Solution Of Exercise 3

January 1, 2021

1. Consider the linear mapping
   $\phi = R^3 \to R^4$
   $$\phi(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}) = \begin{bmatrix} 3x_1 + 2x_2 + x_3 \\ x_1 + x_2 + x_3 \\ x_1 - 3x_2 \\ 2x_1 + 3x_2 + x_3 \end{bmatrix}$$

   (a) Find the transformation matrix A_ϕ

   (b) Determine $rk(A_\phi)$

   (c) Compute the kernel and image of ϕ. What are $dim(ker(\phi)$ and $dim(Im(\phi)$

   Solution:

   From the coefficients on the right, we have $A_\Phi = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & -3 & 0 \\ 2 & 3 & 1 \end{bmatrix}$.

   Using Gaussian elimination, we can compute that $rank(A_\Phi) = 3$. From this we deduce that the kernel is trivial (i.e. only $(0,0,0)$), and clearly $Im(\Phi) = \{(3x_1 + 2x_2 + x_1, x_1 + x_2 + x_3, x_1 - 3x_2, 2x_1 + 3x_2 + x_3)^\intercal : x_1, x_2, x_3 \in R\}$. We have $\dim(\ker(\Phi)) = 0$, and $\dim(Im(\Phi)) = rank(A_\Phi) = 3$.

2. Find the matrix to rotate the vectors

   $x_1 := \begin{bmatrix} 2 \\ 3 \end{bmatrix}, x_2 := \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ by $\pi/6$

Solution:
I will assume we need to rotate these vectors anticlockwise. A rotation about an angle θ is given by the matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Thus for $\theta = 30°$, we have $R = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$.
Therefore, $Rx_1 = \frac{1}{2}\begin{bmatrix} 2\sqrt{3}-3 \\ 2+3\sqrt{3} \end{bmatrix}$ and $Rx_2 = \frac{1}{2}\begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix}$.

3. Are the following mappings linear:

   (a) $\phi : R \to R$
   $x \to \phi(x) = cos(x)$

   (b) $\phi : R^3 \to R^2$
   $x \to \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 3 \end{bmatrix} x$

   Solution

   (a) This is not linear – Φ doesn't even map 0 to 0, indeed!

   (b) We know that any matrix transformation like this is indeed linear. This comes from distributive properties of matrix multiplication.

4. Consider an endomorphism $\phi : R^3 \to R^3$ whose transformation matrix (with respect to the standard basis in R^3 is :
   $A_\phi = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

   (a) Determine $ker(\phi)$ and $Im(\phi)$

   (b) Determine the transformation matrix C_ϕ with respect to the basis
   $B = (\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix})$
   perform a basis change toward the new basis B

   Solution

   (a) Note that $rank(A_\Phi) = 3$, so $\ker(\Phi) = \{0\}$ and $Im(\Phi) = R^3$

(b) Let P be the change of basis matrix from the standard basis of B to R^3. Then $P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix}$.

The matrix C_ϕ is given by $P^{-1} A_\Phi P = \begin{bmatrix} 6 & 9 & 1 \\ -3 & -5 & 0 \\ -1 & -1 & 0 \end{bmatrix}$.

5. Let us consider $b_1, b_2, b'_1, b'_2, 4$ vectors of R^2 expressed in the standard basis of R^2 as:
$b_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, b_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, b'_1 = \begin{bmatrix} 2 \\ -2 \end{bmatrix}, b'_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
and let us define two ordered bases $B = (b_1, b_2), B' = (b'_1, b'_2)$ of R^2

   (a) Show that B and B' are two bases of R^2 and draw those basis vectors.

   (b) . Compute the matrix P_1 that performs a basis change from B' to B

   (c) We consider c_1, c_2, c_3 three vectors of R^3 defined in the standard basis of R^3 as:
   $c_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, c_2 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, c_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$
   and we define $C = (c_1, c_2, c_3)$

      i. Show that C is a basis of R^3, e.g., by using determinants

      ii. Let us call $C' = (c'_1, c'_2, c'_3)$ the standard basis of R^3 . Determine the matrix P_2 that performs the basis change from C to C'

   (d) We consider a homomorphism $\phi : R^2 \to R^3$, such that:
   $\phi(b_1 + b_2) = c_2 + c_3$
   $\phi(b_1 - b_2) = 2c_1 - c_2 + 3c_3$
   where $B = (b_1, b_2$ and $C = c_1, c_2, c_3$ are ordered bases of R^2 and R_3 respectively.
   Determine the transformation matrix A_ϕ of ϕ with respect to the ordered bases B and C

   (e) Determine A', the transformation matrix of ϕ with respect to the bases B' and C'

Solution

(a) Each set B and B' has the correct number of (clearly!) linearly independent vectors, so they are both bases of R^2.

(b) We write the old basis vectors (B') in terms of the new (B), and then transpose the matrix of coefficients. We have $b'_1 = 4b_1 + 6b_2$, and $b'_2 = 0b_1 - b_2$. Thus $P_1 = \begin{bmatrix} 4 & 0 \\ 6 & -1 \end{bmatrix}$.

(c) Let $M = [c_1|c_2|c_3]$, and observe that $\det M = 4 \neq 0$, so the vectors are linearly independent. Since R^3 had dimension 3, and we have three linearly independent vectors, C must indeed be a basis.
Indeed, such an M is the change of basis matrix from C to C' (write the old vectors in terms of the new!) and this is thus the P_2 we require. Thus $P_2 = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ -1 & 2 & -1 \end{bmatrix}$.

(d) Observe that by adding the given results, we find that $\Phi(b_1) = c_1 + 2c_3$; by subtracting, we have $\Phi(b_2) = -c_1 + c_2 - c_3$. Then A_Φ is given by the transpose of the matrix of coefficients, so $A_\Phi = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & -1 \end{bmatrix}$.

(e) We first need to apply P_1 to change from basis B' to B. Then A_Φ will map us to (R^3, C), before P_2 will take us to C'. Remember that matrices are acting like functions here, so they are applied to (column) vectors from right to left. Therefore the multiplication we require is $A' = P_2 A_\Phi P_1$. (This is what part f is asking us to recognise.)

We have $A' = \begin{bmatrix} 0 & 2 \\ -10 & 3 \\ 12 & -4 \end{bmatrix}$.