Derivatives: Part I

Investments

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Lecture Outline

- Financial derivatives: Characteristics
- Forwards and Futures
- Options
  - Pricing: Binomial and Black-Scholes-Merton model

Derivatives: Basics

What is derivative?

Def. Derivative is a financial security whose value depends on other underlying variables.

- What can be the underlying variables?
  - Usually, the price of a traded assets (e.g, equities, interest rates, currencies, commodities etc.)
  - or some properties of asset prices (e.g, volatility etc.)
  - or some events (e.g., default, dividend payout etc.)
  - or weather (e.g. temperature, rainfall), inflation ...
- All variables should be measurable and observable.
- Type of derivatives
  - Futures, Forwards, Swaps, and Options

Where are derivatives?

- Stand-alone instruments
- Corporate securities - warrants, employee stock options
- Embedded in other securities - callable bonds, convertible bonds
- Securitization

:: Supply is fixed except stand-alone instruments.

Where to trade derivatives

1. Exchange-traded market
   - All orders to buy/sell are centralized in one place (via physically or electronically).
   - Contracts are standardized. Safety from counterparty default.
   - Futures and Options are traded.
2. Over-the-counter market
   - No central place to collect all orders
   - Participants contact each other directly or via dealers (a network of dealers).
   - Large institutions such as bank, hedge funds, and corporations are main participants.
   - Contracts are not standardized, so they can be negotiated between traders.
   - Forward, Swaps, Options, and other derivatives are traded.

Why are derivatives useful?

- Derivatives are a means to transfer risk from those that hold it to those that are most willing to bear it $\rightarrow$ excellent risk management tool
- Risk management is often about reducing risk, but it can also be about increasing attractive risk.
- Risk transfer via derivatives allows productive activities to be undertaken that otherwise might not be.
- There is also the potential for abuses, which regulations are designed to discourage.

Forwards and Futures

Futures - Contract Specifications

Def. Futures contract is an **agreement** to buy or sell an asset at a certain time in the future for a certain price.

- Underlying asset: the asset on which futures contract is based
- Futures price: the promised price to trade
- Expiration (or delivery) date: the promised date to trade
- Contract size: amount of asset that will be delivered under one contract (e.g. One futures contract on corn is to buy/sell 5,000 bushels).
- Long v.s. short position
  1. Long position: a trader who agrees to **buy** the underlying asset on futures
  2. Short position: a trader who agrees to **sell** the underlying asset on futures

Futures - Contract Specifications

- Entering long/short position of futures contract **costs nothing** (cf. buying stock or bond).
  - Except some collateral to be put up.
  - This is contrasted with buying/selling in option contracts.

- Futures price changes as a result of supply and demand.

  e.g. If there are more investors who want to buy corn for December delivery, then the futures price increases.

- Futures v.s. Spot price
  - Futures price is for **future** delivery.
  - Spot price is for **immediate** delivery.

Forward - Contract Specifications

Def. Forward contract is an **agreement** to buy or sell an asset at a certain time in the future for a certain price.

- Forward price: the promised price to trade
- How is forward different from futures?
  1. Forward is traded in the OTC market, while futures is traded in the exchange.
  2. Forward is settled only once on the delivery date, while futures is settled every day.

Payoff of Forward

- Forward and futures are very similar to each other, but a **forward** contract is simpler to analyze.

- Let F denote the forward price (the promised price to buy/sell at the contract expiration date T).

- Payoff of forward contract:

$$\begin{cases} \text{long position:} & S_T - F \\ \text{short position:} & F - S_T \end{cases}$$

  where S_T is the spot price of the underlying asset at the expiration.

- A forward contract is settled only once on the expiration date, so cash flow occurs only at the expiration.

Payoff of Futures

- Suppose we enter a **futures** contract that will expire on date T.

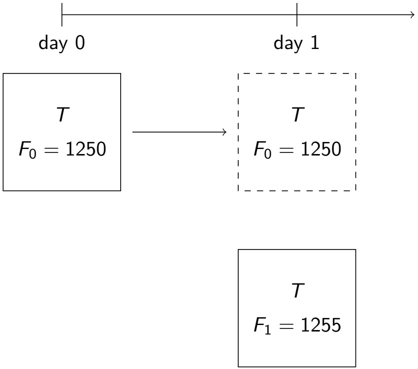
- Unlike forward, the futures will be settled every day.

e.g. Suppose we long futures on gold on day 0 when the futures price is \$1,250 per ounce.

- If later futures prices are as follows, what will be cash flows each day?

Day	Futures price	Daily gain
0	1,250	
1	1,255	?
2	1,248	?
3	1,242	?
⋮	⋮	⋮
T	F_T	?

Payoff of Futures - Closer Look at Day 1



- On day 1, new futures price turns out to be \$1,255.
- The exchange requires the investors from day 0 to abandon the old contract and move to the new one.
- In this settlement, the exchange pays or receives cash to compensate for the price difference.
  - When moving from the old ($F_0 = 1250$) to the new ($F_1 = 1255$), the long-position investor receives \$5.

Payoff of Futures

- On day t, the daily settlement for the long position is as follows.

  1. $$\begin{cases} \text{If } F_t \geq F_{t-1}, & \text{the investor receives } (F_t - F_{t-1}). \\ \text{If } F_t < F_{t-1}, & \text{the investor pays } (F_{t-1} - F_t). \end{cases}$$

     In sum, the daily gain for long position is $F_t - F_{t-1}$.

  2. The investor starts long position in a new contract with F_t.

e.g. Go back to the previous example. Daily gain/loss from the settlement is as follows.

Day	Futures price	Daily gain
0	1,250	
1	1,255	(1,255-1,250) = 5
2	1,248	(1,248-1,255) = -7
3	1,242	(1,242-1,248) = -6
⋮	⋮	⋮
T	F_T	$(F_T - F_{T-1})$

Payoff of Futures - Cumulative Gain

- Suppose we long futures with futures price F_0.
- The futures price on following days turns out to be F_1, F_2, F_3, ..., F_T.
- Assume that the risk-free rate is 0. Then, the cumulative gain from day 1 to day T is

$$
\begin{aligned}
&(F_1 - F_0) + (F_2 - F_1) + (F_3 - F_2) + \ldots + (F_T - F_{T-1}) \\
=&F_T - F_0 \\
=&S_T - F_0
\end{aligned}
$$

- This is the same as payoff for long position in forward contract.

Operation of Margin Accounts

- Recall that futures contracts are traded on the exchange.
- To prevent investors from defaulting on the contracts, the exchange requires investors to set up a margin account.
- When investors enter a position in futures, they are required to deposit **initial margin** (e.g. $3,000 per contract).
- Once the margin account is set up, the gain/loss from daily settlement of futures will be added to/subtracted from the account balance.

Operation of Margin Accounts

- As a result of daily settlements, the balance in the margin account changes time to time.
- During the contract period, investors are also required to maintain the balance at a certain level.
  - **Maintenance margin**: the **minimum** amount that must be maintained during the contract.
  - If the balance in the account falls below the maintenance margin, investors receive a **margin call** from exchange. Then, they need to top up the margin account up to the initial margin.

Operation of Margin Accounts - Example

- On day 0, we long a futures contract on gold at the futures price of \$1,250 per ounce. The contract size is 100 ounce per contract.
- Initial margin is \$3,000 and maintenance margin is \$2,000 per contract.

Day	Futures price	Daily gain	Margin account balance	Margin calls
0	1,250		3,000	
1	1,241	(1,241-1,250)$\times$100 = -900	2,100	
2	1,238	(1,238-1,241)$\times$100 = -300	1,800	1,200
3	1,244	(1,244-1,238)$\times$100 = 600	3,600	
4	1,242	(1,242-1,244)$\times$100 = -200	3,400	
$\vdots$	$\vdots$	$\vdots$		

Delivery of Futures

- There are two types of delivery of futures:
  1. Physical delivery: underlying assets are delivered physically (e.g. commodity)
  2. Cash settlements: final daily gain in futures is paid in cash (e.g. stock index)

- Physical delivery may incur additional costs.
  - storage costs
  - transportation costs
  - to feed and look after livestock

Market Quotes

- Example of futures price quotes

	Open	*High*	*Low*	*Settlement*	*Change*	*Volume*	*Open interest*
Gold 100 oz, $ per oz							
June 2010	1203.80	1216.90	1201.00	1213.40	15.40	194,461	156,156
July 2010	1205.00	1217.50	1202.00	1214.20	15.50	838	714
Aug. 2010	1205.00	1218.70	1202.70	1215.30	15.50	130,676	240,074
Oct. 2010	1208.30	1220.20	1205.30	1217.50	15.60	2,445	21,792
Dec. 2010	1208.80	1222.90	1207.50	1219.90	15.60	7,885	61,497
June 2011	1215.90	1228.00	1215.20	1227.80	15.80	408	13,461

- Prices
  - Open: the price at which contracts were trading at the beginning of the trading day
  - High: the highest price during the day
  - Low: the lowest price during the day
  - Settlement: the price used for calculating daily gain/loss (usually closing price of the day)
  - Open interest: the number of contracts outstanding
    - We count the total number of (net) long positions or (net) short positions for a certain contract.

Hedging Using Futures

- Hedgers participate in futures market to reduce a particular risk facing them (e.g, fluctuations in oil price, foreign exchange rate).

- To hedge a risk, hedgers take a futures position that neutralizes the risk as much as possible.

  1. Short hedge: a hedge that involves a short position in futures
     - when a hedger expects to **sell** an asset in the future

  2. Long hedge: a hedge that involves a long position in futures
     - when a hedger expects to **buy** an asset in the future

Short Hedge - Example

- In May, an oil producer enters a sales contract to sell 1 million barrels of crude oil. The price in the sales contract is the spot price on 15 August.
- Oil futures price for August delivery is \$79 per barrel, and each contract is for delivery of 1,000 barrels.

Q. To hedge the risk, what position on futures should the producer take?
$\Rightarrow$ short 1,000 futures contract.

Short Hedge - Example

What if the spot price of oil on 15 August turns out to be ...

1. \$75 per barrel

$$\text{Total revenue} = \underbrace{75 \times 1M}_{\text{sales contract}} + \underbrace{(79 - 75) \times 1M}_{\text{futures contract}} = 79M$$

2. \$85 per barrel

$$\text{Total revenue} = \underbrace{85 \times 1M}_{\text{sales contract}} + \underbrace{(79 - 85) \times 1M}_{\text{futures contract}} = 79M$$

Hedge - General Case

- Suppose that on date 0, we expect to sell asset A on date T.
- To hedge the risk, we short a certain futures contract.
- Total revenue on date T is

$$S_T + (F_0 - F_T)$$

- Depending on how well the futures contract fits the sales plan, the hedge becomes **perfect** or **imperfect**.

Perfect Hedge

- Perfect hedge means eliminating the risk **completely**, thus leaving no risk.
- The hedge using futures becomes perfect when all of the following conditions are satisfied.
  1. The asset whose price is to be hedged is the same as the asset underlying futures contract.
  2. The delivery date of futures contract is the same as the date to buy/sell the underlying asset.
- In this case, the total revenue is

$$S_T + (F_0 - F_T) = \underbrace{(S_T - F_T)}_{=0} + F_0$$

Imperfect Hedge

- Sometimes, we **cannot** find a futures with the perfect match.

- As a second-best way, we try using an alternative contract with the closest delivery month and on the most similar underlying asset.

- The total revenue is

$$S_T + (F_0 - F_T) = \underbrace{(S_T - F_T)}_{\neq 0} + F_0$$

- This does not eliminate the risk completely.

Cross Hedge

- **Cross hedge** is a case of imperfect hedge where we hedge the price risk of an asset using futures on a different underlying asset.

e.g. An airline that is concerned about the future price of jet fuel uses futures contract on heating oil.

- Hedge ratio $= \frac{\text{size of underlying assets in futures contract}}{\text{size of exposure}}$

  1. In perfect hedge, hedge ratio $= 1$
  2. In cross hedge, hedge ratio is usually not equal to one. Instead, we choose a particular ratio that will result in the best hedge.

Cross Hedge - Minimum Variance Hedge Ratio

- In cross hedge, the hedge ratio is chosen to **minimize the variance** of the value of the hedged position.
- Assume that we have one unit of asset A and shorts futures on h units of underlying asset B.
- The value of the hedging portfolio is ..
  - S_0 at time 0
  - $S_T + h(F_0 - F_T)$ at time T
- The change in the portfolio value is

$$\underbrace{S_T - S_0}_{\Delta S} - h \underbrace{(F_T - F_0)}_{\Delta F}$$

Cross Hedge - Minimum Variance Hedge Ratio

- The variance of the value change is

$$Var(\Delta S) - 2h \times Cov(\Delta S, \Delta F) + h^2 \times Var(\Delta F)$$

- We want to find h such that minimizes the variance.

- To do so, we calculate the derivative of the variance with respect to h and set it equal to 0:

$$h^* = \frac{Cov(\Delta S, \Delta F)}{Var(\Delta F)}$$

- h^* is the minimum variance hedge ratio.

Cross Hedge - Minimum Variance Hedge Ratio

- The hedge ratio can be rewritten as

$$h^* = \rho \frac{\sigma_S}{\sigma_F}$$

where

ρ: the correlation coefficient between ΔS and ΔF

σ_F: the standard deviation of ΔF

σ_S: the standard deviation of ΔS

Cross Hedge - Example

- An airline expects to purchase two million gallons of jet fuel in one month and decides to use heating oil futures for hedging. The standard deviation of futures price is $\sigma_F = 0.0313$, the standard deviation of jut fuel price is $\sigma_S = 0.0263$, and the correlation coefficient is $\rho = 0.928$.

Q1. What is the minimum variance hedge ratio?

Q2. Each of the futures contract is for 42,000 gallons of heating oil. How many contracts does the airline need?

Pricing Forward and Futures

Determination of Forward Prices - Basic Idea

- Investors enter a long or short position in forward contract **at zero cost**.
- In other words, the value of forward contract should be **zero** at the time of initiating the contract.
- Conversely, we can determine the forward price, so that the current value of forward contract becomes zero.

Determination of Forward Prices - Setting

- Assumptions
  - No transaction costs.
  - The market participants have the same tax rate on all net trading profits.
  - The market participants can borrow or lend money at the risk-free interest rate.
  - The market participants take advantage of arbitrage opportunities.
- Notation
  - T: delivery date of contract
  - S_0: spot price of the underlying asset today
  - S_T: spot price of the underlying asset at time T
  - F_0: forward price today
  - r: risk-free rate per annum (with continuous compounding)

Determination of Forward Prices

- Consider an underlying asset that pays no dividends. Its current price is S_0.

- What should be the forward price?

Determination of Forward Prices - Derivation

- Let's consider the following two portfolios:
  1. long forward with F_0 + buy a bond that will pay F_0 at T
  2. buy a stock
- At the contract maturity T, the two portfolios have the same cash flows:
  1. $(S_T - F_0) + F_0$
  2. S_T
- Thus, their current value should be the same:

$$0 + F_0 e^{-rT} = S_0$$

Determination of Forward Prices - Arbitrage

- What if

$$F_0 \neq S_0 e^{rT}?$$

$\Rightarrow$ An arbitrage exists.

e.g. Consider a 3-month forward contract on a stock whose current price is \$40. The 3-month risk-free interest rate is 5% per annum.

1. What if the forward price is 43 ($> 40e^{0.05\times 3/12}$)?

   $\Rightarrow$ There is an arbitrage:

Action	Cash flow in 0	Cash flow in 3 month
buy stock	-40	S_T
short forward	0	$43 - S_T$
sell bond	40	$-40e^{0.05\times 3/12}$
net	0	2.497

Determination of Forward Prices - Arbitrage

2. What if the forward price is 39 ($< 40e^{0.05\times 3/12}$)?

   $\Rightarrow$ There is another arbitrage strategy:

Action	Cash flow in 0	Cash flow in 3 month
sell stock (short selling)	40	$-S_T$
buy forward	0	$S_T - 39$
buy bond	-40	$40e^{0.05\times 3/12}$
net	0	1.503

Determination of Forward Prices - Example

Q. Consider a 1-year forward contract on a stock whose current price is \$50. The forward price is \$51, and the risk-free interest rate is 7% per annum. Is there an arbitrage? If so, show the arbitrage strategy.

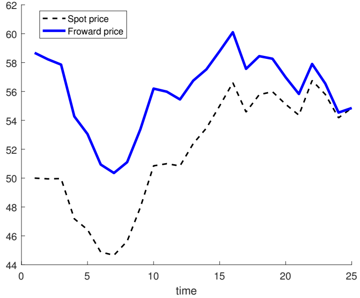
Action	Cash flow in 0	Cash flow in 1 year
net		

Forward and Spot Prices

- Consider a forward contract initiating at time t. Given the maturity date T, the forward price is
$$F_t = S_t e^{r(T-t)}$$

- Thus, the forward and spot prices are usually different. Only at the expiration, they become the same.

- Also, the forward price changes through time.



Dividend Payment and Forward Prices

- Until now, we have assumed that the underlying assets in forward do not pay any dividends.

- What if the underlying asset will pay dividends in the future? Are there changes in forward prices?

⇒ Yes, because...

  - The current price S_0 of the underlying asset includes the future dividends.

  - However, a long/short position in forward will not receive the dividends. Also, the forward payoff is determined by the ex-dividend price.

Determination of Forward Prices - Discrete Dividends

- We consider two different forms of dividend payments.
  1. Discrete dividends: dividends will be paid at certain points in time.
  2. Continuous dividends: dividends will be paid at every instant continuously.

- We first consider the case of discrete dividends.

- Suppose that stock pays dividends until the maturity T. The present value of all future dividends is I.

- The forward price is

$$F_0 = (S_0 - I)e^{rT}$$

Determination of Forward Prices - Discrete Dividends

- Why? Consider the following two portfolios:
  1. long forward with F_0 + buy a bond that will pay $F_0 + Ie^{rT}$ at T
  2. buy a stock

- At the contract maturity T, the two portfolios have the same cash flows:
  1. $(S_T - F_0) + F_0 + Ie^{rT}$
  2. $(S_T + Ie^{rT})$

- The portfolio values are the same at T. Thus, their current values are the same:

$$0 + F_0 e^{-rT} + I = S_0$$

Determination of Forward Prices - Discrete Dividends

Q1. Consider a 9-month forward contract on a corporate bond. The current price of the corporate bond is \$900, and it will pay \$40 coupon in 4 months. The 4-month and 9-month risk-free rates are 3% and 4%, respectively. If there is no arbitrage, what is the forward price?

Answer: The forward price is

$$(900 - 40e^{-0.03\times 4/12})e^{0.04\times 9/12} = 886.60$$

Determination of Forward Prices - Discrete Dividends

Q2. Consider the 9-month forward contract on the corporate bond in Q1. Suppose that the forward price is \$910. Is there an arbitrage? If so, show the arbitrage strategy.

Answer: $886.60 < 910$. Thus, we can think of the following arbitrage strategy:

Action	Cash flow in 0	Cash flow in 4 month	Cash flow in 9 month
buy corporate bond	-900	40	S_T
short forward	0	0	$910 - S_T$
sell 4-month bond	$40e^{-0.03\times 4/12}$	-40	0
sell 9-month bond	$910e^{-0.04\times 9/12}$	0	-910
net	22.707	0	0

Determination of Forward Price - Continuous Dividends

- Some securities pay continuous dividends (e.g, stock index, foreign currency).
  - Once we invest in a stock index, dividends from each individual stock will be paid at different points of time.
  - Having a lot of stocks in the index, we can approximate the index as paying dividends continuously.
- To simplify the argument, we assume that the dividends will be reinvested immediately to buy more shares.

Determination of Forward Price - Continuous Dividends

- What if the underlying asset pays continuous dividends with dividend yield q per annum?

- Forward price is

$$F_0 = S_0 e^{(r-q)T}$$

- Why? Consider the two portfolios:
  1. long forward with F_0 + buy a bond that will pay F_0 at T
  2. buy e^{-qT} share of stock

- The two portfolios will have the same cash flows at T:
  1. $(S_T - F_0) + F_0$
  2. $S_T e^{-qT} e^{qT}$

- Therefore, the two portfolios should have the same present values:

$$0 + F_0 e^{-rT} = S_0 e^{-qT}$$

Determination of Forward Price - Continuous Dividends - Foreign Currency

- If we hold a foreign currency, we receive interests that are paid continuously at the risk-free rate prevailing in the foreign country.

- Thus, foreign currency can be regarded as an asset with continuous dividends.

- Forward price is then

$$F_0 = S_0 e^{(r - r_f)T}$$

  where r_f is the foreign risk-free rate.

Determination of Forward Price - Continuous Dividends - Foreign Currency

Q1. Suppose that the 2-year interest rates in Hong Kong and the United Kingdom are 3% and 1%, respectively, and the spot exchange rate between the British Pound (GBP) and the Hong Kong Dollar (HKD) is 9.65 HKD per GBP. A 2-year forward exchange rate is 11.00. Is there an arbitrage for Hong Kong investors?

Answer: $11.00 > 9.65e^{(0.03-0.01)\times 2}$. Thus, there is an arbitrage. We can consider the following strategy:

Action	Cash flow now	Cash flow in 2 year
buy $e^{-0.01\times 2}$ GBP	$-9.65e^{-0.01\times 2}$	$e^{-0.01\times 2}e^{0.01\times 2}S_T$
short forward	0	$11.00 - S_T$
sell HK bond	$11.00e^{-0.03\times 2}$	-11.00
net	0.900	0

Determination of Forward Price - Commodities

- Storing commodities has costs and benefits
- Forward price with lump-sum storage cost U

$$F_0 = (S_0 + PV(U))e^{rT}$$

- Forward price with proportional storage cost u

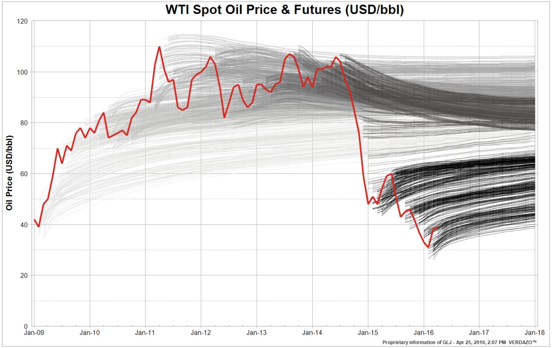
$$F_0 = S_0 e^{(r+u)T}$$

- Forward price with convenience yield y

$$F_0 = S_0 e^{(r-y)T}$$

The share of the forward curve

- Contango is a pattern of forward prices that increases with contract maturity
- Backwardation is a pattern of forward prices over time that decreases with contract maturity



Commodities that cannot be stored

- May be no storage or very limited storage life: electricity, lettuce, strawberries ...
- For non-storable commodities, forward prices can have information about future spot prices because no-arbitrage conditions don't hold.
  - Approach to pricing is to model stochastic future spot prices
  - Also must infer discount rates

Forward vs. Futures Prices

- For the same underlying asset and expiration, the futures and forward prices are very close to each other, but a bit different (due to daily settlement of futures).

- Compare cash flows between forward and futures for a long position:

Day	Forward	Futures
0		
1	0	$F_1 - F_0$
2	0	$F_2 - F_1$
$\vdots$	$\vdots$	$\vdots$
T	$S_T - F_0$	$S_T - F_{T-1}$

- When the risk-free rate is zero, the cumulative gain in futures is the same as the forward payoff. Thus, the forward and futures are the same in cash flows.

  $\Rightarrow$ Futures price = Forward price

Forward vs. Futures Prices

- When the risk-free rate is not zero, the cumulative gain in futures is different from the forward payoff.

Day	Forward	Futures	Interest Factor
0			
1	0	$F_1 - F_0$	$e^{r_1 \times (T-1)/365}$
2	0	$F_2 - F_1$	$e^{r_2 \times (T-2)/365}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
t	0	$F_t - F_{t-1}$	$e^{r_t \times (T-t)/365}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
T	$S_T - F_0$	$S_T - F_{T-1}$	$e^{r_T \times (T-T)/365}$

- Whether the cumulative gain in futures is larger/smaller than the forward payoff depends on the **correlation** between risk-free rate and spot price of underlying asset.

Forward vs. Futures Prices

1. What if the price of the underlying asset is **positively** correlated with the interest rate?

- For a long position, the gain on futures tend to be **larger** than the forward payoff. Why?
  - Suppose that $S_t > S_{t-1}$. Long position is likely to see daily gain ($F_t - F_{t-1} > 0$). This coincides with a larger interest factor due to a higher interest rate.
  - Suppose that $S_t < S_{t-1}$. Long position is likely to see daily loss ($F_t - F_{t-1} < 0$). This coincides with a smaller interest factor due to a lower interest rate.
- Thus, Futures price $>$ Forward price

Forward vs. Futures Prices

2. What if the price of the underlying asset is **negatively** correlated with the interest rate?

- For a long position, the gain on futures tend to be **smaller** than the forward payoff. Why?
  - Suppose that $S_t > S_{t-1}$. Long position is likely to see daily gain ($F_t - F_{t-1} > 0$). This coincides with a smaller interest factor due to a lower interest rate.
  - Suppose that $S_t < S_{t-1}$. Long position is likely to see daily loss ($F_t - F_{t-1} < 0$). This coincides with a larger interest factor due to a higher interest rate.
- Thus, Futures price $<$ Forward price