Solution 6

January 13, 2021

1. Are the following matrices diagonalizable? If yes, determine their diagonal form and a basis with respect to which the transformation matrices are diagonal. If no, give reasons why they are not diagonalizable.

   (a) $A = \begin{bmatrix} 0 & 1 \\ -8 & 4 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

   (c) $A = \begin{bmatrix} 5 & 4 & 2 & 1 \\ 0 & 1 & -1 & -1 \\ -1 & -1 & 3 & 0 \\ 1 & 1 & -1 & 2 \end{bmatrix}$

   (d) $A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$

   Solution:

   (a) If we form the characteristic polynomial, we have $\lambda^2 - 4\lambda + 8$, which has no roots in R. However, if we extend to C, then we will be able to diagonalise the matrix.

   (b) This is a symmetric matrix, and is therefore diagonalisable. Its eigenvalues are 3, and 0 with multiplicity two, so its diagonal form is $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, and a basis of eigenvectors is $\{(1,1,1),(1,-1,0),(1,0,-1)\}$.

(c) Here, we have three distinct eigenvalues, and $\lambda = 4$ has multiplicity two. However, $rank(A - 4I) = 3$, so there is only one linearly independent eigenvector, and this A cannot have a basis of eigenvectors, so it is not diagonalisable.

(d) Again here we have two eigenvectors – $\lambda = 1$ with multiplicity one and $\lambda = 2$ with multiplicity two. However, this time, observe that $rank(A - 2I) = 1$, so there are indeed two linearly independent eigenvectors for this eigenvalue. Thus A is diagonalisable, with diagonal form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, with eigenvectors $\{(3, -1, 3), (2, 1, 0), (2, 0, 1)\}$.

2. Find the SVD of the matrix

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$$

Solution:

(a) First, we compute $A^\mathsf{T} A = \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix}$. We can diagonalise this to find $D = \begin{bmatrix} 25 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{18}} & -\frac{2}{3} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{18}} & \frac{2}{3} \\ 0 & \frac{4}{\sqrt{18}} & \frac{1}{3} \end{bmatrix}$.

We take the square roots of the entries of D to find $\Sigma = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}$, with V equalling our P above.

From here, we compute $u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{5} \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$, and $u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{3} \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{18}} \\ -\frac{1}{\sqrt{18}} \\ \frac{4}{\sqrt{18}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$, giving $U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

It can be checked that $A = U\Sigma V^{\mathsf{T}}$, indeed!

3. Find the singular value decomposition of $A = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$

   Solution: Observe that the eigenvalues of A are complex, so we cannot simply find the eigendecomposition. Proceeding as in the previous question, we have $A^{\mathsf{T}}A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$, which when we perform the eigendecomposition on this new matrix, we obtain $D = \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix}$ and $P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. This P is again our required V, and we have $\Sigma = \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$.

   As before, we can now compute $u_1 = \frac{1}{2\sqrt{2}} \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and similarly $u_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Hence, our U turns out to be the identity matrix.

4. Find the rank-1 approximation of $A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$

   Solution:

   Using our Singular value decomposition from Question 8, we construct $A_1 = \sigma_1 u_1 v_1^{\mathsf{T}} = 5 \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} = \frac{5}{2} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$, and similarly $A_2 = \frac{1}{2} \begin{bmatrix} 1 & -1 & 4 \\ -1 & 1 & -4 \end{bmatrix}$. Then A_1 and A_2 both have rank one, with $A = A_1 + A_2$, as required.

5. Show that for $x \neq 0$
   $\max_x \frac{\|Ax\|_2}{\|x\|_2} = \theta_1$, where θ_1 is the largest singular value of $A \in R^{m\times n}$

   Solution:

The left hand side describes the largest $\|Ax\|_2$ can be, relative to $\|x\|_2$. That is to say, it represents the biggest scaling that can take place under A. If we write $A = U\Sigma V^{\mathsf{T}}$, and then note that U and V^{T} are both change of basis matrices, we deduce that only Σ is doing any scaling. But Σ is an "almost diagonal" matrix, and hence it scales based on its non-zero entries only. Remember that (by convention!) all the entries on the main diagonal of Σ are non-negative. Hence the largest such value represents the biggest scaling factor, which is the right-hand side of the identity.