III) Partial & Total derivatives

(1) Homogeneous function:

(i) Algebraic function:

If $\boxed{f(x,y) = \dfrac{p(x,y)}{q(x,y)}}$, where $p(x,y)$ & $q(x,y)$ are homogeneous functions with degree a & b respectively, then $f(x,y)$ is a homogeneous function with degree $\boxed{n = a-b}$.

Ex (1) $f(x,y) = \dfrac{x^{3/2}+y^{3/2}}{x-y}$ $\left(\because f(x,y) = \dfrac{p(x,y)}{q(x,y)}\right)$

Here, $f(x,y)$ is a homogeneous function with degree $n = \frac{3}{2} - 1 = \frac{1}{2}$

(ii) Transcendental function :-

If $\boxed{f(x,y) = \text{Any transcendental function}\,[g(x,y)]}$, where $g(x,y)$ is a homogeneous function with degree zero, then $f(x,y)$ is a homogeneous function with degree zero.

Ex ① $\boxed{f(x,y) = \tan\left[\dfrac{x^2 - xy}{xy + 3y^2}\right]}$ $\left(\because f(x,y) = \tan(g(x,y))\right)$

Here, $f(x,y)$ is a homogeneous function with degree zero, because $g(x,y) = \dfrac{x^2 - xy}{xy + 3y^2}$ is a homogeneous function with degree zero.

Ex (2) $f(x,y) = \log\left(\frac{x^3 - xy^2}{x+3y}\right)$ $\left(\because f(x,y) = \log[g(x,y)]\right)$

Here, $f(x,y)$ is not a homogeneous function, because $g(x,y)$ is not a homogeneous function with degree zero.

Ex (3) $f(x,y) = e^{\left(\frac{x+y}{x-y^2}\right)}$ $\left(\because f(x,y) = e^{[g(x,y)]}\right)$

Here, $f(x,y)$ is not a homogeneous function, because $g(x,y)$ is not a homogeneous function.

Note (1) If $f(x,y)$ is a homogeneous function with degree 'n' in x & y then $\boxed{f(x,y) = \begin{cases} x^n \cdot \phi\left(\frac{y}{x}\right) \\ y^n \cdot \phi\left(\frac{x}{y}\right) \end{cases}}$

ACE

Ex (1) $f(x,y) = x^2 - 4xy + 3y^2$

$$= \begin{cases} x^2 \cdot \left[1 - 4\left(\frac{y}{x}\right) + 3\left(\frac{y}{x}\right)^2\right] \\ y^2 \cdot \left[\left(\frac{x}{y}\right)^2 - 4\left(\frac{x}{y}\right) + 3\right] \end{cases}$$

$$= \begin{cases} x^n \cdot \phi\left(\frac{y}{x}\right) \\ y^n \cdot \phi\left(x/y\right) \end{cases}$$

Th (1) [Euler's theorem for homogeneous functions]

St: If $u = f(x,y)$ is a homogeneous function with degree 'n' in x & y then

(i) $x \dfrac{\partial u}{\partial x} + y \cdot \dfrac{\partial u}{\partial y} = n \cdot u = n \cdot f$

(ii) $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = n \cdot (n-1) \cdot u = n(n-1) \cdot f$

Note (1) If $\boxed{u = h(x,y) + g(x,y)}$, where $h(x,y)$ & $g(x,y)$ are homogeneous functions with degree m & n respectively

then (i) $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} = (m \cdot h) + (n \cdot g)$

& (ii) $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = \left(m \cdot (m-1) \cdot h\right) + \left(n \cdot (n-1) \cdot g\right)$

Note (2) If $u(x,y)$ is not a homogeneous function but $f(u)$ is a homogeneous function with degree 'n' in x & y then

(i) $x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = n \cdot \frac{f(u)}{f'(u)} = \phi(u)$

(ii) $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \phi(u) \cdot [\phi'(u) - 1]$

(1) If $u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$ then $x\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} =$ ___

(a) $\tan(u)$

(b) $\sec(u)$

(c) $\cot(u)$

(d) $\sin(u)$.

Sol: The given function $u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$ is not a homogeneous function but $f(u) = \sin(u) = \frac{x^2+y^2}{x+y}$ is a homogeneous function with degree $n = 2-1 = 1$. in x & y.

Now, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\cdot\frac{f(u)}{f'(u)} = 1\cdot\frac{\sin(u)}{\cos(u)}$

$\therefore\quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan(u)$.

$dx \rightarrow$ Differential of I.V

$dy \rightarrow$ differential of D.V

$$y = f(x) \longrightarrow \overset{y'}{\frac{dy}{dx}}, \frac{d^2y}{dx^2}, \ldots \longrightarrow \text{Ordinary derivatives}$$

$$u = f(x,y) \begin{cases} \overset{u_x}{\dfrac{\partial u}{\partial x}}, \dfrac{\partial u}{\partial y} \rightarrow u_y \\[2ex] \underset{u_{xx}}{\dfrac{\partial^2 u}{\partial x^2}}, \underset{u_{xy}}{\dfrac{\partial^2 u}{\partial x \partial y}}, \underset{u_{yy}}{\dfrac{\partial^2 u}{\partial y^2}} \end{cases} \longrightarrow \text{Partial derivatives.}$$

(2) If $x = e^u \tan(v)$ & $y = e^u \cdot \sec(v)$ then the value of $\left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}\right) \cdot \left(x \cdot \frac{\partial v}{\partial x} + y \cdot \frac{\partial v}{\partial y}\right)$ is —

(a) 0 ✓

(b) 1

(c) $3/2$

(d) $1/2$.

Sol:- Given $x = e^u \cdot \tan(v)$ — (1)

& $y = e^u \cdot \sec(v)$ — (2)

Dividig (2) by (1), we get

$$\frac{y}{x} = \frac{e^u \cdot \sec(v)}{e^u \cdot \tan(v)}$$

$$\Rightarrow \frac{y}{x} = \frac{1}{\cos(v)} \cdot \frac{1}{\frac{\sin v}{\cos(v)}}$$

$$\Rightarrow \sin(v) = \frac{x}{y}$$

$\Rightarrow \boxed{v = \sin^{-1}\left(\frac{x}{y}\right)}$ is a homogeneous function with degree $n=0$ in x & y

Now, by Euler's theorem, we have

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = n \cdot v = 0 \cdot v = 0$$

$$\therefore \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) \cdot \left(x \cdot \frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right) = (x \cdot u_x + y u_y)(0) = \underline{0}$$

Note:

$$\because \sec^2(v) - \tan^2(v) = 1$$

$$\Rightarrow \left(\frac{y}{e^u}\right)^2 - \left(\frac{x}{e^u}\right)^2 = 1$$

$$\Rightarrow y^2 - x^2 = e^{2u}$$

$$\therefore u = \frac{1}{2}\log(y^2 - x^2)$$

(3) If $u = x^{-2} \cdot \tan\left(\frac{y}{x}\right) + 3y^3 \cdot \sin^{-1}\left(\frac{x}{y}\right)$, $x>0$ & $y>0$

then $x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = $ ___

(a) $6u$

(b) $-6u$

(c) u

(d) $2u$.

Sol: Let $u = h(x,y) + g(x,y)$, where $h = x^{-2} \cdot \tan\left(\frac{y}{x}\right)$ and $g = y^3 \cdot 3\sin^{-1}\left(\frac{x}{y}\right)$ are homogeneous functions with degree $m=-2$ & $n=3$ respectively.

Then $x^2 \cdot u_{xx} + 2xy\, u_{xy} + y^2 \cdot u_{yy} = m\cdot(m-1)\cdot h + n\cdot(n-1)\cdot g$

$$\Rightarrow \quad x^2 \cdot u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = (-2)(-2-1)\cdot h + 3(3-1)g$$

$$\therefore \quad x^2 \cdot u_{xx} + 2xy\, u_{xy} + y^2 \cdot u_{yy} = 6h + 6g = 6(h+g) = \underline{6u}$$

(1) Explicit function

$$\text{Explicit function} \rightarrow \begin{cases} y = f(x) \\ x = f(y) \\ z = f(x, y) \\ \vdots \\ u = f(x_1, x_2, \ldots, x_n) \end{cases}$$

Ex(1) $x - xy + y^3 - 4 = 0$

$$\therefore \; x = f(y) = \frac{4 - y^3}{1 - y}$$

Ex(2) $xy - y^2 z + z x^2 - 4 = 0$

$$\therefore \; z = f(x, y) = \frac{4 - xy}{x^2 - y^2}$$

(2) Implicit function :

$\phi(x, y) = 0$

$\phi(x, y, z) = 0$

$\vdots$

$\phi(x_1, x_2, \ldots, x_n) = 0$

Ex (1) $xy^2 - yz^3 + x^3 y + z - 4 = 0$

Ex (2) $x^2 - xy + y^3 - 4 = 0$

(3) Composite function: [function of function]

If $u = f(x,y)$ is a function of two variable x & y, where $x = \phi(t)$ & $y = \psi(t)$, then the function $u = f(x,y) = f[\phi(t), \psi(t)] = f(t)$ is called a composite function of one independent variable 't'.

Ex ① $u = f(x,y) = x^2 - xy + y^3 - 4$, where $x = t+1$ & $y = t^3 + 3$

(4) Total derivative :-

If $u = f(x, y)$, where $x = \phi(t)$ & $y = \psi(t)$ then the derivative of 'u' w.r.t 't' is called the Total derivative of 'u' w.r.t 't' & it is given by

$$\boxed{\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}}$$

$$\begin{tikzcd} & x \arrow[r] & t \\ u \arrow[ur] \arrow[dr] & & \\ & y \arrow[r] & t \end{tikzcd} \Bigg\} \; u(t) \longrightarrow \frac{du}{dt}$$

(5) Total differential :-

If $u = f(x, y)$ is a function of two independent variables x & y then the Total differential of "u" is given by

$$\boxed{du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy}$$

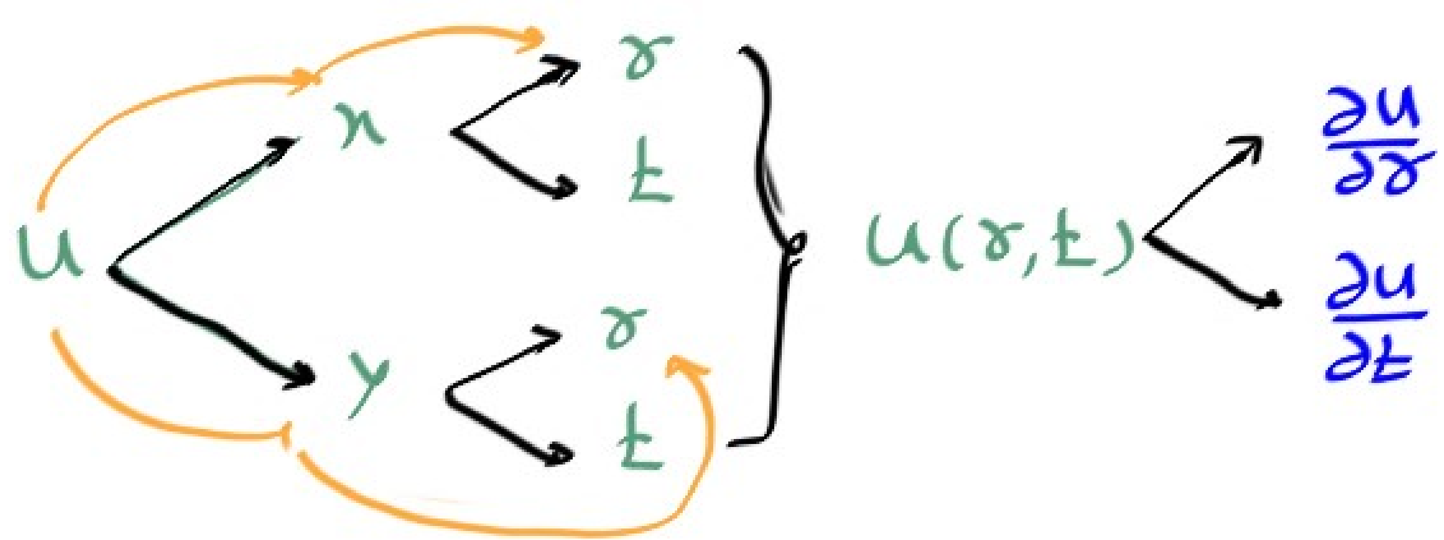
$$\begin{tikzcd} & x \\ u \arrow[ur] \arrow[dr] & \\ & y \end{tikzcd}$$

⑥ Chain rule for partial differentiation :-

If $u = f(x,y)$, where $x = \phi(r,t)$ & $y = \psi(r,t)$.

then (i) $\dfrac{\partial u}{\partial r} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial r}$

& (ii) $\dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial t}$.



(7) Implicit differentiation:-

If $f(x,y)=0$ is an implicit function interm of x & y then $\frac{dy}{dx} = \frac{-\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)} = \frac{-f_x}{f_y}$.

(8) Jacobian of functions :

If $u = u(x,y)$ and $v = v(x,y)$ are two function of two independent variables x and y then the 2^{nd} order determinant $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ is called a Jacobian of two functions u & v w.r.t x & y and it is denoted by $\frac{\partial(u,v)}{\partial(x,y)}$ (or) $J\left(\frac{u,v}{x,y}\right)$ (or) J.

$$\therefore \boxed{J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}}$$

Note (1) If $u = u(x,y,z)$, $v = v(x,y,z)$ & $w = w(x,y,z)$ are three functions of three variables x, y & z then

$$\boxed{J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}}$$

Note (2) If $J = \frac{\partial(u,v)}{\partial(x,y)}$ & $J^* = \frac{\partial(x,y)}{\partial(u,v)}$ then $J \cdot J^* = 1$

(1) If $u = x^3 + xz^2 + y^3 + xyz$, where $x = e^t$, $y = \cos(t)$, $z = t^3$ then $\frac{du}{dt}$ at $t=0$ is ___

(a) 3

(b) −3

(c) 9

(d) 4.

Sol:

$$u \to \left.\begin{matrix} x \longrightarrow t \\ y \longrightarrow t \\ z \longrightarrow t \end{matrix}\right\} u(t) \longrightarrow \frac{du}{dt}.$$

Now, $\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$.

$$\Rightarrow \frac{du}{dt} = (3x^2 + z^2 + yz).(e^t) + (3y^2 + xz)\cdot(-\sin t) + (2xz + xy)\cdot(3t^2)$$

$$\Rightarrow \left(\frac{du}{dt}\right)_{t=0} = (3+0+0)(1) + (0) + (0) \quad (\because \text{At } t=0, x=1, y=1, z=0)$$

$$\therefore \left(\frac{du}{dt}\right)_{t=0} = \underline{\underline{3}}$$

② If $x^3+y^3+3xy=1$ then $\frac{dy}{dx}$ is —

Sol:- Let $f(x,y)=x^3+y^3+3xy-1$

Then $f_x=\frac{\partial f}{\partial x}=3x^2+3y$

& $f_y=\frac{\partial f}{\partial y}=3y^2+3x$.

Now, $\frac{dy}{dx}=\frac{-f_x}{f_y}=\frac{-[3x^2+3y]}{[3y^2+3x]}$

$\therefore\ \frac{dy}{dx}=\frac{-(x^2+y)}{(y^2+x)}$.

③ If $u = x \cdot \log(xy)$, where $\underline{x^3+y^3+3xy=1}$ then $\frac{du}{dx}$ is –

Sol:
Given

$$u \begin{cases} \nearrow x \longrightarrow \cdot \\ \searrow y \longrightarrow \cdot \end{cases} \Big\} \cdot \longrightarrow \frac{du}{dx} = ?$$

$$y = f(x) \longrightarrow \frac{dy}{dx}$$

$$\Rightarrow \quad u \begin{cases} \nearrow x \longrightarrow x \\ \searrow y \longrightarrow x \end{cases} \Big\} \, u(x) \longrightarrow \frac{du}{dx}$$

Now, $\frac{du}{dx} = \frac{\partial u}{\partial x}\frac{dx}{dx} + \frac{\partial u}{\partial y}\frac{dy}{dx}$.

$$\Rightarrow \quad \frac{du}{dx} = \left[x \cdot \frac{1}{xy} \cdot y + 1 \cdot \log(xy)\right] \cdot (1) + \left(x \cdot \frac{1}{xy} \cdot x\right) \cdot \left(\frac{-f_x}{f_y}\right),$$

where $f(x,y) = x^3+y^3+3xy-1$

$$\Rightarrow \quad \frac{du}{dx} = [1+\log(xy)] - \frac{x}{y}\left[\frac{3x^2+3y}{3y^2+3x}\right]$$

$$\therefore \quad \frac{du}{dx} = [1+\log(xy)] - \left(\frac{x^3+xy}{y^3+xy}\right)$$

20 min

④ If $u = \frac{y^2}{x} + x$ & $v = \frac{y^2}{x}$ then $\frac{\partial(u,v)}{\partial(x,y)} = $ ___

(a) $\frac{x}{2y}$

(b) $\frac{2y}{x}$

(c) $\frac{-x}{2y}$

(d) $-\frac{2y}{x}$.

Sol:

Now, $J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$

$$\Rightarrow \quad J = \begin{vmatrix} \left(-\frac{1}{x^2}y^2 + 1\right) & \left(\frac{2y}{x}\right) \\ \left(-\frac{y^2}{x^2}\right) & \frac{2y}{x} \end{vmatrix}$$

$$\Rightarrow \quad J = \left(\cancel{-\frac{2y^3}{x^3}} + \frac{2y}{x}\right) + \left(\cancel{\frac{2y^3}{x^3}}\right)$$

$$\therefore \quad J = \frac{2y}{x}.$$

(5) If $u = 3x + 2y - z$, $v = x - y + z$ & $w = x + 2y - z$ then $\frac{\partial(x,y,z)}{\partial(u,v,w)}$ is _

(a) -2

(b) $\frac{1}{2}$

(c) $-1/2$

(d) 1

Sol: Let $J = \frac{\partial(x,y,z)}{\partial(u,v,w)}$

Then $J^* = \frac{\partial(u,v,w)}{\partial(x,y,z)}$

$$\Rightarrow J^* = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = \begin{vmatrix} 3 & 2 & -1 \\ 1 & -1 & 1 \\ 1 & 2 & -1 \end{vmatrix} = -2$$

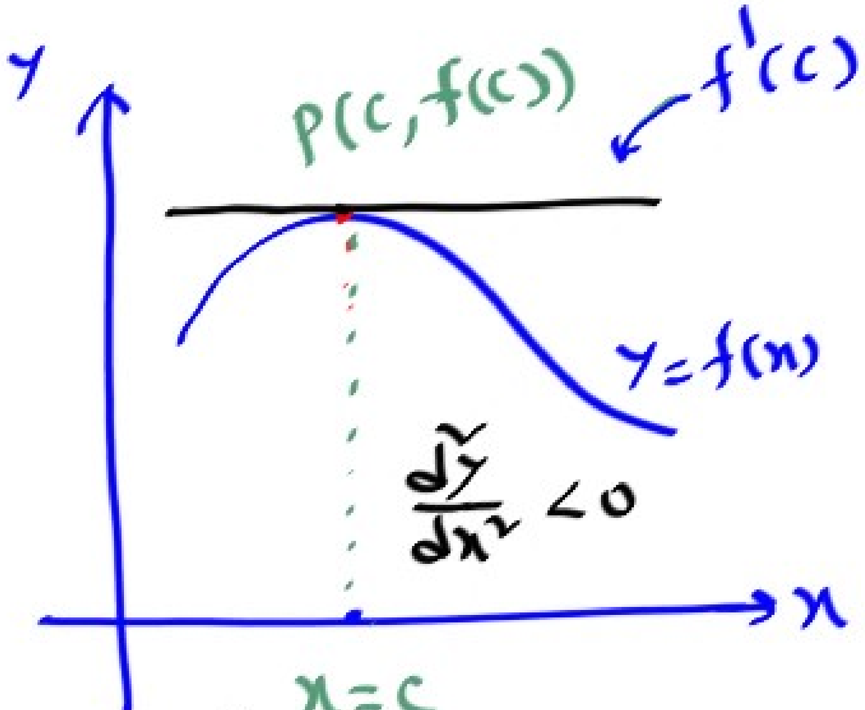
$\because J \cdot J^* = 1$

$\Rightarrow J = \frac{1}{J^*}$

$\therefore J = \frac{1}{-2}$

IV) Maxima & Minima

① $y = f(x)$



$x = c$

↓

$\boxed{f'(c) = 0}$

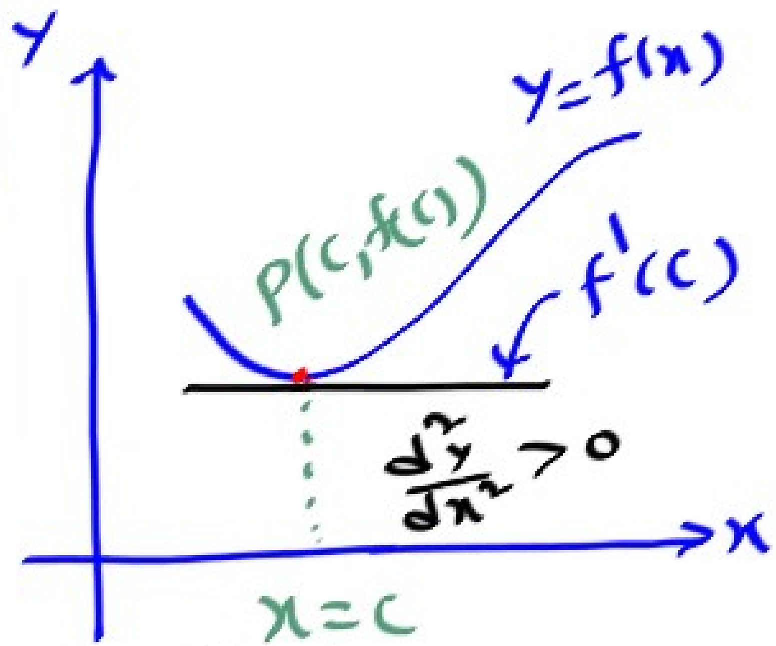
↓

Stationary point

↓

Point of maxima.

② $y = f(x)$



$x = c$

↓

$\boxed{f'(c) = 0}$

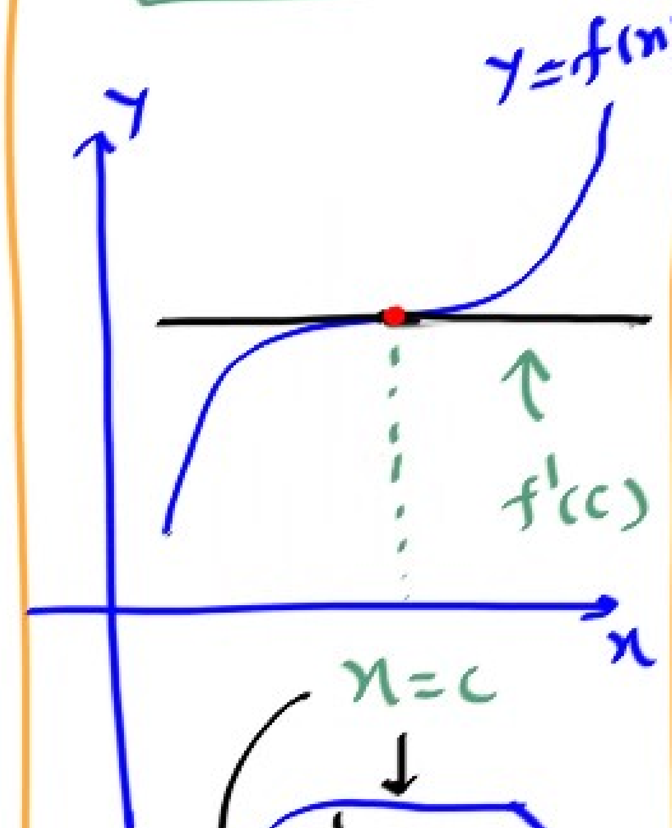
↓

Stationary points

↓

Point of minima

③ $y = f(x)$



$x = c$

↓

$\boxed{f'(c) = 0}$

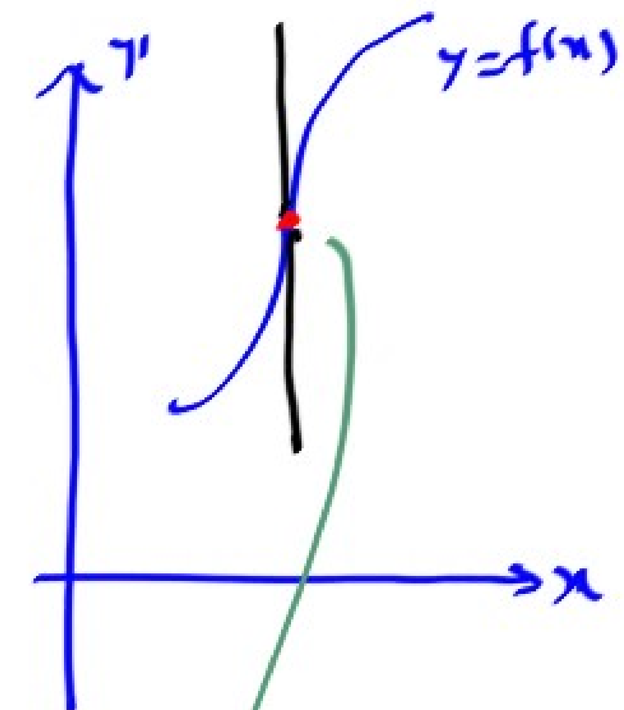
↓

stationary point

↓

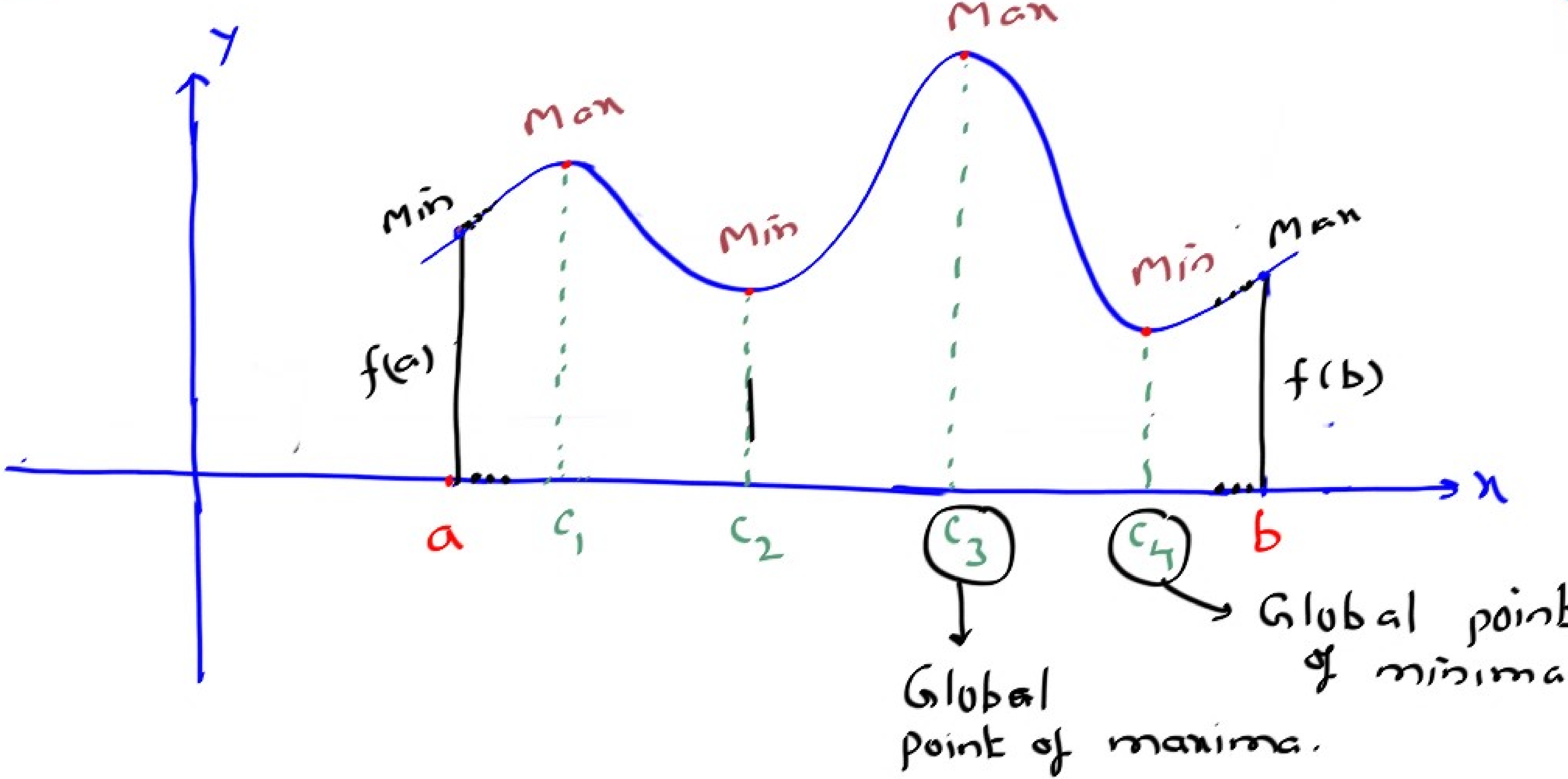
Point of inflection.

④ $y = f(x)$



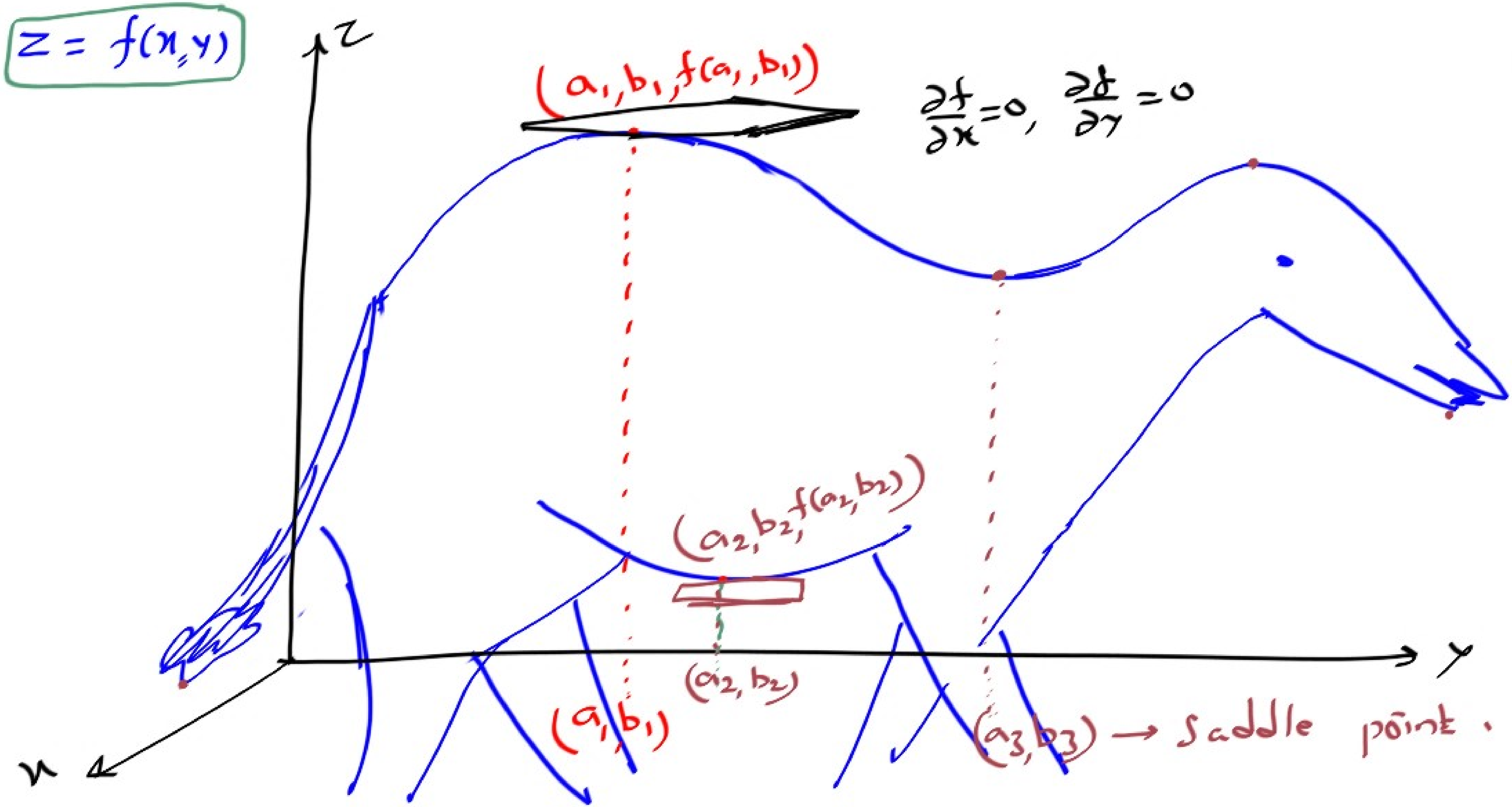
Point of inflection.

(5) $\boxed{y = f(x)}$



y
x
Max
Max
Max
Min
Min
Min
f(a)
f(b)
a
c_1
c_2
c_3
c_4
b
Global point of maxima.
Global point of minima.

⑥ $\boxed{z = f(x,y)}$


z
$(a_1, b_1, f(a_1, b_1))$
$\frac{\partial f}{\partial x} = 0, \ \frac{\partial f}{\partial y} = 0$
$(a_2, b_2, f(a_2, b_2))$
y
x
(a_2, b_2)
(a_1, b_1)
$(a_3, b_3) \rightarrow$ Saddle point.

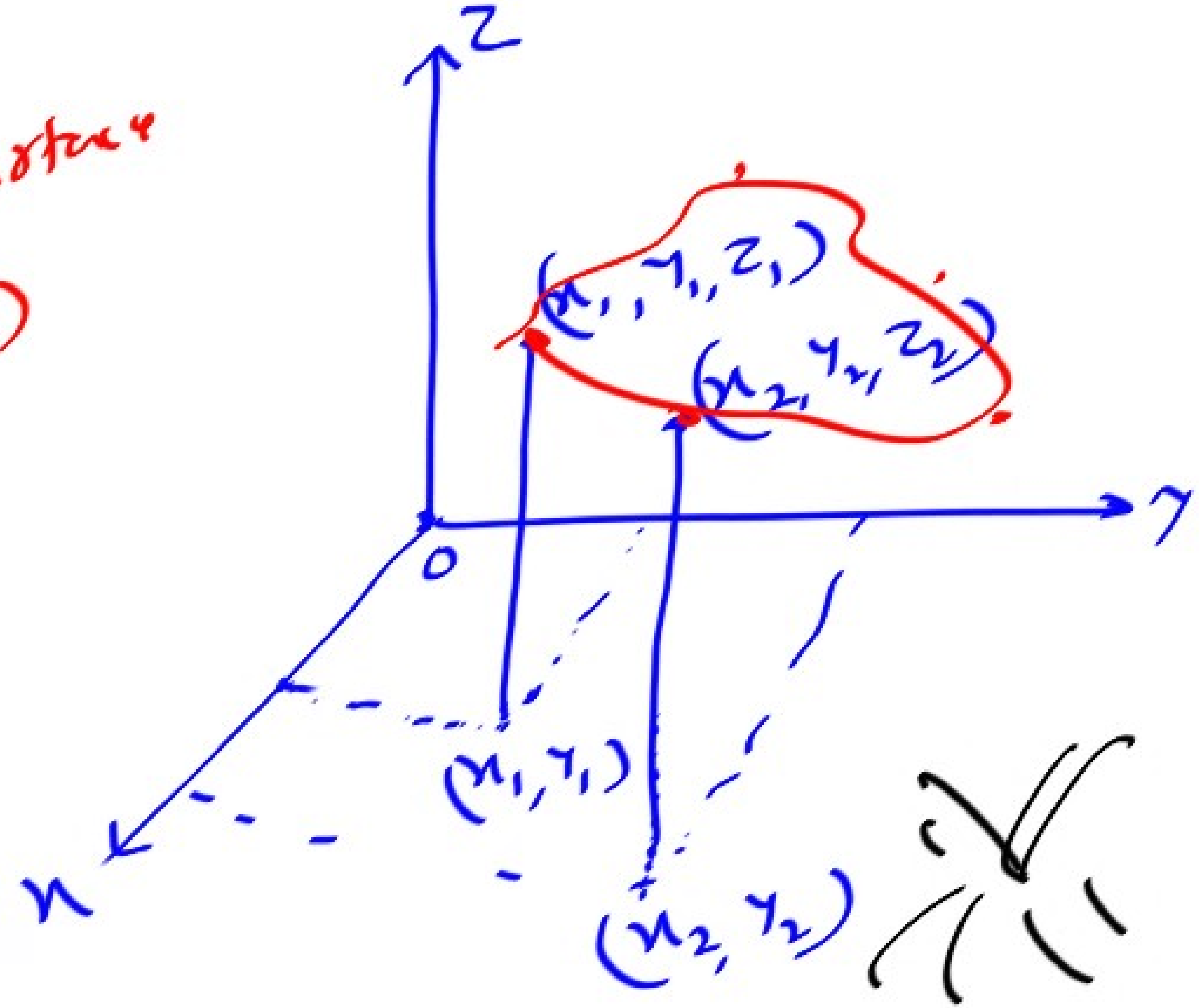
$z_1 = f(x_1, y_1)$

$Z = f(x, y)$ Surface

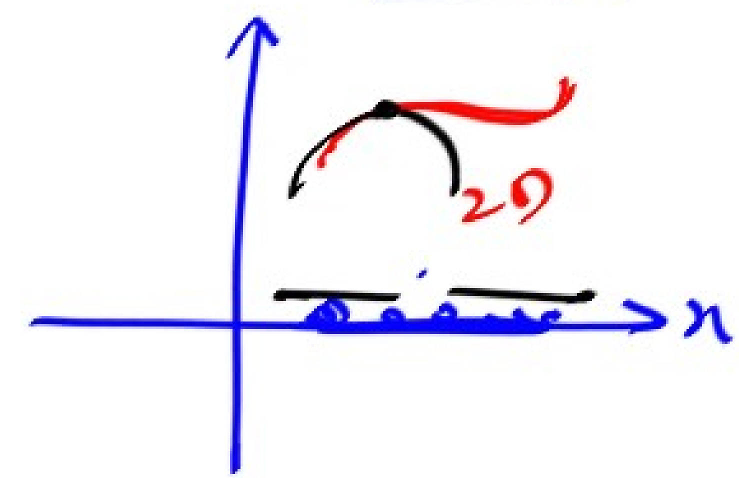
3D

(x, y, z)

$(x, y, f(x, y))$



$y = f(x)$



Procedur for finding the maxima & minima of $y = f(x)$:

Step (1) Find $f'(x)$ & $f''(x)$ for a given function $y = f(x)$.

Step (2) Equate $f'(x)$ to zero for obtaining stationary points $x = a$.

Step (3) Calculate $f''(x)$ at each st. pt. $x = a$ (i.e $f''(a)$)

Step (4) (i) If $f''(a) > 0$ then the function $f(x)$ will have a minimum at $x = a$ & the the minimum value of the function $f(x)$ at $x = a$ is $f(a)$.

(ii) If $f''(a) < 0$ then the function $f(x)$ will have a maximum at $x = a$ & the maximum value of the function $f(x)$ at $x = a$ is $f(a)$.

(iii) If $f''(a) = 0$ then no conclusion by this method.

Note (1)

$f''(a)$
- > 0 $\downarrow$
- < 0 $\uparrow$
- $= 0$ $\longrightarrow$ $f'''(a)$
  - $\neq 0$ $\uparrow\downarrow$
  - $= 0$ $\longrightarrow$ $f^{iv}(a)$
    - > 0 $\downarrow$
    - < 0 $\uparrow$
    - $= 0$ $\longrightarrow$ $f^{v}(a)$
      - $\neq 0$ $\uparrow\downarrow$
      - $= 0$ $\longrightarrow$ $f^{vi}(a)$
        - > 0 $\downarrow$
        - < 0 $\uparrow$
        - $= 0$ $\longrightarrow$