Eigenvalue and Eigenvector

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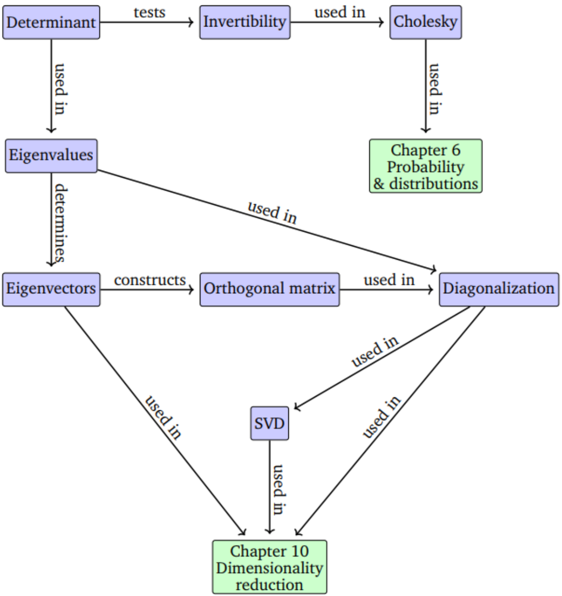
Overview

Roadmap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Determinant

Determinants are only defined for square matrices $A \in \mathrm{R}^{n \times n}$. In this book, we write the determinant as det(A) or sometimes as $|A|$ so that

$$det(A) = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{P}nn \end{vmatrix}$$

$$det(A) : \mathrm{R}^{n \times n} \rightarrow \mathrm{R}$$

For any square matrix $A \in \mathrm{R}^{n \times n}$ it holds that A is invertible if and only if $\det(A) \neq 0$.

For example:

- $n = 1$, $A = a_{11}$, $\det(A) = a_{11}$

$$A^{-1} = \frac{1}{a_{11}}$$

- $n = 2$, $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $\det(A) = a_{11}a_{22} - a_{12}a_{21}$

$$A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Determinant (cont.)

Consider a matrix $A \in R^{n \times n}$. Then, for all j = 1,..,n:

- Expansion along column j

$$det(A) = \sum_{k=1}^{n} (-1)^{k+j} a_{kj} det(A_{k,j})$$

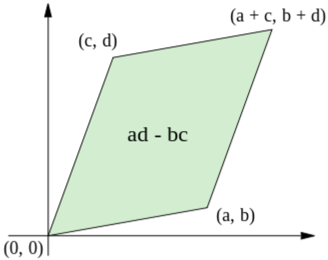
- Expansion along row j

$$det(A) = \sum_{k=1}^{n} (-1)^{k+j} a_{jk} det(A_{j,k})$$

Here $A_{k,j} \in R^{(n1)\times(n1)}$ is the submatrix of A that we obtain when deleting row k and column j.
Proof: a triangular matrix $T \in R^{n \times n}$, the determinant is the product of the diagonal matrix elements

$$det(T) = \prod_{i=1}^{n} T_{ii}$$

Determinants as Measures of Volume



Hình 1: $\begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc$

When is the determinant 0?

Determinant (cont.)

Properties:

- Adding a multiple of a column/row to another one does not change det(A).
- Multiplication of a column/row with $\lambda \in$ R scales det(A) by . In particular, det(A) $= \lambda^n$det(A).
- Swapping two rows/columns changes the sign of det(A).
- Gaussian elimination does not change det(A), det(EA) = det(A).
- The determinant of a matrix product is the product of the corresponding determinants, det(AB) = det(A)det(B)
- displaystyle $\det(A^{-1}) = \frac{1}{det(A)}$

A square matrix $A \in \mathrm{R}^{n \times n}$ has det(A) $\neq$ 0 if and only if rk(A) = n. In other words, A is invertible if and only if it is full rank

Trace

The trace of a square matrix $A \in \mathrm{R}^{nn}$ is defined as

$$tr(A) := \sum_{i=1}^{n} a_{ii}$$

The trace is the sum of the diagonal elements of A.

Properties:

- $\mathrm{tr(A+B) = tr(A) + tr(B)}$
- $\mathrm{tr(}\alpha\ \mathrm{A) =}\ \alpha\mathrm{tr(A)}$
- $\mathrm{tr}(I_n) = \mathrm{n}$
- $\mathrm{tr(AB) = tr(BA)}$

Trace (cont.)

Characteristic Polynomial: For $\lambda \in R$ and a square matrix $A \in R^{n \times n}$

$$p_A(\lambda) := det(A - \lambda I) = c_0 + c_1\lambda + c_2\lambda^2 + ... + c_{n-1}\lambda^{n-1} + (-1)^n\lambda^n$$

$c_0, c_1, ..., c_{n-1} \in R$, is the characteristic polynomial of A. In particular

$$c_0 = det(A)$$
$$c_{n-1} = (-1)^{n-1}tr(A)$$

Proof?

Eigenvalue and eigenvector

Let $A \in \mathrm{R}^{n \times n}$ be a square matrix. Then $\lambda \in \mathrm{R}$ is an eigenvalue of A and $x \in \mathrm{R}^n/\{0\}$ is the corresponding eigenvector of A if

$$Ax = \lambda x$$

The following statements are equivalent:

- is an eigenvalue of A 2 $R^{n \times n}$.
- There exists an $x \in \mathrm{R}^n$ with Ax = x, or equivalently, (A - λI_n)x = 0 can be solved non-trivially, i.e., $x \neq 0$.
- rk(A - λI_n) < n.
- det(A - λI_n) = 0

$\lambda \in \mathrm{R}$ is an eigenvalue of $A \in \mathrm{R}^{n \times n}$ if and only if is a root of the characteristic polynomial $p_A(\lambda)$ of A

Eigenspace and Eigenspectrum: For $A \in \mathrm{R}^{n \times n}$, the set of all eigenvectors of A associated with an eigenvalue spans a subspace eigenspace of R^n, which is called the eigenspace of A with respect to and is denoted eigenspectrum by E_λ. The set of all eigenvalues of A is called the eigenspectrum, or just spectrum spectrum, of A.

Let a square matrix A have an eigenvalue λ_i. The algebraic multiplicity of λ_i is the number of times the root appears in the characteristic polynomial. Then the geometric geometric multiplicity of λ_i is the number of linearly independent eigen vectors associated with λ_i. In other words, it is the dimensionality of the eigenspace spanned by the eigenvectors associated with λ_i.

For example:
$A = I_n \Rightarrow p_I(\lambda) = det(I - \lambda I) = (1 - \lambda)^n = 0 \Rightarrow$ A has one eigenvalue $\lambda = 1$ that occurs n times.

Moreover, Ix = x = 1x holds for all vectors $x \in \mathrm{R}^n/\{0\}$

Eigenvalue and eigenvector (cont.)

Properties:

- A matrix A and its transpose A^T possess the same eigenvalues, but not necessarily the same eigenvectors.
- Symmetric, positive definite matrices always have positive, real eigenvalues

Computing Eigenvalues, Eigenvectors, and Eigenspaces The matrix:

$$A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$$

Steps:

- Find eigenvalues by characteristic polynomial
- Find eigenvectors and eigenspaces

Eigenvalue and eigenvector (cont.)

Given a matrix $A \in \mathrm{R}^{m \times n}$, we can always obtain a symmetric, positive semidefinite matrix $S \in \mathrm{R}^{n \times n}$ by defining

$$S := A^T A$$

If rk(A) = n, then S := $A^T A$ is symmetric, positive definite.

If $A \in \mathrm{R}^{n \times n}$ is symmetric, there exists an orthonormal basis of the corresponding vector space V consisting of eigenvectors of A, and each eigenvalue is real.

Cholesky Decomposition

The Cholesky decomposition/Cholesky factorization provides a square-root equivalent operation on symmetric.

A symmetric, positive definite matrix A can be factorized into a product $A = LL^T$, where L is a lower triangular matrix with positive diagonal elements:

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} = \begin{bmatrix} l_1 & \dots & 0 \\ \dots & \dots & \dots \\ l_{n1} & \dots & l_{nn} \end{bmatrix} \begin{bmatrix} l_1 & \dots & l_{n1} \\ \dots & \dots & \dots \\ 0 & \dots & l_{nn} \end{bmatrix}$$

L is called the Cholesky factor of A, and L is unique.

Exercise: Find the Cholesky decomposition of matrix:

$$\begin{bmatrix} 25 & 15 & 5 \\ 15 & 18 & 0 \\ 5 & 0 & 11 \end{bmatrix}$$