Linear independent

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Overview

Group

(Group). Consider a set $\mathcal{G}$ and an operation $\otimes : \mathcal{G} x \mathcal{G} \rightarrow \mathcal{G}$ defined on $\mathcal{G}$. Then $\mathrm{G} := (\mathcal{G}, \otimes)$ is called a group if the following hold:

- $\forall x, y \in \mathcal{G}, x \otimes y \in \mathcal{G}$
- $\forall x, y, z \in \mathcal{G}, (x \otimes y) \otimes z = x \otimes (y \otimes z)$
- $\exists e \in \mathcal{G} \forall x \in \mathcal{G} : x \otimes e = e \otimes x = x$
- $\forall x \in \mathcal{G} \exists y \in \mathcal{G} : x \otimes y = y \otimes x = e$

If additionally $\forall x, y \in \mathcal{G} : x \otimes y = y \otimes x$ then $\mathrm{G} = (\mathcal{G}, \otimes)$ is an Abelian group

Look at some examples:

- $(\mathbb{R}, +)$
- $(\mathbb{Z}, \cdot)$
- $(\mathbb{R}^n, +)$
- $(\mathbb{R}^{n \times n}, \cdot)$

A real-valued vector space $V = (\mathcal{V}; +; \cdot)$ is a set $\mathcal{V}$ with two operations:

$$+ : \mathcal{V} + \mathcal{V} \to \mathcal{V}$$
$$\cdot : \mathbb{R} \cdot \mathcal{V} \to \mathcal{V}$$

where:

- $(\mathcal{V}; +)$ is an Abelian group
- $\forall \lambda, \gamma \in \mathbb{R}, x, y \in \mathcal{V}$:
  - $\lambda(x + y) = \lambda x + \lambda y$
  - $(\lambda + \gamma)x = \lambda x + \gamma x$
- $\forall \lambda, \gamma \in \mathbb{R}, x \in \mathcal{V} : \lambda(\gamma x) = (\lambda\gamma)x$
- $1 \cdot x = x$

The elements $x \in V$ are called vectors, examples: $\mathcal{V} = \mathbb{R}^n, \mathbb{R}^{m \times n}$.

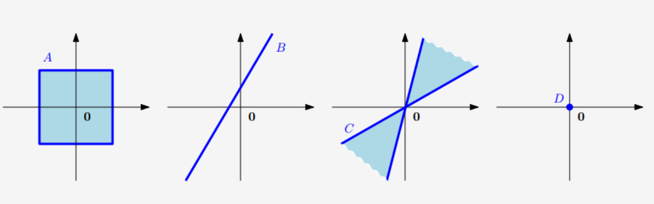
We will denote a vector space $(\mathcal{V}, +, \cdot)$ by V when $+$ and $\cdot$ are the standard vector addition and scalar multiplication. Moreover, we will use the notation $x \in V$ for vectors in $\mathcal{V}$ to simplify notation.
Default, the vectors are column vectors:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} \in \mathbb{R}^{n\times 1} \Rightarrow x^T = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix} \in \mathbb{R}^{1\times n}$$

Vector subspaces

Let V $= (\mathcal{V}, +, \cdot)$ be a vector space and U $\subseteq$ V, U $\neq 0$. Then U $=$ $(\mathcal{U}, +, \cdot)$ is called vector subspace of V (or linear subspace) if U is a vector space with the vector space operations $+$ and $\cdot$ restricted to U $\times$U and R$\times$U. We write U $\subseteq$ V. To determine whether $(\mathcal{U}, +, \cdot)$ is a subspace of V we still do need to show:

- $0 \in \mathcal{U}$
-
  - $\forall \lambda \in \mathbb{R}, x \in \mathcal{U} : \lambda x \in \mathcal{U}$
  - $\forall x, y \in \mathcal{U} : x + y \in \mathcal{U}$



Hình 1: subsets of $\mathbb{R}^2$

Linear Combination

Consider a vector space V and a finite number of vectors $x_1, x_2, ..., x_k \in V$. Then, every $v \in V$ of the form:

$$v = \lambda_1 v_1 + \lambda_2 v_2 + ... + \lambda_k v_k \in V$$

with $\lambda_1, \lambda_2, ..., \lambda_k \in \mathbb{R}$ is a linear combination of vectors $x_1, x_2, ..., x_k$

Linear independent

Consider a vector space V and a finite number of vectors $x_1, x_2, ..., x_k \in V$. If $\exists \lambda_1, \lambda_2, ..., \lambda_k \in \mathbb{R}$ and $\exists \lambda_i \neq 0$ with $i \in [1, k]$:

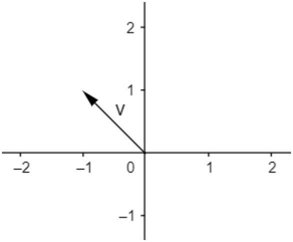
$$\sum_{i=1}^{k} \lambda_i v_i = 0$$

Then, $x_1, x_2, ..., x_k$ are linearly dependent. Otherwise, $x_1, x_2, ..., x_k$ are linearly independent.

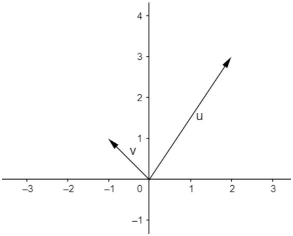
Intuitively, a set of linearly independent vectors consists of vectors that have no redundancy, i.e., if we remove any of those vectors from the set, we will lose something.

Linear independence is one of the most important concepts in linear algebra.

Linear independent (cont.)



(a)



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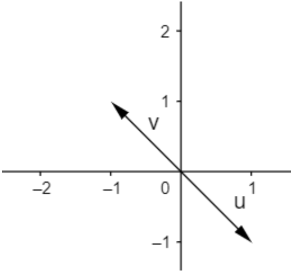

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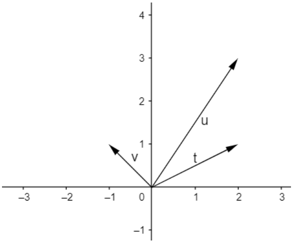

(d)

Linear independent (cont.)

A practical way of checking whether vectors $x_1, x_2, \ldots, x_k \in \mathcal{V}$ are linearly independent is to use Gaussian elimination: Write all vectors as columns of a matrix A and perform Gaussian elimination until the matrix is in row echelon form.

- The pivot columns indicate the vectors, which are linearly independent of the vectors on the left. Note that there is an ordering of vectors when the matrix is built.
- The non-pivot columns can be expressed as linear combinations of the pivot columns on their left

All column vectors are linearly independent if and only if all columns are pivot columns. If there is at least one non-pivot column, the columns (and, therefore, the corresponding vectors) are linearly dependent.

For example: check whether three vectors x_1, x_2, x_3 linearly independent

$$x_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, x_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, x_3 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & 2 & 2 \\ 1 & 1 & 3 \end{bmatrix} \rightarrow A = \begin{bmatrix} \textbf{-1} & 2 & 2 \\ 0 & \mathbf{3} & 5 \end{bmatrix}$$

Hence, x_1, x_2 are linearly independent, x_3 can be displayed by the combination of x_1, x_2

Why pivot columns indicate independent vectors?

Generating set and Span

Consider a vector space $V = (\mathcal{V}; +; \cdot)$ and set of vectors $\mathcal{A} = \{x_1, x_2, ..., x_k\} \in V$.

- If every vector $v \in \mathcal{V}$ can be expressed as a linear combination of $x_1, x_2, ..., x_k$, $\mathcal{A}$ is called a generating set generating set of V.
- The set of all linear combinations of vectors in $\mathcal{A}$ is called the span of $\mathcal{A}$. If $\mathcal{A}$ spans the vector space V, we write V $=$ span$[\mathcal{A}]$ or V $=$ span$[x_1, x_2, ..., x_k]$

Basis

Consider a vector space $V = (\mathcal{V}; +; \cdot)$ and set of vectors $\mathcal{A} \in \mathcal{V}$. A generating set $\mathcal{A}$ of V is called minimal if there exists no smaller set $\tilde{A} \subseteq \mathcal{A} \subseteq \mathcal{V}$ that spans V . Every linearly independent generating set of V is minimal and is called a basis of V.

Let $V = (\mathcal{V}; +; \cdot)$ be a vector space and $\mathcal{B} \subseteq \mathcal{V}, \mathcal{B} \neq \emptyset$. Then, following statements are equal:

- $\mathcal{B}$ is a basis of V
- $\mathcal{B}$ is a minimal generating set of V
- $\mathcal{B}$ is a maximal linearly independent set of vectors in V , i.e., adding any other vector to this set will make it linearly dependent.
- Every vector $x \in \mathcal{V}$ is a linear combination of vectors from B, and every linear combination is unique.

Basis (cont.)

There is no unique basis. However, all bases possess the same number of elements, the basis vectors.

A basis of a subspace $\mathcal{U} = x_1, ..., x_n$ can be found by executing the following steps:

- Write the spanning vectors as columns of a matrix A.
- Determine the row-echelon form of A.
- The spanning vectors associated with the pivot columns are a basis of U.

For example, for a vector subspace $U \subseteq \mathbb{R}^5$ spanned by the vectors:

$$x_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ -1 \\ -1 \end{bmatrix}, x_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 2 \\ -2 \end{bmatrix}, x_3 = \begin{bmatrix} 3 \\ -4 \\ 3 \\ 5 \\ -3 \end{bmatrix}, x_4 = \begin{bmatrix} -1 \\ 8 \\ -5 \\ -6 \\ 1 \end{bmatrix}$$

which vectors x_1, x_2, x_3, x_4 are a basis for $\mathcal{U}$

Rank

The number of linearly independent columns of a matrix $A \in \mathbb{R}^{m \times n}$ equals the number of linearly independent rows and is called the rank rank of A and is denoted by rk(A).

Properties:

- rk(A) $=$ rk(A^T), i.e., the column rank equals the row rank, hint: rk($A^T A$) $=$ rk(A)
- The columns of $A \in \mathbb{R}^{m \times n}$ span a subspace $U \subseteq \mathbb{R}^m$. A basis of U can be found by applying Gaussian elimination to A to identify the pivot columns.
- The rows of $A \in \mathbb{R}^{m \times n}$ span a subspace $U \subseteq \mathbb{R}^n$ with dim(W) $=$ rk(A). A basis of W can be found by applying Gaussian elimination to A.
- For all $A \in \mathbb{R}^{n \times n}$ it holds that A is regular (invertible) if and only if rk(A) $=$ n.

Rank (cont.)

- For all $A \in \mathbb{R}^{m \times n}$ and all $b \in \mathbb{R}^m$ it holds that the linear equation system $\mathrm{Ax} = \mathrm{b}$ can be solved if and only if $\mathrm{rk(A)} = \mathrm{rk(A|b)}$, where $\mathrm{A|b}$ denotes the augmented system.
- A matrix $A \in \mathbb{R}^{m \times n}$ has full rank if its rank equals the largest possible full rank rank for a matrix of the same dimensions. This means that the rank of a full-rank matrix is the lesser of the number of rows and columns, i.e., $\mathrm{rk(A)} = \mathrm{min(m; n)}$. A matrix is said to be rank deficient if it does not rank deficient have full rank.

Find the rank of following matrix: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & -2 & 3 \\ 2 & -3 & 5 \\ 3 & 5 & 0 \end{bmatrix}$