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183. The maximum value of the determinant among all 2×2 real symmetric matrices with trace 14 is ________.

(GATE–14–EC–SET2)

trace of a matrix is sum of principal diagonal elements

$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

$a + c = 14$

$|A| = ac - b^2$

To get max value of $|A|$, we need $b^2 = 0 \Rightarrow b = 0$

$|A| = ac$

max ac such that $a + c = 14$

$\Rightarrow c = 14 - a$

$|A| = a(14 - a)$

$|A| = 14a - a^2 \rightarrow f(a)$

$f'(a) = 14 - 2a = 0$

$a = 7$

$f''(a) = -2$ $\quad f''(7) = -2 < 0$

we get max value at $a = 7$ $\quad |A| = 14 \times 7 - 7^2$

$= 49$

Inverse of a Matrix

- ✓ Only non-singular matrices are invertible.
- **B** is called as Inverse of matrix **A** if

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I} \implies \mathbf{B} = \mathbf{A}^{-1}$$

$$\mathbf{A}^{-1} = \frac{Adj(\mathbf{A})}{\det(\mathbf{A})}$$

where Adj($\mathbf{A}$) is the cofactor matrix transpose.

$$Adj(\mathbf{A}) = (\text{cofactor matrix})^T$$

$A^{-1} = \frac{Adj(A)}{|A|}$

$Adj(A) = \begin{pmatrix} \text{cofactor} \\ \text{matrix} \\ \text{of } A \end{pmatrix}^T$

$\boxed{AA^{-1} = A^{-1}A = I}$

Find the inverse of the following matrix

$$\mathbf{A} = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

Cofactor matrix of A $= \begin{bmatrix} -3 & -6 & -6 \\ 6 & 3 & -6 \\ 6 & -6 & 3 \end{bmatrix}$

$\text{Adj}(A) = \begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix}$, $A^{-1} = \frac{1}{27}\begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix}$.

$|A| = 27$

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If $\mathbf{A}_{n\times n}$ is a non-singular then

$$|\mathbf{A}^{-1}| = \underline{\ 1/|A|\ }$$

$$|Adj(\mathbf{A})| = \underline{\ |A|^{n-1}\ }$$

H/W

$$Adj(Adj(\mathbf{A})) = \underline{\qquad\qquad}$$

H/W

$$|Adj(Adj(\mathbf{A}))| = \underline{\qquad\qquad}$$

$$AA^{-1} = I$$

$$|AA^{-1}| = |I|$$

$$|A|\,|A^{-1}| = 1$$

$$|A^{-1}| = \frac{1}{|A|}$$

$$|AB| = |A||B|$$

$$C = \alpha A$$

$$|C| = \alpha^n |A|$$

$$A^{-1} = \frac{Adj(A)}{|A|}$$

$$Adj(A) = |A|A^{-1}$$

$$|Adj A| = ||A|A^{-1}|$$

$$= |A|^n |A^{-1}|$$

$$= |A|^n \cdot \frac{1}{|A|} = |A|^{n-1}$$

03. If $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ and $B = A^{-1}$, then the element in the second row and third column of B = ______.

(a) 0 (b) $\frac{1}{2}$

(c) $-\frac{1}{2}$ (d) 1

$$A^{-1} = \frac{\text{Adj}(A)}{|A|} = B \Rightarrow \underline{\underline{b_{23}}}$$

Cofactor matrix of A $= \begin{bmatrix} - & - & - \\ - & - & - \\ - & \blacksquare & - \end{bmatrix}_{3\times 3}$

Adj(A) $= \begin{bmatrix} - & - & - \\ - & - & \blacksquare \\ - & - & - \end{bmatrix}_{3\times 3}$ $b_{23} = \frac{\blacksquare}{|A|}$

cofactor of $a_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = -1$

$|A| = 0(\ ) + 0(\ ) + 1(-1)^{3+3}\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2$

$b_{23} = -1/2$

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291. Consider a 2 × 2 matrix $M = [v_1, v_2]$, where, v_1 and v_2 are the column vectors.
Suppose $M^{-1} = \begin{bmatrix} u_1^T \\ u_2^T \end{bmatrix}$, where u_1^T and u_2^T are the row vectors.
Consider the following statements:

Statement 1: $u_1^T v_1 = 1$ and $u_2^T v_2 = 1$

Statement 2: $u_1^T v_2 = 0$ and $u_2^T v_1 = 0$

Which of the following options is correct?

(GATE-19-EE)

(a) Statement 2 is true and statement 1 is false
(b) Both the statements are false
(c) Statement 1 is true and statement 2 is false
(d) Both the statements are true

$$M = [v_1 \ v_2]_{2\times 2} = [\,|\ \ |\,]$$

$$M^{-1} = \begin{bmatrix} u_1^T \\ u_2^T \end{bmatrix} = \begin{pmatrix} \text{---} \\ \text{---} \end{pmatrix}$$

$$[\,=\,][\,|\ |\,]$$

$$u_1^T \cdot v_1 = (\text{———})\left(|\right) = (\ )$$

$$M^{-1} \cdot M = I \qquad M M^{-1} = I$$

$$\overset{M^{-1}}{\begin{bmatrix} u_1^T \\ u_2^T \end{bmatrix}} \overset{M}{[v_1 \ v_2]} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$u_1^T v_1 = 1$ $\quad$ $u_1^T v_2 = 0$

$u_2^T v_1 = 0$ $\quad$ $u_2^T v_2 = 1$

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297. The inverse of the matrix $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$ is

(GATE-19-CE-SET2)

(a) $\begin{bmatrix} -10 & 4 & 9 \\ 15 & -4 & -14 \\ -5 & 1 & 6 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & -\frac{4}{5} & -\frac{9}{5} \\ -3 & \frac{4}{5} & \frac{14}{5} \\ 1 & -\frac{1}{5} & -\frac{6}{5} \end{bmatrix}$

(c) $\begin{bmatrix} -2 & \frac{4}{5} & \frac{9}{5} \\ 3 & -\frac{4}{5} & -\frac{14}{5} \\ -1 & \frac{1}{5} & \frac{6}{5} \end{bmatrix}$

(d) $\begin{bmatrix} 10 & -4 & -9 \\ -15 & 4 & 14 \\ 5 & -1 & -6 \end{bmatrix}$

$$AA^{-1} = I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

a) $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} -10 \\ 15 \\ -5 \end{bmatrix} = -20 + 45 - 20 = -5 \neq 1$

b) $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = 4 - 9 + 4 = -1 \neq 1$

$\begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 10 \\ -15 \\ 5 \end{bmatrix} =$

$20 - 45 + 20$

$= -5 \neq 1$

c) $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix} = -4 + 9 - 4 = 1$

Ans : (C)

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HW

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$$

$|A| = 0$

$\text{Adj } A = \, ?$

Rank of a matrix

For a matrix $A_{m\times n}$

- rank($\mathbf{A}$) denotes the number of nonzero rows in any row echelon form that is row equivalent to $\mathbf{A}$.
- rank($\mathbf{A}$) denotes the number of pivots obtained in reducing $\mathbf{A}$ to a row echelon form with row operations.
- rank($\mathbf{A}$) denotes the size of the largest nonzero minor of $\mathbf{A}$.
- rank($\mathbf{A}$) denotes the number of linearly independent rows or columns of A

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Use determinants to compute the rank of $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 1 \\ 7 & 8 & 9 & 1 \end{bmatrix}_{3\times 4}$

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}_{3\times 3} = 0 \qquad \begin{vmatrix} 1 & 2 & 1 \\ 4 & 5 & 1 \\ 7 & 8 & 1 \end{vmatrix}_{3\times 3} = 0$$

$$\begin{vmatrix} 1 & 3 & 1 \\ 4 & 6 & 1 \\ 7 & 9 & 1 \end{vmatrix}_{3\times 3} = 0 \qquad \begin{vmatrix} 2 & 3 & 1 \\ 5 & 6 & 1 \\ 8 & 9 & 1 \end{vmatrix}_{3\times 3} = 0$$

$R(A) \neq 3$

$$\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = 5-8 = -3 \neq 0$$

$R(A) = 2$

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$A_{3\times4}$

$R(A) \le \min(3,4)$

$\boxed{R(A) \le 3}$

Let $\mathbf{A}$ be $m \times n$ matrix

- ✓ $\text{rank}(\mathbf{A}) \le \min(\text{m,n})$.
- $\text{rank}(\mathbf{A}+\mathbf{B}) \le \text{rank}(\mathbf{A}) + \text{rank}(\mathbf{B})$.
- If $\mathbf{A}$ is $m \times n$ and $\mathbf{B}$ is $n \times p$, then $\text{rank}(\mathbf{AB}) \le \min\{\text{rank}(\mathbf{A}), \text{rank}(\mathbf{B})\}$
- ✓ $\text{rank}(\mathbf{AA}^T) = \text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}^T\mathbf{A})$ $\Rightarrow$ appeared multiple times in GATE
- The rank of a non-zero matrix is non-zero.
- The rank of a null matrix is zero.
- The rank of non-singular matrix is its order. $|A_{n\times n}| \neq 0 \quad R(A) = \underline{\underline{n}}$
- The rank of singular matrix is less than its order. $|A| = 0 \quad R(A) \underline{\underline{< n}}$

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Nullity of a matrix is defined as the difference between order and rank of the matrix.
Nullity of a non-singular matrix of any order is always zero.

01. Let $A = \begin{bmatrix} 3 & P & P \\ P & 3 & P \\ P & P & 3 \end{bmatrix}$. If rank of A is 1, then

P = 3.

$A = \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix}$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$\begin{pmatrix} 3 & 3 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

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04. Suppose that $A_{n\times n}$ is upper triangular matrix such that $a_{ii} = 0, i = 1, 2, ..., n.$
Then rank of A^n = ________.

(a) 0 (b) $n - 1$

(c) 1 (d) n

A is upper triangular matrix

verification $n = 2$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(a)

10. Let $A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$

where a, b, c are non-zero real numbers.

Then Rank of A =

(a) 0 (b) 1

(c) 2 (d) 3

A is skew sym matrix of odd order

$|A| = 0$

$R(A) < 3$

$\begin{vmatrix} 0 & a \\ -a & 0 \end{vmatrix} = a^2 \neq 0$

$R(A) = \underline{\underline{2}}$

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The rank of the matrix

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}_{5\times 5}$$ is ______.

(GATE – 17–EC)

Reduce the given matrix to row echelon

$R(A) = 4$

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277. Consider matrix $A = \begin{bmatrix} k & 2k \\ k^2 - k & k^2 \end{bmatrix}$ and vector $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. The number of distinct real values of k for which the equation Ax = 0 has infinitely many solutions is ________.

(GATE-18-EC)

$AX = 0$

Infinitely many solutions

$R(A) < n$

$R(A) < 2$ $\qquad |A| = 0$

$$\begin{vmatrix} K & 2K \\ K^2 - K & K^2 \end{vmatrix} = 0$$

$K^3 - 2K^3 + 2K^2 = 0$

$-K^3 + 2K^2 = 0$

$-K^2(K-2) = 0$

$K = 0, 0, 2$

Ans = 2

Linearly Independent and Dependent vectors

Ex 1 Consider the vectors $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $v_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$. Are the vectors Linearly independent?

$V_2 = 2V_1$

v_1 & v_2 are Linearly dependent

$V_2 = 2V_1$

$2V_1 - V_2 = 0$

$\alpha_1 V_1 + \alpha_2 V_2 = 0 \Rightarrow$ Homogeneous eqn

$\alpha_1 = 2 \quad \alpha_2 = -1$

$\alpha_1 = 4 \quad \alpha_2 = -2$

$\alpha_1 = 6 \quad \alpha_2 = -3$

$\vdots$

method 2

$|v_1 \; v_2| = \begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} = 0$

v_1 & v_2 are L.D

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Ex 2 Consider the vectors $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. Are the vectors Linearly independent?

$v_3 = v_1 + v_2$

v_1, v_2 & v_3 are L.D

$v_1 + v_2 - v_3 = 0$

$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$

$\alpha_1 = 1 \quad \alpha_2 = 1 \quad \alpha_3 = -1$

$\alpha_1 = 2 \quad \alpha_2 = 2 \quad \alpha_3 = -2$

$\vdots$

method 2

$$|v_1 \; v_2 \; v_3| = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

v_1, v_2 & v_3 are L.D

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Ex 3 Consider the vectors $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Are the vectors Linearly independent?

$$\alpha_1 v_1 + \alpha_2 v_2 = 0$$

$$\alpha_1 = \alpha_2 = 0$$ Trivial solution

method 2

$$|v_1 \ v_2| = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

v_1 & v_2 are Linearly independent

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Ex 4 Consider the vectors $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Are the vectors Linearly independent?

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$$

Only one solution

$$\alpha_1 = \alpha_2 = \alpha_3 = 0$$

Trivial solution

$$|v_1 \ v_2 v_3| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 1 \neq 0$$

v_1, v_2 and v_3 are L.I

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Ex 5 Consider the vectors $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $v_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Are the vectors Linearly independent?

$$\begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}_{3\times 4}$$

$$v_1 + v_2 + v_3 = v_4$$

Rank = 3

Rank = 2

v_1, v_2, v_3 & v_4 are L.D

- Linear Independence: A set of vectors $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is said to be a linearly independent set whenever the only solution for the scalars α_i in the homogeneous equation $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n = \mathbf{0}$ is the trivial solution $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.
- Whenever there is a nontrivial solution for the α (i.e., at least one $\alpha_i \neq 0$), the set $\mathcal{S}$ is said to be a linearly dependent set.

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If the vectors $(1.0,\ -1.0,\ 2.0)$, $(7.0,\ 3.0,\ x)$ and $(2.0,\ 3.0,\ 1.0)$ in $\mathbb{R}^3$ are linearly dependent, the value of x is __________

GATE 2021

$(- - - -)$

$$\begin{vmatrix} 1 & -1 & 2 \\ 7 & 3 & x \\ 2 & 3 & 1 \end{vmatrix} = 0$$

$$1(3-3x) + 1(7-2x) + 2(21-6) = 0$$

$$3-3x+7-2x+30=0$$

$$40 = 5x$$

$$\Rightarrow \underline{\underline{x=8}}$$

R set of real numbers

R^2 $\begin{pmatrix} - \\ - \end{pmatrix}$, R^3 $\begin{pmatrix} - \\ - \\ - \end{pmatrix}$ $(- -)$

15. Consider the following statements:

S1: If $\{X_1, X_2, X_3, X_4\}$ is a linearly independent set of vectors, then the set $\{X_1, X_2, X_3\}$ is linearly independent.

S2: If $\{X_1, X_2, X_3, X_4\}$ is a linearly dependent set of vectors, then the set $\{X_1, X_2, X_3\}$ is linearly dependent.

Which of the following is true?

(a) Only S1

(b) Only S2

(c) Both S1 and S2

(d) Neither S1 nor S2

LU Decomposition

If A is an $n \times n$ matrix such that a zero pivot is never encountered when applying Gaussian elimination with Type III operations, then A can be factored as the product $A = LU$, where the following hold.

L is lower triangular and U is upper triangular.

$l_{ii} = 1$ and $u_{ii} \neq 0$ for each $i = 1, 2, \cdots, n$.

Below the diagonal of L, the entry $l_{i,j}$ is the multiple of row j that is subtracted from row i in order to annihilate the (i, j) position during Gaussian elimination.

U is the final result of Gaussian elimination applied to A.

The decomposition of A into A = LU is called the LU factorization of A, and the matrices L and U are called the LU factors of A.

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Use the LU factorization of A to solve $Ax = b$, where

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 6 & 2 & 1 \\ -2 & 2 & 1 \end{bmatrix}, = \begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix}$$

$\boxed{A = LU}$

L = Lower

U = upper

Consider A

$$\begin{pmatrix} 2 & 1 & 1 \\ 6 & 2 & 1 \\ -2 & 2 & 1 \end{pmatrix}$$

$R_2 \to R_2 - 3R_1$

$R_3 \to R_3 + R_1$

$R_3 - (-1)R_1$

$$\begin{pmatrix} 2 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 3 & 2 \end{pmatrix}$$

$R_3 \to R_3 + 3R_2$

$R_3 - (-3)R_2$

$$\begin{pmatrix} 2 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & -4 \end{pmatrix} = U$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & -3 & 1 \end{pmatrix}$$

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$$LU = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & -3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & -4 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 6 & 2 & 1 \\ -2 & 2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & -3 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$Ax = b$

$LUx = b$

$\underline{LY = b}$

$Ux = \underline{Y}$

$y_1 = b_1$

$3y_1 + y_2 = b_2$

y_2

- To solve a nonsingular system $Ax = b$ using the LU factorization $A = LU$, first solve $Ly = b$ for y with the forward substitution algorithm , and then solve $Ux = y$ for $\mathbf{x}$ with the back substitution procedure.
- The advantage of this approach is that once the LU factors for A have been computed, any other linear system $Ax = \tilde{b}$ can be solved with only n^2 multiplications/divisions and $n^2 - n$ additions/ subtractions.