Capital Asset Pricing Model and Arbitrage Pricing Theory

BUSS254 Investments

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Lecture Outline

- Capital Asset Pricing Model
- Arbitrage Pricing Theory
- Derivation and Extentions

The Capital Asset Pricing Model

Portfolio Theory: Review

- Portfolio risk depends primarily on covariances
  - Not stocks' individual volatilities
- Diversification reduces risk
  - But risk common to all firms cannot be diversified away
- Hold the tangency portfolio T
  - The tangency portfolio has the highest expected return for a given level of risk (i.e., the highest Sharpe ratio)

Equilibrium

- The demand for assets equals supply in equilibrium.
- Suppose all investors hold the same portfolio T; what must T be?
  - T must be the market portfolio
- Proxies for the market portfolio: S&P 500, Russell 2000, MSCI, etc.
  - Value-weighted portfolio of a broad cross-section of stocks

Equilibrium
Example

- Suppose there are three risky assets: A, B, and C. Suppose the tangecy portfolio is

$$\boldsymbol{w} = (w_A, w_B, w_C) = (0.25, 0.50, 0.25)$$

- There are three investors in the economy, 1, 2, and 3, with total wealth of 500, 1000, 1500 billion dollars, respectively. Their asset holdings are:

Investor	Riskless	A	B	C
1	100	100	200	100
2	200	200	400	200
3	-300	450	900	450
Total	0	750	1500	750

Equilibrium

Example

- In equilibrium, total dollar holdings of each asset must equal its market value.
  - Market capitalization of A =\$750
  - Market capitalization of B =\$1,500
  - Market capitalization of C =\$750

- The total market capitalization is $750 + 1,500 + 750 = \$3,000$

- The market portfolio is the tangency portfolio!

$$w_M = \left(\frac{750}{3,000}, \frac{1,500}{3,000}, \frac{750}{3,000} \right) = (0.25, 0.50, 0.25) = w_T$$

The Capital Asset Pricing Model

Implications of T as the Market Portfolio

- Efficient portfolios are combinations of the market portfolio and risk-free asset (e.g. T-bills)
- Expected returns of efficient portfolios satisfy:

$$E(r_p) = r_f + \frac{\sigma_p}{\sigma_m}\left[E(r_m) - r_f\right]$$

  - This yields the required rate of return or cost of capital for efficient portfolios.
  - Trade-off between risk and expected return
  - Multiplier is the ratio of portfolio risk to market risk
- What about other (non-efficient) portfolios?

The Capital Asset Pricing Model

Implications of T as the Market Portfolio

- For any asset, define its market beta as:

$$\beta_i = \frac{Cov(r_i, r_m)}{Var(r_m)} = \rho_{i,m} \frac{\sigma_i}{\sigma_m}$$

- Then the Sharpe-Lintner CAPM implies that (see Appendix for proof):

$$E(r_i) = r_f + \beta_i \left[E(r_m) - r_f \right]$$

  - Risk/reward relation is linear.
  - Beta is the correct measure of risk, not sigma (except for efficient portfolios)
  - Beta measures sensitivity of stock to market movements
  - Examples:
    - $\beta_i = 1 \Rightarrow E(r_i) = E(r_m)$
    - $\beta_i = 0 \Rightarrow E(r_i) = r_f$
    - $\beta_i < 1 \Rightarrow E(r_i) < r - f$

The Capital Asset Pricing Model

Other implications

- β is higher when:

$$\beta_i = \frac{Cov(r_i, r_m)}{Var(r_m)} = \rho_{i,m}\frac{\sigma_i}{\sigma_m}$$

- Remember: $y = \frac{E(r_P)-r_F}{A\sigma_p^2}$, proportion of wealth held in risky assets.
  - Now we have $y = \frac{E(r_M)-r_F}{A\sigma_M^2}$
  - The average of y across all investors is 1: if some borrow, others must have lent. The aggregate wealth in risk-free asset is 0.
  - Therefore, $1 = \overline{y} = \frac{E(r_M)-r_F}{\overline{A}\sigma_M^2}$ and $E(r_M) = r_F + \overline{A}\sigma_M^2$
  - Market risk premium is increasing in r_F, $\overline{A}$ and σ_M^2.

The Capital Asset Pricing Model

Example: Required Return

- Using monthly returns, you estimate that Microsoft's beta is 1.49 (std.err. = 0.18) and Gillette's beta is 0.81 (std.err. = 0.14). If these estimates are a reliable guide going forward, what expected rate of return should you require for holding each stock? $R_f = 0.05$ and $E(R_m) - R_f = 0.06$.

$$
\begin{aligned}
E(R_i) &= R_f + \beta_i \left[E(R_m) - R_f\right] \\
E(R_{GS}) &= 0.05 + 0.81 \times 0.06 = 9.86\% \\
E(R_{MSFT}) &= 0.05 + 1.49 \times 0.06 = 13.94\%
\end{aligned}
$$

- Expected return = Required rate of return = Opportunity cost of capital

The Capital Asset Pricing Model

Beta of a Portfolio

- Betas of arbitrary portfolios of stocks

$$
\begin{aligned}
R_p &= w_1 R_1 + ... + w_n R_n \\
Cov(R_p, R_m) &= Cov(w_1 R_1 + \cdots + w_n R_n, R_m) \\
&= w_1 Cov(R_1, R_m) + \cdots + w_n Cov(R_n, R_m) \\
\frac{Cov(R_p, R_m)}{Var(R_m)} &= w_1 \frac{Cov(R_1, R_m)}{Var(R_m)} + \cdots + w_n \frac{Cov(R_n, R_m)}{Var(R_m)} \\
\beta_p &= w_1 \beta_1 + \cdots + w_n \beta_n
\end{aligned}
$$

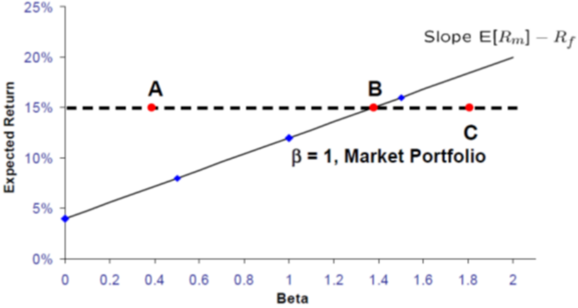
- Therefore,

$$
E(R_p) = R_f + \beta_p \left[E(R_m) - R_f \right]
$$

The Capital Asset Pricing Model

Example: Performance evaluation

- Suppose three mutual funds have the same average return of 15%
- Suppose all three funds have the same volatility of 20%
- Are all three managers equally talented?
- Are all three funds equally attractive?



The Capital Asset Pricing Model

Example: Performance evaluation

- Hedge fund XYZ had an average annualized return of 12.54% and a return standard deviation of 5.50%, and its estimated beta was -0.028. Did the manager exhibit positive performance ability according to the CAPM? If so, what was the manager's alpha? $R_f = 0.05$ and $E(R_m) - R_f = 0.06$.

$$
\begin{aligned}
E(R_i) &= R_f + \beta_i \left[E(R_m) - R_f\right] \\
E(R_{XYZ}) &= 0.05 + (-0.028) \times 0.06 = 4.83\% \\
\alpha_{XYX} &= R_i - E(R_{XYZ}) \\
&= 12.54\% - 4.85\% = 7.71\%
\end{aligned}
$$

Implementing the CAPM

- Security market line must be estimated.
- Identical to estimating index models: Ordinary least square regression
- One unknown parameter: β
- Given return history, β can be estimated by linear regression:

$$
\begin{aligned}
E(R_i) &= R_f + \beta_i \left[E(R_m) - R_f \right] \\
\text{Using history (sample): } R_i &= R_f + \beta_i (R_m - R_f) + \epsilon \\
R_i - R_f &= \alpha_i + \beta_i (R_m - R_f) + \epsilon
\end{aligned}
$$

- The CAPM implies that α_i should be zero.
- Caution: we are using the historical data!

Does the CAPM work?

Predictions

- Market beta determines the expected returns.
- The market is the only source of risk.
- Alpha should be zero for all assets.
- The SML slope = market risk premium.

Does the CAPM work?

Challenges

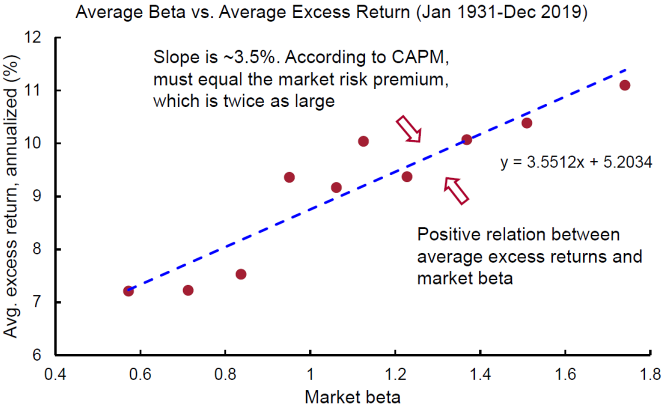
- But testing the CAPM is surprisingly difficult.[1]
  - All assets are not tradable. Cannot observe the market portfolio.
  - We have to estimate betas - estimation error.
  - Alphas and betas may be time-varying.
- Data, in general, does not support the CAPM

[1] The CAPM is formulated in terms of expectations (future values). Market model: $R_i = \alpha_i + \beta_i R_m + e_i$. Take expectation $E(R_i) = \alpha_i + \beta_i E(R_m) \Rightarrow E(R_i) - \alpha_i - \beta_i E(R_m) = 0$. Then $R_i = E(R_i) + \beta_i(R_m - E(R_m)) + e_i$. Plug into the CAPM: $R_i = R_f + \beta_i(R_m - R_f)$, which can be tested.

Does the CAPM work?

Black, Jensen, and Scholes (1972)

- Estimate individual stocks' beta using the past 60 months.
- Create ten portfolios by sorting on the estimated betas.
- For each portfolio, estimate beta and average excess-return.

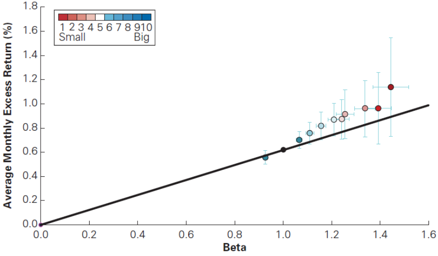


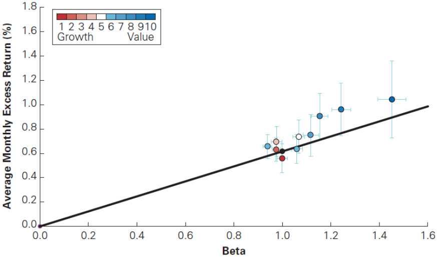
- Implied risk-free rate is much higher than in the data

Does the CAPM work?

Fama and French (1992)

- Firm characteristics seem to predict future returns.
  - Small stocks outperformed large stocks.
  - Stocks with low ratios of market-to-book value outperformed stocks with high ratios.

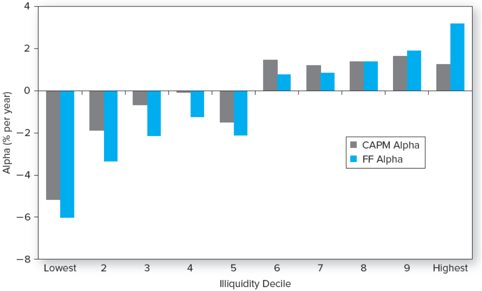




Does the CAPM work?

Pastor and Stambaugh (2003)

- Sort portfolios into deciles based on liquidity beta
- Compute the average alphas of the stocks in each decile using the CAPM and the Fama-French three-factor model.
  - Liquidity risk is a priced factor.
  - The risk premium is associated with it.



Multi-factor models may offer a better description of the risk-return relation?

Summary

- Yet, the central implication of the model is valid: Risk premia is proportional to exposure to systematic risk and independent of firm-specific risk
  - You are only compensated for market risk.
- CAPM still provides useful framework for applications
- Graham and Harvey (2000): 74% of firms use the CAPM to estimate the cost of capital
- Asset management industry uses CAPM for performance attribution
- Pension plan sponsors use CAPM for risk-budgeting and asset allocation

The Arbitrage Pricing Theory

Arbitrage Pricing Theory

- What if there are multiple sources of systematic risk?
- Consider the following a multi-index model:

$$r_i - r_f = a_i + b_{i1} I_1 + b_{i2} I_2 + ... + b_{iL} I_L + e_i$$

- Then the APT implies the following relation:

$$E(r_i) - r_f = b_{i1}[E(I_1) - r_f] + b_{i2}[E(I_2) - r_f] + ... + b_{iL}[E(I_L) - r_f]$$

- Cost of capital depends on L sources of systematic risk

Arbitrage Pricing Theory

Example

- Suppose two factor portfolios, 1 and 2, have expected returns $E(r_1) = 10\%$ and $E(r_2) = 12\%$ and that the risk-free rate is 4%.
- Now consider a well-diversified portfolio, A, with $\beta_{A1} = 0.5$ and $\beta_{A2} = 0.75$.
  - $E(r_A) = r_F + \beta_{A1} E(r_1 - r_F) + \beta_{A2} E(r_2 - r_F) = 0.13$
- Put differently, we can create another portfolio, Q, where long β_{A1} unit of factor portfolio 1 and β_{A2} unit of factor portfolio 2, and $1 - \beta_{A1} - \beta_{A2}$ unit in the risk-free asset.
  - $E(r_Q) = \beta_{A1} E(r_1) + \beta_{A2} E(r_2) + (1 - \beta_{A1} - \beta_{A2}) r_F$
  - Rearranging, we have $E(r_Q) = r_F + \beta_{A1} E(r_1 - r_F) + \beta_{A2} E(r_2 - r_F) = 0.13$
  - No arbitrage condition makes sure that A and Q have the same risk premium.

Arbitrage Pricing Theory

Fama-French 3-factor model

- According to the APT, Fama-French 3-factor model implies:

$$E(r_i) - r_f = b_{iM}[E(r_M) - r_f] + b_{iS}E(SMB) + b_{iH}E(HML)$$

	Single-Factor Model		Three-Factor Model	
	Regression Coefficient	*t*-Statistic	Regression Coefficient	*t*-Statistic
Intercept (alpha)	1.916%	2.065	1.494%	1.790
$r_M - r_f$	1.533	4.865	1.612	5.866
SMB			−0.689	−2.126
HML			−1.133	−3.304
R-square	.286		.455	
Residual std. dev.	6.864%		6.101%	

Estimates of single-index and three-factor Fama-French regressions for Amazon, monthly data, 5 years ending June 2018.

- Assume $r_f = 1\%, E(r_m) - r_f = 6\%, E(SMB) = E(HML) = 2\%$

$$3F : E(r_{Amazon}) = 1\% + (1.612 \times 6\%) + (-0.689 \times 2\%) + (-1.133 \times 2\%) = 7.028\%$$
$$1F : E(r_{Amazon}) = 1\% + (1.533 \times 6\%) = 10.198\%$$

- Amazon offers hedging against the size and value risk factors.
- Note that alpha is ignored. In equilibrium, alpha is zero.

Arbitrage Pricing Theory

- Strengths of the APT
  - Derivation does not require market equilibrium (only no-arbitrage)
  - Allows for multiple sources of systematic risk, which makes sense (assumes factor structure)[2]
- Weaknesses of the APT
  - No theory for what the factors should be
  - Assumption of linearity is quite restrictive Factor structure (index model) required.
  - Approximate relationship

[2] The CAPM can be also generalized to include multifactors. See Appendix 2.

APT in Practice

- The APT framework allows risk to be more tightly controlled, to protect against specific types of risk, or to make specific bets on certain types of risk.
- Assume four influences in the return-generating model
  - I_I: unexpected change in inflation
  - I_S: unexpected change in aggregate sales
  - I_O: unexpected change in oil prices
  - I_M: the return in the S&P index

$$E(r_i) - r_F = \lambda_I b_I + \lambda_S b_S + \lambda_O b_O + \lambda_M b_M$$

Factor	b	λ	Contribution to S&P Expected Excess Return (%)
Inflation	−0.37	−4.32	1.59
Sales growth	1.71	1.49	2.54
Oil prices	0.00	0.00	0.00
Market	1.00	3.96	3.96
Expected excess return for S&P index			8.09

APT in Practice

- Passive management
  - Index fund follows an index. Holding all stocks in the index is costly: e.g., small stocks are illiquid.
    - Suppose you want to follow S&P500. So, you select a portfolio of $N(< 500)$ stocks with $\beta_M = 1$.
    - But S&P500 also has $\beta_S = 1.71$. Your portfolio will fail to track the index if you do not properly control for this risk (when unexpected changes in sales growth are large).
    - You should try to form a portfolio of N stocks so that its β_S is close to 1.71.
    - This process can be done with multi-index model.
  - The APT provides additional insight.
    - $\lambda_I = -4.32$: Market accepts a lower return for an increase in β_I
    - If you want zero sensitivity to inflation, you expect $(-4.37)0.37 = 1.59$ decrease in expected return
    - APT allows you to make specified trade-offs between types of risk and expected returns.

APT in Practice

- Active management
  - If you believe that unexpected inflation will accelerate at a rate above that anticipated by the market ($I_I > 0$), then you can place a bet by increasing your exposure with inflation.
  - The more indexes included in the model, the more active bets you can make.
  - Example:
    - You believe sales will increase 1% more than expected.
    - So, you increase β_I from 1.71 to 2.21 by 0.5.
    - If sales indeed increase as you expected, your return will increase by $(2.21)1\% = 2.21\%$.
    - Of this, $0.5\% = 0.5(1\%)$ is due to your factor bet: excess risk-adjusted return.
    - The rest, 1.71%, would have arisen anyway.

APT in Practice

- Factor investing: active-passive approach
  - Strategy to capture the premiums that result from exposure to systematic risk factors.
  - In equilibrium, there is a positive expected rate of return in excess of the riskless rate associated with, e.g., pervasive market risk or exposure to other factors such as inflation risk.
  - An investor who is less sensitive to these risks may choose to have a higher exposure to them in return for a higher expected premium.
    - Endowments with longer horizons or funds that are naturally "hedged" against certain factors (e.g., a sovereign fund associated with an oil-producing country or liquidity)
  - Allocate across a set of factors with positive risk premia.
  - Factor investing does not seek to predict or "time" the variations in the factors
  - Represents a strategic allocation across a set of factors depending on the investor's risk appetite for exposure to factor risks
  - Challenge: identification of the factors and understanding the economics underlying the historical premia they have generated.

Appendix 1: Derivation of CAPM and APT

Assumptions of the CAPM

1. Individual behavior
   a. Investors are rational, mean-variance optimizers.
   b. Their common planning horizon is a single period.
   c. Investors all use identical input lists, an assumption often termed homogeneous expectations. Homogeneous expectations are consistent with the assumption that all relevant information is publicly available.
2. Market structure
   a. All assets are publicly held and trade on public exchanges.
   b. Investors can borrow or lend at a common risk-free rate, and they can take short positions on traded securities.
   c. No taxes.
   d. No transaction costs

Derivation of the CAPM

Derivation 1

- Key insights:
  - Everyone holds the market portfolio.
  - The market portfolio has the highest Sharpe ratio.
  - In equilibrium, all assets should offer the same Sharpe ratio.
- Consider a portfolio with $1-\alpha$ in security i and α in the market portfolio.

$$E(r_P) = \alpha E(r_M) + (1-\alpha)E(r_i)$$
$$\sigma_P^2 = \alpha^2\sigma_M^2 + (1-\alpha)^2\sigma_i^2 + 2\alpha(1-\alpha)\sigma_{i,M}$$

  - Portfolio P will go through M and i.
  - But it cannot cross the CML.
  - The portfolio frontier should be tangent to the CML at M, i.e., when $\alpha = 1$.

$$\left.\frac{dE(r_P)}{d\sigma_P}\right|_{\alpha=1} = \frac{E(r_M) - r_F}{\sigma_M}$$

Derivation of the CAPM

Derivation 1

$$\frac{dE(r_P)}{d\sigma_P} = \frac{dE(r_P)/d\alpha}{d\sigma_P/d\alpha}$$

$$\text{We have } \frac{dE(r_P)}{d\alpha} = E(r_M) - E(r_i)$$

$$\text{and } 2\sigma_P \frac{d\sigma_P}{d\alpha} = 2\alpha\sigma_M^2 - 2(1-\alpha)\sigma_i^2 + 2(1-2\alpha)\sigma_{i,M}$$

$$\text{Therefore, } \frac{dE(r_P)}{d\sigma_P} = \frac{[E(r_M) - E(r_i)]\sigma_p}{\alpha\sigma_M^2 - (1-\alpha)\sigma_i^2 + (1-2\alpha)\sigma_{i,M}}$$

$$\left.\frac{dE(r_P)}{d\sigma_P}\right|_{\alpha=1} = \frac{[E(r_M) - E(r_i)]\sigma_M}{\sigma_M^2 - \sigma_{i,M}} = \frac{E(r_M) - r_F}{\sigma_M}$$

$$\Rightarrow E(r_M) - E(r_i) = \frac{[E(r_M) - r_F](\sigma_M^2 - \sigma_{i,M})}{\sigma_M^2}$$

$$\Rightarrow E(r_M) - E(r_i) = [E(r_M) - r_F]\left(1 - \frac{\sigma_{i,M}}{\sigma_M^2}\right)$$

$$\boxed{E(r_i) = r_F + [E(r_M) - r_F]\frac{\sigma_{i,M}}{\sigma_M^2}}$$

Derivation of the CAPM

Derivation 2

- Asset i is evaluated within the market portfolio, where the weight on i is w_i.
  - The marginal contribution to the portfolio return :

$$\frac{dE(R_M)}{dw_i} = \frac{d[w_i(E(r_i) - r_F)]}{dw_i} = E(r_i) - r_F$$

  - The marginal contribution to the portfolio risk:

$$\frac{d\sigma_M^2}{dw_i} = \frac{d\left[w_i^2\sigma_i^2 + 2\sum_{j\neq i} w_i w_j \sigma_{i,j}\right]}{dw_i} = 2w_i\sigma_i^2 + 2\sum_{j\neq i} w_j\sigma_{i,j}$$

$$= 2\sum_j w_j\sigma_{i,j} = 2\sigma_{i,M}$$

$$\frac{d\sigma_M}{dw_i} = \frac{1}{2\sigma_M}\frac{d\sigma_M^2}{dw_i} = \frac{\sigma_{i,M}}{\sigma_M}$$

Derivation of the CAPM

Derivation 2

- Continued ..
  - The marginal contribution to the Sharpe ratio:

$$\frac{dE(R_M)/dw_i}{d\sigma_M/dw_i} = \frac{E(r_i) - r_F}{\sigma_{i,M}/\sigma_M}$$

  - The Sharpe ratio is the same for all risky assets:

$$\frac{E(r_i) - r_F}{\sigma_{i,M}/\sigma_M} = \frac{E(r_M) - r_F}{\sigma_{M,M}/\sigma_M}$$

$$\Rightarrow \boxed{E(r_i) - r_F = \frac{\sigma_{i,M}}{\sigma_M^2}(E(r_M) - r_F) = \beta_{i,M}(E(r_M) - r_F)}$$

Derivation of the APT

Derivation 1

- Assumptions:
  1. Security returns can be described by a factor model
  2. There are sufficient securities to diversify away idiosyncratic risk
  3. Well-functioning security markets do not allow arbitrage opportunities to persist.

Derivation of the APT

Derivation 1

- Assume a single index model:

$$R_i = \alpha_i + \beta_i F + e_i, \quad \text{where } E(F_i) = E(e_i) = 0^3$$

- Matrix form: $\boldsymbol{R} = \boldsymbol{\alpha} + \boldsymbol{\beta} F + \boldsymbol{e}$

$$\begin{pmatrix} R_1 \\ \vdots \\ R_N \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix} + \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_N \end{pmatrix} F + \begin{pmatrix} e_1 \\ \vdots \\ e_N \end{pmatrix}$$

- Construct a large portfolio, $A = (w_1, \cdots, w_n)$, such that
  - Zero net investment: $\boldsymbol{w}^T 1 = 0$
  - Zero firm specific risk: $\boldsymbol{w}^T \boldsymbol{e} = 0$
  - Zero exposure to F: $\boldsymbol{w}^T \boldsymbol{\beta} = 0$

$$R_A = \boldsymbol{w}^T \boldsymbol{\alpha} + \boldsymbol{w}^T \boldsymbol{\beta} F + \boldsymbol{w}^T \boldsymbol{e} = \boldsymbol{w}^T \boldsymbol{\alpha}$$

[3] Note that $F = f - E(f)$ so that $E(F) = 0$.

Derivation of the APT

Derivation 1

- No arbitrage assumption. The portfolio does not have any risk. The expected return is the risk-free rate, i.e., the risk premium is zero.

$$\boldsymbol{w}^T 1 = 0 \text{ and } \boldsymbol{w}^T \boldsymbol{\beta} = 0 \Rightarrow \boldsymbol{w}^T \boldsymbol{\alpha} = 0$$
$$\Rightarrow \boldsymbol{\alpha} = \gamma_1 1 + \gamma_2 \boldsymbol{\beta} \quad \text{for all assets}$$

- Risk-free asset: $r_F = \alpha_F = \gamma_1 1 + \gamma_2 \beta_F = \gamma_1$
- Market portfolio: $E(r_M) = \alpha_M = r_F 1 + \gamma_2 1 \Rightarrow \gamma_2 = E(r_M) - r_F$
- Therefore, $\boxed{\alpha_i = E(r_i) = r_F + (E(r_M) - r_F)\beta_i}$
- The CAPM is derived with few assumptions. But this is an approximate relation (only with a large number of assets)
- Can introduce multiple systematic risk factors (other than the market): credit risk, liquidity risk, exchange risk etc.

Derivation of the APT

Derivation 2

- Again, assume a single index model:[4]

$$R_i = \alpha_i + \beta_i F + e_i, \quad \text{where } E(e_i) = 0$$

- Construct a portfolio P, where you long i and short β_i unit of F. (Run a regression to find β_i)

$$R_P = R_i - \beta_i F = \alpha_i + e_i$$

- We removed the factor risk (called "portable alpha" portfolio)
- $E(R_P) = \alpha_i$ and $\sigma_P = \sigma_{e_i}$. Therefore, $SR_P = \alpha_i / \sigma_{e_i}$
- If the market is competitive, SR_P cannot be too large, i.e., bounded.
  - If $SR_P \leq C$, as $\sigma_{e_i} \to 0$, $\alpha_i \to 0$.
  - Therefore, with a large portfolio, α is close to 0.

    $E(R_i) = \beta_i E(F) \Rightarrow \boxed{E(r_i) = r_F + \beta_i (E(r_M) - r_F)}$

[4] Here, $F = f - r_F$, i.e., excess factor return.

Appendix 2: Assumptions and Extentions of the CAPM

Multiperiod Model: Inter-temporal CAPM

- Assumption 1(a): only the mean and variance of wealth matter to investors
  - \$1.1 million and oil price of \$400 vs. \$1 million and oil price of \$40 per barrel?
  - Depends on your energy consumption.
  - Investors care about the stream of consumption that wealth can buy for them.
  - The extra demand for assets that can be used to hedge these "extra-market risks" would increase their prices and reduce their risk premiums relative to the prediction of the CAPM. For example, energy stocks.

- Assumption 1(b): similar extra-market risk factors would arise in a multiperiod model
  - Consider a possible decline in future interest rates.
  - This could reduce the expected income.
  - Assets whose returns will be higher when interest rates fall (e.g., longterm bonds) would hedge this risk and thus command higher prices and lower risk premiums.
  - Such hedging demands affects the asset prices beyond what the CAPM predicts.

Multiperiod Model: Inter-temporal CAPM

- Suppose we can identify K sources of extra-market risk (e.g. interest rate risk, inflation risk) and find K associated hedge portfolios (e.g., bonds, energy stocks).
- Merton's Inter-temporal CAPM generalizes the SML to a multi-index version:

$$E(R_i) = \beta_{i,M} E(R_M) + \sum_{k=1}^{K} \beta_{i,k} E(R_k)$$

  - The risk premium for security i is the sum of the compensation it commands for all of the relevant risk sources to which it is exposed.
  - The first term is the usual risk premium for exposure to market risk.
  - The other terms are risk premiums for each source of extramarket risk times the security beta with respect to that risk source.
  - Thus, this expression generalizes the one-factor SML to a world with multiple sources of systematic risk. Just like the APT, but with based on economic theory.

Consumption CAPM

- Focus directly on consumption (Mark Rubinstein, Robert Lucas, and Douglas Breeden).
  - Ultimately, investors care about how market risk as well as extra-market risks would affect their consumption.
  - Suppose investors make decisions about how much to consume today and save for future (i.e., consume tomorrow).
  - Which asset are riskier?
  - Investors will value additional income more highly during difficult economic times.
  - An asset will therefore be viewed as riskier in terms of consumption if it has positive covariance with consumption growth

$$E(R_i) = \beta_{i,C} E(R_C)$$

  - C is a consumption-tracking portfolio.
  - The higher the consumption β, the higher the risk premium.
  - The model compactly incorporates hedging demands.

Liquidity CAPM

- Assumption 2(d) (no transaction costs) is not obviously true.
  - The liquidity of an asset is the ease and speed with which it can be sold at fair market value.
  - Liquidity cost: bid-ask spread, price impact, immediacy cost
  - A seller must accept the illiquidity discount

- Amihud and Mendelson (1982) focus on the effect of bid-ask spread on risk premium.
  - When trading costs are higher, the illiquidity discount will be greater. Then, the expected rate of return will be higher.
  - Less-liquid securities offer higher average rates of return.
  - The liquidity premium should increase with trading costs (measured by the bid–ask spread) at a decreasing rate.
    - But if an asset is less liquid, it will be held instead by longer-term traders who are less affected by high trading costs.
    - Hence, in equilibrium, investors with long holding periods will, on average, hold more of the illiquid securities

Liquidity CAPM

- So the expected level of liquidity can affect prices, and therefore expected rates of return.
- What about unanticipated changes in liquidity?
  - When liquidity in one stock decreases, it commonly tends to decrease in other stocks at the same time.
  - In other words, variation in liquidity has an important systematic component.
  - Investors demand compensation for exposure to liquidity risk (asset with higher liquidity beta).
  - The liquidity beta measures the sensitivity of a firm's returns to changes in market liquidity.
  - The extra expected return for bearing liquidity risk modifies the CAPM expected return–beta relationship.

References

- BKM, Chapters 9 and 10
- Andrew Lo's lecture note
- Elton, Gruber, Brown, and Goetzmann, Chapters 15 and 16