Derivatives: Part II

Investments

Prof. Ji-Woong Chung

Lecture Outline

- Financial derivatives: Characteristics
- Forwards and Futures
- Options
  - Pricing: Binomial and Black-Scholes-Merton model

Options: Basics

What Are Options?

Def. Option is a **contract** that gives an option-holder (or contract buyer) the **right to buy/sell** an asset in the future for a certain price.

- Comparison to forward
  - Option is similar to forward in that it is a contract to buy/sell an asset in the future.
  - A main difference is that the **buyer of option has the right** to buy/sell as opposed to the obligation in forward contract.
    - Thus, an option buyer can **choose** whether or not to exercise the contract.
  - In contrast, the **seller of option has the obligation** to follow what the buyer decides.

Options - Terminology

- Strike (or exercise) price: the promised price to trade.
- Expiration (or maturity) date: the last date that option can be exercised.
- Option price (or option premium): the price to buy the option contract
- Call vs. Put options
    1. Call option: Holder has the right to **buy** the underlying asset for a fixed price.
    2. Put option: Holder has the right to **sell** the underlying asset for a fixed price.
- European v.s. American options
    1. European option: Option can be exercised **only on the expiration date**.
    2. American option: Option can be exercised **at any time up to the expiration date**.

Investment Using Options - Example

- Suppose that we buy a February Microsoft put option with strike price of 195. Its option price is $9.75.

Q1. If stock price on the expiration date is 210, would you exercise? What is profit?

Q2. If stock price on the expiration date is 180, would you exercise? What is profit?

Option Payoff and Profit - General Result

- Consider a European call option with expiration date T and strike price K.
- The option payoff for a long position will depend on the future stock price S_T on the expiration date:

$$\begin{cases} \text{If } S_T \geq K, & S_T - K \\ \text{If } S_T < K, & 0 \end{cases}$$

- In short, the payoff for the long position in a European call is

$$\max(S_T - K, 0)$$

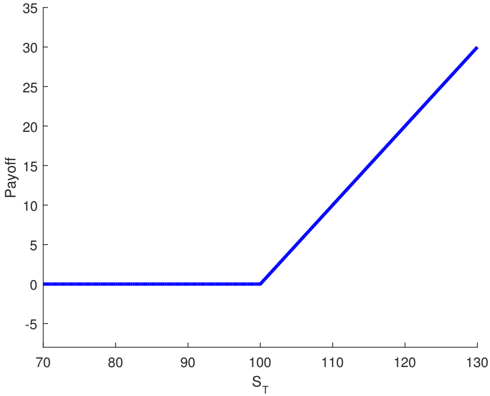
- The option profit is a gain/loss considering the initial cost:

$$\text{profit} = \text{payoff} - \text{option price}.$$

Option Payoff and Profit

e.g. Suppose that an investor buys a European call option on Microsoft shares with a strike price of \$100. The price of an option is \$5.

Payoff to a long position in the call

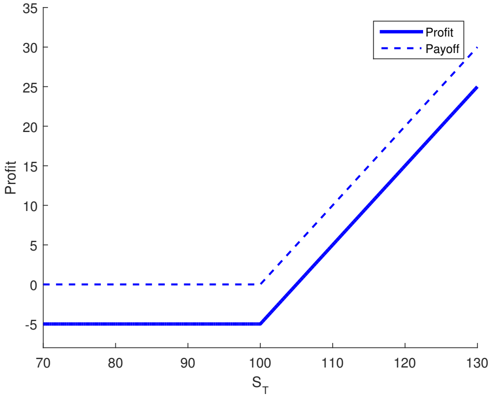


$$\text{Payoff} = \max(S_T - 100, 0)$$

Option Payoff and Profit

e.g. Suppose that an investor buys a European call option on Microsoft shares with a strike price of \$100. The price of an option is \$5.

Profit to a long position in the call



$$\text{Profit} = \max(S_T - 100, 0) - 5$$

Option Payoff and Profit

- Consider a European put option with expiration date T and strike price K.
- The option payoff for the long position will depend on the future stock price S_T on the expiration date:

$$\begin{cases} \text{If } S_T \geq K, & 0 \\ \text{If } S_T < K, & K - S_T \end{cases}$$

- In short, the payoff for the long position in a European put option is

$$\max(K - S_T, 0)$$

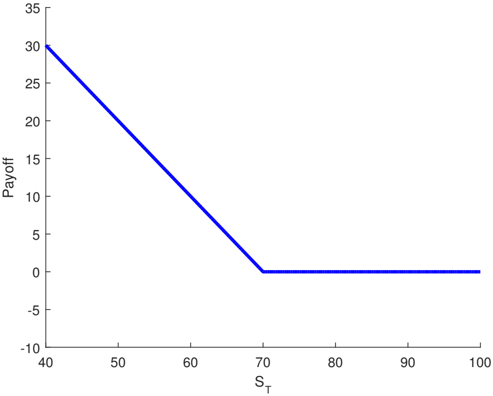
- The option profit is a gain/loss considering the initial cost:

$$\text{profit} = \text{payoff} - \text{option price}.$$

Option Payoff and Profit

e.g Suppose that an investor buys a European put option with a strike price of \$70 to sell IBM shares. The price of an option is \$7.

Payoff to a long position in the put option.

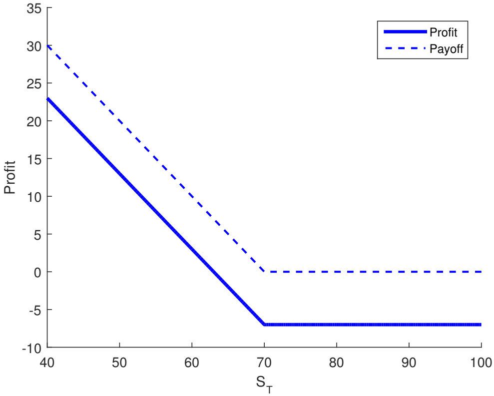


$$\text{Payoff} = \max(70 - S_T, 0)$$

Option Payoff and Profit

e.g Suppose that an investor buys a European put option with a strike price of \$70 to sell IBM shares. The price of an option is \$7.

Profit to a long position in the put option.



$$\text{Profit} = \max(70 - S_T, 0) - 7$$

Option Payoff and Profit

- We have looked at payoffs and profits for long positions in call/put options.
- Note that in profit diagrams, it is standard not to adjust for the time value of the upfront premium payment.
- The payoff for the short position is the opposite of long position.
- Payoff for the short position in a call option

$$-\max(S_T - K, 0) = \min(K - S_T, 0)$$

- Payoff for the short position in a put option

$$-\max(K - S_T, 0) = \min(S_T - K, 0)$$

More on Option Terminology

- In the money / At the money/ Out of the money
  - Options are referred to as in the money (out of the money), when they give positive (negative) payoff if exercised now.
  - They are at the money, when they give zero payoff if exercised now.

	call	put
$S_t > K$	in the money	out of the money
$S_t < K$	out of the money	in the money

- Intrinsic value

  = the maximum of zero and the payoff if options are exercised immediately.
  - Intrinsic value of call option $= \max(S_t - K, 0)$
  - Intrinsic value of put option $= \max(K - S_t, 0)$

Lower/Upper Bounds for Option Prices

Assumptions and Notation

- Assumptions
    1. There are no transaction costs.
    2. All trading profits are subject to the same tax rate.
    3. Borrowing and lending are possible at the risk-free interest rate.

- Notation
    - S_0: Current stock price
    - K: Strike price
    - T: Expiration date
    - S_T: Stock price at expiration
    - r: Risk-free interest rate (continuously compounding)
    - C_0: Value of American call option
    - P_0: Value of American put option
    - c_0: Value of European call option
    - p_0: Value of European put option

European vs American Options

- Consider an American option and a European option with the same strike prices, expiration dates, and underlying assets.

- The American option is always more valuable or as valuable as the European option.

$$c_0 \leq C_0 \quad \text{and} \quad p_0 \leq P_0$$

- This is because the owner of the American option has all exercise opportunities open to the owner of the European option and more.

- This relationships hold for all types of options irrespective of whether the underlying asset pays dividends or not.

Properties of Stock Options

- We consider lower/upper bounds and the put-call parity case by case.

1. Non-dividend-paying stock
   - European options
   - American options

2. Dividend-paying stock
   - Continuous dividend
     - European options
     - American options
   - Discrete dividend
     - European options
     - American options

Upper Bounds for Option Prices - Call

- For call options on non-dividend-paying stock,

$$c_0 \leq S_0 \quad \text{and} \quad C_0 \leq S_0$$

- Why?
  - Let's compare the stock and the European call in terms of time-T cash flow:

    $$\begin{array}{ll} \text{Stock :} & S_T \\ \text{European call :} & \max(S_T - K, 0) \end{array}$$

    Because $S_T \geq \max(S_T - K, 0)$, we conclude that $S_0 \geq c_0$.
  - We can obtain the same inequality by comparing time-t cash flows of American call and stock, where $t \, (t \leq T)$ is the time to exercise the call.

Upper Bounds for Option Prices - Put

- For put options on non-dividend-paying stock,

$$p_0 \leq Ke^{-rT} \quad \text{and} \quad P_0 \leq K$$

- Why?
  - Let's compare the bond and the European put in terms of time-T cash flow:

Bond that will pay K at T:	K
European put :	$\max(K - S_T, 0)$

    Because $K \geq \max(K - S_T, 0)$, we conclude that $Ke^{-rT} \geq p_0$.
  - We can obtain $Ke^{-rt} \geq P_0$ by comparing time-t cash flows of American put and a bond that will pay K at t. Because $0 \leq t \leq T$, we conclude that $K \geq P_0$.

Lower Bounds for Option Prices - European Call

- For a European call on a non-dividend-paying stock

$$c_0 \geq S_0 - Ke^{-rT}$$

- Why? Consider the following two portfolios:
  1. European call + bond that will pay K at T
  2. one share

- At the option expiration T, ① always generates larger cash flows than ②:
  1. $\max(S_T - K, 0) + K = \max(S_T, K)$
  2. S_T

- Hence, under no-arbitrage, current prices should satisfy

$$c_0 + Ke^{-rT} \geq S_0$$

Lower Bounds for Option Prices - European Call

- Combining the fact the option value cannot be negative,

$$c_0 \geq \max(S_0 - Ke^{-rT}, 0)$$

- If the above bound does not hold, an arbitrage exists.
- To make an arbitrage, we sell high and buy low.

Lower Bounds for Option Prices - European Call - Example

Q. Suppose that a call option with $K = \$18$, $r = 10\%$, and $T = 1$ is priced at \$3.00. The current stock price is $S_0 = \$20$. Is there an arbitrage opportunity?

$\Rightarrow$ To prevent any arbitrage, the call price should be between the lower and upper bounds. The upper bound is 20. The lower bound is

$$\max\left(20 - 18e^{-0.1}, 0\right) = 3.713$$

The option price is lower than the lower bound, so an arbitrage opportunity exists. The arbitrage strategy is

Action today	Cash flow now	Cash flow at T	
		$S_T \geq 18$	$S_T < 18$
long call	-3	$(S_T - 18)$	0
buy a bond	-18$e^{-0.1}$	18	18
sell a share (short-selling)	20	-S_T	-S_T
net	0.713	0	18-S_T

Lower Bounds for Option Prices - European Put

- For a European put on a non-dividend-paying stock

$$p_0 \geq Ke^{-rT} - S_0$$

- Why? Consider the following two portfolios:

  3. European put + one share
  4. bond that will pay K at T

- At the option expiration T, ③ always generates larger cash flows than ④:
  3. $\max(K - S_T, 0) + S_T = \max(K, S_T)$
  4. K

- Hence, under no-arbitrage, current prices should satisfy

$$p_0 + S_0 \geq Ke^{-rT}$$

Lower Bounds for Option Prices - European Put

- Combining the fact that the option value cannot be negative

$$p_0 \geq \max(Ke^{-rT} - S_0, 0)$$

Q. Consider a European put option with $K = \$40$, $T = 0.5$ years, when $S_0 = \$37$ and $r = 5\%$. The option price is \$1.00. Is there an arbitrage opportunity?

Put-Call Parity

[European Call and Put]

- European put and call options with the same strike price and the same expiration have a special relationship, **put-call parity**.

$$c_0 + Ke^{-rT} = p_0 + S_0$$

- This implies that the value of a European call (put) option can be deduced from the value of a European put (call).

- This relation always holds when there is no arbitrage. Note that this does not depend on any option pricing model (e.g. binomial model or Black-Scholes-Merton model).

Put-Call Parity - Derivation

[European Call and Put]

- Consider the previous portfolios:
  1. European call + bond that will pay K at T
  3. European put + one share
- At the option expiration T, ① and ③ always generate the same cash flows:
  1. $\max(S_T - K, 0) + K = \max(S_T, K)$
  3. $\max(K - S_T, 0) + S_T = \max(K, S_T)$
- Hence, under no-arbitrage, current prices should satisfy

$$c_0 + Ke^{-rT} = p_0 + S_0$$

Put-Call Parity - Example

[European Call and Put]

Q. Suppose that $S_0 = \$31$, $K = \$30$, $r = 10\%$, $T = 3$ month. The price of a European call is \$3 and the price of a European put is \$2.25. Is there an arbitrage opportunity? If so, find an arbitrage strategy and its profit.

Answer: Let's check whether the put-call parity holds:

1. $c_0 + Ke^{-rT} = 3 + 30e^{-0.1\times 3/12}$=32.26
3. $p_0 + S_0 = 2.25 + 31 = 33.25$

So, the portfolio 3 is overpriced relative to portfolio 1. Then, the arbitrage strategy is

Action today	Today	T	
		$S_T \geq 30$	$S_T < 30$
long call	-3	$(S_T - 30)$	0
buy a bond	-30$e^{-0.1\times 3/12}$	30	30
short put	2.25	0	$-(30 - S_T)$
sell share	31	$-S_T$	$-S_T$
net	0.99	0	0

American Options - Early Exercise

[Non-dividend-paying stock]

- Long position in an American option has the right to exercise earlier than the expiration.
- In a special case, when we long an **American call on a non-dividend-paying stock**, it is **never optimal to exercise early** before the expiration (for reason we will see below).
- Why is early exercise not optimal for this special case?
  - At time t, the option holder has the value

$$C_t = \max \begin{cases} \text{Value of waiting} & \text{if not exercise} \\ S_t - K & \text{if exercise} \end{cases}$$

American Options - Early Exercise

[Non-dividend-paying stock]

- If $C_t > S_t - K$, we can conclude that early exercise is not optimal.
- It turns out that the above inequality holds at any $0 \leq t < T$.
  - As an American call option is worth more or as much as a European call,
    $$C_t \geq c_t$$
  - From the put-call parity of European options, we have
    $$\begin{aligned} c_t &= p_t + S_t - Ke^{-r(T-t)} \\ &= p_t + (S_t - K) + K(1 - e^{-r(T-t)}). \end{aligned}$$
    Thus, $c_t > S_t - K$.
  - Hence, $C_t > S_t - K$. It's better to sell the call.

American Options - Early Exercise

[Non-dividend-paying stock]

- Consider the following two portfolios:
  1. American call + bond that will pay K at T
  2. One share

- At the option expiration T, ① always generates larger cash flows than ②:
  1. $\max(S_T - K, 0) + K = \max(S_T, K)$
  2. S_T

- Hence, under no-arbitrage, at $t < T$,

$$C_t + Ke^{-r(T-t)} \geq S_t \Rightarrow C_t \geq S_t - Ke^{-r(T-t)} \geq S_t - K$$

American Options - Early Exercise

[Non-dividend-paying stock]

- In the case of an American put option, sometimes it is optimal to exercise early.
  - $P_t \leq K$ because the maximum payoff from the option is $K - 0$.
  - Suppose $S_t = 0$, the payoff at t is K and we know $P_t \leq K$.
  - It's better to receive K earlier than later.
  - Hence, it can be desirable to exercise the option at $t \leq T$.

Lower Bounds for American Call

[Non-dividend-paying stock]

- We just proved that the option expiration is the only date that we may exercise an American call on non-dividend-paying stock.
- This means that the European and the American calls will deliver the same cash flows. Thus, $C_0 = c_0$.
- Hence, the lower bound of American call is the same as the lower bound of European call.

Lower Bounds for American Put

[Non-dividend-paying stock]

- For American put option, it is sometimes optimal to exercise early, in particular, when the option is deep in the money.

- At time t, the option holder has the value

$$P_t = \max \begin{cases} \text{Value of waiting} & \text{if not exercise} \\ K - S_t & \text{if exercise} \end{cases},$$

  Thus, we know $P_t \geq K - S_t$.

- Combining the fact that the put option price cannot be negative, the lower bound becomes

$$P_0 \geq \max(K - S_0, 0).$$

Put-Call Parity for American Options

- For American options on non-dividend-paying stocks, the put-call parity is

$$S_0 - K \leq C_0 - P_0 \leq S_0 - Ke^{-rT}$$

- Let's prove the right inequality first and then prove the left inequality.

Put-Call Parity for American Options - Right Inequality

- As $P_0 \geq p_0$, it follows that $C_0 - P_0 \leq C_0 - p_0$. Also, we know that $C_0 = c_0$ for non-dividend-paying stock. Thus,

$$C_0 - P_0 \leq C_0 - p_0 = c_0 - p_0$$

From the put-call parity for European options, we know that $c_0 - p_0 = S_0 - Ke^{-rT}$. Thus,

$$C_0 - P_0 \leq c_0 - p_0 = S_0 - Ke^{-rT},$$

which proves the right inequality.

Put-Call Parity for American Options - Left Inequality

- To prove the left inequality, we consider the following two portfolios:

portfolio A:	American call + bond worth K now
portfolio B:	American put + stock

- We want to prove that the value of the portfolio A is higher than or equal to the value of portfolio B. This will lead to the left inequality, $C_0 - P_0 \geq S_0 - K$.

- In derivation, we consider the two different cases:

  1. case 1: put option is exercised earlier than the expiration.

  2. case 2: put option is not early-exercised.

Put-Call Parity for American Options - Left Inequality - Case 1

- Suppose that the put option is exercised earlier than the expiration, say t ($0 \leq t < T$).
- Then, the portfolio value at time t is

$$\begin{array}{ll} \text{portfolio A:} & C_t + Ke^{rt} \\ \text{portfolio B:} & (K - S_t) + S_t \end{array}$$

- As $C_t \geq 0$ and $e^{rt} \geq 1$, we can conclude that the time-t value of portfolio A is higher than or equal to the value of portfolio B.

Put-Call Parity for American Options - Left Inequality - Case 2

- Suppose that the put option is NOT exercised earlier than the expiration. Then, the put option may or may not be exercised on the expiration T.

- If $S_T \geq K$ on the expiration T,

$$\begin{array}{ll} \text{portfolio A:} & (S_T - K) + Ke^{rT} \\ \text{portfolio B:} & 0 + S_T \end{array}$$

  Thus, the portfolio A has the higher value.

- If $S_T < K$ on the expiration T,

$$\begin{array}{ll} \text{portfolio A:} & 0 + Ke^{rT} \\ \text{portfolio B:} & K - S_T + S_T \end{array}$$

  Again, the portfolio A has the higher value.

Put-Call Parity for American Options - Left Inequality - Case 2

- Thus, we can conclude that the time-T value of portfolio A is higher than the value of portfolio B.

- $C_0 + K \geq P_0 + S_0 \Longrightarrow C_0 - P_0 \geq S_0 - K.$

Properties of Options on Dividend-Paying Stock

- Recall that the underlying asset's dividend payment affects option prices.
- Hence, the bounds and put-call parity should be modified for options on dividend-paying stock.
- For European options on dividend-paying stocks, we can find the result via a shortcut:
  - Starting from the results for non-dividend-paying stocks, we replace S_0 with the ex-dividend component.

Lower/Upper Bounds and Put-Call Parity for European Options - Discrete Dividends

- Suppose that the underlying assets pay discrete dividends. Let D denote the present value of futures dividends until the option expiration.

- For European call,

$$\max\left(S_0 - D - Ke^{-rT}, 0\right) \leq c_0 \leq S_0 - D$$

- For European put,

$$\max\left(Ke^{-rT} - (S_0 - D), 0\right) \leq p_0 \leq Ke^{-rT}$$

- Put-Call parity,

$$c_0 + Ke^{-rT} = p_0 + S_0 - D$$

Lower/Upper Bounds and Put-Call Parity for European Options - Discrete Dividends

Q. A European call option on a stock with $K = 20$ and $T = 3$ is priced at \$9. The current stock price is \$30, and the stock is expected to pay dividend of \$2 in $T = 1$ and $T = 2$. The risk-free interest rate is 3%. What is the price of a European put option with the same strike price and expiration date?

Lower/Upper Bounds and Put-Call Parity for European Options - Continuous Dividends

- Suppose that the underlying assets pay continuous dividends with the dividend yield q per annum.

- For European call,

$$\max\left(S_0 e^{-qT} - Ke^{-rT}, 0\right) \leq c_0 \leq S_0 e^{-qT}$$

- For European put,

$$\max\left(Ke^{-rT} - S_0 e^{-qT}, 0\right) \leq p_0 \leq Ke^{-rT}$$

- Put-Call parity,

$$c_0 + Ke^{-rT} = p_0 + S_0 e^{-qT}$$