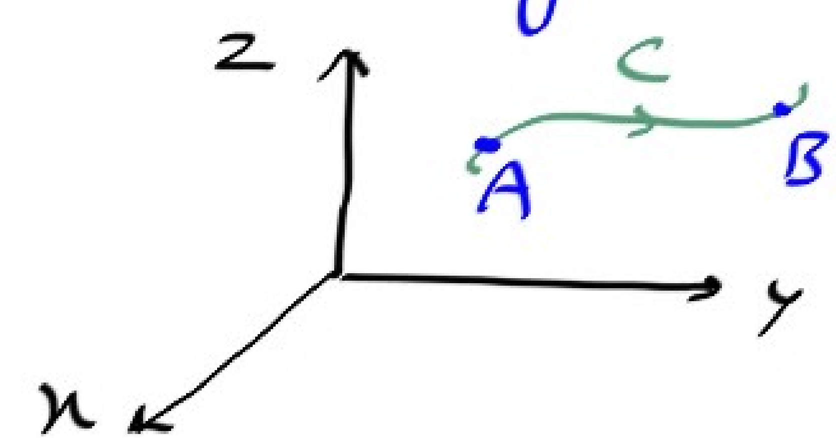
IX) Vector integration

(1) Line integral (L.I):

If a vector function $\bar{f}$ is defined at every point on the curve 'c' from a point A to a point B then the evaluation of integral of a vector function $\bar{f}$ along a curve 'c' from A to B is a line integral of $\bar{f}$ and it is given by

$$\boxed{\int_c \bar{f}\cdot d\bar{r}},$$ where 'c' is the path of integration.

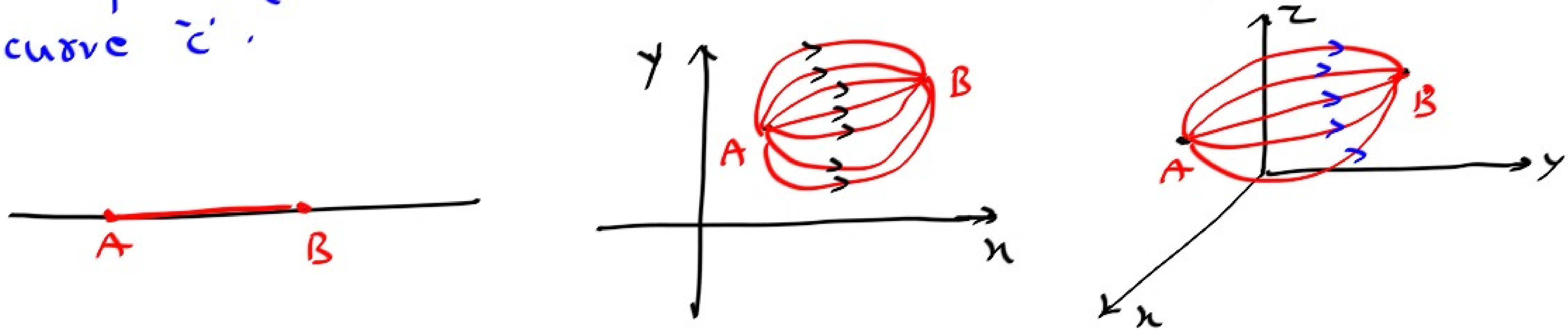


Note (1) [Line integral in Cartesian form]

If $\bar{F} = f_1 \bar{i} + f_2 \bar{j} + f_3 \bar{k}$ & $d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$, where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

then $$\boxed{\int_c \bar{F} \cdot d\bar{r} = \int_c [f_1\,dx + f_2\,dy + f_3\,dz]}$$

Note (2) In general, the value of the line integral depends on path (or) curve 'c' but not on end points of the curve 'c'.



Note (3) If $\bar{f}$ is an irrotational vector (i.e $\nabla \times \bar{f} = \bar{0}$ (or) curl $\bar{f} = \bar{0}$ (or) $\bar{f} = \nabla \phi$) then the value of the line integral depends on end points A & B of the curve (or) path 'c' but not on path 'c'.

$$\boxed{\text{i.e} \quad \int_C \bar{f} \cdot d\bar{r} = \int_A^B (\nabla \phi \cdot d\bar{r}) = \int_A^B d\phi = (\phi)_A^B = \phi(B) - \phi(A)}$$

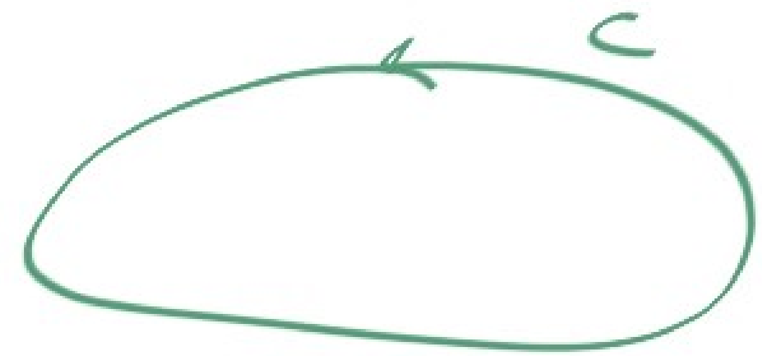
Scalar potential function.

Note (4) If $\bar{F}$ is a force vector acting on a moving particle in the force field then the Total work done by a force vector along a curve 'c' from a point A to a point B is given by line integral.

i.e $\boxed{\text{Work done (W.D)} = \int_C \bar{F} \cdot d\bar{r}}$.

Note (5) If $\bar{f}$ is an irrotational vector (i.e $\nabla \times \bar{f} = \bar{0}$ (or) curl $\bar{f} = \bar{0}$ (or) $\bar{f} = \nabla \phi$) then the value of the line integral along a simple closed curve 'c' is always zero.

i.e $\boxed{\oint_c \bar{f} \cdot d\bar{r} = 0}$



Note (6) If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ (or) $\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x}$ for a given vector function $\bar{f} = f_1 \bar{i} + f_2 \bar{j} = M\bar{i} + N\bar{j}$ then the vector function $\bar{f}$ is an irrotational.

① If $\bar{f} = 2z\,\bar{i} + 2y\,\bar{j} + 2x\,\bar{k}$ then the value of the integral $\int_C \bar{f}\cdot d\bar{r}$ along a straight line from $(0,0,0)$ to $(4,1,-1)$ is

Method -I (General method)

Sol: Let $I = \int_C \bar{f}\cdot d\bar{r}$

Then $I = \int_{(0,0,0)}^{(4,1,-1)} \left[(2z)\,dx + (2y)\,dy + (2x)\,dz\right]$

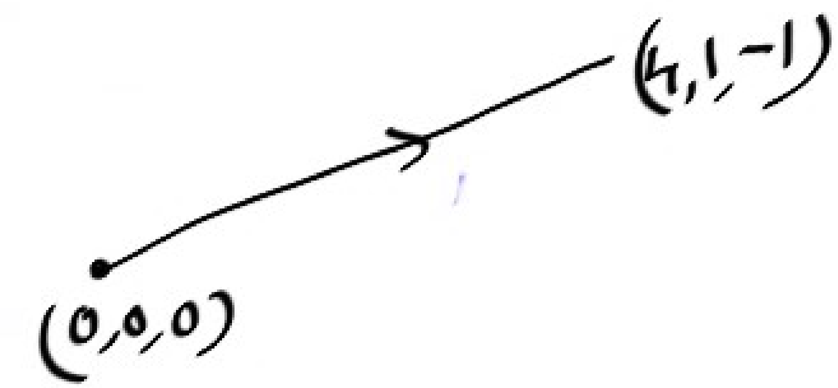
The equation of the straight line is $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$

$\Rightarrow \frac{x-0}{4-0} = \frac{y-0}{1-0} = \frac{z-0}{-1-0} = t$

$\Rightarrow x = 4t,\; y = t,\; z = -t$

$\Rightarrow dx = 4\,dt,\; dy = dt,\; dz = -\,dt$

Here, $y = t$ < for $y = 0 \rightarrow t = 0$; for $y = 1 \rightarrow t = 1$



Now, $I = \int_{t=0}^{1} [(-2t)(4)\,dt + (2t)(dt) + (8t)(-1)\,dt]$

$\Rightarrow \quad I = \int_{t=0}^{1} [-8t + 2t - 8t)\,dt$

$\Rightarrow \quad I = \int_{t=0}^{1} (-14t)\,dt$

$\therefore \quad I = \left(-14\frac{t^2}{2}\right)_0^1 = \underline{-7}$

Method – II

Let $I = \int_C [(2z)\,dx + (2y)\,dy + (2x)\,dz]$

Then $I = \int_C [\,2(z\,dx + x\,dz) + (2y)\,dy\,]$

$\Rightarrow I = \int_C [\,2\;d(xz) + (2y)\,dy\,]$ $\quad (\because d(uv) = u\,dv + v\,du)$

$\Rightarrow I = \left[2\,(xz) + y^2\right]_{A=(0,0,0)}^{B=(4,1,-1)}$

$\Rightarrow I = [\,2(4)(-1) + (1)^2\,] - [\,2(0)(0) + (0)^2\,]$

$\therefore \underline{\underline{I = -7}}$

② If $\bar{F} = (2xy + z^3)\bar{i} + (x^2)\bar{j} + (3xz^2)\bar{k}$ is a force vector then the <u>work done</u> by the force $\bar{F}$ in moving an object in the force field from $(1,-2,1)$ to $(3,1,4)$ is —

Sol: $W \cdot D = \int_C \bar{F} \cdot d\bar{r}$

$$\Rightarrow W \cdot D = \int_C \left[(2xy + z^3)\,dx + (x^2)\,dy + (3xz^2)\,dz\right]$$

$$\Rightarrow W \cdot D = \int_C \left[(2xy\,dx + x^2\,dy) + (z^3\,dx + 3xz^2\,dz)\right]$$

$$\Rightarrow W \cdot D = \int_C \left[d(x^2y) + d(xz^3)\right]$$

$$\Rightarrow W \cdot D = (x^2y + xz^3)_{(1,-2,1)}^{(3,1,4)}$$

$$\therefore W \cdot D = (\qquad) - (\qquad) = \underline{\underline{202}}$$

③ If $f = 2x^3 + 3y^2 + 4z$ then the value of line integral

GATE $\int_C [(\text{grad} f) \cdot d\bar{r}]$ evaluated over curve C formed

by the segments $(-3,-3,2) \rightarrow (2,-3,2) \rightarrow (2,6,2) \rightarrow (2,6,-1)$ is —

Sol:- Let $I = \int_C [(\text{grad} f) \cdot d\bar{r}]$

Then $I = \int_C (\nabla f \cdot d\bar{r})$

$\Rightarrow I = \int_C df = (f)_{A=(-3,-3,2)}^{B=(2,6,-1)}$

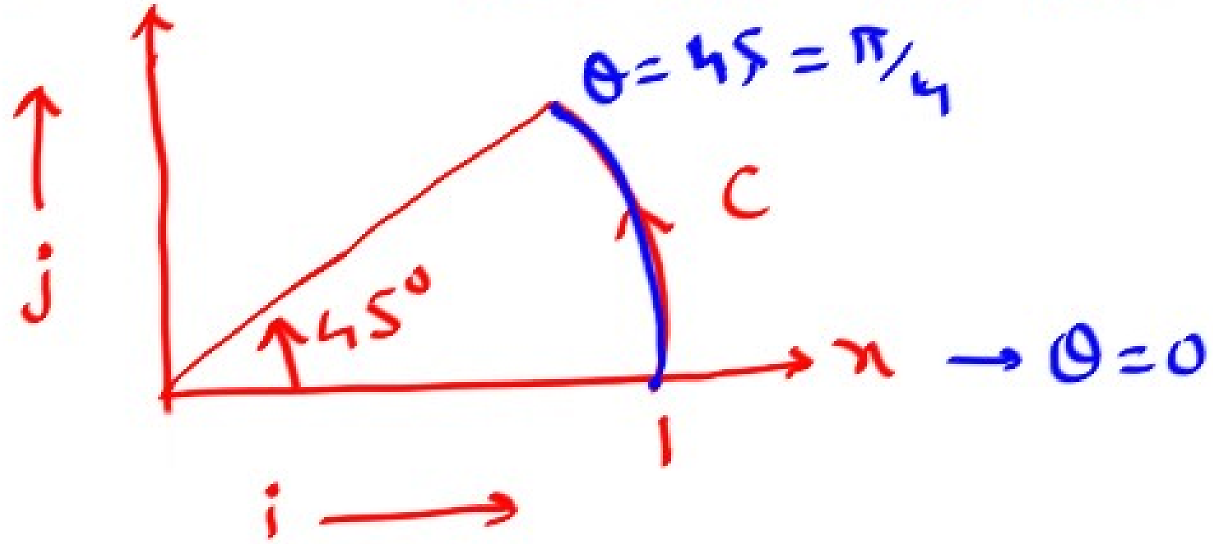
$\Rightarrow I = (2x^3 + 3y^2 + 4z)_{(-3,-3,2)}^{(2,6,-1)}$

$\therefore I = (\qquad) - (\qquad) = \_\_\_$

$\nabla\phi \cdot d\bar{r} = d\phi$

f

(4) The vector function $f(\bar{r}) = -x\bar{i} + y\bar{j}$ is defined over a circular arc 'c' shown in the figure



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The line integral of $\int_c f(\bar{r}) \cdot d\bar{r}$ is —

Sol: Let $I = \int_C f(\bar{r}) \cdot d\bar{r}$

Then $I = \int_C [(-x)\,dx + (y)\,dy]$

The parametric equations of a circle $x^2+y^2=r^2$ are

$x = r\cos\theta$, $y = r\sin\theta$

$\Rightarrow x = \cos\theta$ & $y = \sin\theta$

$\Rightarrow dx = -\sin\theta\, d\theta$ & $dy = \cos\theta\, d\theta$

Here: $\theta = 0$ to $\theta = \pi/4$.

Now, $I = \int_{\theta=0}^{\pi/4} [(-\cos\theta)(-\sin\theta)\,d\theta + (\sin\theta)(\cos\theta)\,d\theta]$

$\Rightarrow I = \int_{\theta=0}^{\pi/4} (2\sin\theta \cdot \cos\theta)\,d\theta = \int_0^{\pi/4} \sin(2\theta)\,d\theta$

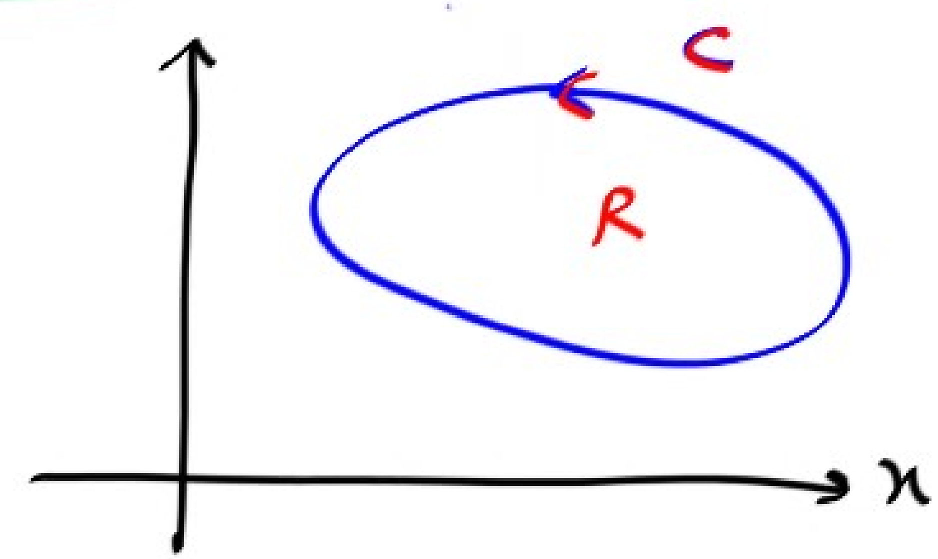
$\therefore I = \left(-\frac{\cos(2\theta)}{2}\right)_0^{\pi/4} = \frac{1}{2}[1-0] = 1/2$

Th ① [Green's theorem in a plane] (Line integral = Double integral)

St: If R is a closed region of xy-plane bounded by one or more simple curves 'C' and $M(x,y)$, $N(x,y)$, $\frac{\partial M}{\partial y}$ & $\frac{\partial N}{\partial x}$ are continuous functions in a region R

then $$\boxed{\oint_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy}$$, where $\bar{f} = M\bar{i} + N\bar{j}$
$= f_1\bar{i} + f_2\bar{j}$

$$\int_C \bar{f}\cdot d\bar{r} = \int_C (f_1\,dx + f_2\,dy)$$



(1) The value of $\int[(x+y)\,dx+(x^2)\,dy]$, where R is the triangle with vertices at (0,0), (2,0) & (2,4), is —

(a) 20

(b) $\frac{20}{9}$

(c) −10

(d) $\frac{20}{3}$.

Sol: By a G.T, we have

$$\boxed{\oint_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy}$$

Now, $\oint_C [(x+y)\,dx + (x^2)\,dy] = \iint_R \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(x+y)\right] dx\,dy$

$$= \iint_R [2x-1]\,dx\,dy$$

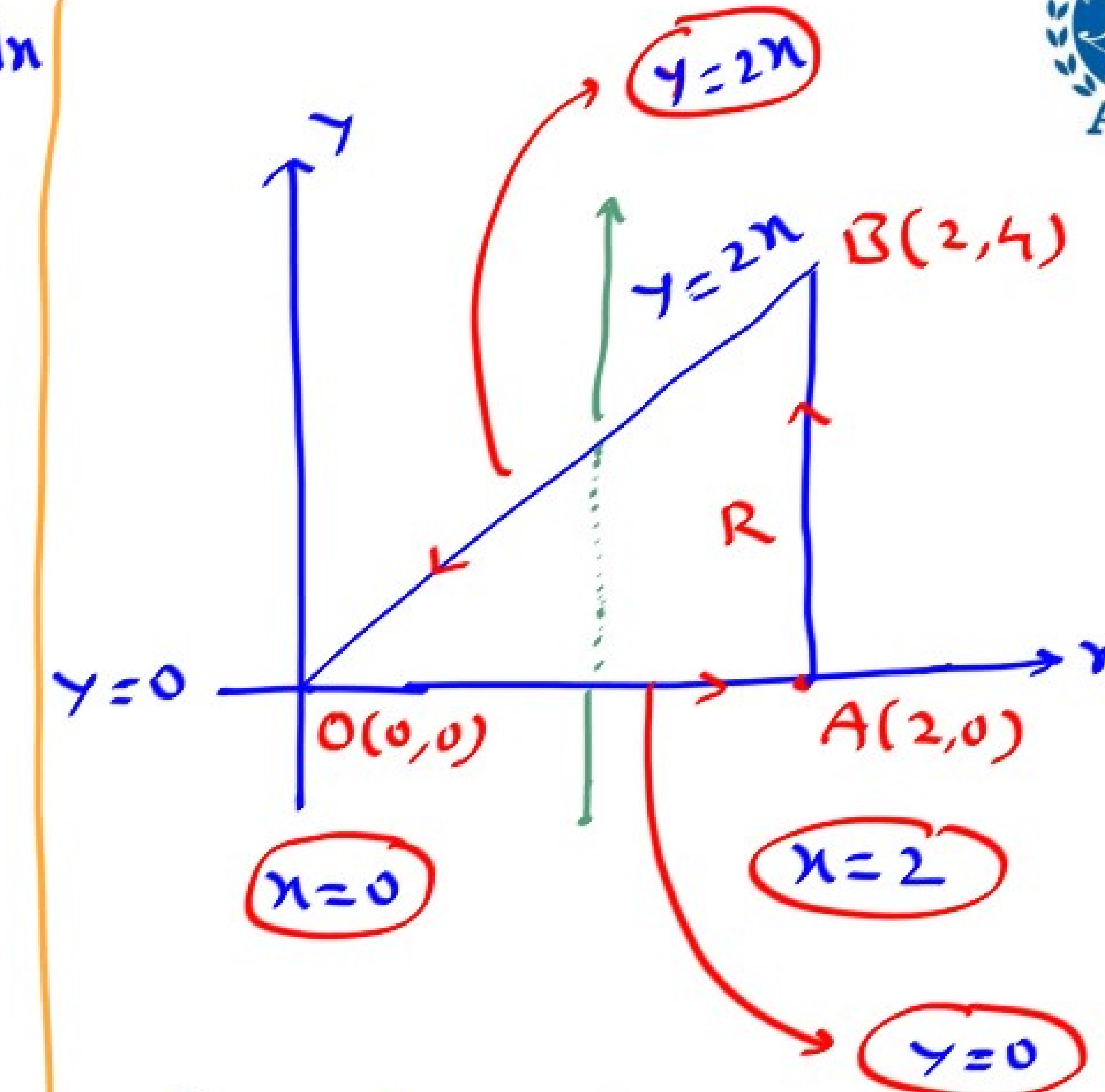
$$\Rightarrow \oint_C \left[(x+y)\,dx+(x^2)\,dy\right] = \int_{x=0}^{2}\left[\int_{y=0}^{2x}(2x-1)\,dy\right]dx$$

$$\Rightarrow \quad '' \quad = \int_{x=0}^{2}(2x-1)\,(y)_0^{2x}\,dx$$

$$\Rightarrow \quad '' \quad = \int_{x=0}^{2}(2x-1)\,(2x-0)\,dx$$

$$\Rightarrow \quad '' \quad = \left(\frac{4x^3}{3}-\frac{2x^2}{2}\right)_0^2$$

$$\therefore \oint_C \left[(x+y)\,dx+(x^2)\,dy\right] = \frac{20}{3}$$



$$\int_C = \int_{OA} + \int_{AB} + \int_{BO}.$$

(2) The value of the line integral $\int_c [(2x-y)\,dx + (x+3y)\,dy]$ over the ellipse $x^2+4y^2=4$ is –

(a) π

(b) 2π

(c) 4π

(d) 3π.

Sol: By a G.T, we have

$$\boxed{\oint_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy}$$

Now, $\oint_C [(2x-y)\,dx + (x+3y)\,dy] = \iint_R \left[\frac{\partial}{\partial x}(x+3y) - \frac{\partial}{\partial y}(2x-y)\right] dx\,dy$

$\Rightarrow$ " $= \iint_R (1+1)\,dx\,dy = (2)\left[\iint_R 1\,dx\,dy\right]$

$\Rightarrow$ " $= 2$ (Area of the plane region R), where R is $\frac{x^2}{2^2}+\frac{y^2}{1^2}=1$

$\therefore \oint_C [(2x-y)\,dx + (x+3y)\,dy] = 2(\pi ab)_{a=2,\,b=1} = \underline{4\pi}$, $\left(\because \frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \to A = \pi ab\right)$

(3) If $\bar{A} = \nabla\phi$ then the value of $\int_C \bar{A}\cdot d\bar{r}$, where 'c' is $\frac{x^2}{4} + \frac{y^2}{9} = 1$, is –

Sol:– Given that $\bar{A} = \nabla\phi$

$\Rightarrow$ $\bar{A}$ is an irrotational vector

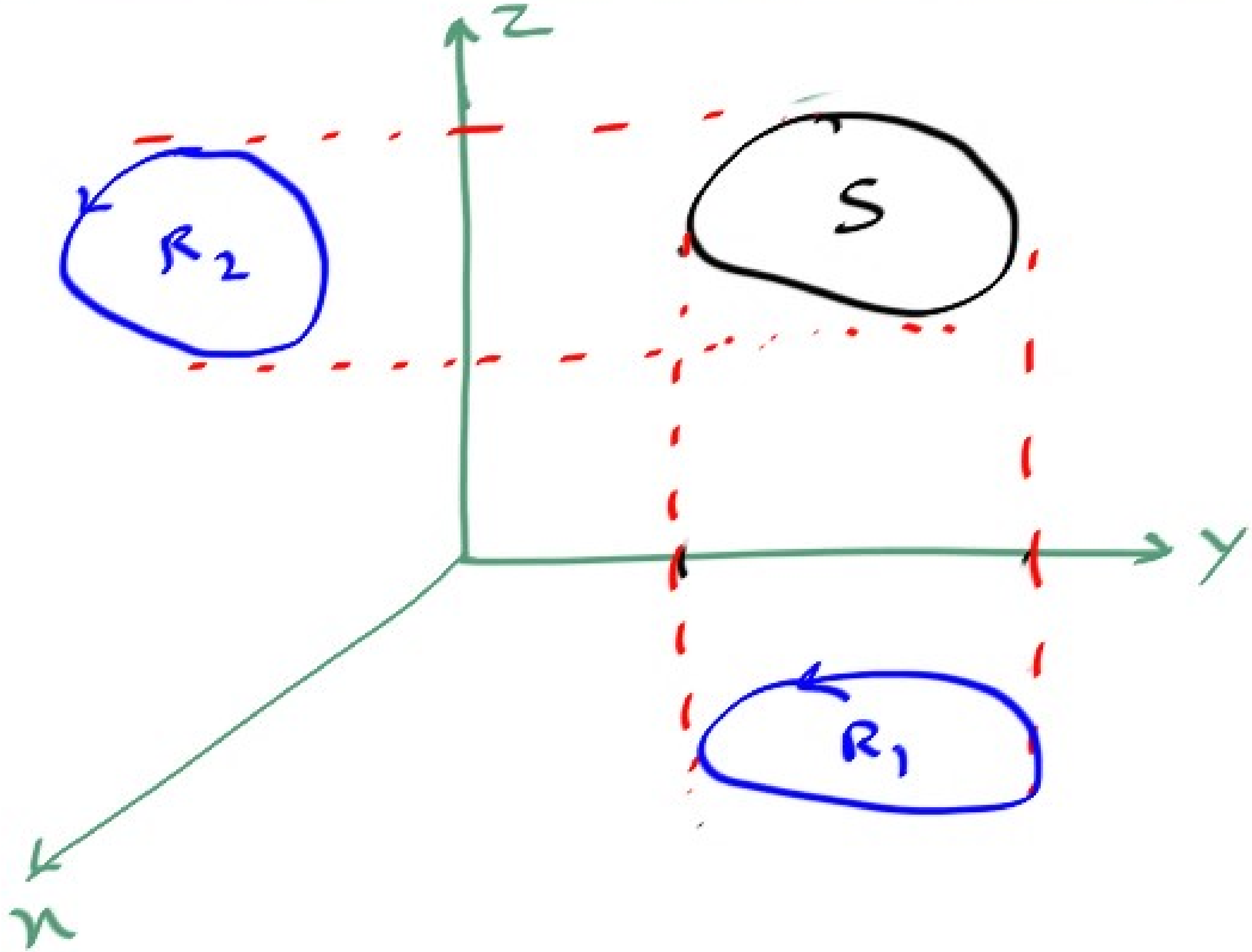
$\therefore$ $\oint_C \bar{A}\cdot d\bar{r} = 0$

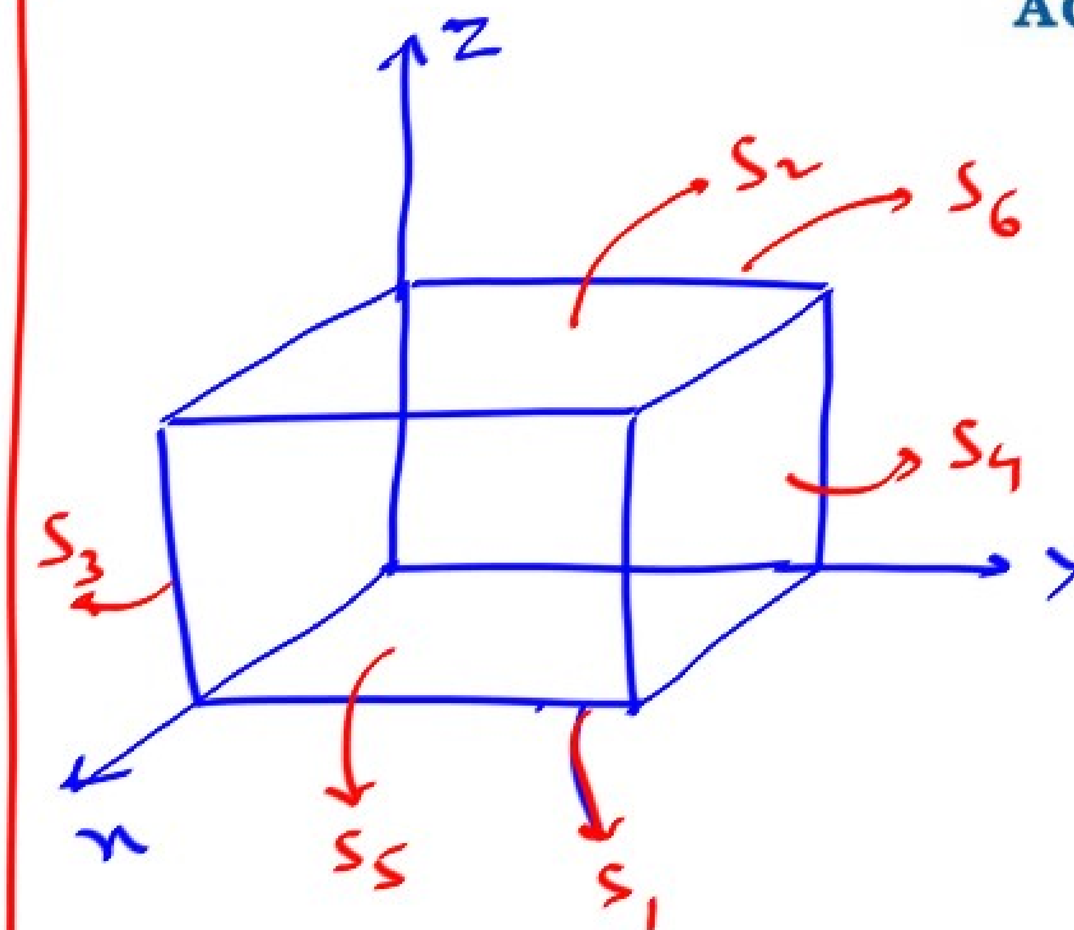
② **Surface integral (S.I) :-**

If a vector function $\bar{f}$ is defined at every point on the surface 'S' then the evaluation of integral of a vector function $\bar{f}$ over surface 'S' is a surface integral of $\bar{f}$ over 'S' & it is given by

$\boxed{\iint_S (\bar{f} \cdot \bar{n})\, ds}$ (or) $\boxed{\int_S (\bar{f} \cdot \bar{n})\, ds}$ (or) $\boxed{\iint_S (\bar{f} \cdot d\bar{s})}$, where $\bar{n}$ is outward drawn unit normal vector to the surface 'S' and $d\bar{s} = \bar{n}\, ds$

Method of evaluation of surface integral:-





$$\iint_S = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6}$$

(i) If R_1 is the projection of surface 'S' on xy-plane

then $$\boxed{\iint_S (\bar{f}\cdot\bar{n})\, ds = \iint_{R_1} (\bar{f}\cdot\bar{n}) \frac{dx\,dy}{|\bar{n}\cdot\bar{k}|}}$$

(ii) If R_2 is the projection of surface 'S' on xz-plane

then $$\boxed{\iint_S (\bar{f}\cdot\bar{n})\, ds = \iint_{R_2} (\bar{f}\cdot\bar{n}) \frac{dx\,dz}{|\bar{n}\cdot\bar{j}|}}$$

(iii) If R_3 is the projection of surface 'S' on yz-plane

then $$\boxed{\iint_S (\bar{f}\cdot\bar{n})\, ds = \iint_{R_3} (\bar{f}\cdot\bar{n}) \frac{dy\,dz}{|\bar{n}\cdot\bar{i}|}}$$

Th ② [Gauss-Divergence Theorem] (Surface integral = Volume integral)

St: If 'S' is a closed surface enclosing volume 'V' of the region and $\bar{F} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$ is a vector defined & differentiable at every point on the region 'V' then

$$\boxed{\oint\!\!\!\oint_S (\bar{F}\cdot\bar{n})\, ds = \iiint_V (\operatorname{div}\bar{F})\, dx\,dy\,dz}$$

(or)

$$\boxed{\oint\!\!\!\oint_S [f_1\, dy\,dz + f_2\, dx\,dz + f_3\, dx\,dy] = \iiint_V (\operatorname{div}\bar{F})\, dx\,dy\,dz}$$

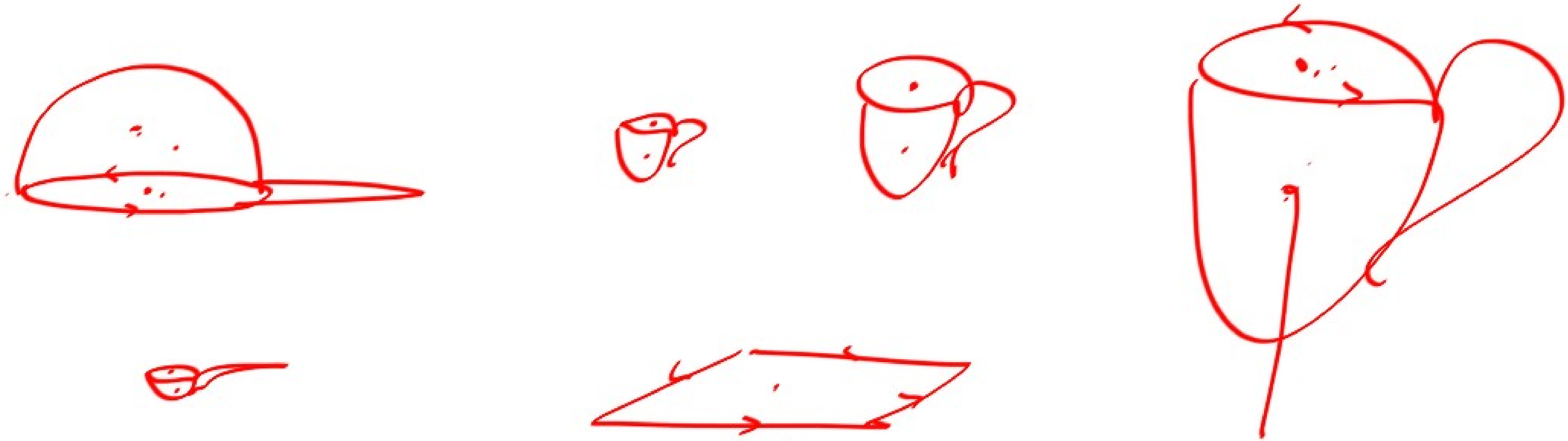
, where $\bar{n}$ is outward drawn unit normal vector to the surface 'S'.

Th ③ [Stoke's Theorem] (Line integral = Surface integral)

St: If 'S' is an open two-sided surface bounded by a simple closed curve and $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$ is a vector defined & differentiable at every point on the surface 'S' then

$$\boxed{\oint_C \bar{f}\cdot d\bar{r} = \iint_S (\operatorname{curl}\bar{f}\cdot\bar{n})\, ds}$$

, where $\bar{n}$ is outward drawn unit normal vector to the surface 'S'.



20 min'

① The value of $\iint_S (\bar{f}\cdot\bar{n})\,ds$, where S is the surface of the cylinder $x^2+y^2=16$, $-1 \le z \le 1$ & $\bar{f} = x\bar{i} + y\bar{j} + z\bar{k}$, is —

Sol: By a G.D.T, we have

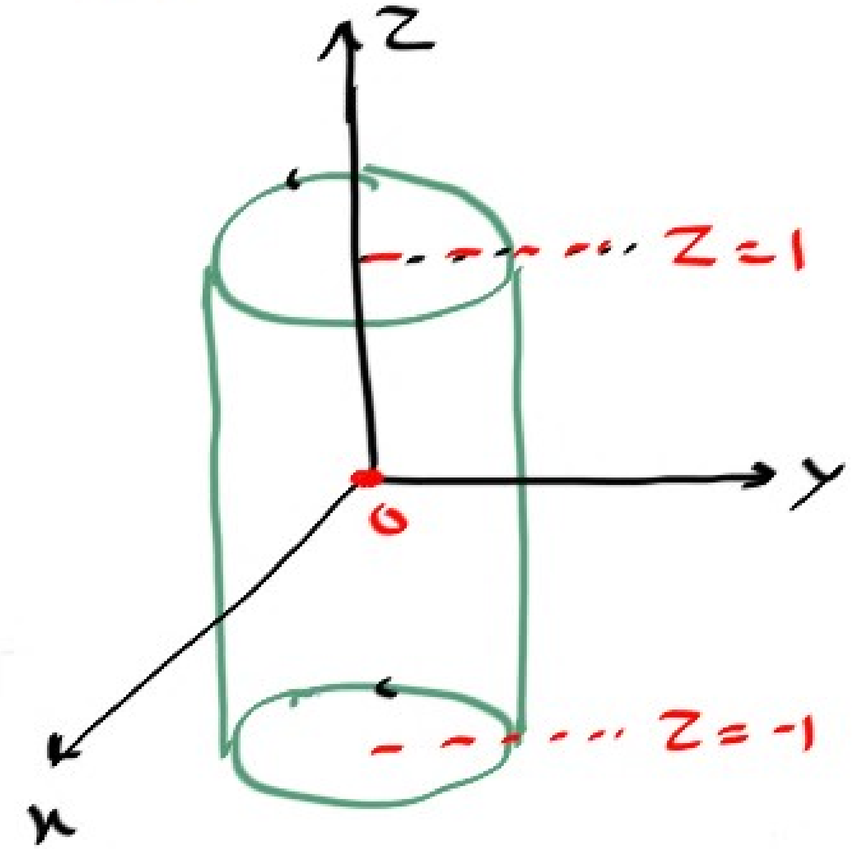
$$\boxed{\oiint_S (\bar{f}\cdot\bar{n})\,ds = \iiint_V (\operatorname{div}\bar{f})\,dv}$$

Now, $\oiint_S (\bar{f}\cdot\bar{n})\,ds = \iiint_V \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) dx\,dy\,dz$

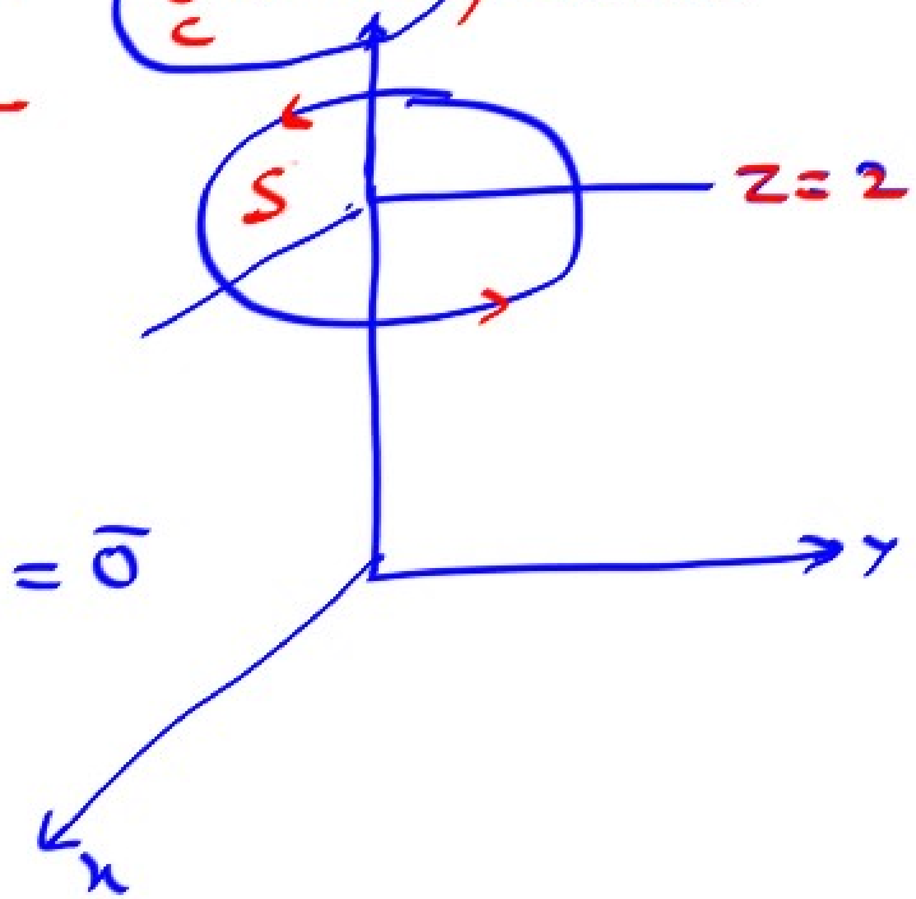
$\Rightarrow \oiint_S (\bar{f}\cdot\bar{n})\,ds = \iiint_V (1+1+1)\,dx\,dy\,dz$

$\Rightarrow$ " $= (3)\left(\iiint_V 1\,dx\,dy\,dz\right)$

$\therefore \oiint_S (\bar{f}\cdot\bar{n})\,ds = 3(\pi r^2 h)_{r=4,\,h=2} = \underline{96\pi}$



(2) Let $\bar{f} = (x^2+yz)\,\bar{i} + (y^2+xz)\,\bar{j} + (z^2+xy)\,\bar{k}$ be the differentiable vector point function. Then the value of $\int_C \bar{f}\cdot d\bar{r}$, where 'c' is the curve $x^2+y^2=4,\ z=2$ is —



Sol:- By a S.T, we have

$$\boxed{\oint_C \bar{f}\cdot d\bar{r} = \iint_S (\text{curl}\,\bar{f}\cdot\bar{n})\,ds}$$

Now, $\text{curl}\,\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2+yz) & (y^2+xz) & (z^2+xy) \end{vmatrix} = 0\bar{i} - 0\bar{j} + 0\bar{k} = \bar{0}$

$\Rightarrow \bar{f}$ is an irrotational vector

$\therefore \oint_C \bar{f}\cdot d\bar{r} = \iint_S (\bar{0}\cdot\bar{n})\,ds = \iint_S (0)\,ds = 0$

(3) The value of the surface integral $\iint_S \left(\underbrace{xy}_{f_1}\, dy\,dz + \underbrace{yz}_{f_2}\, dz\,dx + \underbrace{zx}_{f_3}\, dx\,dy \right)$ where 'S' is the surface bounded by $x=0, x=4, y=0, y=3,$ $z=0$ & $z=4$, is –

Sol:– By G.D.T, we have

$$\boxed{\iint_S (\bar{f}\cdot\bar{n})\, ds = \iiint_V (\operatorname{div} \bar{f})\, dv}$$

Now, $\iint_S (\bar{f}\cdot\bar{n})\, ds = \iiint_V \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \right) dx\, dy\, dz$

$$\Rightarrow \quad '' \quad = \int_{x=0}^{4} \left[\int_{y=0}^{3} \left\{ \int_{z=0}^{4} (y+z+x)\, dz \right\} dy \right] dx$$

$$\Rightarrow \quad '' \quad = \int_{x=0}^{4} \left[\int_{y=0}^{3} \left(yz + \frac{z^2}{2} + xz \right)_0^4 dy \right] dx$$

$$\Rightarrow \iint_S (\bar{f}\cdot\bar{n})\, ds = \int_{x=0}^{4} \left[\left(4\frac{y^2}{2} + 8y + 4xy\right)_0^3 \right] dx$$

$$\Rightarrow \iint_S (\bar{f}\cdot\bar{n})\, ds = \left(18x + 24x + 12\frac{x^2}{2}\right)_0^4$$

$$\therefore \iint_S (\bar{f}\cdot\bar{n})\, ds = \underline{\underline{264}}$$

(4) Evaluate the integral $\oint_C \bar{f} \cdot d\bar{r}$, where 'C' is the boundary of the upper half of surface of the sphere $x^2+y^2+z^2=1$ above the xy-plane and $\bar{f} = (2x-y)\bar{i} - yz^2\bar{j} - y^2z\bar{k}$.

Sol: By a S.T, we have

$$\boxed{\oint_C \bar{f} \cdot d\bar{r} = \iint_S (\text{curl}\,\bar{f} \cdot \bar{n})\, ds}$$

Now, $\text{curl}\,\bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2x-y) & (-yz^2) & (-y^2z) \end{vmatrix} = (-2yz+2yz)\bar{i} - (0-0)\bar{j} + (0+1)\bar{k}$

$\Rightarrow \text{curl}\,\bar{f} = \bar{k}$

Let $\phi = x^2+y^2+z^2-1=0$ be the equation of surface 'S'.

Then $\bar{n} = \frac{\nabla\phi}{|\nabla\phi|} = x\bar{i} + y\bar{j} + z\bar{k}$

$$\Rightarrow \text{curl}\,\bar{F}\cdot\bar{n} = (\bar{k})\cdot(x\bar{i}+y\hat{j}+z\bar{k})$$

$$\Rightarrow \text{curl}\,\bar{F}\cdot\bar{n} = z$$

Let R be the projection of the surface 'S' on xy-plane.

Then $\oint_C \bar{F}\cdot d\bar{r} = \iint_S (\text{curl}\,\bar{F}\cdot\bar{n})\, ds$

$$\Rightarrow \quad \text{''} \quad = \iint_R \cancel{(\text{curl}\,\bar{F}\cdot\bar{n})} \frac{dx\,dy}{\cancel{|\bar{n}\cdot\bar{k}|}}$$

$$\Rightarrow \quad \text{''} \quad = \iint_R 1\, dx\,dy$$

$$\therefore \oint_C \bar{F}\cdot d\bar{r} = (\pi r^2)_{r=1} = \underline{\pi}$$