Exercise 3

January 1, 2021

1. Show that $\langle ., . \rangle$ define for all $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \in R^2$ and $y = \begin{bmatrix} y_1 & y_2 \end{bmatrix}^T \in R^2$:
   $\langle x, y \rangle := x_1 y_1 - (x_1 y_2 + x_2 y_1) + 2(x_2 y_2)$ is an inner product

2. Compute the distance between:

   $$x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, y = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} \text{ using}$$

   (a) $\langle x, y \rangle := x^T y$

   (b) $\langle x, y \rangle := x^T A y, A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

3. Compute the angle between:
   $x = \begin{bmatrix} 1 & 2 \end{bmatrix}^T, y = \begin{bmatrix} -1 & -1 \end{bmatrix}^T$ using:

   (a) $\langle x, y \rangle := x^T y$

   (b) $\langle x, y \rangle := x^T B y, B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$

4. Consider the Euclidean vector space R^5 with the dot product. A subspace $U \subseteq R^5$ are given by:

$$U = span[\begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 5 \\ 0 \\ 7 \end{bmatrix}], x = \begin{bmatrix} -1 \\ -9 \\ -1 \\ 4 \\ 1 \end{bmatrix}$$

(a) Determine the orthogonal projection $\pi_U(x)$ of x onto U

(b) Determine the distance $d(x, U)$

5. Consider R^3 with the inner product:

$$\langle x, y \rangle := x^T \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} y$$

Furthermore, we define e_1, e_2, e_3as the standard/canonical basis in R^3

(a) Determine the orthogonal projection $\pi_U(e_2)$ of e_2 onto:
$U = span \begin{bmatrix} e_1 & e_3 \end{bmatrix}$
Hint: Orthogonality is defined through the inner product

(b) Compute the distance $d(e_2, U)$

(c) Draw the scenario: standard basis vectors and $\pi_U(e_2)$

(d) Let V be a vector space and π an endomorphism of V

6. Let V be a vector space and π an endomorphism of V

(a) Prove that π is a projection if and only if $id_V - \pi$ is a projection, where id_V is the identity endomorphism on V

(b) Assume now that π is a projection. Calculate $Im(id_V - \pi$ and $\ker(id_V - \pi$ as a function of $Im(\pi)$ and $ker(\pi)$