3$:
a. A = $(\lambda,\ \lambda + \mu^3,\ \lambda - \mu^3)$ — $\lambda, \mu \in \mathbb{R}$
b. B = $(\lambda^2,\ -\lambda^2,\ 0)$ — $\lambda \in \mathbb{R}$
c. C = $(\xi_1,\xi_2,\xi_3) \in \mathbb{R}^3$ — $\xi_2 \in \mathbb{Z}$

Solution:
a. A subspace since it satisfies all of 3 conditions, non-empty subspace contains 0, For general λ_1, λ_2, μ_1, μ_2
, the conditions of closure under addition and scalar multiplication are satisfied.
b. Not a subspace since it does not satisfies the scalar multiplication with c = -1
c. This is not a subspace of R^3 since it fails the condition of closure under scalar multiplication for any number $\in \mathbb{R} \notin \mathbb{Z}$. For instance, c = 0.5

Problem 3: Consider two subspaces of $\mathbb{R}^4$ and determine a basis of $U_1 \cap U_2$

$$U_1 = \text{Span}(\begin{bmatrix} 1 \\ 1 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix})$$

$$U_2 = \text{Span}(\begin{bmatrix} -1 \\ -2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \\ -2 \\ -1 \end{bmatrix})$$

Solution:

$$\text{a} \begin{bmatrix} 1 \\ 1 \\ -3 \\ 1 \end{bmatrix} + \text{b} \begin{bmatrix} 2 \\ -1 \\ 0 \\ -1 \end{bmatrix} + \text{c} \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \text{d} \begin{bmatrix} -1 \\ -2 \\ 2 \\ 1 \end{bmatrix} + \text{e} \begin{bmatrix} 2 \\ -2 \\ 0 \\ 0 \end{bmatrix} + \text{f} \begin{bmatrix} -3 \\ 6 \\ -2 \\ -1 \end{bmatrix}$$

$$\leftrightarrow \text{a} \begin{bmatrix} 1 \\ 1 \\ -3 \\ 1 \end{bmatrix} + \text{b} \begin{bmatrix} 2 \\ -1 \\ 0 \\ -1 \end{bmatrix} + \text{c} \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \text{ - d} \begin{bmatrix} -1 \\ -2 \\ 2 \\ 1 \end{bmatrix} \text{ - e} \begin{bmatrix} 2 \\ -2 \\ 0 \\ 0 \end{bmatrix} \text{ - f} \begin{bmatrix} -3 \\ 6 \\ -2 \\ -1 \end{bmatrix} = 0$$

$$\leftrightarrow \begin{bmatrix} 1 & 2 & -1 & 1 & -2 & 3 \\ 1 & -1 & 1 & 2 & 2 & -6 \\ -3 & 0 & -1 & -2 & 0 & 2 \\ 1 & -1 & 1 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = 0$$

Apply row reduction, $A_1 = \begin{bmatrix} 1 & 2 & -1 & 1 & -2 & 3 \\ 1 & -1 & 1 & 2 & 2 & -6 \\ -3 & 0 & -1 & -2 & 0 & 2 \\ 1 & -1 & 1 & -1 & 0 & 1 \end{bmatrix}$

$$\leftrightarrow \begin{bmatrix} 1 & 2 & -1 & 1 & -2 & 3 \\ 0 & 3 & -2 & -1 & -4 & 9 \\ 0 & 6 & -4 & 1 & -6 & 11 \\ 0 & 3 & -2 & 2 & -2 & 2 \end{bmatrix} \leftrightarrow \begin{bmatrix} 1 & 2 & -1 & 1 & -2 & 3 \\ 0 & 3 & -2 & -1 & -4 & 9 \\ 0 & 0 & 0 & -3 & -2 & 7 \\ 0 & 0 & 0 & -3 & -2 & 7 \end{bmatrix} \leftrightarrow \begin{bmatrix} 1 & 2 & -1 & 1 & -2 & 3 \\ 0 & 1 & \frac{-2}{3} & \frac{-1}{3} & \frac{-4}{3} & 3 \\ 0 & 0 & 0 & 1 & \frac{2}{3} & \frac{-7}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\leftrightarrow \begin{bmatrix} 1 & 0 & 1 & 0 & \frac{-4}{3} & \frac{8}{3} \\ 0 & 1 & -2 & 0 & \frac{-10}{3} & \frac{20}{3} \\ 0 & 0 & 0 & 1 & \frac{2}{3} & \frac{-7}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore, we have:

a + c + $\frac{-4}{3}$e + $\frac{8}{3}$f = 0 $\rightarrow$ a = -c + $\frac{4}{3}$e - $\frac{8}{3}$f

b - 2c + $\frac{-10}{3}$e + $\frac{20}{3}$f = 0 $\rightarrow$ b = 2c + $\frac{10}{3}$e - $\frac{20}{3}$f

d + $\frac{2}{3}$e + $\frac{-7}{3}$f = 0 $\rightarrow$ d = $\frac{-2}{3}$e + $\frac{7}{3}$f

$$\begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \begin{bmatrix} -c + 4\frac{}{3}\text{e - } \frac{8}{3}\text{f} \\ 2c + 10\frac{}{3}\text{e - } \frac{20}{3}\text{f} \\ c \\ \text{-}2\frac{}{3}\text{e + } \frac{7}{3}\text{f} \\ e \\ f \end{bmatrix} = \begin{bmatrix} -c \\ 2c \\ c \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 4\frac{}{3}\text{e} \\ 10\frac{}{3}\text{e} \\ 0 \\ \text{-}2\frac{}{3}\text{e} \\ e \\ 0 \end{bmatrix} + \begin{bmatrix} -8\frac{}{3}\text{f} \\ -20\frac{}{3}\text{f} \\ 0 \\ 7\frac{}{3}\text{f} \\ 0 \\ f \end{bmatrix}$$

Therefore, the span of $U_1 \cap U_2$ is ($\begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 4_{\overline{3}} \\ 10_{\overline{3}} \\ 0 \\ \text{-}2_{\overline{3}} \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} \text{-}8_{\overline{3}} \\ \text{-}20_{\overline{3}} \\ 0 \\ 7_{\overline{3}} \\ 0 \\ 1 \end{bmatrix}$)

Problem 4: Which of the following sets are linearly independent

(a) $x_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$, $x_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$, $x_3 = \begin{bmatrix} 3 \\ -3 \\ 8 \end{bmatrix}$)

(b) $x_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $x_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, $x_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$)

Solution:

(a) $\begin{bmatrix} 2 & 1 & 3 \\ -1 & 1 & -3 \\ 3 & -2 & 8 \end{bmatrix} \leftrightarrow \begin{bmatrix} 1 & -1 & 3 \\ 2 & 1 & 3 \\ 3 & -2 & 8 \end{bmatrix} \leftrightarrow \begin{bmatrix} 1 & -1 & 3 \\ 0 & -3 & 3 \\ 0 & -1 & 1 \end{bmatrix} \leftrightarrow \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \leftrightarrow \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

Note that the last column does not have a pivot, then the set x_1, x_2, x_3 are not independent.

(b) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \leftrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \leftrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Note that every column have pivots, then the set x_1, x_2, x_3 are independent.

Problem 5: Prove: rank(A) = rank($\mathbb{A}^\intercal$)

Solution: Prove that A^TAx $= 0 \leftrightarrow x^T A^T$Ax $= 0 \leftrightarrow (Ax)^T$Ax $= 0 \leftrightarrow$ length of $Ax = 0 \leftrightarrow$ Ax $= 0$. The rank is the number of linearly independent columns and it has been proven that Ax $= 0$. As a result, rank(A) = rank(A^TA) $\leq$ rank(A^T) (Since the matrix A^TA is composed by the matrix multiplication between the every row of matrix A^T and column of matrix A, so the rank of A^TA will be always less or equal than rank of A^T. Redo the process above again but replace A with A^T, we have that rank(A^T) $\leq$ rank($(A^T)^T$), which is rank(A). As a result, rank(A) = rank(A^T)