Linear Algebra

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Reference Book: Matrix Analysis and applied Linear Algebra by Carl D Meyer
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For ECE students vector space

For CSE students LU decomposition

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Imagine there are three friends. They go to a market

Friend 1 $3\text{apples} + 2\text{bananas} + 1\text{orange} = 39$ — (1)

Friend 2 $2\text{apples} + 3\text{bananas} + 1\text{orange} = 26$ — (2)

Friend 3 $1\text{apples} + 2\text{bananas} + 3\text{orange} = 25$ — (3)

How to find cost of each fruit??

solve (1) and (2) to eliminate x (4)

Solve (1) and (3) to eliminate x (5)

solve (4) and (5) to eliminate y and get z

Elimination of variables

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Solve the following linear system of equations

$$2x + y + z = 1$$

$$6x + 2y + z = -1$$

$$-2x + 2y + z = 7$$

$$\underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 6 & 2 & 1 \\ -2 & 2 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix}}_{b}$$

$$[A|b] = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 6 & 2 & 1 & -1 \\ -2 & 2 & 1 & 7 \end{pmatrix}$$

$R_2 \to R_2 - 3R_1$, $R_3 \to R_3 + R_1$

$$\begin{pmatrix} 2 & 1 & 1 & 1 \\ 0 & -1 & -2 & -4 \\ 0 & 3 & 2 & 8 \end{pmatrix}$$

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$$\begin{pmatrix} 2 & 1 & 1 & 1 \\ 0 & -1 & -2 & -4 \\ 0 & 3 & 2 & 8 \end{pmatrix}$$

$R_3 \to R_3 + 3R_2$

$$\begin{pmatrix} 2 & 1 & 1 & 1 \\ 0 & -1 & -2 & -4 \\ 0 & 0 & -4 & -4 \end{pmatrix}$$

$$-4z = -4$$

$$z = 1$$

$$-y - 2z = -4$$

$$-y - 2 = -4$$

$$y = 2$$

$$2x + y + z = 1$$

$$2x + 2 + 1 = 1$$

$$x = -1$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

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Pivots elements

$$\begin{pmatrix} 2 & 1 & 1 & 1 \\ 0 & -1 & -2 & -4 \\ 0 & 0 & -4 & -4 \end{pmatrix}$$

$\Rightarrow$ This is in Row echelon form.

Columns containing pivot elements are called Basic Columns.

Columns not containing pivot elements are called Non Basic Columns.

$\text{Rank}(A) = 3$

$\text{Rank}(A|b) = 3$

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Solve the following system of linear equations

$$\begin{aligned} x_1 + 2x_2 + x_3 + 3x_4 + 3x_5 &= 5 \\ 2x_1 + 4x_2 \qquad\quad + 4x_4 + 4x_5 &= 6 \\ x_1 + 2x_2 + 3x_3 + 5x_4 + 5x_5 &= 9 \\ 2x_1 + 4x_2 \qquad\quad + 4x_4 + 7x_5 &= 9 \end{aligned}$$

$$[A|b] = \begin{bmatrix} 1 & 2 & 1 & 3 & 3 & 5 \\ 2 & 4 & 0 & 4 & 4 & 6 \\ 1 & 2 & 3 & 5 & 5 & 9 \\ 2 & 4 & 0 & 4 & 7 & 9 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$

$R_4 \to R_4 - 2R_1$

$$\begin{bmatrix} 1 & 2 & 1 & 3 & 3 & 5 \\ 0 & 0 & -2 & -2 & -2 & -4 \\ 0 & 0 & 2 & 2 & 2 & 4 \\ 0 & 0 & -2 & -2 & 1 & -1 \end{bmatrix}$$

$R_3 \to R_3 + R_2$, $R_4 \to R_4 - R_2$

$$\begin{pmatrix} 1 & 2 & 1 & 3 & 3 & 5 \\ 0 & 0 & -2 & -2 & -2 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 3 \end{pmatrix}$$

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$$\begin{pmatrix} 1 & 2 & 1 & 3 & 3 & 5 \\ 0 & 0 & -2 & -2 & -2 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 3 \end{pmatrix}$$

$R_3 \longleftrightarrow R_4$

$$\begin{pmatrix} 1 & 2 & 1 & 3 & 3 & 5 \\ 0 & 0 & -2 & -2 & -2 & -4 \\ 0 & 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$3x_5 = 3 \Rightarrow x_5 = 1$

$-2x_3 - 2x_4 - 2\underline{\underline{x_5}} = -4$

$-2x_3 - 2x_4 - 2 = -4$

$-2x_3 - 2x_4 = -2$

$x_3 + x_4 = 1$

Non basic columns of A are 2 and 4

$\Rightarrow$ x_2 and x_4 are called free variables.

Let $x_4 = 1 \Rightarrow x_3 + 1 = 1$

$\Rightarrow x_3 = 0$

$x_1 + 2x_2 + x_3 + 3x_4 + 3x_5 = 5$

$x_1 + 2x_2 + 0 + 3 + 3 = 5$

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$x_1 + 2x_2 = -1$

let $x_2 = 0 \Rightarrow x_1 = -1$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

Let us general solution

$x_5 = 1$

$x_3 + x_4 = 1$

Let $x_4 = K_1 \Rightarrow x_3 = 1 - K_1$

$x_1 + 2x_2 + x_3 + 3x_4 + 3x_5 = 5$

$x_1 + 2x_2 + 1 - K_1 + 3K_1 + 3 = 5$

$x_1 + 2x_2 = 1 - 2K_1$

Let $x_2 = K_2 \Rightarrow x_1 = 1 - 2K_1 - 2K_2$

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$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 - 2K_1 - 2K_2 \\ K_2 \\ 1 - K_1 \\ K_1 \\ 1 \end{pmatrix} = K_1 \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} + K_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$K_1 = K_2 = 0$ $\quad X = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix},$ $\quad K_1 = 1 \quad K_2 = 2$ $\quad X$

The System has infinitely many solutions.

$$\begin{pmatrix} 1 & 2 & 1 & 3 & 3 & 5 \\ 0 & 0 & -2 & -2 & -2 & -4 \\ 0 & 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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Solve the following system of linear equations

$$
\begin{aligned}
x_1 + 2x_2 + x_3 + 3x_4 + 3x_5 &= 0\\
2x_1 + 4x_2 \qquad\quad + 4x_4 + 4x_5 &= 0\\
x_1 + 2x_2 + 3x_3 + 5x_4 + 5x_5 &= 0\\
2x_1 + 4x_2 \qquad\quad + 4x_4 + 7x_5 &= 0
\end{aligned}
$$

$$\begin{pmatrix} x_1\\ x_2\\ x_3\\ x_4\\ x_5 \end{pmatrix} = K_1\begin{pmatrix} -2\\ 0\\ -1\\ 1\\ 0 \end{pmatrix} + K_2\begin{pmatrix} -2\\ 1\\ 0\\ 0\\ 0 \end{pmatrix} + \cancel{\begin{pmatrix} 1\\ 0\\ 1\\ 0\\ 1 \end{pmatrix}}$$

$$\begin{pmatrix} x_1\\ x_2\\ x_3\\ x_4\\ x_5 \end{pmatrix} = K_1\begin{pmatrix} -2\\ 0\\ -1\\ 1\\ 0 \end{pmatrix} + K_2\begin{pmatrix} -2\\ 1\\ 0\\ 0\\ 0 \end{pmatrix}$$

(1) $K_1 = K_2 = 0$ $\quad X = \begin{pmatrix} 0\\ 0\\ 0\\ 0\\ 0 \end{pmatrix}$ zero solution

(2) $K_1 = 1$ $\quad K_2 = 1$ $\quad X = \begin{pmatrix} -4\\ 1\\ -1\\ 1\\ 0 \end{pmatrix}$

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Summary :

Ex 1

$$\begin{pmatrix} 2 & 1 & 1 & 1 \\ 0 & -1 & -2 & -4 \\ 0 & 0 & -4 & -4 \end{pmatrix}$$

$0\ 0\ 0\ \alpha$ $\quad$ $\underline{\alpha \neq 0}$

Number of variables $= n = 3$

Rank (A) = 3 $\quad$ $R(A|b) = 3$ $\quad$ Rank(A) $= r = 3$

Number of free variables $= n - r = 3 - 3 = 0$

Unique solution $\quad$ b is non basic column

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Ex2

$$\begin{pmatrix} 1 & 2 & 1 & 3 & 3 & 5 \\ 0 & 0 & -2 & -2 & -2 & -4 \\ 0 & 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$0 \; 0 \; 0 \; 0 \; 0 \; \alpha$

$\alpha \neq 0$

Number of variables $= n = 5$

Rank (A) $= 3$ Rank(A|b) $= 3$ Rank (A) $= r = 3$

Number of free variables $= n - r = 5 - 3 = 2$

Infinitely many solutions.

b is non basic column

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Ex 3

$$\begin{pmatrix} 1 & 2 & 1 & 3 & 3 & 5 \\ 0 & 0 & -2 & -2 & -2 & -4 \\ 0 & 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$0\ 0\ 0\ 0\ 0\ \alpha$

$\alpha \neq 0$

$n = 5$

$\text{Rank}(A) = 3$, $R(A|b) = 4$ $\qquad \text{Rank}(A) \neq \text{Rank}(A|b)$

$0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 \neq 1$

NO solution , b is a basic column

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m linear equations in n unknowns is given by

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \quad E_1$$
$$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \quad E_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m \quad E_m$$

$$A_{m \times n} X_{n \times 1} = b_{m \times 1}$$

The above equations in matrix form can be written as $\mathbf{Ax} = \mathbf{b}$.

$$\underset{m \times n}{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$A_{m \times n}$

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Let $S = \begin{Bmatrix} E_1 \\ E_2 \\ \vdots \\ E_m \end{Bmatrix}$ be the linear system with m equations and n unknowns.

For a linear system S , each of the following three elementary operations results in an equivalent system S'.

(1) Interchange the i^{th} and j^{th} equations. That is, if

Type I

$$S = \begin{Bmatrix} E_1 \\ E_2 \\ \vdots \\ E_i \\ E_j \\ \vdots \\ E_m \end{Bmatrix} \qquad S' = \begin{Bmatrix} E_1 \\ E_2 \\ \vdots \\ E_j \\ E_i \\ \vdots \\ E_m \end{Bmatrix}$$

(2) Replace the i^{th} equation by a nonzero multiple of itself. That is

$$S = \begin{Bmatrix} E_1 \\ E_2 \\ \vdots \\ E_i \\ E_j \\ \vdots \\ E_m \end{Bmatrix} \qquad S' = \begin{Bmatrix} E_1 \\ E_2 \\ \vdots \\ \alpha E_i \\ E_j \\ \vdots \\ E_m \end{Bmatrix} \qquad \underline{\alpha \neq 0}$$

Type $\underline{\text{II}}$

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(3) Replace the j^{th} equation by a combination of itself plus a multiple of the i^{th} equation. That is,

$$S = \begin{Bmatrix} E_1 \\ E_2 \\ \vdots \\ E_i \\ E_j \\ \vdots \\ E_m \end{Bmatrix} \qquad S' = \begin{Bmatrix} E_1 \\ E_2 \\ \vdots \\ E_i \\ E_j + \alpha E_i \\ \vdots \\ E_m \end{Bmatrix} \qquad \text{Type } \underline{\text{III}}$$

A typical structure for a matrix in row echelon form is illustrated below with the pivots circled.

$$\begin{bmatrix} * & * & * & * & * & * & * & * \\ 0 & 0 & * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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What is a Consistent System?

A System which has atleast one solution is called Consistent system

Inconsistent System: A System with no solution is called Inconsistent System.

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Consistency

Each of the following is equivalent to saying that $[\mathbf{A}|\mathbf{b}]$ is consistent.

- In row reducing $[\mathbf{A}|\mathbf{b}]$, a row of the following form never appears:

$$(0\ 0\ .\ .\ 0|\alpha) \quad \text{where } \alpha \neq 0$$

$$0x_1 + 0x_2 + \cdots 0x_n \neq \alpha$$

- $\mathbf{b}$ is a nonbasic column in $[\mathbf{A}|\mathbf{b}]$.
- $\text{rank}[\mathbf{A}|\mathbf{b}] = \text{rank}\,(\mathbf{A})$.
- $\mathbf{b}$ is a combination of the columns in $\mathbf{A}$.

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$\mathbf{b}$ is a combination of the columns in $\mathbf{A}$.

$$2x + y + z = 1$$
$$6x + 2y + z = -1$$
$$-2x + 2y + z = 7$$

$$-1\begin{pmatrix}2\\6\\-2\end{pmatrix} + 2\begin{pmatrix}1\\2\\2\end{pmatrix} + 1\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\-1\\7\end{pmatrix}$$