Exercise 3

December 28, 2020

1. Consider the linear mapping
   $\phi = R^3 \to R^4$
   $$\phi(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}) = \begin{bmatrix} 3x_1 + 2x_2 + x_3 \\ x_1 + x_2 + x_3 \\ x_1 - 3x_2 \\ 2x_1 + 3x_2 + x_3 \end{bmatrix}$$
   (a) Find the transformation matrix A_ϕ
   (b) Determine $rk(A_\phi)$
   (c) Compute the kernel and image of ϕ. What are $dim(ker(\phi)$ and $dim(Im(\phi)$

2. Find the matrix to rotate the vectors
   $$x_1 := \begin{bmatrix} 2 \\ 3 \end{bmatrix}, x_2 := \begin{bmatrix} 0 \\ -1 \end{bmatrix} \text{ by } \pi/6$$

3. Are the following mappings linear:
   (a) $\phi : R \to R$
   $x \to \phi(x) = cos(x)$
   (b) $\phi : R^3 \to R^2$
   $$x \to \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 3 \end{bmatrix} x$$

4. Consider an endomorphism $\phi : R^3 \to R$ whose transformation matrix (with respect to the standard basis in R^3 is :

$$A_\phi = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

(a) Determine $ker(\phi)$ and $Im(\phi)$

(b) Determine the transformation matrix C_ϕ with respect to the basis
$$B = (\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} , \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} , \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix})$$
perform a basis change toward the new basis B

5. Let us consider b_1, b_2, b'_1, b'_2, 4 vectors of R^2 expressed in the standard basis of R^2 as:
$$b_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, b_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, b'_1 = \begin{bmatrix} 2 \\ -2 \end{bmatrix}, b'_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
and let us define two ordered bases $B = (b_1, b_2), B' = (b'_1, b'_2)$ of R^2

   (a) Show that B and B' are two bases of R^2 and draw those basis vectors.

   (b) . Compute the matrix P_1 that performs a basis change from B' to B

   (c) We consider c_1, c_2, c_3 three vectors of R^3 defined in the standard basis of R^3 as:
$$c_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, c_2 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, c_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$
and we define $C = (c_1, c_2, c_3)$

      i. Show that C is a basis of R^3, e.g., by using determinants

      ii. Let us call $C' = (c'_1, c'_2, c'_3)$ the standard basis of R^3 . Determine the matrix P_2 that performs the basis change from C to C'

   (d) We consider a homomorphism $\phi : R^2 \to R^3$, such that:
$\phi(b_1 + b_2) = c_2 + c_3$
$\phi(b_1 - b_2) = 2c_1 - c_2 + 3c_3$
where $B = (b_1, b_2$ and $C = c_1, c_2, c_3$ are ordered bases of R^2 and R_3 respectively.
Determine the transformation matrix A_ϕ of ϕ with respect to the ordered bases B and C

(e) Determine A', the transformation matrix of ϕ with respect to the bases B' and C'