HOMOGENEOUS SYSTEMS: A system of m linear equations in n unknowns

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = 0$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = 0$$

in which the right-hand side consists entirely of 0s is said to be a homogeneous system.
Let $A_{m\times n}$ be the coefficient matrix for a homogeneous system of m linear equations in n unknowns, and suppose rank(A)=r.

$x_1 = x_2 = \cdots = x_n = 0$ (Trivial solution)

Homogeneous Sys are always consistent.

$[A|b] =$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 + 2R_1$

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$v_1 = \begin{pmatrix}1\\0\end{pmatrix} \quad v_2 = \begin{pmatrix}0\\2\end{pmatrix}$

- The unknowns that correspond to the positions of the basic columns (i.e., the pivotal positions) are called the basic variables, and the unknowns corresponding to the positions of the nonbasic columns are called the free variables.
- There are exactly r basic variables and n-r free variables.
- To describe all solutions, reduce A to a row echelon form using Gaussian elimination, and then use back substitution to solve for the basic variables in terms of the free variables. This produces the general solution that has the form

$$\mathbf{x} = x_{f_1}\mathbf{h}_1 + x_{f_2}\mathbf{h}_2 + \dots + x_{f_{n-r}}\mathbf{h}_{n-r}$$

where $x_{f_1}, x_{f_2}, \dots, x_{f_{n-r}}$ are the free variables and $\mathbf{h}_1, \mathbf{h}_2, \dots, \mathbf{h}_{n-r}$ are $n \times 1$ columns.
- These $n - r$ solutions are said to linearly independent.
- A homogeneous system possesses a unique solution (the trivial solution) if and only if rank (A) = n i.e., if and only if there are no free variables

$X = k_1 h_1 + k_2 h_2$

$$\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix} = k_1\begin{pmatrix}-2\\0\\-1\\1\\0\end{pmatrix} + k_2\begin{pmatrix}-2\\1\\0\\0\\0\end{pmatrix}$$

$v_1 = \begin{pmatrix}1\\0\end{pmatrix} \quad v_2 = \begin{pmatrix}2\\0\end{pmatrix}$

$v_2 = 2v_1$

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Homogeneous system

Unique solution

$x_1 = x_2 = \cdots = x_n = 0$ Trivial Solution | zero solution

$R(A) = n$

Infinitely many solution

$R(A) < n$

Non trivial solution | Non zero solution

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NON-HOMOGENEOUS SYSTEMS

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m$$

a system of m linear equations in n unknowns is said to be non homogeneous whenever $b_i \neq 0$ for at least one i.
Unlike homogeneous systems, a non homogeneous system may be inconsistent.

Inconsistent : $R(A) \neq R(A|b)$

To describe the set of all possible solutions of a consistent non homogeneous system, construct a general solution by exactly the same method used for homogeneous systems as follows.

- Use Gaussian elimination to reduce the associated augmented matrix $[\mathbf{A}|\mathbf{b}]$ to a row echelon form $[\mathbf{E}|\mathbf{c}]$.
- Identify the basic variables and the free variables.
- Apply back substitution to $[\mathbf{E}|\mathbf{c}]$ and solve for the basic variables in terms of the free variables.
- Write the result in the form

$$\mathbf{x} = \mathbf{p} + x_{f_1}\mathbf{h}_1 + x_{f_2}\mathbf{h}_2 + \ldots + x_{f_{n-r}}\mathbf{h}_{n-r}$$

where $x_{f_1}, x_{f_2}, \ldots, x_{f_{n-r}}$ are the free variables and $\mathbf{p}, \mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_{n-r}$ are n $\times$ 1 columns. This is the general solution of the non-homogeneous system.

$AX = b$

$AX = 0$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 - 2k_1 - 2k_2 \\ k_2 \\ 1 - k_1 \\ k_1 \\ 1 \end{pmatrix} = k_1 \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$x_{f_1} h_1 + x_{f_2} h_2 + P$

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The general solution of the non-homogeneous system is given by a particular solution plus the general solution of the associated homogeneous system.

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Non Homogeneous System

No solution

$R(A) \neq R(A|b)$

Unique Solution

$R(A) = R(A|b) = n$

Infinitely many solutions

$R(A) = R(A|b) < n$

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11. Consider the linear equations:

$$x - 2y + z = 3,$$

$$2x + \alpha z = -2,$$

$$-2x + 2y + \alpha z = 1.$$

In order to have unique solution to this linear system of equations the value of α should not be equal to

(a) $\frac{-2}{3}$ (b) $\frac{2}{3}$

(c) $\frac{4}{3}$ (d) $\frac{-4}{3}$

For unique solution

$R(A) = R(A|b) = \underline{\underline{3}}$

$$\begin{pmatrix} 1 & -2 & 1 & 3 \\ 2 & 0 & \alpha & -2 \\ -2 & 2 & \alpha & 1 \end{pmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 + 2R_1$

$$\begin{pmatrix} 1 & -2 & 1 \\ 0 & 4 & \alpha - 2 \\ 0 & -2 & \alpha + 2 \end{pmatrix}$$

$R_3 \to R_3 + \frac{1}{2} R_2$

$\alpha + 2 + \frac{1}{2}(\alpha - 2)$

$\frac{2\alpha + 4 + \alpha - 2}{2}$

$$\begin{pmatrix} 1 & -2 & 1 \\ 0 & 4 & \alpha - 2 \\ 0 & 0 & \frac{3\alpha + 2}{2} \end{pmatrix}$$

$\frac{3\alpha + 2}{2} \neq 0$

$3\alpha + 2 \neq 0$

$\alpha \neq -2/3$

(a)

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MCQ

12. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 4 & 3 & 10 \end{bmatrix}$, then which of the following is ***not*** true?

(a) Rank of A = 2 True

(b) The system AX = O has infinitely many solutions, where $X = [x\ y\ z]^T$ True

(c) The system AX = B has a unique solution False

(d) A^{-1} does not exist

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 4R_1$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -5 & -2 \\ 0 & -5 & -2 \end{pmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -5 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

$R(A) = \underline{\underline{2}}$

$AX = 0 \quad \underline{\underline{R(A) < n}}$

$AX = B$

$R(A) = R(A|b) = \underline{\underline{n}}$

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13. Consider the system $\begin{cases} kx + y + z = 1 \\ x + ky + z = 1 \\ x + y + kz = 1 \end{cases}$.

If the system has a unique solution then which of the following is true?

(a) k = 1 and k = –2 (b) k ≠ 1 and k ≠ – 2

(c) k = 1 and k ≠ –2 (d) k ≠ 1 and k = –2

$AX = b$

$R(A) = R(A|b) = 3$

$$[A|b] = \begin{bmatrix} K & 1 & 1 & 1 \\ 1 & K & 1 & 1 \\ 1 & 1 & K & 1 \end{bmatrix}$$

a and c are eliminated

for $K=1$ $R(A) = R(A|b) = 1$ $K \neq 1$

$R_3 \to R_3 + R_2 + R_1$

$$\begin{pmatrix} K & 1 & 1 & 1 \\ 1 & K & 1 & 1 \\ 2+K & 2+K & 2+K & 3 \end{pmatrix}$$

For unique solution $K \neq -2$

(b)

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$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 2 & 1 & 0 & 1 \\ 6 & 3 & 4 & 7 \\ 4 & 2 & 1 & 3 \end{bmatrix}$$

14. Let AX = B be a system of three equations in three variables x, y and z. The augmented matrix of the system is given by

$$[A \mid B] = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}.$$

Which of the following is ***not*** true?

(a) Rank of [A | B] = 2 True

(b) The system AX = B has infinitely many solutions True

(c) The system AX = O has unique solution False

(d) Rank of A = 2 True

$R_2 \to R_2 - 3R_1, \quad R_3 \to R_3 - 2R_1$

$$\begin{pmatrix} 2 & 1 & 0 & 1 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$R_3 \to R_3 - \frac{1}{4}R_2$

$$\begin{pmatrix} 2 & 1 & 0 & 1 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$R(A) = 2 \quad R(A|b) = 2$

which of the following is true

a, b, d

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15. The following system of equations

$$x + y + z = 3,$$
$$x + 2y + 3z = 4,$$
$$x + 4y + kz = 6$$

has infinitely many solutions when

(a) $k \neq 0$ (b) $k = 0$

(c) $k = 7$ (d) $k \neq 7$

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$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & K & 6 \end{array}\right)$$

$R_2 \to R_2 - R_1$ $R_3 \to R_3 - R_1$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & K-1 \end{pmatrix}$$

$R_3 \to R_3 - 3R_2$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & K-7 \end{pmatrix}$$

$K = 7 \Rightarrow R(A) = 2$

Infinitely many solutions

16. Consider the matrix $A = \begin{bmatrix} k & k & k \\ 0 & k-1 & k-1 \\ 0 & 0 & k^2-1 \end{bmatrix}$. 3×3

If the system AX = O has only one independent solution then k = ______.

(a) 0, –1 (b) –1, 1

(c) 0, 1 (d) 0, 1, –1

Given $n - r = 1$

$3 - r = 1$

$\boxed{r = 2}$

verification

$k = 1$

$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$R(A) = 1$

eliminate options with $k = 1$

b, c, d are eliminated

Ans (a)

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18. Consider the following system of equations:

$3x + 2y = 1,\ \ 4x + 7z = 1,$

$x + y + z = 3,\ x - 2y + 7z = 0$

Which of the following is true?

(a) The system has no solution

(b) The system has infinitely many solutions

(c) The system has unique solution

(d) The rank of the augmented matrix of the system is 2

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$$\begin{pmatrix} 1 & 1 & 1 & 3 \\ 4 & 0 & 7 & 1 \\ 3 & 2 & 0 & 1 \\ 1 & -2 & 7 & 0 \end{pmatrix}$$

$R_2 \to R_2 - 4R_1$

$R_3 \to R_3 - 3R_1$

$R_4 \to R_4 - R_1$

$$\begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -4 & 3 & -11 \\ 0 & -1 & -3 & -8 \\ 0 & -3 & 6 & -3 \end{pmatrix}$$

$R_2 \leftrightarrow R_3$

$$\begin{matrix} 1 & 1 & 1 & 3 \\ 0 & -1 & -3 & -8 \\ 0 & -4 & 3 & -11 \\ 0 & -3 & 6 & -3 \end{matrix}$$

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$$\begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -1 & -3 & -8 \\ 0 & -4 & 3 & -11 \\ 0 & -3 & 6 & -3 \end{pmatrix}$$

$R_3 \to R_3 - 4R_2$

$R_4 \to R_4 - 3R_2$

$$\begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -1 & -3 & -8 \\ 0 & 0 & 15 & 21 \\ 0 & 0 & 15 & 21 \end{pmatrix}$$

$R_4 \to R_4 - R_3$

$$\begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -1 & -3 & -8 \\ 0 & 0 & 15 & 21 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$R(A) = R(A|b) = \underline{\underline{3}}$

unique $\underline{\underline{\text{solution}}}$

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227. If the following system has non – trivial solution

$$px + qy + rz = 0$$
$$qx + ry + pz = 0$$
$$rx + py + qz = 0$$

Then which one of the following Options is TRUE? **(GATE – 15 – CS – Set 3)**

(a) $p - q + r = 0$ or $p = q = -r$
(b) $p + q - r = 0$ or $p = -q = r$
(c) $p + q + r = 0$ or $p = q = r$
(d) $p - q + r = 0$ or $p = -q = -r$

$$A = \begin{pmatrix} p & q & r \\ q & r & p \\ r & p & q \end{pmatrix}$$

If $p = q = r$

$$A = \begin{pmatrix} p & p & p \\ p & p & p \\ p & p & p \end{pmatrix} \rightarrow \begin{pmatrix} p & p & p \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$R(A) = 1 < 3$

$p = q = r$ leads to infinitely many $\underline{\text{solutions}}$

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$$A = \begin{pmatrix} p & q & r \\ q & r & p \\ r & p & q \end{pmatrix}$$

$R_3 \to R_3 + R_2 + R_1$

$$\begin{pmatrix} p & q & r \\ q & r & p \\ r+q+p & p+r+q & q+p+r \end{pmatrix}$$

$p+q+r=0$

$R(A) < 3$

$p+q+r=0$ leads to infinitely many solutions

$p = q = -r$

$$\begin{pmatrix} p & p & -p \\ p & -p & p \\ -p & p & p \end{pmatrix}$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 + R_1$

$$\begin{pmatrix} p & p & -p \\ 0 & -2p & 2p \\ 0 & 2p & 0 \end{pmatrix}$$

$R_3 \to R_3 + R_2$

$$\begin{pmatrix} p & p & -p \\ 0 & -2p & 2p \\ 0 & 0 & 2p \end{pmatrix}$$

Nontrivial solution

(C)

17. Let AX = B be a system of three equations in three variables x, y and z. If A has three linearly independent columns and B is a linear combination of the columns of A, then which of the following is true?

(a) The system has unique solution

(b) The system has infinitely many solutions

(c) The system has no solution

(d) The system AX = O has non-zero solution

$AX = B$

Given B is linear combination of columns of A

$\Rightarrow$ $AX = B$ is Consistent

Let us solve this after covering Linearly independent and dependent vector Concept.

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268. Let $c_1, \ldots, c_n$ be scalars, not all zero, such that $\sum_{i=1}^{n} c_i a_i = 0$ where a_i are column vectors in R^n. Consider the set of linear equations

$Ax = b$

Where $A = [a_1, \ldots, a_n]$ and $b = \sum_{i=1}^{n} a_i$.

The set of equations has

(GATE – 17–CSIT)

(a) a unique solution at $x = J_n$ where J_n denotes a n-dimensional vector of all 1

(b) no solution

(c) infinitely many solutions

(d) finitely many solutions

$b = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$

b is linear combination of columns of A

$\Rightarrow Ax = b$ is consistent

eliminate (b) and (d)

Let us solve this after covering Linearly independent and dependent vector Concept.

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17. Find the values of k, for which the following system of linear equations has an infinite number of solutions?

$x_1 - x_2 + 2x_3 = 7$, $x_1 + x_2 - x_3 = 1$, $-x_1 + kx_2 + 3x_3 = 0$

(a) –7/3 & 4/3 (b) –7/3 & –4/3

(c) 7 & 4 (d) –7 & –4

Question not properly framed