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MATH 3550 - Differential Equations
Project 4 - Lorenz Equations and Chaotic Orbits

Description

Edward Lorenz was a meteorologist who studied the behavior of weather systems. While evaluating simple weather forecast equations, he noticed that miniscule changes to his initial conditions lead to drastic changes in the predictions. This "chaotic" behavior resulted in long-term predictions being difficult, or impossible,to make. Thus, the study of chaos was born. Lorenz' equations (1) have constraints of: $\sigma, r, b < 0$ and functions $X(\tau)$, $Y(\tau), Z(\tau)$ that respectively quantify convective motion, measure the temperature difference between ascending and descending currents, and reflect deviation from a linear temperature profile across a layer.

$$\begin{pmatrix} X'(\tau) \\ Y'(\tau) \\ Z'(\tau) \end{pmatrix} = \begin{pmatrix} -\sigma X + \sigma Y \\ -XZ + rX - Y \\ XY - bZ \end{pmatrix} \tag{1}$$

In this paper, mirroring the work of Lorenz, we found the equilibrium points of the nonlinear system (1), linearized the system using the Jacobian matrix to find the eigenvalues, and used the eigenvalues to determine the behavior around the fixed points. Then, using MATLAB, we reproduced Lorenz' results, using the same parameters and initial values, to solve the system (1) with the improved Euler's method. This reproduced the graphs that Lorenz originally found. Using our code, we were then able to examine chaos within subcritical and supercritical conditions, as well as changes to the parameters, on system behavior.

Linearization

In order to effectively analyze this set of nonlinear equations, we must linearize the system and describe behavior around the equilibrium points. To find the equilibria, we set $X'(\tau) = Y'(\tau) = Z'(\tau) = 0$ to find the null surfaces.

$$\begin{array}{ccc} -\sigma X + \sigma Y = 0 & & X = Y \\ -XZ + rX - Y = 0 & \rightarrow & -Y(Z - r + 1) = 0 \\ XY - bZ = 0 & & XY = bZ \end{array}$$

We see that in the second line, $Y = 0$ and $Z = r - 1$ both satisfy the equation. Leading us to these null surface equations:

$$\begin{gathered} X = Y \\ Y = 0 \ or \ Z = r - 1 \\ Y^2 = bZ \end{gathered} \tag{2}$$

The solutions to which are the intersection of the null surfaces. Since $Y = 0$ and $Z = r - 1$ are both solutions of $Y'(\tau) = 0$, they act together as a system of null surfaces. As a result, each equilibrium point only intersects either $Y = 0$ or $Z = r - 1$.

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sqrt{b(r-1)} \\ \sqrt{b(r-1)} \\ r-1 \end{pmatrix}, \begin{pmatrix} -\sqrt{b(r-1)} \\ -\sqrt{b(r-1)} \\ r-1 \end{pmatrix} \tag{3}$$

Next, we'll consider how b and r affect the number of equilibrium points. Since $b > 0$, the square root of b will always be real and, thus, not affect the number of points. However, r can be split into two domains when analyzing $\sqrt{r-1}$: $0 < r < 1$ and $r \geq 1$. When $0 < r < 1$, the value of the radical is imaginary; therefore, there can only be one equilibrium point, $(0, 0, 0)$. When $r \geq 1$, $\sqrt{r-1}$ will once again output a real number. As a result, the equilibrium points are the same as equation (3).

Recall that the Jacobian matrix for a general 3x3 matrix and its characteristic equation are:

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{pmatrix}$$

$$det(J - \lambda I) = 0 \tag{4}$$

After finding the partial derivatives with respect to each function in (1), we get the general Jacobian matrix for the Lorenz equations:

$$J = \begin{pmatrix} -\sigma & \sigma & 0 \\ -Z + r & -1 & -X \\ Y & X & -b \end{pmatrix}$$

Using the general Jacobian matrix, we can analyze the linear behavior surrounding the fixed points by substituting them into the matrix:

$$J \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix} \tag{5}$$

$$J \begin{pmatrix} \sqrt{b(r-1)} \\ \sqrt{b(r-1)} \\ r-1 \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & -\sqrt{b(r-1)} \\ \sqrt{b(r-1)} & \sqrt{b(r-1)} & -b \end{pmatrix} \tag{6}$$

$$J\begin{pmatrix}-\sqrt{b(r-1)}\\-\sqrt{b(r-1)}\\r-1\end{pmatrix}=\begin{pmatrix}-\sigma & \sigma & 0\\1 & -1 & \sqrt{b(r-1)}\\-\sqrt{b(r-1)} & -\sqrt{b(r-1)} & -b\end{pmatrix} \tag{7}$$

Stability of the First Fixed Point

Stability largely depends on the types and values of eigenvalues for a given equilibrium point. By finding eigenvalues of the matrices above, we can accurately classify the stability of the fixed point. Since these eigenvalues are also dependent on multiple variables, we will analyze if a change in those variables will change the types of eigenvalues and the system stability at this point. Using the general characteristic equation (4) and the Jacobian linearization (5), we find:

$$(\lambda+b)(\lambda^2+(\sigma+1)\lambda+\sigma(1-r))=0 \tag{8}$$

Immediately, we notice that $\lambda_1 = -b$. Using the general quadratic formula on the second half of (8), we can find the other two eigenvalues:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\lambda=\frac{-(\sigma+1)\pm\sqrt{(\sigma+1)^2-4\sigma(1-r)}}{2}$$

Which simplifies to:

$$\lambda_2,\lambda_3=\frac{-(\sigma+1)\pm\sqrt{(\sigma-1)^2+4\sigma r}}{2} \tag{9}$$

Notice that for all values of r and σ, the eigenvectors λ_2 and λ_3 are real numbers. We can further classify the eigenvectors by analyzing the trace and determinant of the second half of (8), where:

$$tr=-(\sigma+1) \tag{10}$$

$$det=\sigma(1-r) \tag{11}$$

When $r < 1$, the determinant (11) is positive and the trace (10) is negative, indicating that the eigenvectors λ_2 and λ_3 are real negative numbers. Recall that $\lambda_1 = -b$ is also a negative real number; therefore, when $r < 1$, the fixed point is a stable sink. However, when $r > 1$, (11) is negative - forcing λ_2 and λ_3 to have positive and negative values. Because of the mixture of positive and negative eigenvalues, the point $(0,0,0)$ when $r > 1$, is an unstable saddle.

At $r = 1$, a fork occurs, and we see the emergence of the other two equilibrium points. Since the other two points continue to exist for $r \geq 1$, we call this a supercritical pitchfork bifurcation.

Stability of the Second and Third Fixed Points

For the other equilibrium points, the analysis is not as simple. Defining the characteristic equation as a function of an eigenvalue allows us to find limits to what the values can be. Then, using different forms of the characteristic equations, we can find key equations that will eventually help determine the other eigenvalues. After that, we evaluate how r affects the systems and if there is another bifurcation. For a general 3x3 matrix A, its characteristic equation is:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$det(A - \lambda I) = 0$$

$$\lambda^3 - tr(A)\lambda^2 + (a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - a_{13}a_{31} - a_{12}a_{21} - a_{23}a_{32})\lambda - det(A) = 0 \quad (12)$$

Applying this characteristic equation (12) to the Jacobian linearization matrices for the other two fixed points, (6) and (7), we get the same equation for both of them:

$$\lambda^3 + (\sigma + b + 1)\lambda^2 + b(\sigma + r)\lambda + 2\sigma b(r - 1) = 0 \quad (13)$$

As a result, the eigenvalues found will apply for both fixed points. Further, since we know that the eigenvalues, λ_1, λ_2, and λ_3 are solutions, or roots, to equations (12) and (13), we can write another form of the characteristic equation:

$$(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) = 0$$

$$\lambda^3 - (\lambda_1 + \lambda_2 + \lambda_3)\lambda^2 + (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)\lambda - \lambda_1\lambda_2\lambda_3 = 0 \quad (14)$$

From all of the equations, (12), (13), and (14), we can deduce two facts:

$$tr(A) = \lambda_1 + \lambda_2 + \lambda_3 = -(\sigma + b + 1) \quad (15)$$

$$det(A) = \lambda_1\lambda_2\lambda_3 = -2\sigma b(r - 1) \quad (16)$$

To find the first eigenvalue, we must set the characteristic equation (13) as a function of λ:

$$f(\lambda) = \lambda^3 + (\sigma + b + 1)\lambda^2 + b(\sigma + r)\lambda + 2\sigma b(r - 1) \quad (17)$$

Notice that $f(0) = 2\sigma b(r - 1)$, and recall that σ and b are positive real numbers and r must be greater than 1 in order for the second and third equilibrium points to be real. Thus $f(0)$ returns a positive real number, and for all positive λ values, $f(\lambda) > 0$. To find a negative output for the function (17), we set:

$$f(-(\sigma+b+r))$$

$$(-(\sigma+b+r))^3+(\sigma+b+1)(-(\sigma+b+r))^2-b(\sigma+r)(\sigma+b+r)+2\sigma b(r-1)$$

$$-(\sigma+b+r)^2(r-1)-b(\sigma^2+\sigma b+2\sigma+br+r^2) \tag{18}$$

Since σ and b are positive real numbers and $r > 1$, we see that (18) is in fact less than zero. Using the Intermediate Value Theorem, we can conclude that there exists a value, λ_1, that satisfies $f(\lambda_1) = 0$, and it has to be a negative real number.

$$f(-(\sigma+b+r)) \leq f(\lambda_1) \leq f(0)$$

$$-(\sigma+b+r) \leq \lambda_1 \leq 0 \tag{19}$$

To see how the other two eigenvalues behave, we have to find the critical value of r, r_C, where λ_2 and λ_3 switch from being real to complex numbers. If we suppose that λ_2 and λ_3 are purely imaginary and conjugates of each other, we can see that:

$$\lambda_2+\lambda_3=0$$

$$tr(A)=\lambda_1=-(\sigma+b+1) \tag{20}$$

Plugging the eigenvalue from (20) back into the function of lambda (17), we can solve for the critical r value that makes the equations above true.

$$f(-(\sigma+b+1))=-b(\sigma+r_c)(\sigma+b+1)+2\sigma b(r_c-1)=0 \tag{21}$$

$$r_c=\frac{\sigma(\sigma+b+3)}{\sigma-b-1} \tag{22}$$

If we then map the equation (21) to a function of r, we can analyze how r affects λ.

$$g(r)=f(-(\sigma+b+1))=-b(\sigma+r)(\sigma+b+1)+2\sigma b(r-1)$$

We can also use the first derivative to show us the direction of the slope.

$$g'(r)=\sigma b-b^2-b$$

$$g'(r)=b(\sigma-b-1) \tag{23}$$

Notice that $g(r_C) = f(\lambda_1) = 0$ where $\lambda_1 = -(\sigma+b+1)$ and that r is inversely proportional to λ_1 from (21). Looking at equation (23), we know that σ has to be greater than $b+1$ in order for $g'(r) > 0$. Thus, as r increases past r_c, $g(r)$ also increases. Resulting in λ_1 decreasing beyond

$-(\sigma + b + 1)$, where $-(\sigma + b + r) \leq \lambda_1 \leq -(\sigma + b + 1)$, in order for $f(\lambda_1)$ to stay equal to zero.

Recall equation (15). If λ_1 is more negative than the trace, λ_2 and λ_3 have to satisfy $\lambda_2 + \lambda_3 > 0$ in order to balance out the trace. However, since we proved that the eigenvalues cannot be positive real numbers in equation (19), the only solutions for λ_2 and λ_3 is that they are complex conjugates with positive real parts. Thus when $r > r_c$, the equilibrium points are unstable spiral saddles. Conversely if $r < r_c$, λ_1 has to increase beyond $-(\sigma + b + 1)$, where $-(\sigma + b + 1) \leq \lambda_1 \leq 0$. As a result, λ_2 and λ_3 are negative real solutions, and the fixed points are stable.

Because the value of r can change the type of eigenvalues, there is another bifurcation. This particular one is called the Hopf bifurcation - where complex conjugate eigenvalues become purely imaginary [1]. Since our eigenvalues become purely imaginary at $r = r_c$ and then become complex conjugates at $r > r_c$, we can classify r_c as a subcritical Hopf bifurcation.

Our findings for constants σ, b and a varying r are summarized in Table 1.

	$0 < r < 1$	$1 < r$	$1 < r < r_c$	$r_c < r$
Fixed Point	$(0, 0, 0)$		$(\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1)$	
$\lambda_1, \lambda_2, \lambda_3$ (sign)	-,-,-	-,+,-	-,-,-	-,+,+
Real vs Complex	real, real, real	real, real, real	real, real, real	real, complex, complex
Stability	stable sink	unstable saddle	stable	unstable spiral saddle

Table 1: Fixed point characteristics related to r values.

Reproducing Lorenz' results

Let $\sigma = 10$, $b = 8/3$, and $r = 28$. Then, using equation (22), the critical value of r, r_c, can be calculated:

$$r_c = \frac{470}{19} \approx 24.7368$$

The eigenvalues $\lambda_{1,1}$, $\lambda_{1,2}$, and $\lambda_{1,3}$ for the first fixed point, (0,0,0), can be approximated to -8/3, -22.83, and 11.83 using equation (9):

$$\lambda_{1,1} = -8/3$$
$$\lambda_{1,2} = \tfrac{1}{2}(-11 + \sqrt{1201})$$
$$\lambda_{1,3} = \tfrac{1}{2}(-11 - \sqrt{1201})$$

Using Table 1, since $r > 1$ and there are two negative eigenvalues, we know there is an unstable saddle at (0,0,0).
For the second two fixed points, $(\pm\sqrt{b(r-1)},\ \pm\sqrt{b(r-1)},\ r-1)$, the eigenvalues can be found using a system derived from equations (13) and (14).

$$\lambda_1 + \lambda_2 + \lambda_3 = -(\sigma + b + 1)$$
$$\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = b(\sigma + r)$$
$$\lambda_1\lambda_2\lambda_3 = -2\sigma b(r-1)$$

Using the σ, b, and r values above, this system is:

$$\lambda_1 + \lambda_2 + \lambda_3 = -\tfrac{41}{3}$$
$$\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = \tfrac{304}{3}$$
$$\lambda_1\lambda_2\lambda_3 = -1440$$

Solving the system gives the eigenvalues $\lambda_{(2,3),1}$, $\lambda_{(2,3),2}$, and $\lambda_{(2,3),3}$ which were approximated using Wolfram Alpha [2].

$$\lambda_{(2,3),1} \approx -13.8546$$
$$\lambda_{(2,3),2} \approx 0.9396 + 10.1945i$$
$$\lambda_{(2,3),3} \approx 0.9396 - 10.1945i$$

Since there are two complex eigenvalues and one real eigenvalue, we know that there is an unstable spiral saddle at $(\pm\sqrt{b(r-1)},\ \pm\sqrt{b(r-1)},\ r-1)$ as seen in Table 1.

Using the improved-Euler's method in MATLAB, the original system (1) can be solved with the initial condition $(X(0),\ Y(0),\ Z(0)) = (0,1,0)$ when $t = 15$ and $dt = 1x10^{-7}$.

```
x(15) = -11.086959, y(15) = -5.701489, z(15) = 35.571307
```

Lorenz' original values can be replicated using a step size of $dt = 0.01$ [3]:

```
x(15) = -8.856067, y(15) = -3.065535, z(15) = 33.534128
```

The numerical solutions of the system can be seen plotted in Figure 1.

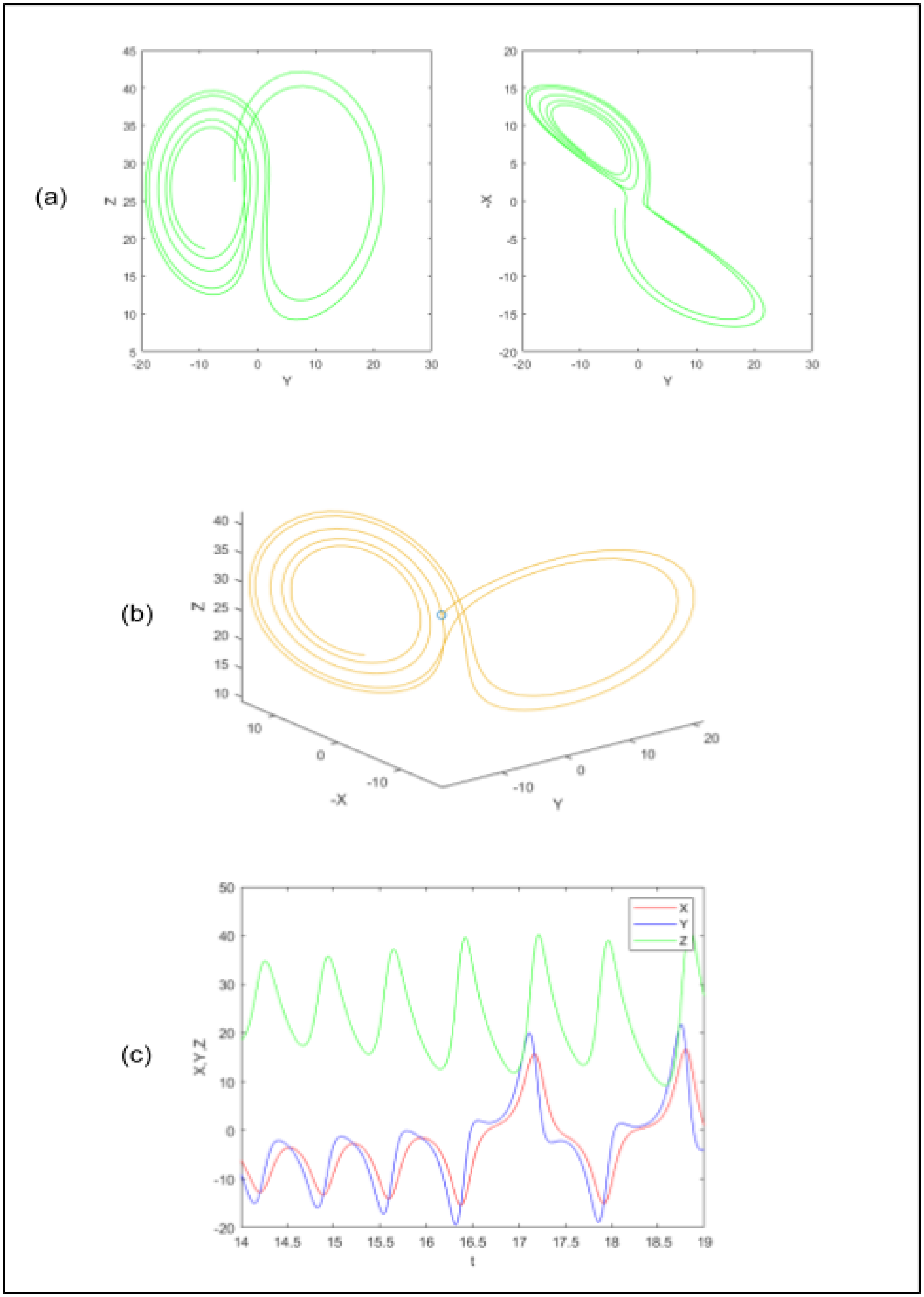

Figure 1: (a) Solutions to convection system (1) for $14 \le t \le 19$. (b) Solutions to convection system (1) in three dimensions for $14 \le t \le 19$. (c) X, Y, Z $vs.$ t for $14 \le t \le 19$.

Exploring System Behavior for Different Variables and Initial Conditions

Let $\sigma = 10$ and $b = 8/3$. When $r < 1$, all trajectories approach the origin of $(0,0,0)$ but never touch as $t \to \infty$. Thus, the origin can be seen as globally stable. At $r = 1$, a supercritical pitchfork bifurcation occurs. For $1 < r < r_c$, assuming $\sigma > b+1$, the system is stable and spirals towards its fixed point attractors. Once r reaches 24.7368, the value of r_c, the stability of the system is lost through a subcritical Hopf bifurcation, and the trajectory repels away from one non-origin fixed point to the other in an unstable spiral saddle, looping between the points sporadically and indefinitely.

Modifying the conditions for b and σ will change the characteristics of the system. When $\sigma > b+1$ and the values of b and σ are changed, the system will retain its characteristics but lose its stability at a different critical value of r_c as shown in equations (22) and (23). When $\sigma < b+1$, the value of r_c becomes negative. Although $r = 28$ is larger than r_c, the system is stable and spirals towards its fixed point attractors, behaving similarly to when $r < r_c$ and $\sigma > b+1$. This appears to be the case for all positive values of r. When $r = 1$, a supercritical pitchfork bifurcation occurs.

To visualize the sensitivity of the system (1) to slight changes in initial conditions for subcritical and supercritical r values, we prepared the graphs shown in Figure 2. For a subcritical value of $r = 10$, a slight change in initial conditions does not affect the final results at $t = 20$. In contrast, for a supercritical value of $r = 40$, a slight variation in initial conditions changes the final results significantly.

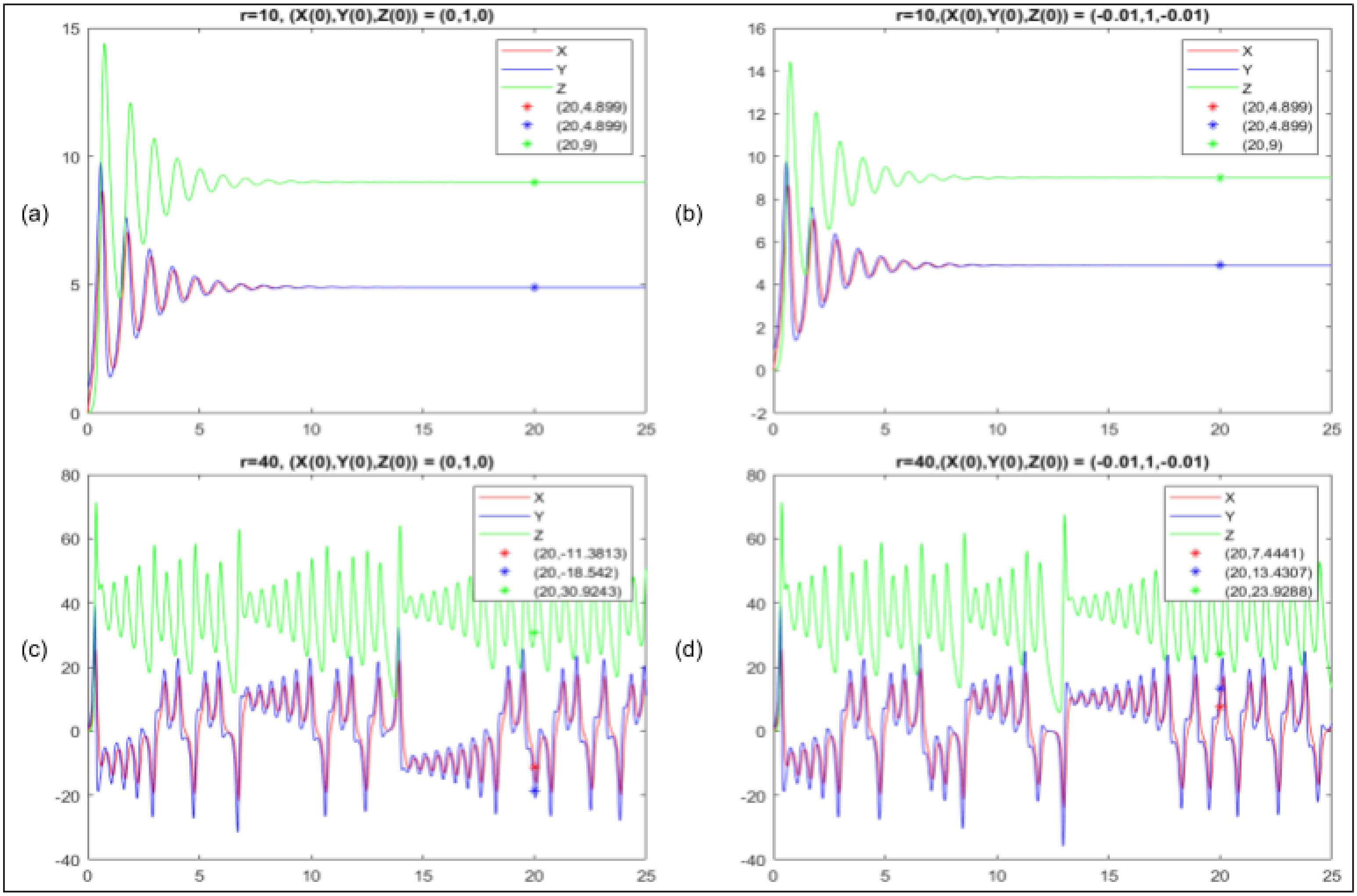

Figure 2: Parts (a)-(d) show the effect of a small change in initial conditions on (X, Y, Z) at $t = 20$ for subcritical ($r = 10$) and supercritical ($r = 40$) values of r.

The information in the Figure 2 is summarized in Table 2:

Type of r (subcritical or supercritical)	**Figures** (Figure 2)	**Difference in initial conditions** $(\Delta X(0),\ \Delta Y(0),\ \Delta Z(0))$	**Difference at t=20** $(\Delta X(20),\ \Delta Y(20),\ \Delta Z(20))$
subcritical ($r < r_c$)	(a) and (b)	$(-0.01,\ 0,\ -0.01)$	$(0,\ 0,\ 0)$
supercritical ($r > r_c$)	(c) and (d)	$(-0.01,\ 0,\ -0.01)$	(18.8254 , 31.9547, 6.9955)

Table 2: Difference in outcome related to r values at t=20.

This shows how, in supercritical conditions, a slight change in the initial condition produces a large difference in the final result. Thus, showing the existence of mathematical chaos.

References

[1] G. van der Heijden, Hopf Bifurcation https://www.ucl.ac.uk/~ucesgvd/hopf.pdf (Encyclopedia of Nonlinear Science, New York, 2004).

[2] Wolfram Research, Inc., https://www.wolframalpha.com/input/?i=solve+a%2Bb%2Bc%3D-41%2F3%2C+ab%2Bac%2Bbc%3D304%2F3%2C+abc%3D-1440 (Wolfram|Alpha Knowledgebase, Champaign, 2018).

[3] E. N. Lorenz, Deterministic Nonperiodic Flow, *Journal of the Atmospheric Sciences* **20**, 130–141 (1963).

Appendix

```
% project4.m
% Henry Chu, Katie Mueller, Ben Schaefer
% Ver 3.0
%
% Lorenz Equations
% dx/dt = -s*X + s*Y
% dy/dt = -X*Z + r*X - Y
% dz/dt = X*Y - b*Z

% clears command line, variables, and closes figures
clc, clear, close all;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% default variables
s       = 10;   % 10
b       = 8/3;  % 8/3
r       = 28;   % 28
dt      = .01;  % .01
stop    = 25;   % stopping time

% initial conditions
x0      = 0;
y0      = 1;
z0      = 0;

[x,y,z,t] = iemLorenz(s, b, r, dt, stop, x0, y0, z0);

plotAll(x,y,z,t, 'Our Results');

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Reproduction of Lorenz Graphs
figure('Name', "Lorenz' Results", 'NumberTitle', 'off');
tauInd = 15/dt + 1;   % add 1 because 0 index is 1
fprintf("x(15) = %f, y(15) = %f, z(15) = %f\n\n", x(tauInd), y(tauInd), z(tauInd));

% calculating indices
startTime = 14;
endTime = 19;
startInd = startTime/dt + 1;
endInd = endTime/dt + 1;

subplot(1,2,1);
plot(y(startInd:endInd), z(startInd:endInd), '-g');
xlabel("Y");
ylabel("Z");

subplot(1,2,2);
plot(y(startInd:endInd), -x(startInd:endInd), '-g');
xlabel("Y");
ylabel("-X")

figure('Name', "Lorenz Animation", 'NumberTitle', 'off');
comet3(y(startInd:endInd), -x(startInd:endInd), z(startInd:endInd), 0.1);
xlabel('Y');
ylabel('-X');
zlabel('Z');
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Part 3 - play with variable values
% call [x,y,z,t] = iemLorenz(sigma, b, r, dt, stopping time, x0, y0, z0)
% then plotAll(x,y,z,t, figure name)
% default values are above where
%   s = 10, b = 8/3, r = 28, dt = .01, stopping time = 25,
%   x0 = 0, y0 = 1, z0 = 0
%   rc = 24.7

%%% variable r & initial conditions
[x2,y2,z2,t2] = iemLorenz(s, b, r, dt, stop, -.01, y0, 0.002);
plotAll(x2,y2,z2,t2, 'r=28 w/Different Initial Conditions');

[x2,y2,z2,t2] = iemLorenz(s, b, 10, dt, stop, x0, y0, z0);
plotAll(x2,y2,z2,t2, 'r=10');

[x2,y2,z2,t2] = iemLorenz(s, b, 10, dt, stop, -.01, y0, 0.002);
plotAll(x2,y2,z2,t2, 'r=10 w/Different Initial Conditions');

%%% variable b
% [x2,y2,z2,t2] = iemLorenz(s, .1, r, .001, 100, x0, y0, z0);
% plotAll(x2,y2,z2,t2, 'b=.1');

% [x3,y3,z3,t3] = iemLorenz(s, 10, r, dt, stop, x0, y0, z0);
% plotAll(x3,y3,z3,t3, 'b=10');

%%% variable sigma
% might want to look at sigma < b+1
% [x2,y2,z2,t2] = iemLorenz(8/3, b, r, dt, 10, x0, y0, z0);
% plotAll(x2,y2,z2,t2, 's=8/3');

% [x3,y3,z3,t3] = iemLorenz(100, b, r, dt, stop, x0, y0, z0);
% plotAll(x3,y3,z3,t3, 's=100');

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function plotAll(x,y,z,t, name)
    % 4 graphs
    figure('Name', name, 'NumberTitle', 'off');
    subplot(2,2,1);
    % plot x,y,z vs t
    plot(t,x,'-r',t,y,'-b',t,z,'-g');
    title('X,Y,Z vs t');
    legend('X', 'Y', 'Z');

    subplot(2,2,2);
    plot(x,y,'-m');
    title('Y vs X');

    subplot(2,2,3);
    plot(x,z,'-y');
    title('Z vs X');

    subplot(2,2,4);
    plot(y,z,'-c');
    title('Z vs Y');
```

```
    % Animated 3D graph
    aniName = strcat(name, ' - Animated');
    figure('Name', aniName, 'NumberTitle', 'off');
    comet3(x,y,z, 0.1);
end

function [x,y,z,t] = iemLorenz(s, b, r, dt, stop, x0, y0, z0)

    % setup
    t   = (0:dt:stop);
    x   = zeros(size(t));
    y   = zeros(size(t));
    z   = zeros(size(t));
    x(1) = x0;
    y(1) = y0;
    z(1) = z0;

    % critical r
    rc  = (s*(s+b+3))/(s-b-1);

    % Improved Euler's Method
    for k = 2:length(t)
        % m = f(t(k-1), y(k-1))
        mx = -s*x(k-1) + s*y(k-1);
        my = -x(k-1)*z(k-1) + r*x(k-1) - y(k-1);
        mz = x(k-1)*y(k-1) - b*z(k-1);

        % Regular Euler's Method
        % y(k) = y(k-1) + m*dt
        x(k) = x(k-1) + mx*dt;
        y(k) = y(k-1) + my*dt;
        z(k) = z(k-1) + mz*dt;

        % n = f(t, Euler's Method)
        nx = -s*x(k) + s*y(k);
        ny = -x(k)*z(k) + r*x(k) - y(k);
        nz = x(k)*y(k) - b*z(k);

        % Result
        % y(k) = y(k-1) + dt/2*(m+n)
        x(k) = x(k-1) + dt/2*(mx+nx);
        y(k) = y(k-1) + dt/2*(my+ny);
        z(k) = z(k-1) + dt/2*(mz+nz);
    end

    fprintf('Critical r:    %f\n', rc);
    fprintf('Fixed Point 1: (0,0,0)\n');
    fprintf('Fixed Point 2: (%f, %f, %f)\n', sqrt(b*(r-1)), sqrt(b*(r-1)), r-1);
    fprintf('Fixed Point 3: (%f, %f, %f)\n', -sqrt(b*(r-1)), -sqrt(b*(r-1)), r-1);
    fprintf('Final value:   (%f, %f, %f)\n\n', x(length(t)), y(length(t)), z(length(t)));
end
```