Exercise 2

December 26, 2020

1. Consider set G of 3×3 matrices defined as follows:

   $$G = \{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \in R^{3\times3} | x, y, z \in R \}$$

   We define $\cdot$ as the standard matrix multiplication.
   Is (G,$\cdot$) a group? If yes, is it Abelian? Justify your answer

   Solution:
   Let $A_1 = \begin{bmatrix} 1 & x_1 & y_1 \\ 0 & 1 & z_1 \\ 0 & 0 & 1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 1 & x_2 & y_2 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{bmatrix} \in G$
   Then $A_1 A_2 = \begin{bmatrix} 1 & x_1 + x_2 & y_1 + x_1 z_2 + y_2 \\ 0 & 1 & z_1 + z_2 \\ 0 & 0 & 1 \end{bmatrix} \in G$, so we have closure.

   Associativity follows from the associativity of standard matrix multiplication.

   Letting $x = y = z = 0$, observe that the identity is in G

   Finally, if we take $x_2 = -x_1$, $z_2 = -z_1$, and $y_2 = -y_1 - x_1 z_2$, then observe that $A_1 A_2 = I_3$, and thus inverses are of the required form! Therefore, G is a group.

   The group is not abelian, e.g. take $x_1 = z_2 = 1$ and everything else to be 0. Then multiplying these matrices in the other order (i.e. $x_2 = z_1 = 1$) gives a different answer.

2. Which of the following sets are subspaces of R^3:

   (a) $A = \{(\lambda, \lambda + \mu^3, \lambda - \mu^3) | \lambda, \mu \in R\}$
   (b) $B = \{(\lambda^2, -\lambda^2, 0) | \lambda \in R)\}$
   (c) $C = \{(\xi_1, \xi_2, \xi_3) \in R^3 | \xi_2 \in Z\}$

   Solution:

   (a) We can relabel μ^3 as ν, so ν can be any real number, and then we have $A = \{(\lambda, \lambda + \nu, \lambda - \nu)^\mathsf{T} : \lambda, \nu \in R\}$. This has a basis of $\{(1, 1, 1)^\mathsf{T}, (0, 1, -1)^\mathsf{T}\}$, so it is a subspace of R^3.
   (b) We cannot do the same trick as before, since the square of a real number is always at least zero. Clearly $(1, -1, 0)^\mathsf{T} \in B$, but -1 times this vector, i.e. $(-1, 1, 0)^\mathsf{T} \notin B$, and thus B is not a subspace.
   (c) This is not a subspace. Observe that $(0, 1, 0)^\mathsf{T} \in D$, so if D were a subspace, then any (real!) multiple should be in D also. However, $\frac{1}{2}(0, 1, 0)^\mathsf{T} \notin D$.

3. Consider two subspaces of R^4:

   U_1 $=Span[\begin{bmatrix} 1 \\ 1 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}], U_2 =$Span$[\begin{bmatrix} -1 \\ -2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \\ -2 \\ -1 \end{bmatrix}]$

   Determine a basis of $U_1 \cap U_2$

   Solution:
   We write the given vectors as $v_1, \ldots v_6$ from left to right. Firstly, observe that $\dim(U_1) = 2$ and $\dim(U_2) = 2$ (compute the rank of $[v_1|v_2|v_3]$, then $[v_4|v_5|v_6]$). Since we can write $v_3 = \frac{1}{3}(v_1 - 2v_2)$ and $v_6 = -v_4 - 2v_5$, we need not consider v_3 and v_6 any further.

   Now, if we find the rank of $[v_1|v_2|v_4|v_5]$, we get 3, so $\dim(U_1 + U_2) = 3$. Therefore, $\dim(U_1 \cap U_2) = 2 + 2 - 3 = 1$. Hence, to find a basis of $U_1 \cap U_2$, we need only find any non-zero vector in the space.

   Let $0 \neq v \in U_1 \cap U_2$. Then we can write $v = \alpha_1 v_1 + \alpha_2 v_2$, and $v = \alpha_4 v_4 + \alpha_5 v_5$. Subtracting these equations, we have $0 = \alpha_1 v_1 + \alpha_2 v_2 - \alpha_4 v_4 - \alpha_5 v_5$.

Remember we want a non-zero solution for v, and observe that the rank of $[v_1|v_2|v_4]$ is 3 (i.e. these three vectors are linearly independent). Hence we can take $\alpha_5 = 9$, say (this means we don't have fractions later, but there's no way to know this a priori!), and solve for the other α_i's. Using Gaussian elimination, we obtain $\alpha_1 = 4$, $\alpha_2 = 10$ and $\alpha_4 = -6$. Thus

$$v = 4v_1 + 10v_2 = -6v_4 + 9v_5 = \begin{bmatrix} 24 \\ -6 \\ -12 \\ -6 \end{bmatrix}.$$

Finally, we can write our basis of $U_1 \cap U_2$ as just $\{v\}$.

4. Are the following sets of vectors linearly independent?

(a) $x_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, x_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, x_3 = \begin{bmatrix} 3 \\ -3 \\ 8 \end{bmatrix}$

(b) $x_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, x_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, x_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

Solution:

(a) If we form a matrix out of these three vectors and compute its determinant, we get zero. Thus, the vectors are not linearly independent.

(b) Let's form the equation $\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0$. These three vectors are linearly independent if and only if the *only* solution to this equation is $\alpha_1 = \alpha_2 = \alpha_3 = 0$.

Looking at the third component, we have that $\alpha_1 \cdot 1 + \alpha_2 \cdot 0 + \alpha_3 \cdot 0 = 0$, that is to say, $\alpha_1 = 0$.

Next, look at the second component. We already know $\alpha_1 = 0$, so we have $\alpha_2 \cdot 1 + \alpha_3 \cdot 0 = 0$, that is to say $\alpha_2 = 0$ also.

Finally, look at the first component. We have that $\alpha_3 \cdot 1 = 0$, so all of the α_i's are zero. Therefore, our three vectors are linearly independent.

5. Proof: rank(A) = rank(A^T)

   Define $rank(A)$ to mean the column rank of A:
   col $rank(A) = dim\{Ax : x \in R^n\}$
   First show that $A^t Ax = 0$ if and only if $Ax = 0$. This is standard linear algebra: one direction is trivial, the other follows from:

   $A^T Ax = 0 \Rightarrow x^t A^T Ax = 0 \Rightarrow (Ax)^T (Ax) = 0 \Rightarrow Ax = 0$

   Therefore, the columns of $A^T A$ satisfy the same linear relationships as the columns of A. It doesn't matter that they have different number of rows. They have the same number of columns and they have the same column rank. (This also follows from the rank+nullity theorem, if you have proved that independently (i.e. without assuming row rank = column rank)

   Therefore, col $rank(A)$ =col $rank(A^T A) \leq$ col $rank(A^T)$ Now simply apply the argument to At to get the reverse inequality, proving col $rank(A)$ =col $rank(A^T)$. Since col $rank(A)$ is the row $rank$ of A, we are done