Calculus (2-4)M

Sub Topics

I) Mean Value Theorems

II) Taylor & Maclaurin Series

III) Partial & Total derivatives

IV) Maxima & Minima

V) Definite integrals

VI) Improper integrals

VII) Multiple integrals

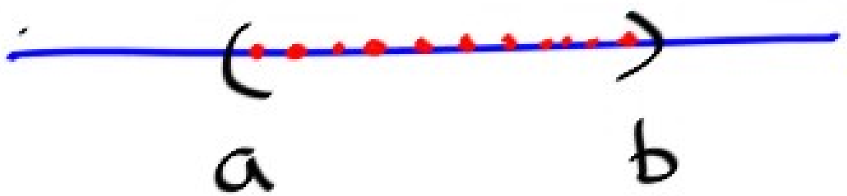
VIII) Vector Differentiation

IX) Vector Integration.

I) Mean Value Theorems

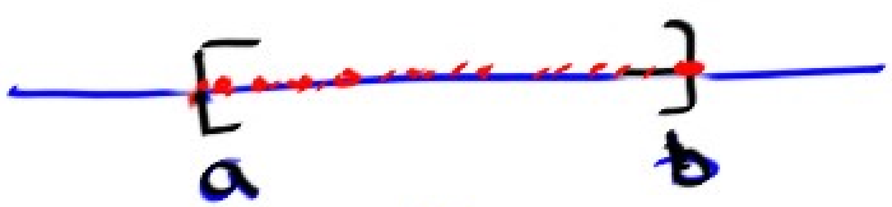
(1) Intervals:

(I) open interval:



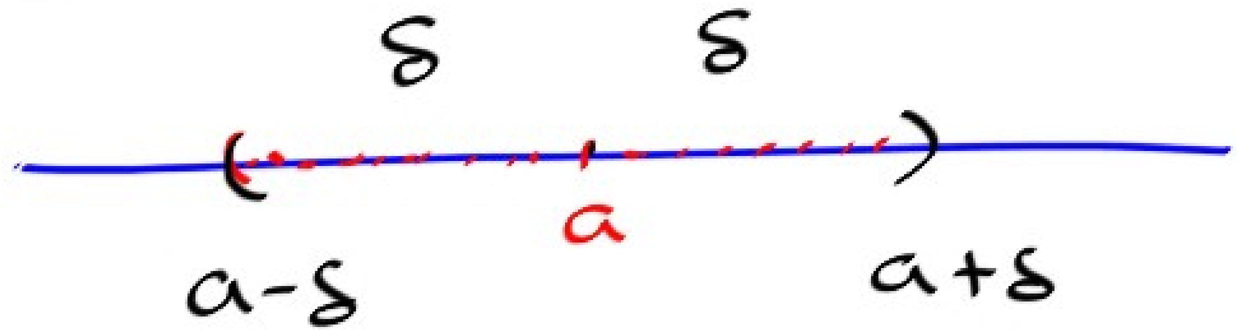
$$(a,b) = \{x : a < x < b\}$$

(II) closed interval:



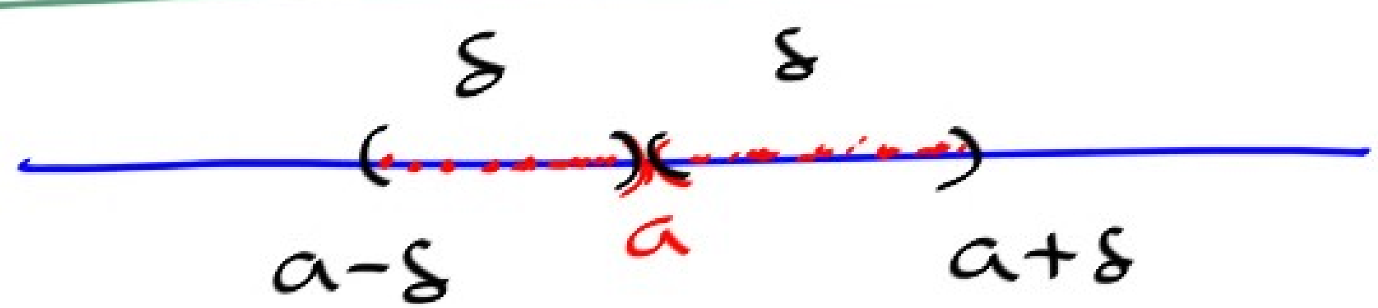
$$[a,b] = \{x : a \le x \le b\}.$$

III) Neighbour-hood (nbd) of a point 'a' :



$$N_\delta(a) = (a-\delta, a+\delta) = \{ x : a-\delta < x < a+\delta \}$$

IV) Deleted nbd of a point 'a' :



$$N_\delta(a) - \{a\} = (a-\delta, a) \cup (a, a+\delta) = \{ x : a-\delta < x < a+\delta \ \& \ x \neq a \}$$

(2) Limit of a function $y=f(x)$ at a point $x=a$:

Let $y=f(x)$ be defined in the deleted nbd of a point $x=a$.

A function $y=f(x)$ is said to have a limit 'l' at $x=a$ if

$$\boxed{\underset{x\to a^-}{Lt} f(x) = l = \underset{x\to a^+}{Lt} f(x)}$$

Note (1) $L.H.L = f(a^-) = \underset{x\to a^-}{Lt} f(x)$

Note (2) $R.H.L = f(a^+) = \underset{x\to a^+}{Lt} f(x)$.

(3) Continuity:

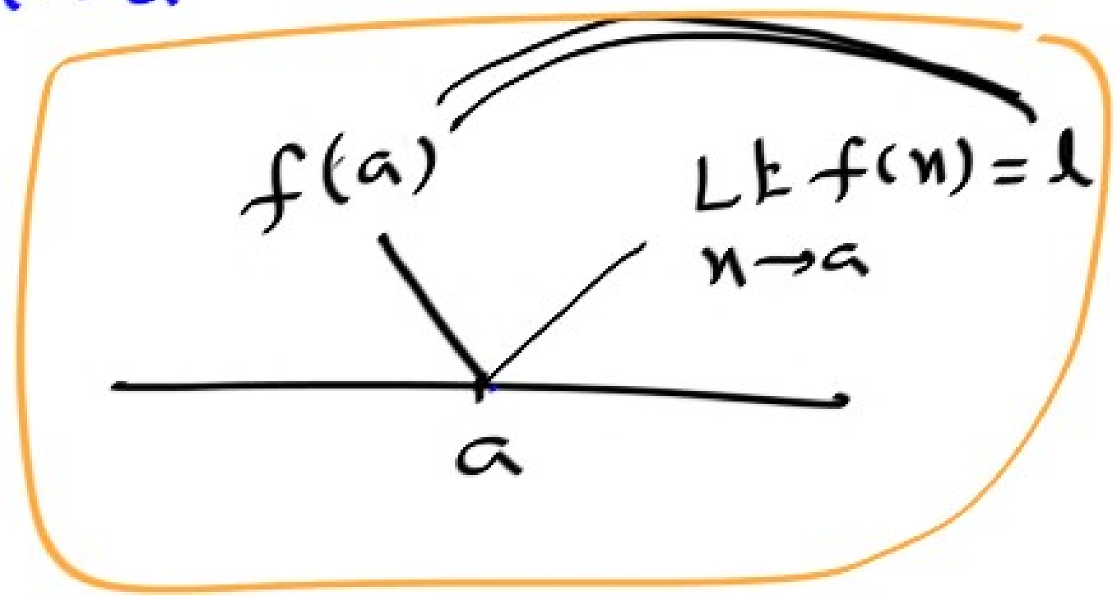
(I) Continuity of a function $f(x)$ at a point $x=a$:

A function $y=f(x)$ is said to be continuous at $x=a$

if (i) $f(a)$ is defined

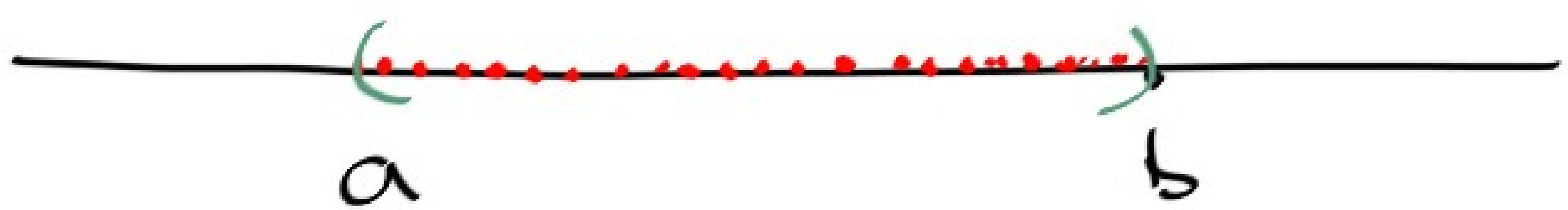
(ii) $\underset{x \to a^-}{Lt} f(x) = \underset{x \to a^+}{Lt} f(x)$ (i.e $\underset{x \to a}{Lt} f(x)$ exists)

(iii) $\underset{x \to a}{Lt} f(x) = f(a)$.



(II) Continuity of a function $f(x)$ on (a,b) :

A function $y = f(x)$ is said to be continuous on (a,b) if $f(x)$ is continuous at every point between a and b.



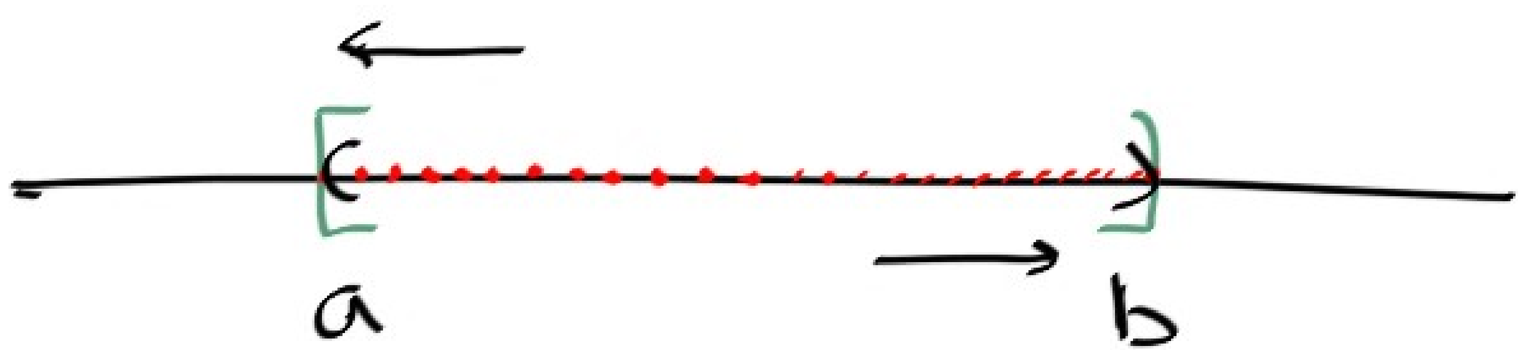
(III) Continuity of a function $f(x)$ on $[a,b]$:

A function $y = f(x)$ is said to be continuous on $[a,b]$ if

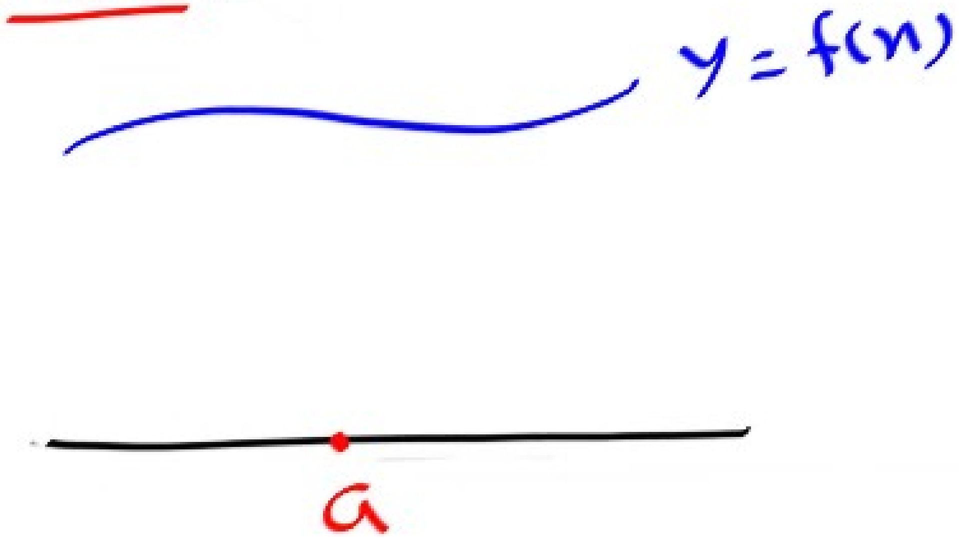
(i) $f(x)$ is continuous on (a,b)

(ii) $f(x)$ is right continuous at $x = a$ (i.e $\underset{x \to a^+}{Lt} f(x) = f(a)$)

(iii) $f(x)$ is left continuous at $x = b$ (i.e $\underset{x \to b^-}{Lt} f(x) = f(b)$)

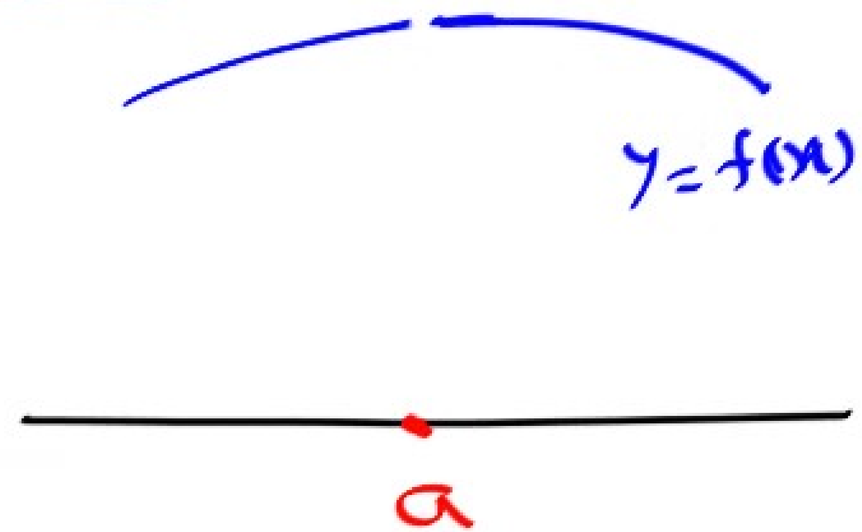


Note (1)



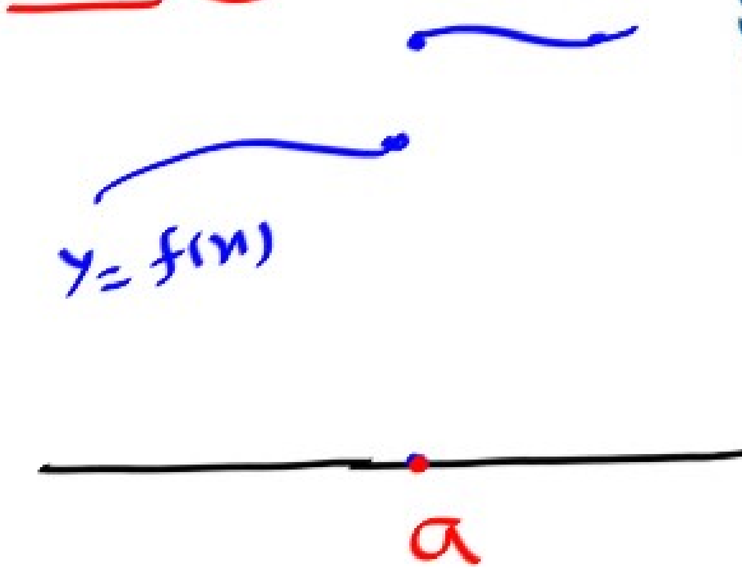
$f(x)$ is continuous at $x = a$

Note (2)



$f(x)$ is not continuous at $x = a$ (i.e discontinuous)

Note (3)



$f(x)$ is not continuous at $x = a$ (i.e discontinuous)

(4) Differentiability :-

(I) Differentiability of a function $f(x)$ at a point $x=a$:-

If a function $y=f(x)$ is defined in the nbd of a point $x=a$ and $\boxed{\underset{x \to a}{Lt} \frac{f(x)-f(a)}{x-a} \text{ exists and finite}}$ then the finite limit value is called the derivative of $f(x)$ at a point $x=a$ and it is denoted by "$f'(a)$".

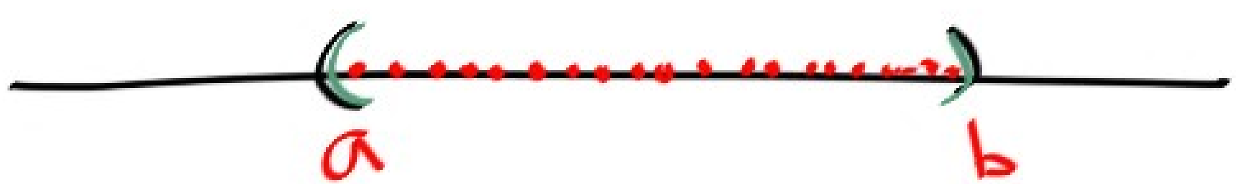
$$\therefore \boxed{f'(a) = \underset{x \to a}{Lt} \frac{f(x)-f(a)}{x-a}}$$

If the derivative of $f(x)$ exists at $x=a$ (i.e $f'(a)$ exists) then the function $f(x)$ is called a differentiable function at $x=a$ and the process of finding the derivative is called differentiation.

Note ① $f'(a) = \mathop{Lt}\limits_{x\to a} \dfrac{f(x)-f(a)}{x-a}$ exists $\iff \underbrace{\mathop{Lt}\limits_{x\to a^-} \dfrac{f(x)-f(a)}{x-a}}_{L.H.D} = \underbrace{\mathop{Lt}\limits_{x\to a^+} \dfrac{f(x)-f(a)}{x-a}}_{R.H.D.}$

II) Differentiability of $f(x)$ on (a,b):

A function $y=f(x)$ is said to be differentiable on (a,b) if $f(x)$ is differentiable at every point between a & b.



III) Differentiability of a function $f(x)$ on $[a,b]$:

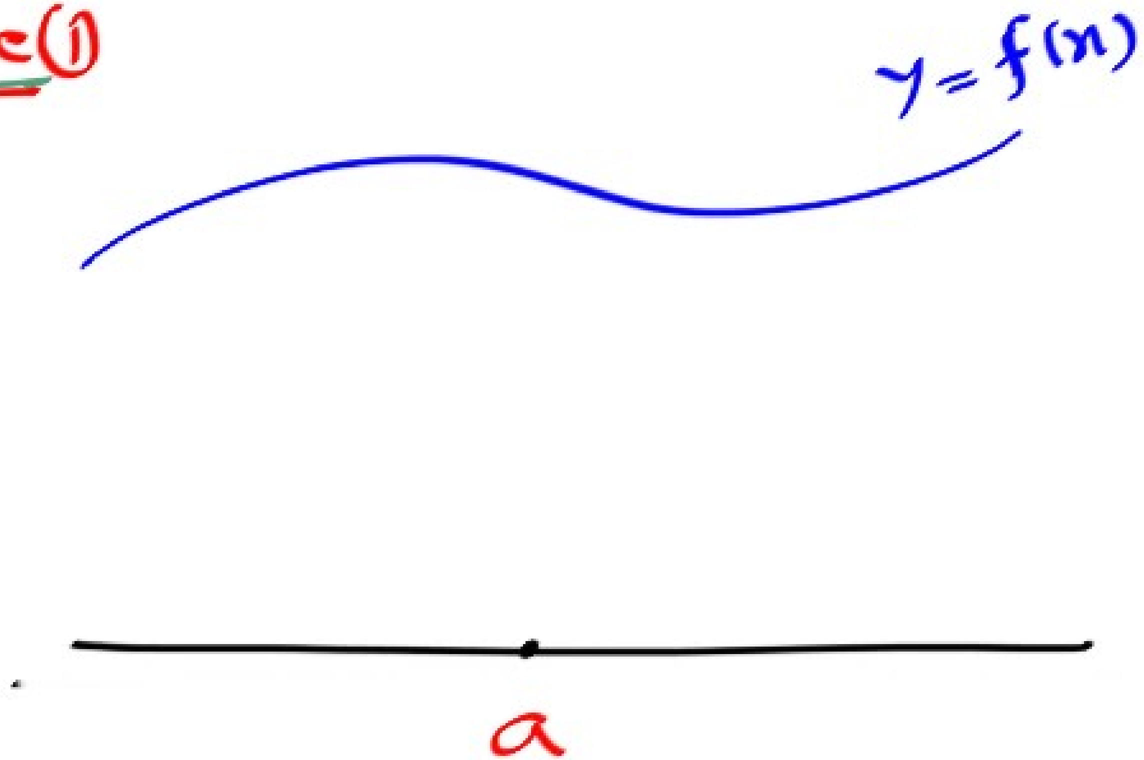
A function $y=f(x)$ is said to be differentiable on $[a,b]$

if (i) $f(x)$ is differentiable on (a,b).

(ii) $f(x)$ is right differentiable at $x=a$

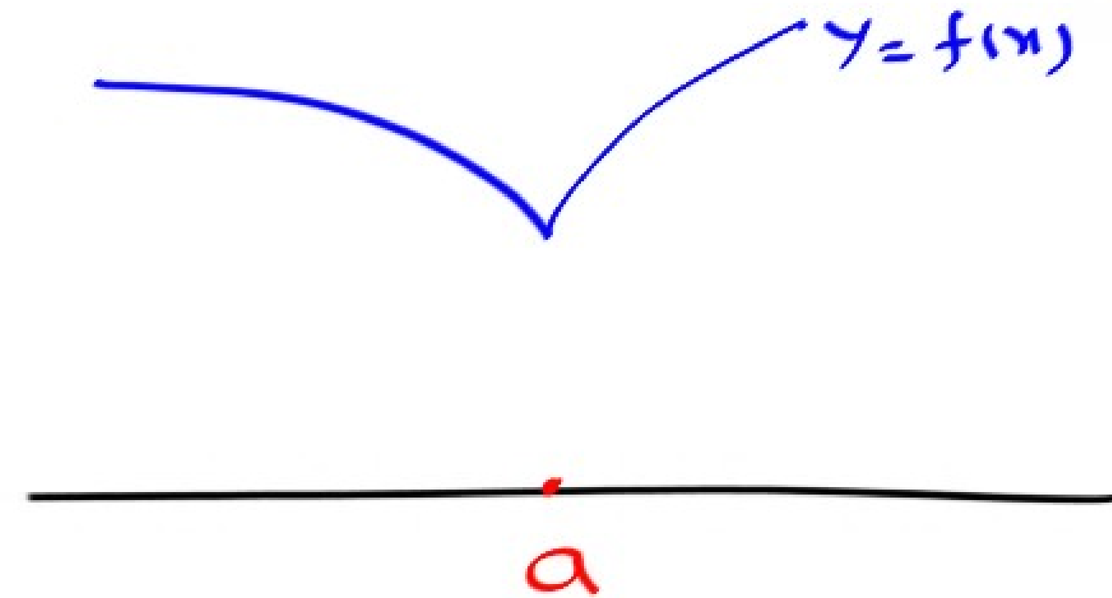
(iii) $f(x)$ is left differentiable at $x=b$.

Note (1)



$f(x)$ is differentiable at $x = a$

Note (2)



$f(x)$ is not differentiable at $x = a$.

① If $f(x) = \begin{cases} x & , x \le 1 \\ 2x-1 & , x > 1 \end{cases}$ then at $x=1$ which of the following is true ?

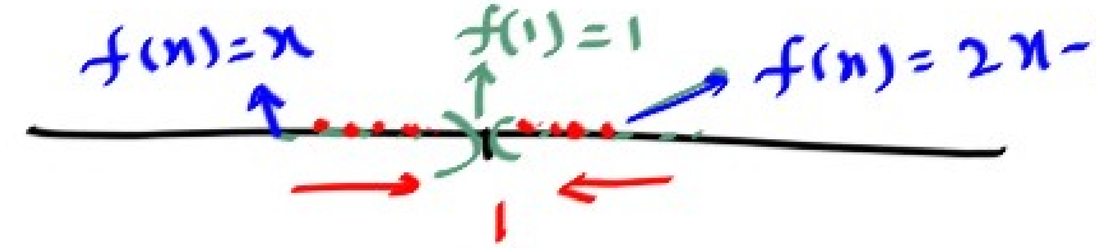
(a) $f(x)$ is continuous but not differentiable

~~(b)~~ $f(x)$ is continuous & differentiable

~~(c)~~ $f(x)$ is neither continuous nor differentiable

~~(d)~~ $f(x)$ is differentiable but not continuous.

Sol: Given $f(x) = \begin{cases} x, & x \le 1 \\ 2x-1, & x > 1 \end{cases}$



Consider $L.H.L = \underset{x \to 1^-}{Lt} f(x) = \underset{x \to 1}{Lt}\, x = 1$

and $R.H.L = \underset{x \to 1^+}{Lt} f(x) = \underset{x \to 1}{Lt} (2x-1) = 2-1 = 1$

Here, $L.H.L = 1 = R.H.L$

$\therefore \underset{x \to 1}{Lt} f(x)$ exists & $\underset{x \to 1}{Lt} f(x) = 1$

Here, $f(1) = 1$ & $\underset{x \to 1}{Lt} f(x) = 1 = f(1)$

$\therefore f(x)$ is continuous at $x = 1$

$x \to a^- \Rightarrow x < a$

$x \to a^+ \Rightarrow x > a$

Consider, $L.H.D = \underset{x\to 1^-}{Lt} \frac{f(x)-f(1)}{x-1} = \underset{x\to 1}{Lt} \frac{x-1}{x-1} = \underset{x\to 1}{Lt} 1 = 1$

and $R.H.L = \underset{x\to 1^+}{Lt} \frac{f(x)-f(1)}{x-1} = \underset{x\to 1}{Lt} \frac{(2x-1)-1}{x-1} = \underset{x\to 1}{Lt} \frac{2(x-1)}{x-1} = 2$

Here, $L.H.D = 1 \neq R.H.D = 2$

$\Rightarrow$ $f'(1)$ does not exist

$\Rightarrow$ $f(x)$ is not differentiable at $x=1$.

$\therefore$ $f(x)$ is continuous but not differentiable at $x=1$

Hence, option (a) is true.

$$f(x) = \begin{cases} | \;\; | \\ [\;\;] \\ \alpha \\ \begin{cases} \underline{\quad}, (\;\;) \\ \underline{\quad}, (\;\;) \\ \underline{\quad}, (\;\;) \end{cases} \end{cases}$$

Note (1) Limit exist $\begin{cases} \text{Continuous} \begin{cases} \text{Differentiable} \\ \text{Not differentiable} \end{cases} \\ \text{Not continuous} \end{cases}$

Note (2) Differentiability $\Rightarrow$ Continuity $\Rightarrow$ Limit exists.

Note (3) $\boxed{e^{ax}, \cos(ax), \sin(ax), \cosh(ax), \sinh(ax)}$ and every polynomial of the form $\boxed{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \; (\because n \in N)}$ are everywhere defined, continuous, differentiable & also integrable.

Note (4) If $f(x)$ & $g(x)$ are continuous functions

then (i) $f+g$ is also continuous

(ii) $f-g$ is " "

(iii) $f \cdot g$ is " "

(iv) $\frac{f}{g}$ $(\because g \neq 0)$ is " "

Note (6) If $f(x)$ & $g(x)$ are differentiable functions

then (i) $f+g$ is also differentiable

(ii) $f-g$ is also "

(iii) $f \cdot g$ is " "

(iv) $\frac{f}{g}$ $(\because g \neq 0)$ is also differentiable elemen

$2x^3 - 4x^9 + 7x^2 + 4$

$3x^5 - 7x^2 + 9x^{5/3} - 2$

$e^{(\,)}$, log( ), trig & hyp

Transcendental f

$5/2 \notin \underline{N}$

Th (1) [Rolle's Theorem (R.T)]

St: Let $f(x)$ be defined on $[a,b]$ such that

(i) $f(x)$ is continuous on $[a,b]$

(ii) $f(x)$ is differentiable on (a,b)

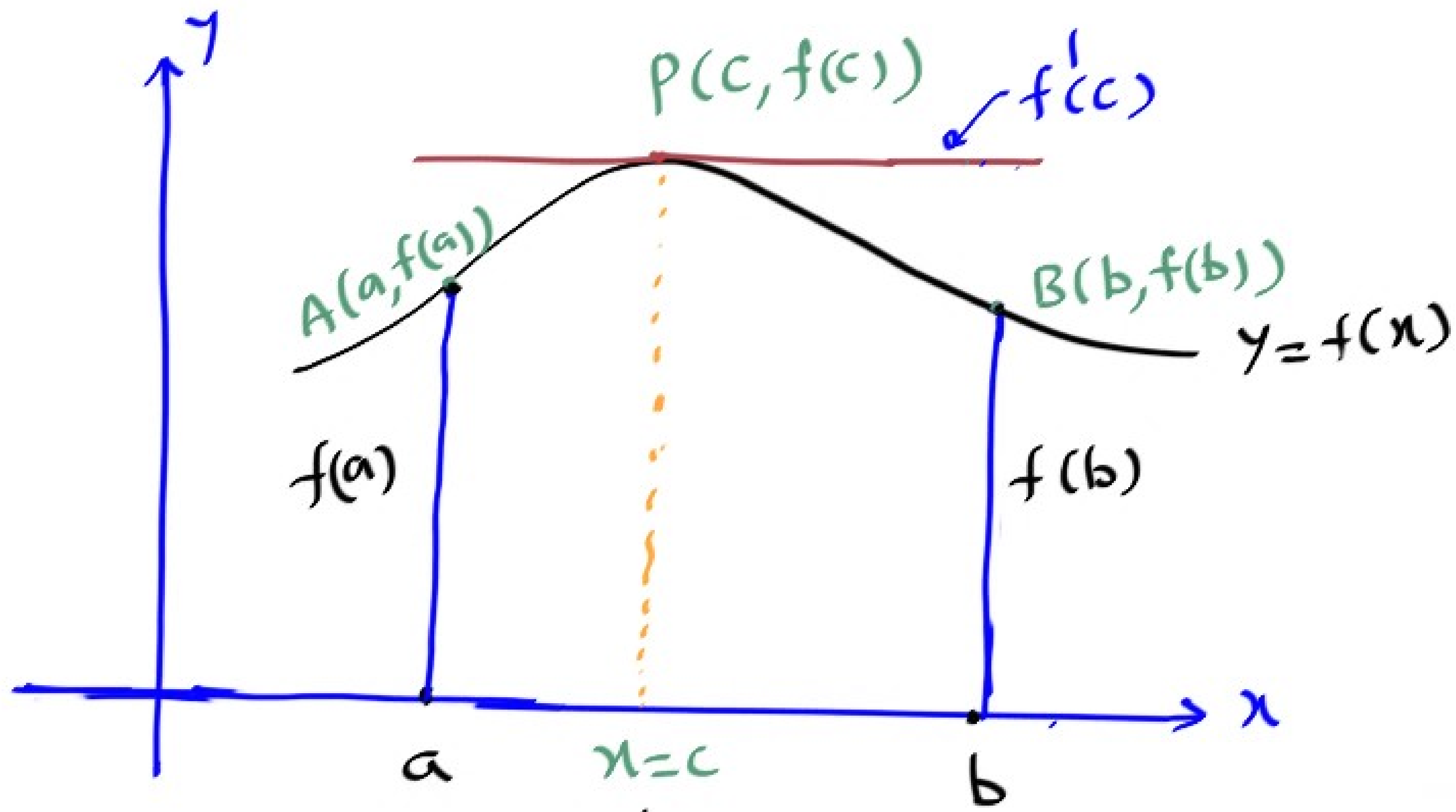
(iii) $f(a) = f(b)$.

Then $\exists$ at least one real number 'c'

in (a,b) $\ni$ $\boxed{f'(c) = 0}$.

$f(x)$ — cont $[a,b]$, diff (a,b), $f(a) = f(b)$

$f'(x) = 0$



$f'(c) = 0$

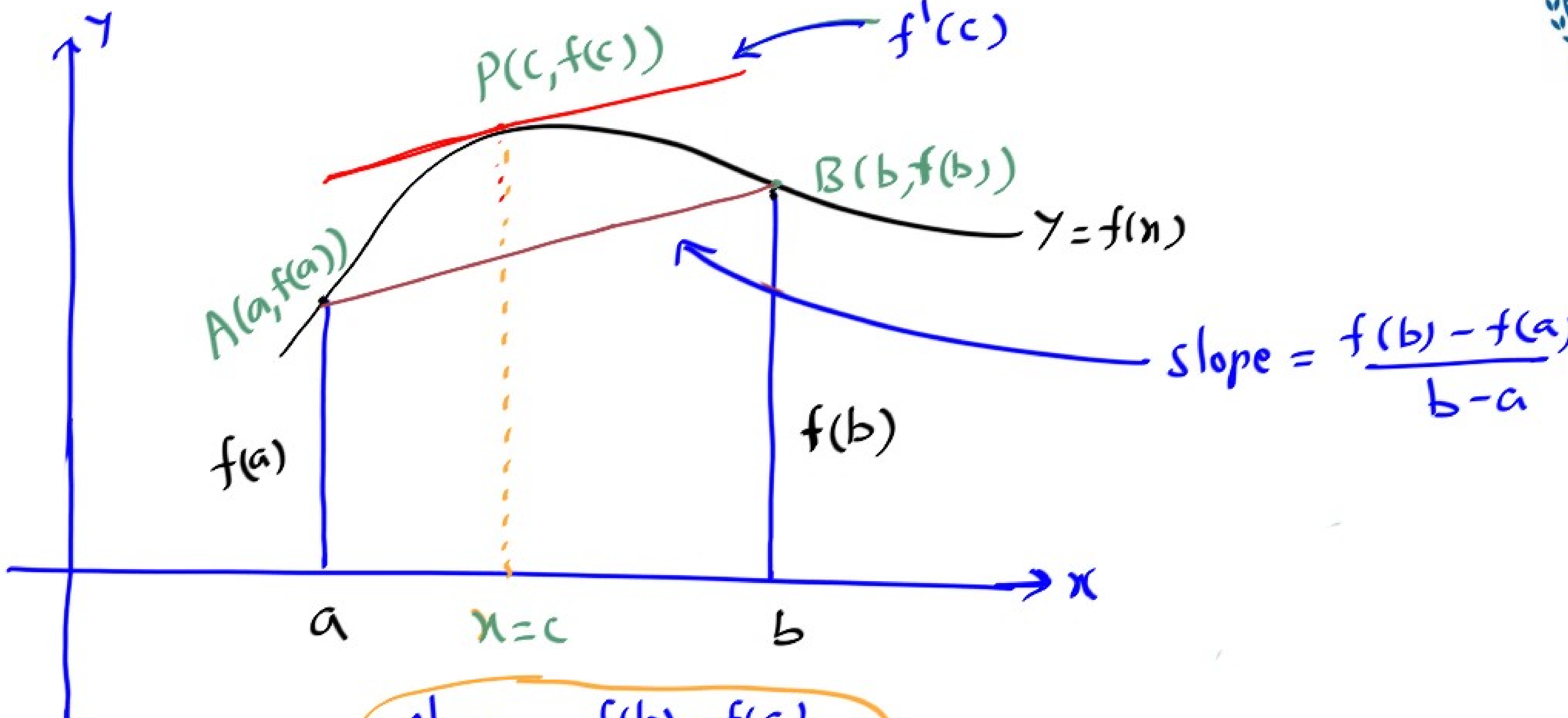
Th (2) [Lagrange's Mean Value Theorem (L.M.V.T)] → 1st M.V.T for differential calculus
→ Mean Value Theorem.

St :- If a function f(x) defined on [a, b] is

(i) continuous on [a, b]

(ii) differentiable on [a, b]

then ∃ at least one real number 'c' in (a, b) ∋ $\boxed{f'(c) = \frac{f(b) - f(a)}{b-a}}$



$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Th(3) [Cauchy's Mean Value Theorem (C.M.V.T)] $\rightarrow$ 2^{nd} M.V.T for differential calculus.

St: Let $f(x)$ & $g(x)$ be defined on $[a,b]$

$\ni$ (i) $f(x)$ & $g(x)$ are continuous on $[a,b]$

(ii) $f(x)$ & $g(x)$ are differentiable on (a,b)

& (iii) $g'(x) \neq 0$.

Then $\exists$ at least one real number 'c' in (a,b) $\ni$

$$\boxed{\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}}$$

GATE

① Let $f(x) = x^2 - 2x + 2$ be a continuous function on $x \in [1,3]$. The point $\underline{x}$ at which the tangent of $f(x)$ becomes parallel to the straight line joining $f(1)$ & $f(3)$ is —

(a) 3

(b) 1

(c) 0

(d) 2.

Sol: Given $f(x) = x^2 - 2x + 2$ in $[a,b] = [1,3]$

$\Rightarrow f'(x) = 2x - 2$

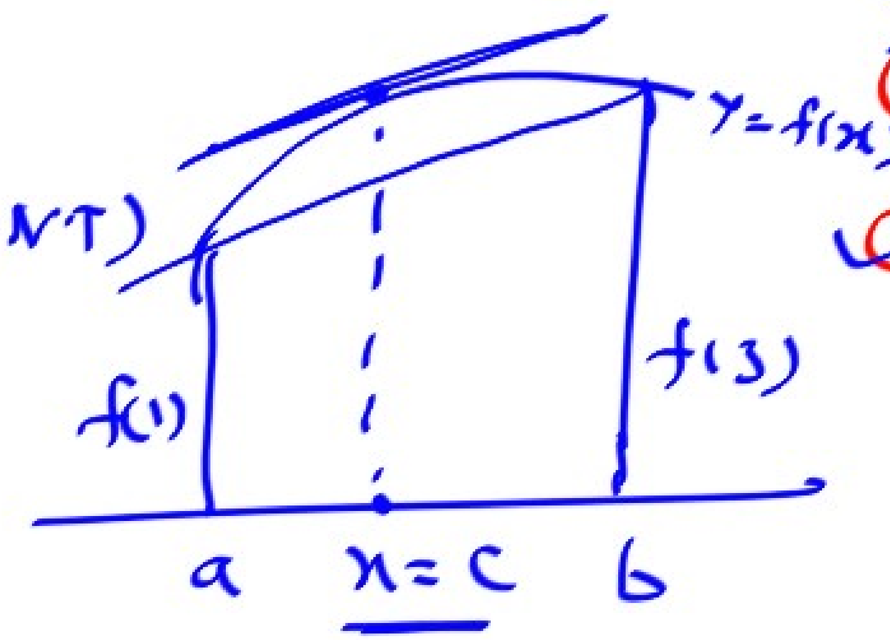
Here, $f(1) = 1 \neq f(3) = 5$

Consider $f'(c) = \dfrac{f(3) - f(1)}{3-1}$ (by L.M.V.T)

$\Rightarrow 2c - 2 = \dfrac{5-1}{3-1}$

$\Rightarrow 2c - 2 = 2$

$\therefore c = 2 \in (1,3).$



(2) If $f(0) = 2$ & $f'(x) = \frac{1}{5-x^2}$ then the lower & upper bounds of $f(1)$ estimated by mean value theorem are —

GATE

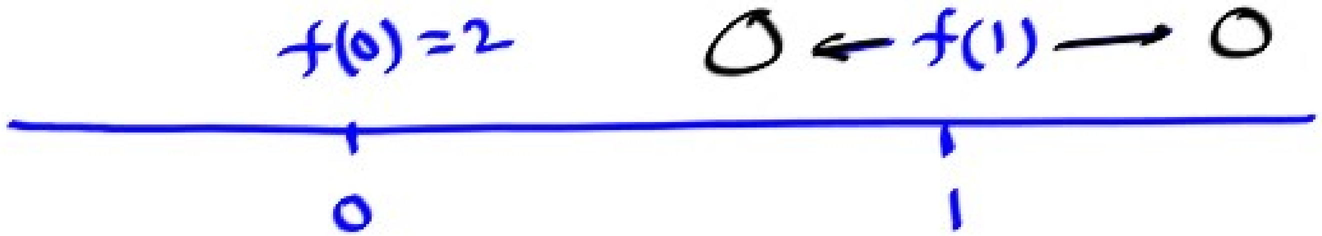
(a) 1.9, 2.2

(b) 2.2, 2.25

(c) 2.25, 2.5

(d) none of these

Sol:



Let $f(x)$ be defined on $[a,b] = [0,1] \ni f'(x) = \frac{1}{5-x^2}$

Then by L.M.V.T, $\exists\ c \in (0,1) \ni f'(c) = \frac{f(1)-f(0)}{1-0}$

$$\Rightarrow \frac{1}{5-c^2} = \frac{f(1)-2}{1-0}$$

$$\Rightarrow f(1) = 2 + \frac{1}{5-c^2}$$

$\because\ c \in (0,1)$

$\Rightarrow 0 < c < 1$

$\Rightarrow 0^2 < c^2 < 1^2$

$\Rightarrow -0^2 > -c^2 > -1^2$

$\Rightarrow 5-0^2 > 5-c^2 > 5-1^2$

$\Rightarrow \dfrac{1}{5-0^2} < \dfrac{1}{5-c^2} < \dfrac{1}{5-1^2}$

$\Rightarrow 2+\dfrac{1}{5} < 2+\dfrac{1}{5-c^2} < 2+\dfrac{1}{4}$

$\therefore\ \dfrac{11}{5} < f(1) < \dfrac{9}{4}$ (or) $f(1) \in (2.2, 2.25)$

(3) Suppose that $f: R \to R$ is a continuous function

GATE

on the interval $[-3, 3]$ and a differentiable function in the interval $(-3, 3)$ $\ni$ for every x in the interval $f'(x) \leq 2$. If $f(-3) = 7$ then $f(3)$ is at most ____

Sol:-

Let $f(x)$ be define on $[-3,3]$ $\ni$ $f'(x) \leq 2$

Then by L.M.V.T, $\exists$ $c \in (-3,3)$ $\ni$ $f'(c) = \dfrac{f(3) - f(-3)}{3-(-3)}$

$$\Rightarrow f'(c) = \frac{f(3) - 7}{6} \quad \text{--- (1)}$$

$$\because f'(x) \leq 2$$

$$\Rightarrow \frac{f(3) - 7}{6} \leq 2 \quad (\text{or}) \quad f(3) - 7 \leq 12$$

$$\therefore f(3) \leq \underline{19}$$

(4) The equation $\sin(x)+2\sin(2x)+3\sin(3x)=\frac{8}{\pi}$ has at least one root in

(a) $\left(\frac{\pi}{2},\pi\right)$

(b) $\left(0,\frac{\pi}{2}\right)$

(c) $\left(\pi,\frac{3\pi}{2}\right)$

(d) none of these.

Sol:- Let $f'(x)=\sin(x)+2\sin(2x)+3\sin(3x)-\frac{8}{\pi}$

Then $f(x)=-\cos(x)-\cos(2x)-\cos(3x)-\frac{8}{\pi}x+k$

Here, (i) $f(x)$ is continuous on $[0,\pi/2]$

(ii) $f(x)$ is differentiable on $(0,\pi/2)$

& (iii) $f(0)=-3+k=f(\pi/2)$

$\therefore$ By a R.T, the given equation will have at least one root in $(0,\pi/2)$.

$\left(\frac{\pi}{2}, \pi\right)$

$\downarrow$

$f(\pi/2) = 0 + 1 - 0 - 4 + k = -3 + k$

$f(\pi) = \cancel{1} - \cancel{1} + 1 - 8 + k = -7 + k$

$f(0) = -1 - 1 - 1 + 0 + k = -3 + k$

$f(x)$ — C → [a, b] ✓ ; D — (a, b) ✓ ; $f(a) = f(b)$

$\downarrow$

$f'(x) = 0$

II) Taylor & Maclaurin Series

① Taylor series:

If a function $f(x)$ has all order derivatives at $x=a$ (i.e $f'(a), f''(a), f'''(a), \ldots\ldots$ exist) then the function $f(x)$ can be expressed as a power series in "+ve" integer powers of $(x-a)$ (or) about the point $x=a$ and it is called a Taylor series expansion of $f(x)$ about $x=a$ which is given by

$$f(x) = f(a) + (x-a)\cdot f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots\cdots + \frac{(x-a)^n}{n!} f^{(n)}(a) + \cdots$$

(or) $$f(x) = \sum_{n=0}^{\infty} a_n \cdot (x-a)^n$$, where $a_n = \frac{f^{(n)}(a)}{n!}$

Note (1) $\boxed{P_1(x) = f(a) + (x-a)\cdot f'(a)}$ is called a first order Taylor polynomial (or) linear approximation of $f(x)$ about a point $x=a$ (or) around $x=a$.

Note (2) $\boxed{P_2(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a)}$ is called a 2^{nd} order polynomial (or) a quadratic approximation of $f(x)$ about $x=a$ (or) around $x=a$.

Note (3) The coefficient of $(x-a)^n$ in the Taylor series expansion of $f(x)$ about $x=a$ is given by $a_n = \frac{f^{(n)}(a)}{n!}$

② Maclaurin Series :-

The Taylor series expansion of $f(x)$ about origin (i.e $a=0$) is called a Maclaurin series expansion of $f(x)$ and it is given by

$$\boxed{f(x) = f(0) + x \cdot f'(0) + \frac{x^2}{2!} f''(0) + \cdots\cdots + \frac{x^n}{n!} \cdot f^{(n)}(0) + \cdots}$$

(or)

$$\boxed{f(x) = \sum_{n=0}^{\infty} a_n \cdot x^n}, \text{ where } a_n = \frac{f^{(n)}(0)}{n!}$$

(1) The Taylor series expansion of $f(x) = e^{\sin(x)}$ about $x=0$ is

Sol: Given $f(x) = e^{\sin(x)}$ & $a=0$

The Taylor series expansion of $f(x)$ about $x=a$ is given by

$$f(x) = f(a) + (x-a)\cdot f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots\cdots \quad \text{— (1)}$$

Now, $f(x) = e^{\sin(x)}$ & $f(a) = f(0) = e^{\sin(0)} = e^0 = 1$

$\Rightarrow f'(x) = e^{\sin(x)} \cdot \cos(x)$ & $f'(a) = f'(0) = e^0 \cdot 1 = 1$

$\Rightarrow f''(x) = e^{\sin(x)} \cdot \cos^2(x) + e^{\sin(x)} \cdot (-\sin x)$ & $f''(a) = f''(0) = 1 + 0 = 1$

$\vdots$

Substituting above all in (1), we get:

$$f(x) = (1) + (x-0)\cdot(1) + \frac{(x-0)^2}{2!}(1) + \cdots$$

$$\therefore e^{\sin(x)} = 1 + x + \frac{x^2}{2!} + \cdots$$

② In the Taylor series expansion of $f(x) = \log(\sec x)$ about $x=0$, the coefficient of x^4 is ____

(a) $\frac{1}{12}$

(b) $\frac{1}{4}$

(c) 12

(d) 14.

Sol: Given $f(x) = \log(\sec x)$ & $a=0$

$\Rightarrow f'(x) = \frac{1}{\sec(x)} \sec(x) \cdot \tan(x) = \tan(x)$

$\Rightarrow f''(x) = \sec^2(x)$.

$\Rightarrow f'''(x) = 2 \sec(x) \cdot \sec(x) \cdot \tan x = 2\sec^2(x) \cdot \tan(x)$.

$\Rightarrow f^{iv}(x) = 2\left[2 \sec(x) \cdot \sec(x) \tan x \cdot \tan x + \sec^2(x) \sec^2(x)\right]$

$\Rightarrow f^{iv}(0) = 2[0+1] = 2$

$\therefore$ The coefficient of x^4 (or) $(x-0)^4$ is given by

$$a_4 = \frac{f^{iv}(0)}{4!} = \frac{2}{4!} = \frac{1}{12}.$$

(3) Let $f(x) = e^{x+x^2}$ for real x. From among the following choose the Taylor series approximation of $f(x)$ around $x=0$, which includes all the powers of x less than (or) equal to 3.

(a) $1+x+x^2+x^3$

(b) $1+x+\frac{3}{2}x^2+\frac{7}{6}x^3$

(c) $1+x+\frac{3}{2}x^2+x^3$

(d) $1+x+3x^2+7x^3$

Sol: Let $f(x) = e^{x+x^2}$ & $a=0$.

Then the 3rd order approximation of $f(x)$ about (or) around $x=a$ is given by

$$\boxed{P_3(x) = f(a) + (x-a)\cdot f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a).} \quad (1)$$

Now, $f(x) = e^{x+x^2}$ & $f(a) = f(0) = 1$

$\Rightarrow f'(x) = e^{x+x^2}(1+2x)$ & $f'(0) = 1$

$\Rightarrow f''(x) = e^{x+x^2}(1+2x)^2 + e^{x+x^2}(0+2)$ & $f''(a) = f''(0) = 1+2 = 3$

$1+4x^2+4x$

$\Rightarrow f'''(x) = \left[e^{x+x^2}(1+2x)^3 + e^{x+x^2} \cdot 2(1+2x)(2)\right] + e^{x+x^2}(2)\cdot(1+2x)$

$1^3 + 3(1)(2x) + 3(1)(2x)^2 + (2x)^3$

& $f'''(0) = 1+4+2 = 7$

Substituting above all in ①, we get

$$P_3(x) = (1) + (x-0)\cdot(1) + \frac{(x-0)^2}{2!}(3) + \frac{(x-0)^3}{3!}(7)$$

$$\therefore P_3(x) = 1 + x + \frac{3}{2}x^2 + \frac{7}{6}x^3$$