Exercise 11

February 4, 2021

1. Consider the time-series model

   $x_{t+1} = Ax_t + w, w \sim \mathcal{N}(0, Q)$
   $y_t = Cx_t + v, v \sim \mathcal{N}(0, R)$

   Where w, v are i.i.d. Gaussian noise variables. Further, assume that $p(x_0) = \mathcal{N}(\mu_0, \Sigma_0)$

   (a) What is the form of $p(x_0, x_1, .., x_T))$? Justify your answer (you do not have to explicitly compute the joint distribution).

   (b) Assume that $p(x_t|y_1, ..., y_t) = \mathcal{N}(\mu_t, \Sigma_t)$

      i. Compute $p(x_{t+1}|y_1, ..., y_t)$

      ii. Compute $p(x_{t+1}, y_{t+1}|y_1, ..., y_t)$

      iii. At time $t + 1$, we observe the value $y_{t+1} = \hat{y}$. Compute the conditional distribution $p(x_{t+1}|y_1, ..., y_{t+1})$

2. Consider a Gaussian random variable $x \sim \mathcal{N}(x|\mu_x, \Sigma_x)$ where $x \in R^D$ Furthermore, we have
   $y = Ax + b + w$
   where $y \in R^E, A \in R^{E\times D}, b \in R^E$, and $w \sim \mathcal{N}(w|0, Q)$ is independent Gaussian noise. "Independent" implies that x and w are independent random variables and that Q is diagonal.

   (a) Write down the likelihood $p(y|x)$

   (b) The distribution $p(y) = \int p(y|x)p(x)dx$ is Gaussian. Compute the mean μ_y and the covariance Σ_y. Derive your result in detail.

(c) The random variable y is being transformed according to measurement mapping $z = Cy + v$, where $z \in R^F, C \in R^{F \times E}, v \sim \mathcal{N}(v|0, R)$ is independent Gaussian noise.

   i. Write down $p(z|y)$

   ii. Compute the mean μ_z and the covariance Σ_z. Derive your result in detail.

(d) Now, a value $\hat{y}$ is measured. Compute the posterior distribution $p(x|\hat{y})$

3. Given a continuous random variable X, with cdf $F_X(x)$. Show that the random variable $Y := F_X(X)$ is uniformly distributed.