Exercise 4

January 6, 2021

1. Show that $\langle .,. \rangle$ define for all $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \in R^2$ and $y = \begin{bmatrix} y_1 & y_2 \end{bmatrix}^T \in R^2$:
   $\langle x, y \rangle := x_1 y_1 - (x_1 y_2 + x_2 y_1) + 2(x_2 y_2)$ is an inner product

   Solution:
   We must show that $\langle \cdot, \cdot \rangle$ is a bilinear, symmetric, positive definite map.

   It is a calculation to show that $\langle \alpha x + \beta x', y \rangle = \alpha \langle x, y \rangle + \beta \langle x', y \rangle$, and similarly for the second argument, and so $\langle \cdot, \cdot \rangle$ is bilinear.

   We have $\langle y, x \rangle = y_1 x_1 - (y_1 x_2 + y_2 x_1) + 2 y_2 x_2 = \langle x, y \rangle$ indeed, so we have symmetry.

   Finally, $\langle x, x \rangle = x_1^2 - 2x_1 x_2 + 2x_2^2 = (x_1 - x_2)^2 + x_2^2$. Clearly, if $x = 0$, then $\langle x, x \rangle = 0$. If $x \neq 0$, then either $x_2 \neq 0$, in which case $\langle x, x \rangle > 0$; or $x_2 = 0$ with $x_1 \neq 0$, so $\langle x, x \rangle > 0$ again (since it's the sum of two squares of real numbers). Therefore $\langle \cdot, \cdot \rangle$ is an inner product indeed!

2. Compute the distance between:

   $$x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, y = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} \text{ using}$$

   (a) $\langle x, y \rangle := x^T y$

   (b) $\langle x, y \rangle := x^T A y, A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

Solution:

We have $d(x,y) = \sqrt{\langle (x-y),(x-y)\rangle} = \sqrt{\left\langle \begin{bmatrix}2\\3\\3\end{bmatrix}, \begin{bmatrix}2\\3\\3\end{bmatrix} \right\rangle}$.

(a) Thus $d(x,y) = \sqrt{22}$.

(b) Here, $d(x,y) = \sqrt{47}$.

3. Compute the angle between:
$x = \begin{bmatrix}1 & 2\end{bmatrix}^T, y = \begin{bmatrix}-1 & -1\end{bmatrix}^T$ using:

(a) $\langle x,y\rangle := x^T y$

(b) $\langle x,y\rangle := x^T By, B = \begin{bmatrix}2 & 1\\1 & 3\end{bmatrix}$

Solution:

We have that the angle ω between the two vectors is given by $\omega = \cos^{-1}\left(\frac{\langle x,y\rangle}{\|x\|\|y\|}\right)$, where $\|x\| = \sqrt{\langle x,x\rangle}$.

(a) Here, $\|x\| = \sqrt{6}$ and $\|y\| = \sqrt{2}$. Also, $\langle x,y\rangle = -3$. Thus $\omega = \cos^{-1}\left(\frac{-3}{\sqrt{12}}\right) = \frac{5\pi}{6}$.

(b) Now, $\|x\| = \sqrt{18}$ and $\|y\| = \sqrt{7}$. Also, $\langle x,y\rangle = -11$. Thus $\omega = \cos^{-1}\left(\frac{-11}{\sqrt{126}}\right) = \cos^{-1}\left(\frac{-11\sqrt{14}}{42}\right)$.

4. Consider the Euclidean vector space R^5 with the dot product. A subspace $U \subseteq R^5$ are given by:

$$U = span[\begin{bmatrix}0\\-1\\2\\0\\2\end{bmatrix}, \begin{bmatrix}1\\-3\\1\\-1\\2\end{bmatrix}, \begin{bmatrix}-3\\4\\1\\2\\1\end{bmatrix}, \begin{bmatrix}-1\\-3\\5\\0\\7\end{bmatrix}], x = \begin{bmatrix}-1\\-9\\-1\\4\\1\end{bmatrix}$$

(a) Determine the orthogonal projection $\pi_U(x)$ of x onto U

(b) Determine the distance $d(x,U)$

Solution:

(a) Let $v_1, \ldots, v_4$ be the four vectors defined in the question. Observe that $rank[v_1|v_2|v_3|v_4] = 3$. Moreover, $rank[v_1|v_2|v_3] = 3$, so these three vectors form a basis of U. Let B be this matrix of basis vectors, i.e. $B = [v_1|v_2|v_3]$.

Now we compute $B^\mathsf{T} B = \begin{bmatrix} 9 & 9 & 0 \\ 9 & 16 & -14 \\ 0 & -14 & 31 \end{bmatrix}$ and $B^\mathsf{T} x = \begin{bmatrix} 9 \\ 23 \\ -25 \end{bmatrix}$.

Next, we solve $B^\mathsf{T} B \lambda = B^\mathsf{T} x$ for λ. Using Gaussian elimination, we obtain $\lambda = \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}$.

Finally, $\pi_U(x)$ is given by $B\lambda = \begin{bmatrix} 1 \\ -5 \\ -1 \\ -2 \\ 3 \end{bmatrix}$.

(b) By construction, $d(x, U) = d(x, \pi_U(x))$. This is given by $\|x - \pi_U(x)\| = \left\| \begin{bmatrix} -2 \\ -4 \\ 0 \\ 6 \\ -2 \end{bmatrix} \right\|$.

This norm is given by the square root of the dot product of the vector with itself, so $d(x, U) = \sqrt{60} = 2\sqrt{15}$.

5. Consider R^3 with the inner product:

$$\langle x, y \rangle := x^T \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} y$$

Furthermore, we define e_1, e_2, e_3as the standard/canonical basis in R^3

(a) Determine the orthogonal projection $\pi_U(e_2)$ of e_2 onto:
$U = span \begin{bmatrix} e_1 & e_3 \end{bmatrix}$

Hint: Orthogonality is defined through the inner product

(b) Compute the distance $d(e_2, U)$

(c) Draw the scenario: standard basis vectors and $\pi_U(e_2)$

Solution:

(a) We require $\langle e_1, e_2 - \pi_U(e_2)\rangle = 0$ and $\langle e_3, e_2 - \pi_U(e_2)\rangle = 0$. That is to say, $\left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -\lambda_1 \\ 1 \\ -\lambda_2 \end{bmatrix} \right\rangle = 0$, and $\left\langle \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -\lambda_1 \\ 1 \\ -\lambda_2 \end{bmatrix} \right\rangle = 0$.
Computing the first gives us $\lambda_1 = \frac{1}{2}$, while the second gives $\lambda_2 = -\frac{1}{2}$.

Therefore, $\pi_U(e_2) = \begin{bmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix}$.

(b) Part b We have $d(e_2, U) = \|e_2 - \pi_U(e_2)\| = \sqrt{\left\langle \begin{bmatrix} -\frac{1}{2} \\ 1 \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ 1 \\ \frac{1}{2} \end{bmatrix} \right\rangle} =$
1.

6. Let V be a vector space and π an endomorphism of V

(a) Prove that π is a projection if and only if $id_V - \pi$ is a projection, where id_V is the identity endomorphism on V

(b) Assume now that π is a projection. Calculate $Im(id_V - \pi$ and $\ker(id_V - \pi$ as a function of $Im(\pi)$ and $ker(\pi)$

Solution

(a) Let $x \in V$. Observe that $((id - \pi) \circ (id - \pi))(x) = (id - \pi)(x - \pi(x)) = (x - \pi(x)) - (\pi(x) - \pi^2(x)) = x - 2\pi(x) + \pi^2(x)$. Hence $(id - \pi)$ is a projection if and only if $x - \pi(x) = x - 2\pi(x) + \pi^2(x)$. This happens if and only if $\pi(x) = \pi^2(x)$, that is to say, where π is a projection.

(b) We have $Im(id-\pi) = \{(id-\pi)(x) : x \in V\} = \{x-\pi(x) : x \in V\}$. Observe that $\pi(x - \pi(x)) = \pi(x) - \pi^2(x) = 0$, since π is a projection. Thus, $Im(id - \pi) \subseteq \ker \pi$.

Now, suppose $k \in \ker \pi$. Then $k - \pi(k) = k$, so $k \in Im(id - \pi)$. Thus $\ker \pi \subseteq Im(id - \pi)$. Therefore, we have that $Im(id - \pi) = \ker \pi$.

We have that $\ker(id - \pi) = \{x \in V : (id - \pi)(x) = 0\} = \{x \in V : x = \pi(x)\}$. Clearly, $\ker(id - \pi) \subseteq Im\pi$.

Take $x \in Im\pi$. Then there exists some $y \in V$ such that $\pi(y) = x$. Observe that $(id - \pi)(x) = x - \pi(x) = \pi(y) - \pi^2(y) = 0$, since π is a projection. Hence $Im\pi \subseteq \ker(id-\pi)$. Therefore we have that $\ker(id - \pi) = Im\pi$.