# Solution 5

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1. Compute the determinant using the Laplace expansion (using the first row) and the Sarrus Rule for

   $$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 0 & 2 & 4 \end{bmatrix}$$

   Solution:

   Laplace Expansion:
   We have $\det A = 1(4\cdot 4 - 6\cdot 2) - 3(2\cdot 4 - 6\cdot 0) + 5(2\cdot 2 - 4\cdot 0) = 0$.
   Sarrus Rule
   We have $\det A = 1\cdot 4\cdot 4 + 2\cdot 2\cdot 5 + 0\cdot 3\cdot 6 - 5\cdot 4\cdot 0 - 6\cdot 2\cdot 1 - 4\cdot 3\cdot 2 = 0$.

2. Compute the following determinant efficiently:

   $$\begin{bmatrix} 2 & 0 & 1 & 2 & 0 \\ 2 & -1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 & 2 \\ -2 & 0 & 2 & -1 & 2 \\ 2 & 0 & 0 & 1 & 1 \end{bmatrix}$$

   Solution:
   Perform Gaussian elimination to "fix" the first columns. I have used only the rule allowing me to add a multiple of a row to a different row, which doesn't change the determinant. We have $\begin{bmatrix} 2 & 0 & 1 & 2 & 0 \\ 0 & -1 & -1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 3 & 1 & 2 \\ 0 & 0 & -1 & -1 & 1 \end{bmatrix}$.

From here, we can either continue with Gaussian elimination to get our matrix into upper triangular form, then multiply the entries on the diagonal together (remembering to take into account any elementary operations which would change the determinant!), or we can simply compute the determinant of the lower-right $3 \times 3$ matrix, since this is quick to do by hand (it is $-3$). Thus the determinant of the overall matrix is $2 \cdot -1 \cdot -3 = 6$.

3. Compute the eigenspaces of $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -2 & 2 \\ 2 & 1 \end{bmatrix}$

   Solution:

   (a) If we solve the equation $\det(A - \lambda I) = 0$ for $\lambda$, we obtain $\lambda = 1$ only. (Or, indeed, we can observe that $A$ is in lower triangular form, so the eigenvalues are the entries on the main diagonal.)

   The space $E_1 = \{x \in R^2 : (A - I)(x) = 0\} = span\{(0, 1)\}$.

   (b) Again, solving $\det(B - \lambda I) = 0$, we find that $\lambda = 2$ or $\lambda = -3$. We then have $E_2 = span\{(1, 2)\}$ and $E_{-3} = span\{(-2, 1)\}$.

4. Compute the eigenspaces of $A = \begin{bmatrix} 0 & -1 & 1 & 1 \\ -1 & 1 & -2 & 3 \\ 2 & -1 & 0 & 0 \\ 2 & -1 & 1 & 0 \end{bmatrix}$

   Solution:
   If we take $\det(A - \lambda I)$ and simplify, we have $(\lambda - 2)(\lambda - 1)(\lambda + 1)^2$., so we have three eigenvalues. For $\lambda = 2, 1$, our eigenspace will certainly have dimension 1. For $\lambda = -1$, it could (at this stage!) have dimension 1 or 2.

   Observe that $(1, 0, 1, 1)$ is an eigenvector with eigenvalue 2, and $(1, 1, 1, 1)$ is an eigenvector with eigenvalue 1. Thus $E_2 = span\{(1, 0, 1, 1)\}$ and $E_1 = span\{(1, 1, 1, 1)\}$.

   Now observe that $rank(A + I) = 3$, so there can only be one linearly independent eigenvector with eigenvalue -1. Note that $(0, 1, 1, 0)$ will do. Hence $E_{-1} = span\{(0, 1, 1, 0)\}$.

5. Diagonalizability of a matrix is unrelated to its invertibility. Determine for the following four matrices whether they are diagonalizable and/or invertible $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

   Solution:
   Only the first two are diagonalisable – they are diagonal, so there is nothing to prove! The other two, however, are not diagonalisable – each only has one linearly independent eigenvector, whereas we need there to exist a basis of eigenvectors to yield diagonalisability.

   Only the first and third matrices are invertible – the determinants are non-zero, while the other two matrices have determinant zero.

   This tells us that diagonalisability is indeed unrelated to invertibility!

6. Compute the eigenspaces of the following transformation matrices. Are they diagonalizable?

   (a) $A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

   Solution:

   (a) The characteristic polynomial is $(5 - \lambda)(1 - \lambda)^2$. Now, $rank(A - I) = 2$, so there is only one linearly independent eigenvector for $\lambda = 1$. Hence $A$ is not diagonalisable.

      We have $E_5 = span\{(1, 1, 0)\}$, and $E_1 = span\{(-3, 1, 0)\}$.

   (b) The characteristic polynomial here is $-\lambda^3(1 - \lambda)$, so the eigenvalues are 1, and 0 (with multiplicity 3). Observe that $rank(A - 0I) = rankA = 1$, so there are three linearly independent eigenvectors for the eigenvalue 0. With the other eigenvector for $\lambda = 1$, we will have a basis of eigenvectors, and hence $A$ will be diagonalisable.

We have $E_1 = span\{(1,0,0,0)\}$, and $E_0 = span\{(1,-1,0,0),(0,0,1,0),(0,0,0,1)\}$.

7. Show that for any $A \in R^{n\times m}$ the matrices $A^T A$ and $AA^T$ possess the same nonzero eigenvalues.

   Solution:

   We need only show this in one direction, since we can replace $A$ with $A^{\mathsf{T}}$ below, and the argument still hold true.

   Suppose $\lambda \neq 0$ is an eigenvalue of $A^{\mathsf{T}}A$. Then there is some vector $x \neq 0$ such that $A^{\mathsf{T}}Ax = \lambda x$. Thus, $AA^{\mathsf{T}}Ax = \lambda Ax$. Therefore, $Ax$ (equivalently $\lambda x$, or, indeed just $x$ itself!) is an eigenvector of $AA^{\mathsf{T}}$, with eigenvalue $\lambda$.