Supplementary material for Lesson 4

1 Derivative of a Likelihood

Recall that the likelihood function is viewed as a function of the parameters θ and the data y are considered fixed. Thus, when we take the derivative of the log-likelihood function, it is with respect to θ only.

Example: Consider the normal likelihood where only the mean μ is unknown:

$$\begin{aligned} f(\boldsymbol{y}|\mu) &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2\sigma^2}(y_i-\mu)^2\right] \\ &= \frac{1}{(\sqrt{2\pi\sigma^2})^n} \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu)^2\right] \end{aligned}$$

which yields the log-likelihood

$$\begin{aligned} \ell(\mu) &= -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu)^2 \\ &= -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\left[\sum_{i=1}^{n}(y_i^2 - 2\mu y_i + \mu^2)\right] \\ &= -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\left[\sum_{i=1}^{n} y_i^2 - 2\mu\sum_{i=1}^{n} y_i + n\mu^2\right] . \end{aligned}$$

We take the derivative of $\ell(\mu)$ with respect to μ to obtain

$$\begin{aligned} \frac{d\ell(\mu)}{d\mu} &= \frac{2\sum_{i=1}^{n} y_i}{2\sigma^2} - \frac{2n\mu}{2\sigma^2} \\ &= \frac{\sum_{i=1}^{n} y_i}{\sigma^2} - \frac{n\mu}{\sigma^2} . \end{aligned} \tag{1}$$

If we set the expression in (1) equal to 0, we obtain the MLE $\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} y_i = \bar{y}$.

2 Products of Indicator Functions

Because $0 \cdot 1 = 0$, the product of indicator functions can be combined into a single indicator function with a modified condition.

Examples: $I_{\{x<5\}} \cdot I_{\{x\geq 0\}} = I_{\{0\leq x<5\}}$, and $\prod_{i=1}^{n} I_{\{x_i<2\}} = I_{\{x_i<2 \text{ for all } i\}} = I_{\{\max_i x_i<2\}}$.