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Matrix Algebra

- Addition of Matrices
- Multiplication of Matrices
- Types of matrices
- Determinants
- Inverse of a matrix

Addition of Matrices

Addition of Matrices is possible only when the matrices are of same order.
Element wise addition is used to add two matrices of same order.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2} \qquad B = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \qquad A+B = \begin{bmatrix} a+x & b+y \\ c+z & d+w \end{bmatrix}$$

Matrix Multiplication

The product AB is defined only when the number of columns of Matrix A is equal to the number of rows of matrix B

$A_{m\times n} \times B_{n\times p} = (AB)_{m\times p}$

matrix multiplication possible

$(AB)_{m\times p}$

$B_{n\times p}\ A_{m\times n}$

matrix multiplication not possible.

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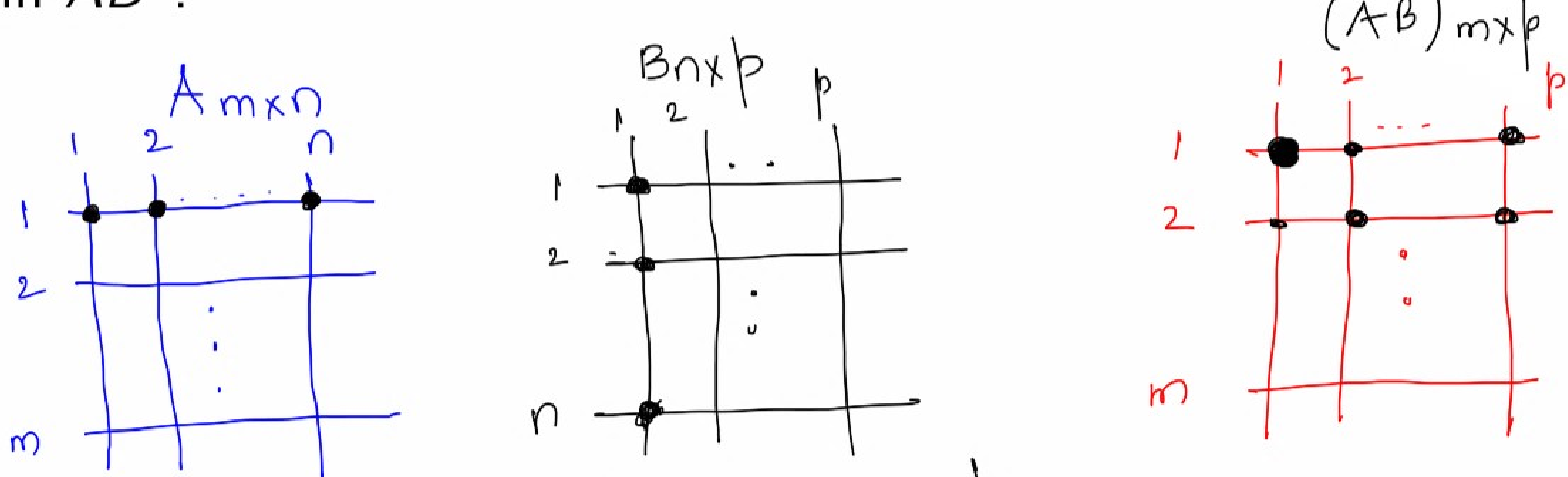
Consider the two matrices $A_{m\times n}$ and $B_{n\times p}$. The total number of multiplications and additions needed to obtain AB ?



Total multiplications $= mpn = m\times n\times p$

Total additions $= m(n-1)p$

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05. What is the minimum number of multiplications involved in computing the matrix product PQR? Matrix P has 4 rows and 2 columns, matrix Q has 2 rows and 4 columns and matrix R has 4 rows and 1 column ________. **(GATE – 2013)**

$P_{4\times2}$, $Q_{2\times4}$, $R_{4\times1}$

$PQR \rightarrow PQ \rightarrow (PQ)R$

$PQR \rightarrow QR \rightarrow P(QR)$

$(QR)P \neq PQR$

case 1

$P_{4\times2} \cdot Q_{2\times4} = (PQ)_{4\times4}$

$4\times2\times4 = 32$

$(PQ)_{4\times4} \cdot R_{4\times1} = PQR$ $\quad 4\times4\times1 = 16$

48

case 2

$Q_{2\times4} \cdot R_{4\times1} = (QR)_{2\times1}$

$2\times4\times1 = 8$

$P_{4\times2} \cdot (QR)_{2\times1} = PQR$ $\quad 4\times2\times1 = 8$

16

Ans = 16

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Types of matrices

Idempotent matrix: An Idempotent matrix is a matrix which when multiplied by itself yields itself. $A^2 = A$

If $\mathbf{A}^2 = \mathbf{I}$ for a square matrix $\mathbf{A}$ of order n, then $\mathbf{A}$ is called an involutary matrix. $A^2 = I \Rightarrow A \cdot A = I \Rightarrow A = A^{-1}$

Nilpotent matrix: A Nilpotent matrix is a square matrix $\mathbf{N}$ such that $\mathbf{N}^k = 0$ for some integer k. The smallest such k is called the degree or index of $\mathbf{N}$.

Ex $N \neq 0$ $N^2 \neq 0$ $N^3 = 0$ $N^4 = N^3 . N = 0$, $N^5 = 0$

N is nilpotent of index 3

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A square matrix **A** is said to be an Orthogonal matrix if

$$AA^T = A^T A = I \quad \text{or} \quad A^{-1} = A^T$$

$$\mathbf{A} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$C_1 = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \quad C_2 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$C_1^T C_2 = \underset{1\times 2}{\begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}} \underset{2\times 1}{\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}} = \frac{-1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}$$

$$= 0$$

C_1 and C_2 are orthogonal to each other.

$\|C_1\|$ = Euclidean norm or length of vector

$$\|C_1\| = \sqrt{\left(-\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = 1$$

$$\|C_2\| = 1$$

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$$\mathbf{A} = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$C_1 = \begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix} \quad C_2 = \begin{pmatrix} 2/3 \\ 1/3 \\ -2/3 \end{pmatrix} \quad C_3 = \begin{pmatrix} 2/3 \\ -2/3 \\ 1/3 \end{pmatrix}$$

$$C_2^T C_3 = \begin{pmatrix} 2/3 & 1/3 & -2/3 \end{pmatrix}\begin{pmatrix} 2/3 \\ -2/3 \\ 1/3 \end{pmatrix} = 0$$

$$\|C_1\| = \sqrt{\frac{1}{9} + \frac{4}{9} + \frac{4}{9}}$$

$$= 1$$

$$\|C_2\| = 1 \qquad \|C_3\| = 1$$

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The columns of orthogonal matrix are perpendicular to each other.

The length or norm of each column or row is equal to 1

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$$\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$c_1^T c_2 = (\cos\theta \quad \sin\theta) \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = 0$$

$$\|c_1\| = \sqrt{\cos^2\theta + \sin^2\theta} = \underline{1}$$

$$\|c_2\| = \sqrt{(-\sin\theta)^2 + \cos^2\theta} = \underline{1}$$

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a_{ij} = Element belonging to i^{th} row and j^{th} column

- Symmetric matrix: A square matrix is said to be Symmetric if

$$A^T = A \quad \text{or} \quad a_{ij} = a_{ji}$$

- Skew Symmetric matrix: A square matrix is said to be Skew Symmetric if

$$A^T = -A \quad \text{(or)} \quad a_{ij} = -a_{ji}$$

- For Skew Symmetric matrix, the principal diagonal elements are always zero.

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- Conjugate of a matrix: The conjugate of a matrix **A** is obtained by replacing elements of **A** with their corresponding conjugates and is denoted by $\overline{\mathbf{A}}$.

If $\mathbf{A}=\begin{bmatrix} 1-2i & 3i \\ i & 4+6i \end{bmatrix}$ then $\overline{\mathbf{A}}$ is ____.

$z = a + ib$

$\bar{z} = a - ib$

$\bar{A} = \begin{bmatrix} 1+2i & -3i \\ -i & 4-6i \end{bmatrix}$

$A = \begin{bmatrix} 2 & 3 \\ 4 & 2+i \end{bmatrix}$

$\bar{A} = \begin{pmatrix} 2 & 3 \\ 4 & 2-i \end{pmatrix}$

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- Transposed conjugate of a matrix: If $\overline{\mathbf{A}}$ is a conjugate matrixof $\mathbf{A}$ then $\overline{\mathbf{A}}^T$ is called transposed conjugate of $\mathbf{A}$ and is denoted as $\mathbf{A}^\theta$.

  If $\mathbf{A}=\begin{bmatrix} 4+i & i \\ 2-3i & 5 \end{bmatrix}$ then $\mathbf{A}^\theta$ is ____.

$\overline{A} = \begin{pmatrix} 4-i & -i \\ 2+3i & 5 \end{pmatrix}$ $\quad \overline{A}^T = \begin{pmatrix} 4-i & 2+3i \\ -i & 5 \end{pmatrix}$

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Sym : $A^T = A$

skew sym : $A^T = -A$

- Hermitian matrix: $\bar{A}^T = A$
- Skew Hermitian matrix: $\bar{A}^T = -A$

Identify Hermitian and skew Hermitian matrix.

$$\mathbf{A} = \begin{bmatrix} 2 & 1-3i \\ 1+3i & 4 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} -i & 2+i \\ -(2-i) & 0 \end{bmatrix}$$

- Unitary matrix: $A\bar{A}^T = \bar{A}^T A = I$

orthogonal matrix

$AA^T = A^TA = I$

$$\mathbf{A} = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$$

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07. The number of $n \times n$ symmetric matrices possible with the entries chosen from the set $\{0, 1, 2, \ldots\ldots, q-1\}$ is

(a) $q^{n\left(\frac{n+1}{2}\right)}$ (b) $q^{n\left(\frac{n-1}{2}\right)}$

(c) $(q-1)^{\left(n^2\right)}$ (d) $q^{\left(n^2\right)}$

$\{0,1\} \to 2$ elements

$\{0,1,2\} \to 3$ elements

$\{0,1,2,\ldots q-1\} \to q$ elements

$$\begin{bmatrix} q & q \\ \_ & \_ \\ & q \\ \_ & \_ \end{bmatrix}$$

$\underline{\underline{2 \times 2}}$

verification $n=2$

a) $q^{2(3/2)} = q^3$

b) $q^{2(1)/2} = q$ X

c) $(q-1)^{2^2} = (q-1)^4$ X

q^3 $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$

$\begin{matrix} a & d \\ \_ & \_ \\ d & c \end{matrix}$

d) q^4 X

$$\begin{pmatrix} 1 & 5 & 6 \\ 5 & 2 & 7 \\ 6 & 7 & 3 \end{pmatrix}$$

$\{0, 1, 2, \ldots q-1\}$ q elements

$$\begin{bmatrix} - & - \\ - & - \end{bmatrix}_{2\times 2}$$

Principal diagonal elements $= 2$

Above P.D elements $= 1$

$$\overline{\quad 3 \quad} \Rightarrow q^{3}$$

$$\begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}_{3\times 3}$$

P.D elements $= 3$

Above P.D elements $= 3$

$$\overline{\quad 6 \quad} \Rightarrow q^{6}$$

$$\begin{bmatrix} - & - & \cdot & \cdot & \cdot & - \\ - & - & \cdot & \cdot & \cdot & - \\ \cdot & & & & & \\ \cdot & & & & & \\ - & - & \cdot & \cdot & \cdot & - \end{bmatrix}_{n\times n}$$

P.D elements $= n$

Above P.D elements $= \dfrac{n^2-n}{2}$

$$n + \frac{n^2-n}{2} = \frac{n^2+n}{2} \Rightarrow q^{\frac{n^2+n}{2}}$$

Determinant

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

minor

cofactor

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{vmatrix} = 1(-1)^{1+1}\begin{vmatrix} 2 & 3 \\ 4 & 7 \end{vmatrix} + 1(-1)^{1+2}\begin{vmatrix} 1 & 3 \\ 1 & 7 \end{vmatrix} + 1(-1)^{1+3}\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix}$$

$$= 2 - 4 + 2 = 0$$

Singular matrix

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For the matrix given below, verify the Determinant using all rows and all columns.

$$\mathbf{A} = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

Cofactor matrix of A $= \begin{bmatrix} -3 & -6 & -6 \\ 6 & 3 & -6 \\ 6 & -6 & 3 \end{bmatrix}_{3\times 3}$

Cofactor of $a_{11} = (-1)^{1+1}\begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} = -3$

cofactor of $a_{12} = (-1)^{1+2}\begin{vmatrix} 2 & -2 \\ 2 & 1 \end{vmatrix} = -6$

cofactor of $a_{13} = (-1)^{1+3}\begin{vmatrix} 2 & 1 \\ 2 & -2 \end{vmatrix} = -6$

$$\mathbf{A} = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

Cofactor matrix of A $= \begin{bmatrix} -3 & -6 & -6 \\ 6 & 3 & -6 \\ 6 & -6 & 3 \end{bmatrix}$

Det using 1st row $= (-1)(-3) + (-2)(-6) + (-2)(-6) = 27$

Det using 2nd row $= (2)(6) + (1)(3) + (-2)(-6) = 27$

Det using 3rd row $= (2)(6) + (-2)(-6) + (1)(3) = 27$

Det using 1st column $= (-1)(-3) \quad (2)(6) + (2)(6) = 27$

Det using 2nd column $= (-2)(-6) + (1)(3) + (-2)(-6) = 27$

Det using 3rd column $= (-2)(-6) + (-2)(-6) + (1)(3) = 27$

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$$\mathbf{A} = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

Cofactor matrix of A $= \begin{bmatrix} -3 & -6 & -6 \\ 6 & 3 & -6 \\ 6 & -6 & 3 \end{bmatrix}$

Sum of product of elements of 1st row with cofactors from 2nd row

$$(-1)(6) + (-2)(3) + (-2)(-6) = 0$$

Sum of product of elements of 1st row with cofactors from 3rd row

$$(-1)(6) + (-2)(-6) + (-2)(3) = 0$$

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Determinant

- Determinant of a square matrix is the sum of product of elements of a row or a columns with their cofactors.
- The cofactor of $\mathbf{A}_{n\times n}$ associated with the (i,j)-position is defined as

$$\mathbf{A}_{ij} = (-1)^{i+j} M_{ij}$$

  where M_{ij} is the (n-1)$\times$(n-1) minor obtained by deleting the i^{th} row and j^{th} column of $\mathbf{A}$.
- A minor determinant (or simply a minor) of $\mathbf{A}_{m\times n}$ is defined to be the determinant of any k$\times$k sub matrix of $\mathbf{A}$.

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- Effects of Row Operations: Let **B** be the matrix obtained from $\mathbf{A}_{n\times n}$ by one of the three elementary row operations:
  - Type I: Interchange rows i and j.
  - Type II: Multiply row i by $\alpha \neq 0$.
  - Type III: Add α times row i to row j.

Ex Type I

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ x & y & z \end{pmatrix}$$

$R_1 \longleftrightarrow R_2$

$$B = \begin{pmatrix} d & e & f \\ a & b & c \\ x & y & z \end{pmatrix}$$

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Type II $\quad R_3 \to \alpha R_3 \,, \quad \alpha \neq 0$

$$B = \begin{pmatrix} a & b & c \\ d & e & f \\ \alpha x & \alpha y & \alpha z \end{pmatrix}$$

$C = \alpha \underline{A}$

$$C = \begin{pmatrix} \alpha a & \alpha b & \alpha c \\ \alpha d & \alpha e & \alpha f \\ \alpha x & \alpha y & \alpha z \end{pmatrix}$$

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Type $\underline{\text{III}}$ $\quad R_3 \to R_3 + \alpha R_1$

$$B = \begin{pmatrix} a & b & c \\ d & e & f \\ x+\alpha a & y+\alpha b & z+\alpha c \end{pmatrix}$$

$R_3 \to \underline{\underline{\beta}} R_3 + \alpha R_1$

change the Det $\underline{\underline{\text{value}}}$

$R_3 \to R_3 + R_2 + R_1 \Rightarrow$ Det won't change

$\begin{pmatrix} & \end{pmatrix}_{n\times n} \Rightarrow R_1 \to R_1 + R_2 + \cdots R_n$

Det won't change

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The value of det($\mathbf{B}$) is as follows:

- det($\mathbf{B}$) $= -|A|$ for Type I operations.
- det($\mathbf{B}$) $= \alpha|A|$ for Type II operations.
- det($\mathbf{B}$) $= |A|$ for Type III operations.

$C = \alpha A$

$|C| = \alpha^n |A|$

Properties of Determinants

- Transposition doesn't alter determinants.

$$det(\mathbf{A}) = det(\mathbf{A}^T)$$

- $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$ for all n×n matrices.
- $\mathbf{A}_{n\times n}$ is nonsingular if and only if $\det(\mathbf{A}) \neq 0$.
- $\mathbf{A}_{n\times n}$ is singular if and only if $\det(\mathbf{A}) = 0$.
- Triangular Determinants

$$\begin{vmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ 0 & t_{22} & \cdots & t_{2n} \\ \vdots & \vdots & \ddots & \cdots \\ 0 & 0 & \cdots & t_{nn} \end{vmatrix} = t_{11}t_{22}\cdots t_{nn}$$

$|A+B|$ may not be equal to $|A| + |B|$

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- The determinants of a triangular matrix or a Diagonal is the product of its principal diagonal elements.
- If each element of a row or column in a square matrix is zero then the value of its determinant is zero.
- If two rows or two columns of a square matrix are identical then the determinant is zero.
- The determinant of a skew symmetric matrix of odd order is always zero.
- The determinant of an orthogonal matrix is either 1 or -1.

$$\begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow 0(\,) + 0(\,) + 0(\,) = 0$$

$$\begin{pmatrix} a & b & c \\ d & e & f \\ d & e & f \end{pmatrix} \Rightarrow R_3 \to R_3 + (-1)R_2 \quad \begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 0 \end{pmatrix}$$

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The determinant of matrix A is 5 and determinant of matrix B is 40. The determinant of matrix AB is
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$$|AB| = |A|\,|B|$$

$$= 5 \times 40$$

$$= \underline{\underline{200}}$$

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02. If $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -3 & 3 \\ 1 & 3 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$,

then $|AB - BA| =$ ______.

A and B are symmetric $A^T = A$, $B^T = B$

$C = AB - BA$

$C^T = (AB - BA)^T = (AB)^T - (BA)^T$

$= B^T A^T - A^T B^T$

$C^T = BA - AB$

$= -(AB - BA)$

$= -C$

$C^T = -C$

AB − BA is skew sym matrix of odd order

$\Rightarrow |AB - BA| = \underline{\underline{0}}$

A/W

06. If A and B are symmetric matrices of order 3×3, then consider the following:

S_1: $AB = BA$

S_2: $AB - BA$ is singular

Which of the following is true?

(a) S_1 is true and S_2 is false

(b) S_1 is false and S_2 is true

(c) Both S_1 and S_2 are true

(d) Both S_1 and S_2 are false

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08. Let A be a square matrix of order n – 1.

The elements of A are defined by

$$a_{ij} = \begin{cases} n-1, & \text{for } i=j \\ -1 & \text{for } i \neq j \end{cases}.$$

The determinant of A = ______.

(a) n^{n-1}

(b) n^{n-2}

(c) $(n-1)^{n-2}$

(d) $(n-1)^{n-3}$

$\underline{\underline{n=3}}$ $A_{\underline{\underline{2\times 2}}}$

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}_{2\times 2} \longrightarrow |\,| = 4-1 = \underline{\underline{3}}$$

a) 3^2 X

b) 3^1

c) $(2)^1$ X

d) 2^0 X

$\underline{\underline{\text{Ans}}}$ (b)

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Q.Let A be an $m \times n$ matrix and B an $n \times m$ matrix. It is given that $\det(I_m + AB) = \det(I_n + BA)$, where I_k is the $k \times k$ identity matrix. Using the above property, the determinant of the matrix given below is GATE-13

$$\begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

$R_1 \to R_1 + R_2 + R_3 + R_4$

$$\begin{vmatrix} 5 & 5 & 5 & 5 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{vmatrix} = 5 \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{vmatrix}$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$R_4 \to R_4 - R_1$

$$= 5 \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 5(1)(1)(1)(1)$$

$$= \underline{\underline{5}}$$

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Consider the matrix

$$J_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{6\times 6}$$

Which is obtained by reversing the order of the columns of the identity matrix I_6. Let $P = I_6 + \alpha J_6$, where α is a non – negative real number. The value of α for which det $(P) = 0$ is ______.

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$$\underline{I_6} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$\underline{|P| = 0}$

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \underline{\alpha} \\ 0 & 1 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 1 & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 1 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 1 & 0 \\ \alpha & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$\Rightarrow \underline{\alpha = 1}$