20/04/2021

4) The soln. of homogeneous differential eqn.

$x(y\,dx + x\,dy)\cos\left(\frac{y}{x}\right) = y(x\,dy - y\,dx)\sin\left(\frac{y}{x}\right)$ is

[GATE CE-2015]

Soln. Given:

$$x(y\,dx + x\,dy)\cos\left(\frac{y}{x}\right) = y(x\,dy - y\,dx)\sin\left(\frac{y}{x}\right)$$

$$d(xy) = \frac{y}{x}(x\,dy - y\,dx)\tan\left(\frac{y}{x}\right)$$

Dividing both sides by x^2

$$\frac{d(xy)}{x\cancel{^2}} = \frac{y}{\cancel{x}}\left(\frac{x\,dy - y\,dx}{x^2}\right)\tan\left(\frac{y}{x}\right)$$

$$\frac{d(xy)}{(xy)} = \tan\left(\frac{y}{x}\right) d\left(\frac{y}{x}\right)$$

$$\int \frac{d(xy)}{(xy)} = \int \tan\left(\frac{y}{x}\right) d\left(\frac{y}{x}\right)$$

$$\ln(xy) = \ln(\sec(\tfrac{y}{x})) + \ln(c)$$

$$\ln(xy) = \ln[c\sec(\tfrac{y}{x})]$$

$$xy = c\sec(\tfrac{y}{x})$$

$$\therefore xy\cos(\tfrac{y}{x}) = c$$

$$\int \tan(\theta)\,d\theta = \ln(\sec\theta)$$

$$= -\ln(\cos\theta)$$

Linear Differential Eqn.

I. The general form is

$$\frac{dy}{dx} + Py = Q, \text{ where } P, Q \rightarrow f(x) \text{ (or) const.}$$

$$IF = e^{\int P\,dx}$$

General soln. is $y(IF) = \int Q(IF)\,dx + c$

II. The general form is

$$\frac{dx}{dy} + Px = Q, \quad \text{where } P, Q \rightarrow f(y) \text{ (or) const.}$$

$$IF = e^{\int P\,dy}$$

General soln. is $x(IF) = \int Q(IF)\,dy + c$

Note

1)

P	$\frac{1}{x}$	$\frac{2}{x}$	$\frac{3}{x}$	$\frac{4}{x}$	....	....
IF	x	x^2	x^3	x^4	....	....

2)

P	$\frac{1}{y}$	$\frac{2}{y}$	$\frac{3}{y}$	$\frac{4}{y}$	....	....	....
IF	y	y^2	y^3	y^4	....	....	....

problems

1) If $y(x)$ satisfies the differential eqn.

$(\sin x)\frac{dy}{dx} + y\cos x = 1$ subjected to the conditions $y\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$, then $y\left(\frac{\pi}{6}\right)$ is

A) $\frac{\pi}{3}$ B) 0 C) $\frac{\pi}{6}$ D) $\frac{\pi}{2}$

[GATE ME – 2021]

Soln. $\dfrac{dy}{dx} + y\cot x = \operatorname{cosec} x$

Here: $P = \cot x$, $Q = \operatorname{cosec} x$

$$\boxed{e^{\ln(f(x))} = f(x)}$$

$$IF = e^{\int \cot x\,dx} = e^{\ln(\sin x)} = \sin x$$

General soln. $y(IF) = \int Q(IF)\,dx$

$$y\sin x = \int \cancelto{1}{(\operatorname{cosec} x)(\sin x)}\,dx$$

$$y\sin x = x + c \rightsquigarrow (1)$$

$$y\left(\frac{\pi}{2}\right) = \frac{\pi}{2}, \quad (1) \Rightarrow \frac{\pi}{2}\,\overbrace{\sin\left(\frac{\pi}{2}\right)}^{1} = \frac{\pi}{2} + c$$

$$\Rightarrow \frac{\pi}{2} = \frac{\pi}{2} + c$$

$$\Rightarrow c = 0$$

$$(1) \Rightarrow y\sin x = x$$

when $x = \frac{\pi}{6}$, $\quad y\sin\left(\frac{\pi}{6}\right) = \frac{\pi}{6}$

$$y\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

$$\therefore \quad y = \frac{\pi}{3}$$

2) consider the initial value problem below. the value of y at $x = \ln(2)$ (round off to 3 decimal places) is ________ [GATE EE-2020]

$$\frac{dy}{dx} = 2x - y, \quad y(0) = 1$$

Soln.

$$\frac{dy}{dx} = 2x - y$$

$$\frac{dy}{dx} + y = 2x \quad [P = 1,\ Q = 2x]$$

$$IF = e^{\int dx} = e^{x}$$

General soln. is $y e^{x} = \int \underbrace{2x}_{\text{polynomial}}\ \underbrace{e^{x}}_{\sin x,\ \sinh x,\ \cos x,\ \cosh x,\ e^{x},\ e^{-x}} dx \rightsquigarrow (1)$

	Diff	Int.
$\oplus$	$2x$	e^x
$\ominus$	2	e^x
$\oplus$	0	e^x

$$(1) \Rightarrow y\, e^{x} = 2x\, e^{x} - 2\, e^{x} + c$$

$$\Rightarrow \quad y = 2x - 2 + c\, e^{-x} \rightsquigarrow (2)$$

$y(0)=1,$ $\quad (2) \Rightarrow 1 = -2 + c$

$$\Rightarrow 3 = c$$

$$(2) \Rightarrow y = 2x - 2 + 3e^{-x}$$

At $x = 2$, $y = 2\ln(2) - 2 + 3e^{-\ln(2)}$

$\therefore \quad y = 0.886$

3) The value of y as $t \to \infty$ for an initial value of $y(1) = 0$, for the differential eqn.

$(4t^2+1)\frac{dy}{dt} + 8ty = t$ is

A) 1 B) $\frac{1}{2}$ C) $\frac{1}{4}$ ✓D) $\frac{1}{8}$

4) $x\frac{dy}{dx}+y=x^4$ with $y(1)=\frac{6}{5}$ is

[GATE ME-2009]

Ans $y=\frac{x^4}{5}+\frac{1}{x}$

5) $x^2\frac{dy}{dx}+2xy=2\frac{\ln x}{x}$ with $y(1)=0$ then $y(e)$ is

[GATE ME-2005]

<u>Soln.</u>

$$\frac{dy}{dx} + \underbrace{\frac{2}{x}}_{P} y = \underbrace{\frac{2\ln x}{x^3}}_{Q} \quad \left[\frac{dy}{dx} + Py = Q\right]$$

$$IF = x^2$$

General soln. is $y(x^2) = \int \left(\frac{2\ln x}{x^{\cancel{3}}}\right) (\cancel{x^2})\, dx$

$$x^2 y = \int \frac{2\ln x}{x} dx$$

put $\ln x = t$

$$\frac{1}{x}\,dx = dt$$

$$\Rightarrow \quad x^2 y = \int 2t\,dt$$

$$\Rightarrow \quad x^2 y = t^2 + c$$

$$\Rightarrow \quad x^2 y = (\ln x)^2 + c \longrightarrow (1)$$

$$y(1) = 0, \quad (1) \Rightarrow (1)^2(0) = (\cancelto{0}{\ln(1)})^2 + c$$

$$\Rightarrow \quad 0 = c$$

$$(1) \Rightarrow \quad x^2 y = (\ln x)^2$$

when $x = e$, $e^2 y = (\ln e)^2 = 1$

$$\therefore y = \frac{1}{e^2}$$

6) The soln. of the differential eqn.

$$(1+y^2)\,dx = (\tan^{-1}y - x)\,dy = 0 \text{ is}$$

[ESE-2020 prelims]

Soln.

$$\frac{dx}{dy} = \frac{\tan^{-1}y - x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2} - \frac{x}{1+y^2}$$

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2} \quad \left[\frac{dx}{dy} + Px = Q\right]$$

$$P = \frac{1}{1+y^2}, \quad Q = \frac{\tan^{-1}y}{1+y^2}$$

$$IF = e^{\int P\,dy} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$$

General soln. is $x\,(IF) = \int Q\,(IF)\,dy$

$$x\, e^{\tan^{-1} y} = \int \frac{\tan^{-1} y}{1+y^2}\, e^{\tan^{-1} y}\, dy$$

put $\tan^{-1} y = t$

$\frac{1}{1+y^2} dy = dt$

$$= \int t\, e^{t}\, dt$$

$$\Rightarrow x e^{\tan^{-1} y} = t e^{t} - e^{t} + c$$

$$\Rightarrow x e^{t} = (t-1) e^{t} + c$$

$$\Rightarrow x = (t-1) + \frac{c}{e^{t}}$$

$$\Rightarrow x = (t-1) + c e^{-t}$$

$$\therefore x = (\tan^{-1} y - 1) + c e^{-\tan^{-1} y}$$

	Diff.	Int.
$\oplus$	t	e^{t}
$\ominus$	1	e^{t}
$\oplus$	0	e^{t}

7) $\frac{dx}{dt} = 10 - 0.2x, \quad x(0) = 1$ [GATE EC-2015]

<u>Ans</u> $x = 50 - 49e^{-0.2t}$

8) $\frac{dy}{dx}(x + y^2) = y$ <u>Ans</u> $x = y^2 + cy$

9) $\frac{dy}{dx} = \frac{1}{e^{-y} - x}$ <u>Ans</u> $x = (c+y)e^{-y}$

Eqns. Reducible to Linear Differential Eqns.

A differential eqn. in the form

Differentiate w.r.t. x

$$\underbrace{f'(y)\frac{dy}{dx}}_{\mathrm{I}} + P\underbrace{f(y)}_{\mathrm{II}} = Q. \qquad \left.\begin{matrix} P \\ Q \end{matrix}\right\} f(x) \text{ (or) const.}$$

can be converted to LDE by putting $f(y) = t$

Problems

1) The soln. of $\tan y \frac{dy}{dx} + \tan x = \cos y \cos^2 x$ is

Soln. Multiplying both sides by $\sec y$

Differentiate w.r.t. x

$$\underbrace{\sec y \tan y \frac{dy}{dx}}_{\mathrm{I}} + \underbrace{\boxed{(\sec y)} \tan x}_{\mathrm{II}} = \cos^2 x$$

$$\frac{dt}{dx} + \underbrace{(\tan x)}_{P}t = \underbrace{\cos^2 x}_{Q}$$

put $\sec y = t$

$$\sec y \tan y \frac{dy}{dx} = \frac{dt}{dx}$$

$$IF = e^{\int \tan x \, dx} = e^{\ln(\sec x)} = \sec x$$

General soln. is

$$t \sec x = \int (\cos^2 x)(\sec x)\, dx$$

$$\frac{1}{\cos x}$$

$$\sec y \sec x = \int \cos x \, dx$$

$$\sec y \sec x = \sin x + c$$

$$\therefore \quad \sec y = (\sin x + c)\cos x$$

2) The soln. of $\frac{dy}{dx} + \tan x \tan y = \cos x \sec y$

Ans $\sin y = (c + x)\cos x$

3) The integrating factor of the eqn.

$$\sec^2 y \frac{dy}{dx} + x \tan y = x^3 \text{ is}$$

A) $e^{\frac{x^2}{2}}$ B) $e^{-\frac{x^2}{2}}$ C) $e^{\frac{x}{2}}$ D) $e^{-\frac{x}{2}}$

[ISRO EE-2013]

Differentiate w.r.t. x

Soln. Given : $\underbrace{\sec^2 y \frac{dy}{dx}}_{\mathrm{I}} + \underbrace{x \boxed{\tan y}}_{\mathrm{II}} = x^3$

$$\underbrace{\frac{dt}{dx}+\underbrace{x}_{P}\,t=\underbrace{x^3}_{Q}}\ \text{(LDE)}$$

put $\tan y = t$

$$\sec^2 y\,\frac{dy}{dx}=\frac{dt}{dx}$$

$$IF = e^{\int x\,dx} = e^{\frac{x^2}{2}}$$

General soln. is

$$t\,e^{\frac{x^2}{2}}=\int x^3\,e^{\frac{x^2}{2}}\,dx$$

$$t\, e^{\frac{x^2}{2}} = \int x^2\, e^{\frac{x^2}{2}} (x\,dx)$$

put $\dfrac{x^2}{2} = z$

$x^2 = 2z$

$\cancel{2}x\,dx = \cancel{2}dz$

$x\,dx = dz$

$$t\, e^{z} = \int 2z\, e^{z}\, dz \rightsquigarrow (1)$$

Diff.		Int.
$+$	$2z$	e^z
$-$	2	e^z
$+$	0	e^z

$$(1) \Rightarrow t e^{z} = 2z e^{z} - 2 e^{z} + c$$

$$\Rightarrow t = 2z - 2 + c e^{-z}$$

$$\therefore \tan y = x^2 - 2 + c e^{-\frac{x^2}{2}} \;//$$

4) $\frac{dy}{dx} + x(\sin(2y)) = x^3 \cos^2 y$

Ans $\tan y = \frac{1}{2}(x^2 - 1) + c e^{-x^2}$

Bernoulli's Differential Eqn.

(Eqns. Reducible to LDE)

The general form is

$$\frac{dy}{dx} + Py = Qy^n \rightsquigarrow (1) \quad (n \neq 0, 1)$$

i) If $n=0$ then eqn. (1) reduces to LDE

ii) If $n=1$ then eqn. (1) reduces to variable separable form.

iii) If 'n' is any number other than 0 and 1 then divide both sides of eqn. (1) by y^n

$$\frac{1}{y^n}\frac{dy}{dx}+\frac{Py}{y^n}=\frac{Q\cancel{y^n}}{\cancel{y^n}}$$

$$y^{-n}\frac{dy}{dx}+Py^{1-n}=Q$$

$$\frac{1}{(1-n)}\frac{dt}{dx}+Pt=Q$$

$$\boxed{\frac{dt}{dx}+P(1-n)t=Q(1-n)}$$

which is LDE in t.

put $y^{1-n}=t$

Differentiate w.r.t. x

$$(1-n)\,y^{-n}\frac{dy}{dx}=\frac{dt}{dx}$$

$$y^{-n}\frac{dy}{dx}=\frac{1}{(1-n)}\frac{dt}{dx}$$

Note The differential eqn.

$\frac{dy}{dx} + Py = Qy^n$ can be reduced to LDE

Remember

$$\boxed{\frac{dt}{dx} + P(1-n)t = Q(1-n)}$$ where $t = y^{1-n}$

Problems

1) Consider the differential eqn. given below.

$$\frac{dy}{dx} + \frac{x}{1-x^2}\, y = x\sqrt{y} \quad \text{[GATE EC-2021]}$$

The integrating factor of the differential eqn. is

A) $(1-x^2)^{-\frac{3}{4}}$

B) $(1-x^2)^{-\frac{1}{4}}$

C) $(1-x^2)^{-\frac{1}{2}}$

D) $(1-x^2)^{-\frac{3}{2}}$

Given: $\frac{dy}{dx}+\underbrace{\left(\frac{x}{1-x^2}\right)}_{P}y=\underbrace{x}_{Q}\,y^{\overbrace{\tfrac{1}{2}}^{}_{n}}$ $\left[\frac{dy}{dx}+Py=Qy^n\right]$

$$\frac{dt}{dx}+P(1-n)t=Q(1-n)\quad,\quad t=y^{1-n}=y^{1-\frac{1}{2}}=y^{\frac{1}{2}}$$

$$1-n=1-\tfrac{1}{2}=\tfrac{1}{2}$$

$$\frac{dt}{dx}+\left(\frac{x}{1-x^2}\right)\left(\tfrac{1}{2}\right)t=x\,\tfrac{1}{2}$$

$$\frac{dt}{dx} + \underbrace{\frac{x}{2(1-x^2)}}_{P}\, t = \underbrace{\frac{x}{2}}_{Q} \qquad \left[\frac{dt}{dx} + Pt = Q\right]$$

$$IF = e^{\int \frac{x}{(1-x^2)\cdot 2}dx} = e^{\frac{1}{2}\int \frac{-2x}{-2(1-x^2)}dx}$$

$$= e^{-\frac{1}{4}\int \frac{-2x \rightsquigarrow f'(x)}{(1-x^2) \rightsquigarrow f(x)}dx}$$

$$IF = e^{-\frac{1}{4}\ln(1-x^2)}$$

$$= e^{\ln(1-x^2)^{-\frac{1}{4}}}$$

$$= (1-x^2)^{-\frac{1}{4}}$$

$$\boxed{\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)|}$$

$$\boxed{e^{\ln f(x)} = f(x)}$$