183. The maximum value of the determinant among all 2 × 2 real symmetric matrices with trace 14 is ________.

(GATE–14–EC–SET2)

$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

trace $(A) = a + c = 14$

$|A| = ac - b^2$

max $ac - b^2$

such that $a + c = 14$

To get the maximum value of $ac - b^2$, we need to take $b = 0$

max ac

$a + c = 14 \Rightarrow c = 14 - a$

$a(14 - a) = 14a - a^2$

$f(a) = 14a - a^2$

$f'(a) = 14 - 2a = 0$

$\Rightarrow a = 7$

$f''(a) = -2$

$f''(7) < 0$

$\therefore$ max value occurs when $a = 7$

$\Rightarrow 14(7) - 7^2$

$= 49$

Inverse of a Matrix

- Only non-singular matrices are invertible.
- **B** is called as Inverse of matrix **A** if

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I} \implies \mathbf{B} = \mathbf{A}^{-1}$$

$$\mathbf{A}^{-1} = \frac{Adj(\mathbf{A})}{\det(\mathbf{A})}$$

where Adj(**A**) is the cofactor matrix transpose.

$$Adj(\mathbf{A}) = (\text{cofactor matrix})^T$$

$$A^{-1} = \frac{Adj(A)}{|A|}$$

$$Adj(A) = \begin{pmatrix}\text{cofactor matrix} \\ \text{of} \quad A\end{pmatrix}^T$$

$$\boxed{AA^{-1} = A^{-1}A = I}$$

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Find the inverse of the following matrix

$$\mathbf{A} = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

Cofactor matrix of A $= \begin{bmatrix} -3 & -6 & -6 \\ 6 & 3 & -6 \\ 6 & -6 & 3 \end{bmatrix}_{3\times 3}$

$$\text{Adj}(A) = \begin{pmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{pmatrix}, \quad A^{-1} = \frac{\text{Adj}\,A}{|A|} = \frac{1}{27}\begin{pmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{pmatrix}$$

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If $\mathbf{A}_{n\times n}$ is a non-singular then

$$|\mathbf{A}^{-1}| = \underline{\quad 1/|A| \quad}$$

$$|Adj(\mathbf{A})| = \underline{\quad |A|^{n-1} \quad}$$

$$Adj(Adj(\mathbf{A})) = \underline{\qquad\qquad}$$

$$|Adj(Adj(\mathbf{A}))| = \underline{\qquad\qquad}$$

H/W

$$AA^{-1} = I$$

$$|AA^{-1}| = |I|$$

$$|A||A^{-1}| = 1$$

$$|A^{-1}| = \frac{1}{|A|}$$

$$|AB| = |A||B|$$

$$C = \alpha A_{n\times n}$$

$$|C| = \alpha^n |A|$$

$$A^{-1} = \frac{Adj\ A}{|A|} \Rightarrow Adj(A) = |A|\ A^{-1}$$

$$|Adj(A)| = \big|\,|A|\ A^{-1}\big|$$

$$= |A|^n |A^{-1}|$$

$$= |A|^n \frac{1}{|A|} = |A|^{n-1}$$

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03. If $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ and $B = A^{-1}$, then the element in the second row and third column of B = ______.

(a) 0 (b) $\frac{1}{2}$

(c) $-\frac{1}{2}$ (d) 1

cofactor matrix of A $= \begin{bmatrix} - & - & - \\ - & - & - \\ - & \text{[redacted]} & - \end{bmatrix}_{3\times 3}$

b_{23}

Adj A $= \begin{bmatrix} - & - & - \\ - & - & \text{[redacted]} \\ - & - & - \end{bmatrix}_{3\times 3}$

$A^{-1} = \dfrac{\text{Adj}(A)}{|A|}$

cofactor of $a_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = -1$

$|A| = 0(\;) + 0(\;) + 1(-1)^{3+3} \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2$

Ans: $b_{23} = \frac{-1}{2}$

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291. Consider a 2×2 matrix $M = [v_1, v_2]$, where, v_1 and v_2 are the column vectors.

Suppose $M^{-1} = \begin{bmatrix} u_1^T \\ u_2^T \end{bmatrix}$, where u_1^T and u_2^T are the row vectors.

Consider the following statements:

Statement 1: $u_1^T v_1 = 1$ and $u_2^T v_2 = 1$

Statement 2: $u_1^T v_2 = 0$ and $u_2^T v_1 = 0$

Which of the following options is correct?

(GATE-19-EE)

(a) Statement 2 is true and statement 1 is false

(b) Both the statements are false

(c) Statement 1 is true and statement 2 is false

(d) Both the statements are true

$$M = [v_1 \ v_2] = \begin{bmatrix} | & | \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} u_1^T \\ u_2^T \end{bmatrix} = \begin{bmatrix} — \\ — \end{bmatrix}$$

$$\underline{M^{-1}M = I} \quad , \quad \underline{MM^{-1} = I}$$

$$\begin{pmatrix} u_1^T \\ u_2^T \end{pmatrix} \begin{pmatrix} v_1 & v_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$u_1^T v_1 = 1 \qquad u_1^T v_2 = 0$$

$$u_2^T v_1 = 0 \qquad u_2^T v_2 = 1$$

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297. The inverse of the matrix $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$ is

(GATE-19-CE-SET2)

(a) $\begin{bmatrix} -10 & 4 & 9 \\ 15 & -4 & -14 \\ -5 & 1 & 6 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & -\frac{4}{5} & -\frac{9}{5} \\ -3 & \frac{4}{5} & \frac{14}{5} \\ 1 & -\frac{1}{5} & -\frac{6}{5} \end{bmatrix}$

(c) $\begin{bmatrix} -2 & \frac{4}{5} & \frac{9}{5} \\ 3 & -\frac{4}{5} & -\frac{14}{5} \\ -1 & \frac{1}{5} & \frac{6}{5} \end{bmatrix}$

(d) $\begin{bmatrix} 10 & -4 & -9 \\ -15 & 4 & 14 \\ 5 & -1 & -6 \end{bmatrix}$

$A \cdot A^{-1} = I$

a) $\begin{bmatrix} \cancel{2 \quad 3 \quad 4} \\ 4 \quad 3 \quad 1 \\ 1 \quad 2 \quad 4 \end{bmatrix} \begin{bmatrix} -10 \\ 15 \\ -5 \end{bmatrix} = -20+45-20 = 5 \neq 1$ ✗

b) $\begin{bmatrix} \cancel{2 \quad 3 \quad 4} \\ 4 \quad 3 \quad 1 \\ 1 \quad 2 \quad 4 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = 4-9+4 = -1 \neq 1$ ✗

d) $\begin{bmatrix} \cancel{2 \quad 3 \quad 4} \\ 4 \quad 3 \quad 1 \\ 1 \quad 2 \quad 4 \end{bmatrix} \begin{bmatrix} 10 \\ -15 \\ 5 \end{bmatrix} = 20-45+20 = -5$

c) $\begin{bmatrix} \cancel{2 \quad 3 \quad 4} \\ 4 \quad 3 \quad 1 \\ 1 \quad 2 \quad 4 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix} = -4+9-4 = 1$

$\boxed{\text{Ans} = C}$

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Rank of a matrix

For a matrix $A_{m\times n}$

- rank($\mathbf{A}$) denotes the number of nonzero rows in any row echelon form that is row equivalent to $\mathbf{A}$.
- rank($\mathbf{A}$) denotes the number of pivots obtained in reducing $\mathbf{A}$ to a row echelon form with row operations.
- rank($\mathbf{A}$) denotes the size of the largest nonzero minor of $\mathbf{A}$.
- rank($\mathbf{A}$) denotes the number of linearly independent rows or columns of A

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Use determinants to compute the rank of $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 1 \\ 7 & 8 & 9 & 1 \end{bmatrix}_{3\times 4}$

4x4

minors

$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0$

$\begin{vmatrix} 1 & 2 & 1 \\ 4 & 5 & 1 \\ 7 & 8 & 1 \end{vmatrix} = 0$

$\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}_{2\times 2} = -3 \neq 0$

$\Rightarrow \text{Rank}(A) = 2$

$\begin{vmatrix} 1 & 3 & 1 \\ 4 & 6 & 1 \\ 7 & 9 & 1 \end{vmatrix} = 0$

$\begin{vmatrix} 2 & 3 & 1 \\ 5 & 6 & 1 \\ 8 & 9 & 1 \end{vmatrix} = 0$

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$A_{3\times4}$ $R(A_{3\times4}) \leq \min(3,4)$

$\Rightarrow R(A_{3\times4}) \leq 3$

Let **A** be $m \times n$ matrix

- rank($\mathbf{A}$) $\leq$ min(m,n). ✓
- rank($\mathbf{A}+\mathbf{B}$) $\leq$ rank($\mathbf{A}$) + rank($\mathbf{B}$).
- If $\mathbf{A}$ is $m \times n$ and $\mathbf{B}$ is $n \times p$, then rank($\mathbf{AB}$) $\leq$ min {rank($\mathbf{A}$), rank($\mathbf{B}$)}
- rank($\mathbf{A}\mathbf{A}^T$) = rank($\mathbf{A}$) = rank($\mathbf{A}^T\mathbf{A}$) ✓ $\longrightarrow$ GATE Questions
- The rank of a non-zero matrix is non-zero.
- The rank of a null matrix is zero.
- The rank of non-singular matrix is its order.
- The rank of singular matrix is less than its order.

$|A_{n\times n}| = 0$ $\quad R(A) < n$

$|A_{4\times4}| \neq 0$

$R(A) = 4$

$|A_{n\times n}| \neq 0$ $\quad R(A) = n$

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01. Let $A = \begin{bmatrix} 3 & P & P \\ P & 3 & P \\ P & P & 3 \end{bmatrix}$. If rank of A is 1, then

P = ______.

$P = 3$

$A = \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix}$

$\begin{pmatrix} -3 & -3 \\ 3 & -3 \end{pmatrix}$

$9 + 9 = \underline{\underline{18}}$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$R(A) = \underline{1}$

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04. Suppose that $A_{n\times n}$ is upper triangular matrix such that $a_{ii} = 0$, $i = 1, 2, \ldots\ldots., n$.

Then rank of A^n = ________.

(a) 0 (b) $n - 1$

(c) 1 (d) n

verification

$\underline{n=2}$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$A^2 = O$ $\quad R(A^2) = 0$

(a)

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10. Let $A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$ $a_{ij} = -a_{ji}$

where a, b, c are non-zero real numbers.

Then Rank of A =

(a) 0 (b) 1

(c) 2 (d) 3

A is a skew sym matrix of odd order

$|A| = 0$

$R(A) < \underline{\underline{3}}$

$\begin{vmatrix} 0 & a \\ -a & 0 \end{vmatrix} = a^2 \neq 0$

$R(A) = \underline{\underline{2}}$

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The rank of the matrix

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$ is ______.

5x5

(GATE – 17–EC)

Hint : Reduce the matrix to row echelon form

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277. Consider matrix $A = \begin{bmatrix} k & 2k \\ k^2 - k & k^2 \end{bmatrix}_{2\times 2}$ and vector $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. The number of distinct real values of k for which the equation Ax = 0 has infinitely many solutions is ________.

(GATE-18-EC)

$AX = 0$

$R(A) < n$

$R(A) < 2$

when $R(A_{n\times n}) < n \Rightarrow |A_{n\times n}| = 0$

$$\begin{vmatrix} K & 2K \\ K^2 - K & K^2 \end{vmatrix} = 0$$

$K^3 - 2K^3 + 2K^2 = 0$

$-K^3 + 2K^2 = 0$

$-K^2(K-2) = 0$

$K = 0, 0, 2$

Ans $\underline{\underline{2}}$

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Linearly Independent and Dependent vectors

Ex 1 Consider the vectors $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $v_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$. Are the vectors Linearly independent?

method 1

$V_2 = 2V_1$

v_1 & v_2 are Linearly dependent

$2V_1 - V_2 = 0$

$\alpha_1 V_1 + \alpha_2 V_2 = 0$

$\alpha_1 = 2 \quad \alpha_2 = -1$

$\alpha_1 = 4 \quad \alpha_2 = -2$

$\alpha_1 = 6 \quad \alpha_3 = -3$

$\vdots$

method 2

$|V_1 \ V_2| = \begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} = 0$

V_1 & V_2 are L.D

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Ex 2 Consider the vectors $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. Are the vectors Linearly independent?

$v_3 = v_1 + v_2$

v_1, v_2 & v_3 are L.D

$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$

$\alpha_1 = 1 \quad \alpha_2 = 1 \quad \alpha_3 = -1$

$\alpha_1 = 2 \quad \alpha_2 = 2 \quad \alpha_3 = -2$

$\alpha_1 = 3 \quad \alpha_2 = 3 \quad \alpha_3 = -3$

$\vdots$

method 2

$|v_1 \; v_2 \; v_3| = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{vmatrix} = 0$

v_1, v_2 & v_3 are L.D

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Ex 3 Consider the vectors $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Are the vectors Linearly independent ?

$$\alpha_1 V_1 + \alpha_2 V_2 = 0$$

$$\alpha_1 = \alpha_2 = 0$$

method 2

$$|V_1 V_2| = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

V_1 & V_2 are Linearly independent

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Ex 4 Consider the vectors $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Are the vectors Linearly independent?

$$|v_1 \quad v_2 \quad v_3| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$$

v_1, v_2 & v_3 are L.I.

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Ex 5 Consider the vectors $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, v_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Are the vectors Linearly independent?

$$[v_1 \quad v_2 \quad v_3 \quad v_4] = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$\text{Rank} = \underline{\underline{3}}$

The given set is L.D

$v_1 + v_2 + v_3 = v_4$

$v_1 + v_2 + v_3 - v_4 = 0$

$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4 = 0$

$\alpha_1 = \alpha_2 = \alpha_3 = 1, \; \alpha_4 = -1$

$\alpha_1 = \alpha_2 = \alpha_3 = 2, \; \alpha_4 = -2$

$\vdots$

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- Linear Independence: A set of vectors $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is said to be a linearly independent set whenever the only solution for the scalars α_i in the homogeneous equation $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n = \mathbf{0}$ is the trivial solution $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.
- Whenever there is a nontrivial solution for the α (i.e., at least one $\alpha_i \neq 0$), the set $\mathcal{S}$ is said to be a linearly dependent set.

$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$

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If the vectors $(1.0,\ -1.0,\ 2.0)$, $(7.0,\ 3.0,\ x)$ and $(2.0,\ 3.0,\ 1.0)$ in $\mathbb{R}^3$ are linearly dependent, the value of x is __________

GATE 2021

$$x = 8$$

$$\begin{vmatrix} 1 & -1 & 2 \\ 7 & 3 & x \\ 2 & 3 & 1 \end{vmatrix} = 0$$

$$1(3-3x) + 1(7-2x) + 2(21-6) = 0$$

$$3-3x+7-2x+30 = 0$$

$$-5x = -40$$

$$\Rightarrow x = 8$$

15. Consider the following statements:

S1: If $\{X_1, X_2, X_3, X_4\}$ is a linearly independent set of vectors, then the set $\{X_1, X_2, X_3\}$ is linearly independent.

S2: If $\{X_1, X_2, X_3, X_4\}$ is a linearly dependent set of vectors, then the set $\{X_1, X_2, X_3\}$ is linearly dependent.

Which of the following is true?

(a) Only S1

(b) Only S2

(c) Both S1 and S2

(d) Neither S1 nor S2

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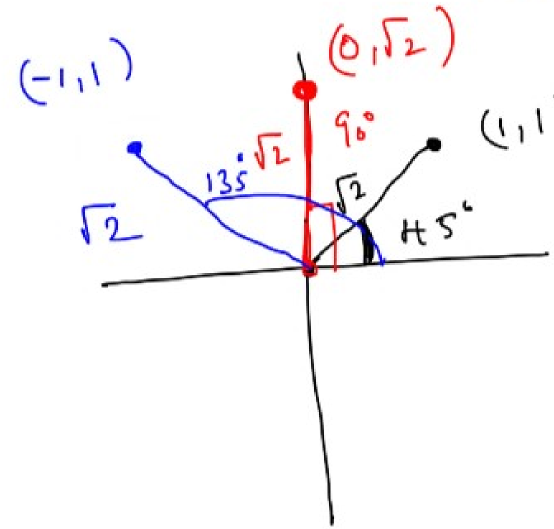
Consider a vector $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and let $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Find Ax when $\theta = 45^\circ, 90^\circ$.



when $\theta = 45^\circ$

$$AX = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{2}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix}$$

when $\theta = 90^\circ$

$$AX = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

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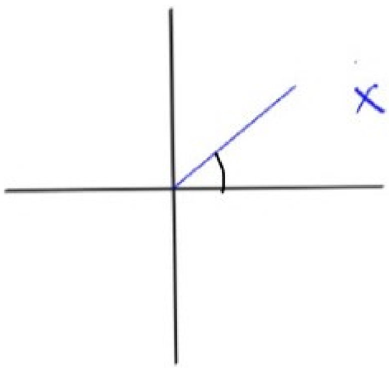
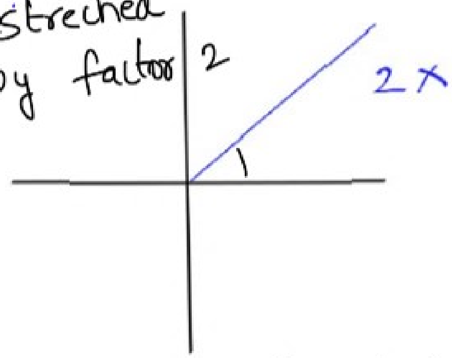
$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is rotating the vector x by angle θ in anti clock wise direction

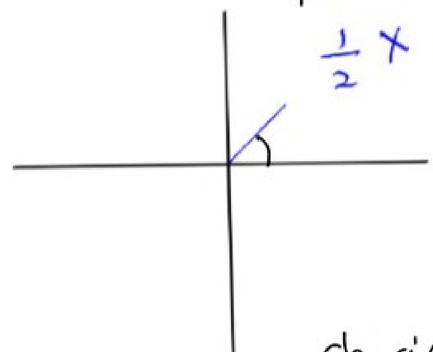
Introduction to Eigen values and Eigen vectors



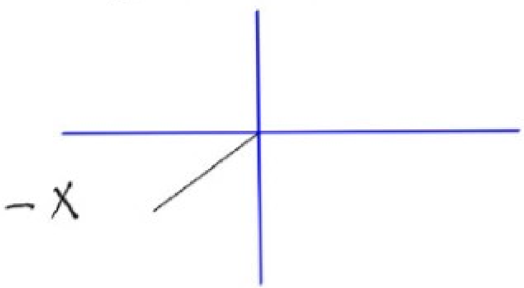
streched by factor 2



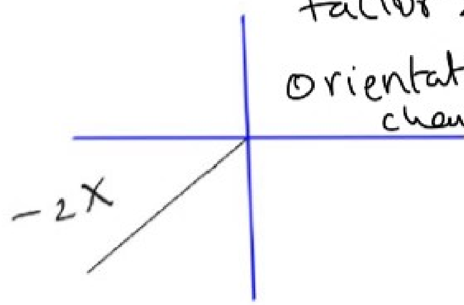
shrinked by factor 2



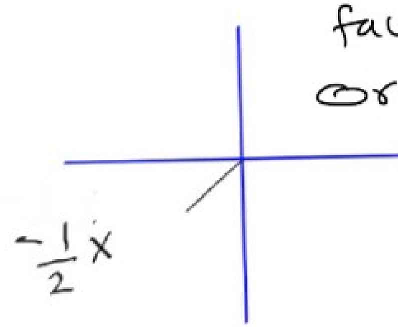
Orientation is changed

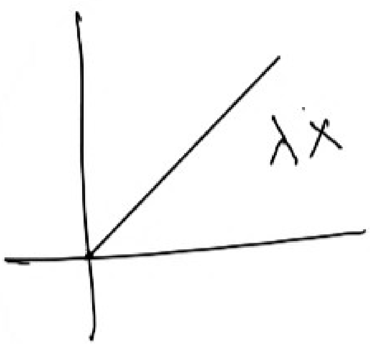


streched by factor 2
orientation is changed



shrinked by factor 2
orientation is changed.





λx indicates stretch or shrink of vector in the direction of x $(\lambda > 0)$

If λ is negative, there would be change in orientation

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Eigen value and Eigen vector problem:

$A_{n\times n} \Rightarrow$ matrix

$X_{n\times 1} \neq 0$ Non zero vector

$\lambda \rightarrow$ Scalar

$AX = \lambda X$

$AX - \lambda X = 0$

$(A - \lambda I)X = 0$

$BX = 0$

$R(B) < n$

$R(A - \lambda I) < n$

$\left(A_{n\times n} X_{n\times 1}\right)_{n\times 1}$

$|A - \lambda I| = 0$