Portfolio Theory

BUSS254 Investments

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Lecture Outline

- Portfolio Theory: Overview
  - 2-asset case
  - N-asset case
- Derivation
- Index Model
- Black-Litterman model (optional)

Portfolio Theory: Overview

Portfolio

- A portfolio is simply a specific combination of securities.

$$\boldsymbol{w} = \begin{bmatrix} w_1 & w_2 & \dots & w_n \end{bmatrix},$$

where $w_i = N_i P_i / \sum N_j P_j$ for security i.

- Portfolio weights can sum to 0 (dollar-neutral portfolios), and weights can be positive (long positions) or negative (short positions).

Portfolio Theory

- How should one form a portfolio?
- Too broad a question
- As a theory, let's make simplifying assumptions
- And let's see what we can learn from the theory

Capital Allocation Decision

Utility function

- Which portfolio would you choose?

Portfolio	Risk premium	Expected return	Risk (Std.Dev.)
L	2%	7%	5%
M	4	9	10
H	8	13	20

- Investors prefer higher expected return and lower volatility.
- Assign a welfare score, or utility, to competing portfolios on the basis of the expected return and risk of those portfolios. For example,

$$U = E(r) - \frac{1}{2}A\sigma^2$$

- A: an index of the investor's risk aversion
  - More risk-averse investors (who have larger values of A) penalize risky investments more severely.
  - Risk-neutral $A = 0$, risk-tolerant $A < 0$, risk-averse $A > 0$.

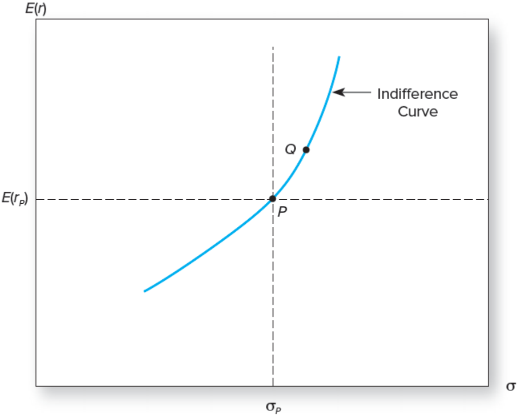
Capital Allocation Decision

Indifference Curve

- Portfolio $A \succ$ portfolio B if (one inequality is strict)

$$E(r_A) \geq E(r_B) \text{ and } \sigma_A \leq \sigma_B$$

- The indifference curve: all portfolio $(E(r), \sigma)$ points with the same U



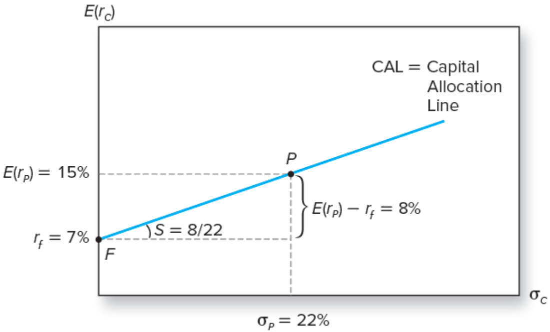
Capital Allocation Decision

Capital allocation line

- How to allocate capital between risk-free asset (F) and risky asset (P) to form a *complete* portfolio (C)?
- Suppose we invest $y\%$ in the risky portfolio. Then,

$$E(r_C) = yE(r_P) + (1-y)r_F = r_F + y[E(r_P) - r_F]$$
$$\text{and } \sigma_C = y\sigma_P$$

- Capital Allocation Line: the set of all feasible portfolios resulting from different values of y, i.e., $E(r_C) = r_F + \frac{[E(r_P)-r_F]}{\sigma_P}\sigma_C$

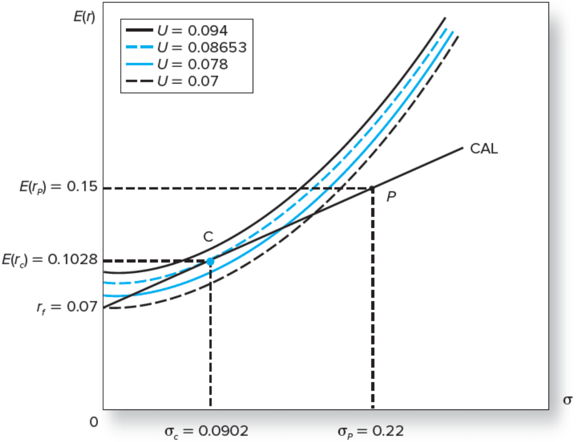


Capital Allocation Decision

Complete portfolio

- Individuals will choose a portfolio, C, that maximizes her utility.

$$\max_y E(r_C) - \frac{1}{2}A\sigma_C^2 = r_F + y[E(r_P) - r_F] - \frac{1}{2}Ay^2\sigma_P^2 \Rightarrow y^* = \frac{E(r_P) - r_F}{A\sigma_P^2}$$



When A=4

Risk and Reward

- Now, the question is how to form a portfolio of risky assets
- Assumptions
  - Investors like higher expected returns and dislike risk.
  - Investors care only about expected returns and volatility.
- How can we choose portfolio weights to optimize the risk/reward characteristics of the overall portfolio?

Mean-Variance Analysis

- Mean and variance for individual return of i:

$$
\begin{aligned}
E(r_i) &= \mu_i \\
var(r_i) &= E[(r_i - \mu_i)^2] = \sigma_i^2
\end{aligned}
$$

- Mean and variance for portfolio returns:

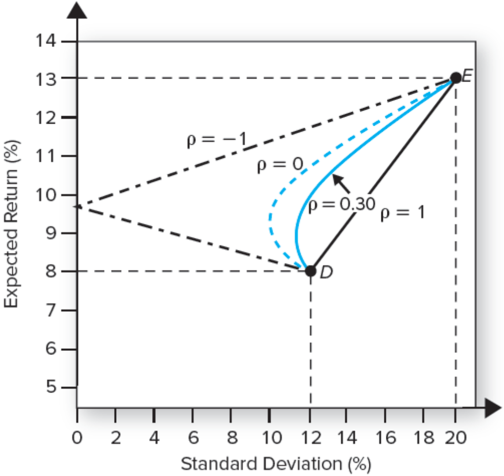
$$
\begin{aligned}
r_p &= w_1 r_1 + w_2 r_2 + \cdots + w_n r_n \\
E(r_p) &= w_1 \mu_1 + w_2 \mu_2 + \cdots + w_n \mu_n = \mu_p \\
var(r_p) &= E[(r_p - \mu_p)^2] \\
&= E[(w_1(r_1 - \mu_1) + ... + w_n(r_n - \mu_n))^2]
\end{aligned}
$$

- Portfolio variance is the weighted sum of all the variances and covariances.

$$
\sum E[w_i w_j (r_i - \mu_i)(r_j - \mu_j)] = \sum w_i w_j cov(r_i, r_j)
$$

Two Assets

- Excel spreasheet `"Portfolio Choice using Excel.xlsx"`



Two Assets

- Key lessons:
  1. $E(r_p) = \sum_{i=1}^{N} w_i E(r_i)$.
  2. $\sigma_p \leq \sum_{i=1}^{N} w_i \sigma_i$. The equality holds under perfect correlation.
  3. σ_p is smaller if correlation is lower.
  4. The mean-variance graph is non-linear.
  5. With a risk-free asset, portfolio plots along a straight line (capital allocation line).
  6. With a risk-free asset, one should hold the same portfolio of "risky assets" no matter what your tolerance for risk.

NB Markowitz's theory tells us nothing about where the prices, returns, variances or covariances come from.

N Assets

- Consider an equal-weighted portfolio, $w = 1/n$.
- Then

$$\begin{aligned}
\sigma_p^2 &= \sum_{i=1}^{n} \frac{\sigma_i^2}{n^2} + \frac{1}{n^2} \sum_{i \neq j} \sigma_{i,j} \\
&= \frac{1}{n}\text{Average variance} + \frac{n-1}{n}\text{Average covariance} \\
&\to \text{Average covariance as } n \uparrow \infty
\end{aligned}$$

- For portfolios with many stocks, the variance is determined by the average covariance among the stocks
  - In a well-diversified portfolio, covariances are more important than variances
  - A stock's covariance with other stocks determines its contribution to the portfolio's overall variance
  - Investors should care more about the risk that is common to many stocks; risks that are unique to each stock can be diversified away

N Assets

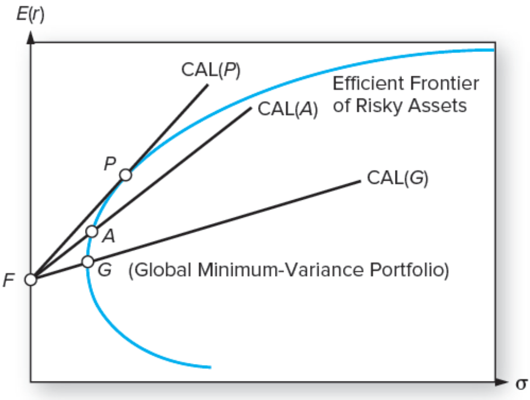
- The average stock has a monthly standard deviation of 10% and the average correlation between stocks is 0.40. If you invest the same amount in each stock, what is variance of the portfolio? What if the correlation is 0.0? 1.0?

$$\sigma_{i,j} = \rho\sigma_i\sigma_j = (0.40)(0.10)(0.10) = 0.004$$
$$\sigma_p^2 = \frac{1}{n}0.10^2 + \frac{n-1}{n}0.004 \approx 0.004 \text{ if } n \text{ is large}$$
$$\sigma_p = \sqrt{0.004} = 6.3\%$$

- Eventually, diversification benefits reach a limit.
- Remaining risk is known as systematic or market risk.
- Due to common factors that cannot be diversified.

N Assets

- Excel spreasheet "`Portfolio Choice using Excel.xlsx`"



N Assets

- Key steps:
  1. Derive a set of feasible portfolios (minimum variance boundary)
  2. The efficient frontier is the top half of the boundary
  3. Introduce a risk-free asset
  4. Derive the tangency portfolio: all investors should hold exactly the same stock portfolio
  5. The tangency portfolio has the highest possible Sharpe ratio of any portfolio.
  6. All efficient portfolios are combinations of the riskless asset and a unique portfolio of stocks, called the tangency portfolio.
  7. The efficient frontier becomes a straight line
  8. Capital allocation depends on personal preference

Summary

- Diversification reduces risk.
  - The standard deviation of a portfolio is less than the average standard deviation of the individual stocks in the portfolio.
- In diversified portfolios, covariances among stocks are more important than individual variances.
  - Only systematic risk matters.
- Investors should try to hold portfolios on the efficient frontier.
  - These portfolios maximize expected return for a given level of risk.
- With a riskless asset, all investors should hold the tangency portfolio.
  - This portfolio maximizes the trade-off between risk and expected return.

Portfolio Theory: Algebra

Assumptions

- We allow short selling.
  - With a short selling constraint, the algebraic derivation becomes more complicated. Numerically solved.
- We allow risk-free lending/borrowing
  - When there is no risk-free asset, we can "assume" two arbitary risk-free rates and find two optimal portfolios.
  - The efficient frontier is a set of all linear combinations of the two optimal portfolios (Merton, 1973)

Two Assets

Setting

- Asset 1: $E(r_1) = \mu_1$ and $var(r_1) = \sigma_1^2$
- Asset 2: $E(r_2) = \mu_2$ and $var(r_2) = \sigma_2^2$
- $\mu_1 \neq \mu_2$ and $\sigma_1^2 \neq \sigma_2^2$
- Correlation: ρ
- Portfolio: $(w, 1-w)$

$$\mu_p = w\mu_1 + (1-w)\mu_2$$
$$\sigma_p^2 = w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2\sigma_1\sigma_2\rho w(1-w)$$

Two Assets

Without risk-free asset

- Problem: minimize risk given level of expected return

$$
\begin{aligned}
\min_{w} \quad & w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2\sigma_1\sigma_2\rho w(1-w) \\
\text{s.t.} \quad & w\mu_1 + (1-w)\mu_2 = r \text{ and } w + (1-w) = 1
\end{aligned}
$$

- The solution to this problem is $w^* = \frac{r-\mu_2}{\mu_1-\mu 2}$. Substituting w^* into the objective function, the general equation for the relationship between portfolio risk and return is:

$$
\begin{aligned}
\sigma_r^2 &= \sigma_1^2\left(\frac{r-\mu_2}{\mu_1-\mu_2}\right)^2 + \sigma_2^2\left(\frac{\mu_1-r}{\mu_1-\mu_2}\right)^2 + 2\rho\sigma_1\sigma_2\left(\frac{r-\mu_2}{\mu_1-\mu_2}\right)\left(\frac{\mu_1-r}{\mu_1-\mu_2}\right) \\
&= ar^2 + br + c \qquad \text{(Hyperbola)}
\end{aligned}
$$

Two Assets

Without risk-free asset; Three special cases

$$\sigma_r^2 = \sigma_1^2 \left(\frac{r-\mu_2}{\mu_1-\mu_2}\right)^2 + \sigma_2^2 \left(\frac{\mu_1-r}{\mu_1-\mu_2}\right)^2 + 2\rho\sigma_1\sigma_2 \left(\frac{r-\mu_2}{\mu_1-\mu_2}\right)\left(\frac{\mu_1-r}{\mu_1-\mu_2}\right)$$

- Let $\alpha = \left(\frac{r-\mu_2}{\mu_1-\mu_2}\right)$
  1. When $\rho = 1$,
  $$\sigma_r^2(\rho=1) = \sigma_1^2\alpha^2 + \sigma_2^2(1-\alpha)^2 + 2\sigma_1\sigma_2\alpha(1-\alpha) = (\sigma_1\alpha + \sigma_2(1-\alpha))^2$$
  2. When $\rho = 0$,
  $$\sigma_r^2(\rho=0) = \sigma_1^2\alpha^2 + \sigma_2^2(1-\alpha)^2$$
  3. When $\rho = -1$,
  $$\sigma_r^2(\rho=-1) = \sigma_1^2\alpha^2 + \sigma_2^2(1-\alpha)^2 - 2\sigma_1\sigma_2\alpha(1-\alpha) = (\sigma_1\alpha - \sigma_2(1-\alpha))^2$$
- Therefore,
$$\sigma_r^2(\rho=1) \geq \sigma_r^2(\rho=0) \geq \sigma_r^2(\rho=-1)$$

N Assets

Without risk-free asset: Problem

- Problem

$$\min_{w} \; \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}w_iw_j$$

$$\text{s.t.} \quad \sum_{i=1}^{n}\mu_iw_i = r \text{ and } \sum_{i=1}^{n}w_i = 1$$

- Form the Lagrangian with Lagrange multipliers u and v

$$\mathcal{L} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}w_iw_j - v\left(\sum_{i=1}^{n}\mu_iw_i - r\right) - u\left(\sum_{i=1}^{n}w_i - 1\right)$$

- Setting $\partial\mathcal{L}/\partial w_i = 0$ for all i, $\partial\mathcal{L}/\partial v = 0$, and $\partial\mathcal{L}/\partial u = 0$

$$\sum_{j=1}^{n}\sigma_{ij}w_i - v\mu_i - u = 0$$

- Can solve the $n+2$ equations in $n+2$ variables.

N Assets

Without risk-free asset: Solution with target r

- We have a system of linear equations. Matrix formulation:

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} & \mu_1 & 1 \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} & \mu_2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \sigma_{21} & \sigma_{n2} & \cdots & \sigma_{nn} & \mu_n & 1 \\ \mu_1 & \mu_2 & \cdots & \mu_n & 0 & 0 \\ 1 & 1 & \cdots & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_1 \\ \vdots \\ w_n \\ -v \\ -u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ r_p \\ 1 \end{bmatrix}$$

$$Cx = b \longrightarrow x = C^{-1}b$$

- As long as C is invertible.

N Assets

Without risk-free asset: Two fund theorem

- Suppose:
  - $\boldsymbol{x}^{(1)}$ is optimal for r_1 with (v_1, u_1)
  - $\boldsymbol{x}^{(2)}$ is optimal for r_2 with (v_2, u_2)
- Consider
  - A return $(1-\beta)r_1 + \beta r_2 = r$, i.e., $\beta = \frac{(r-r_1)}{(r_2-r_1)}$.
  - A portfolio $\boldsymbol{y} = (1-\beta)\boldsymbol{x}^{(1)} + \beta \boldsymbol{x}^{(2)}$.
  - Set $v = (1-\beta)v_1 + \beta v_2$ and $u = (1-\beta)u_1 + \beta u_2$.

$$
\begin{aligned}
&\sum_{j=1}^{n} \sigma_{ij} y_i - v\mu_i - u \\
&= (1-\beta)\left(\sum_{j=1}^{n} \sigma_{ij} x_i^{(1)} - v_1\mu_i - u_1\right) + \beta\left(\sum_{j=1}^{n} \sigma_{ij} x_i^{(2)} - v_2\mu_i - u_2\right) = 0
\end{aligned}
$$

- $\boldsymbol{y}$ satisfies the FOC and is optimal for r.
- **Two fund theorem**: Any efficient portfolio is a linear comibnation of two different efficient portfolios.

N Assets

Without risk-free asset: Efficient Frontier

- The optimal portfolio for target return r.

$$
\begin{aligned}
\boldsymbol{y}^* &= \left(\frac{r_2 - r}{r_2 - r_1}\right) \boldsymbol{x}^{(1)} + \left(\frac{r - r_1}{r_2 - r_1}\right) \boldsymbol{x}^{(2)} \\
&= r\left(\frac{\boldsymbol{x}^{(2)} - \boldsymbol{x}^{(1)}}{r_2 - r_1}\right) + \left(\frac{r_2 \boldsymbol{x}^{(1)} - r_1 \boldsymbol{x}^{(2)}}{r_2 - r_1}\right) \\
y_i^* &= r g_i + h_i \text{ for each } i
\end{aligned}
$$

- Therefore,

$$
\begin{aligned}
\sigma_r^2 &= \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} (r g_i + h_i)(r g_j + h_j) \\
&= r^2 \left(\sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} g_i g_j \right) + 2r \left(\sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} g_i + h_j \right) + \left(\sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} h_i h_j \right)
\end{aligned}
$$

- The n-asset frontier has the same structure as the 2-asset frontier (Hyperbola).

N Assets

With risk-free asset: Problem

- Introduce a risk-free asset, yielding r_f. The weight on the risk-free asset, w_0.
- The mean-variance problem

$$\max \left(r_f w_0 + \sum_{i=1}^{n} \mu_i w_i \right) - \tau \left(\sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} w_i w_j \right)$$

$$\text{s.t. } w_0 + \sum_{i=1}^{n} w_i = 1$$

$$\Rightarrow \max r_f + \sum_{i=1}^{n} (\mu_i - r_f) w_i - \tau \left(\sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} w_i w_j \right)$$

Let $\hat{\mu}_i = \mu - r_f$, excess return of asset i.

N Assets

With risk-free asset: Solution

- The optimal portfolio satisfies:

$$\hat{\mu}_i - 2\tau \sum_{j=1}^{n} \sigma_{ij} w_j = 0, \quad \forall i$$

- Matrix formulation

$$\begin{bmatrix} \hat{\mu}_1 \\ \hat{\mu}_2 \\ \vdots \\ \hat{\mu}_n \end{bmatrix} - 2\tau \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\Rightarrow \hat{\boldsymbol{\mu}} - 2\tau \boldsymbol{\Sigma} \boldsymbol{w} = 0$$

$$\Rightarrow \boldsymbol{w}(\tau) = \frac{1}{2\tau} \boldsymbol{\Sigma}^{-1} \hat{\boldsymbol{\mu}}$$

N Assets

With risk-free asset: One fund theorem

- Note that $\boldsymbol{w}(\tau)^T \boldsymbol{w}(\tau) \neq 1$
- Divide $\boldsymbol{w}$ by the sum of its component.

$$s^* = \left(\frac{1}{2\tau} 1^T \boldsymbol{\Sigma}^{-1} \hat{\boldsymbol{\mu}} \right)^{-1} \frac{1}{2\tau} \boldsymbol{\Sigma}^{-1} \hat{\boldsymbol{\mu}} = \left(1^T \boldsymbol{\Sigma}^{-1} \hat{\boldsymbol{\mu}} \right)^{-1} \boldsymbol{\Sigma}^{-1} \hat{\boldsymbol{\mu}}$$

- s^* is independent of τ!
- **One-fund theorem**: All efficient portfolios in a market with a risk-free asset are linear combination of the risk-free asset and s^*, a straight-line.
- Invest w_0 in risk-free and $1 - w_0$ in s^*.
  - $\mu_p = w_0 r_f + (1 - w_0)\mu_s^*$ and $\sigma_p = (1 - w_0)\sigma_s^*$
  - $\mu_p = r_f + \left(\frac{\mu_s^* - r_f}{\sigma_s^*} \right) \sigma_p$ (Efficient frontier - a line)
- s^* is the tangency portfolio.
- s^* is the portfolio that maximizes the Sharpe ratio.

N Assets

Without risk-free asset: Efficient frontier

- We already saw that the efficient frontier without a risk-free asset is a hyperbola shape.
- Alternatively,
  - Assume two risk-free assets with $r_f^{(1)}$ and $r_f^{(2)}$.
  - Derivate two optimal portfolios, $s^{*(1)}$ and $s^{*(2)}$
  - The efficient frontier without a risk-free asset is a set of all linear combinations of $s^{*(1)}$ and $s^{*(2)}$ (Two-fund Theorem)
- We can, of course, solve the original optimization problem.

$$\min \ \frac{1}{2}\boldsymbol{w}^T\boldsymbol{\Sigma}\boldsymbol{w}$$
$$\text{s.t. } w^T\boldsymbol{\mu} = r \text{ and } w^T 1 = 1$$

N Assets

Alternative derivation: With risk-free asset

- Alternative derivation: with risk-free asset

$$\min \ \frac{1}{2}\boldsymbol{w}^T \boldsymbol{\Sigma} \boldsymbol{w}$$
$$\text{s.t. } w^T \hat{\boldsymbol{\mu}} = r$$

$$\mathcal{L} = \frac{1}{2}\boldsymbol{w}^T \boldsymbol{\Sigma} \boldsymbol{w} - u\left(w^T \hat{\boldsymbol{\mu}} - r\right)$$
$$\text{FOC: } \boldsymbol{\Sigma}\boldsymbol{w} = u\hat{\boldsymbol{\mu}} \Rightarrow \boldsymbol{w} = u\boldsymbol{\Sigma}^{-1}\hat{\boldsymbol{\mu}}$$
$$\text{Since } 1^T \boldsymbol{w} = 1,\ u = \frac{1}{1^T \boldsymbol{\Sigma}^{-1}\hat{\boldsymbol{\mu}}}$$
$$\Rightarrow \boldsymbol{w} = (1^T \boldsymbol{\Sigma}^{-1}\hat{\boldsymbol{\mu}})^{-1}\boldsymbol{\Sigma}^{-1}\hat{\boldsymbol{\mu}}$$

- We can also find the portfolio of risky assets by maximizing Sharp ratio = Tangency portfolio

N Assets

Global Minimum Variance Portfolio: Without risk-free asset

- Global Minimum Variance Portfolio

$$\min \ \frac{1}{2}\boldsymbol{w}^T\boldsymbol{\Sigma}\boldsymbol{w}$$
$$\text{s.t. } w^T 1 = 1$$

$$\mathcal{L} = \frac{1}{2}\boldsymbol{w}^T\boldsymbol{\Sigma}\boldsymbol{w} - u\left(w^T 1 - 1\right)$$
$$\text{FOC: } \boldsymbol{\Sigma}\boldsymbol{w} = u1 \Rightarrow \boldsymbol{w} = u\boldsymbol{\Sigma}^{-1}1$$
$$\text{Since } 1^T\boldsymbol{w} = 1, \ u = \frac{1}{1^T\boldsymbol{\Sigma}^{-1}1}$$
$$\Rightarrow \boldsymbol{w} = (1^T\boldsymbol{\Sigma}^{-1}1)^{-1}\boldsymbol{\Sigma}^{-1}1$$

- The GMVP with a risk-free asset?

Summary

- Without risk-free asset
  - For a given target return, set up a system of FOC equations and solve for weights.
  - Two-fund (seperation) theorem
  - The efficient frontier: derived the shape, not equations (cumbersome).

- With risk-free asset
  - The efficent frontier is a line.
  - Found the tangecy portfolio. One-fund theorem.
  - The efficient frontier without a risk-free rate can be derived using the two-fund theorem.

- Additional contraints can be introduced.
  - Short-selling constraint: $w_i \geq 0$
  - Leverage constraint: $w_i \leq L$

Limitations

- Parameter estimation
  - The true μ and Σ unknown
  - We estimate them (e.g. using historical returns) $\Rightarrow$ Statistical error is introduced.
  - Σ has $n(n+1)/2$ data!
  - Portfolio is very sensitive to inputs.
- Often, we get negative weights.
  - Short sales are risky and not always feasible.
  - Sometime, unrealistically large negative numbers.
- Is variance an appropriate risk measure?

Developments

- Improve parameter estimation
  - Shrinkage methods (e.g., James and Stein, 1961, Ledoit and Wolf, 2004)
  - Use subjective views: Black-Litterman method
  - Non-parametric methods, machine learning methods etc.
- Improve optimization strategy
  - Leverage constraint
  - Robust target return contraint etc.
- Alternative measures of risk
  - Value-at-Risk, Expected shortfall etc.

Index Model

Index (Factor) Model

Definition

- How many inputs do we need?
  - N expected returns
  - N variances
  - $N(N-1)/2$ covariances

- Assume the following structure for asset returns (statistical model as opposed to economic model):

$$R_i = \text{own news} + \text{market news} + \text{random shocks}$$

- This seems a reasonable approximation of reality. Let's give more structure to it. R is an excess return. Consider single index for now.[1]

$$R_i = a_i + b_i R_m + e_i,$$

$$\text{where } E(e_i) = 0, cov(e_i, R_m) = 0, cov(e_i, e_j) = E(e_i e_j) = 0,$$

[1] Originally, this model was called the Sharpe's Diagonal model

Index (Factor) Model

Characteristics of index model

$$R_i = a_i + b_i R_m + e_i$$

- Now how many inputs do we need?
  - $E(R_i) = a_i + b_i E(R_m)$
  - $var(R_i) = b_i^2 \sigma_m^2 + \sigma_{ei}^2$ (market-wide + firm-specific)
  - $cov(R_i, R_j) = b_i b_j \sigma_m^2$

  $\Rightarrow$ Now we need $3N + 2$ inputs.

- Portfolio characteristics
  - $R_p = \sum_{i=1}^n w_i R_i$
  - $E(R_p) = \sum_{i=1}^n w_i E(R_i) = \sum_{i=1}^n w_i a_i + \sum_{i=1}^n w_i b_i E(R_m)$
    $= a_p + b_p E(R_m)$
  - $$\begin{aligned} var(R_p) &= \sum_{i=1}^n \sum_{j=1}^n w_i w_j \sigma_{ij} = \sum_{i=1}^n w_i^2 \sigma_i^2 + \sum_{i \neq j} w_i w_j \sigma_{ij} \\ &= \sum_{i=1}^n w_i^2 b_i^2 \sigma_m^2 + \sum_{i=1}^n \sigma_{ei}^2 + \sum_{i \neq j} w_i w_j b_i b_j \sigma_m^2 \\ &= (\sum_{i=1}^n w_i^2 b_i^2 + \sum_{i \neq j} w_i w_j b_i b_j) \sigma_m^2 + \sum_{i=1}^n w_i^2 \sigma_{ei}^2 \\ &= (\sum_{i=1}^n w_i b_i \sum_{j=1}^n w_j b_j) \sigma_m^2 + \sum_{i=1}^n w_i^2 \sigma_{ei}^2 = b_p^2 \sigma_m^2 + \sum_{i=1}^n w_i^2 \sigma_{ei}^2 \end{aligned}$$
  - When n is large, $\sigma_p = b_p \sigma_m = (\sum_{i=1}^n w_i b_i) \sigma_m$
  - b_i is the individual securities' contribution to the portfolio risk.

Index (Factor) Model

Summary

- The model simplifies a return generating process, putting a restriction on the sources of return uncetainty.
- Note that this model ignores cross-asset correlation other than through R_m.
  - Suppose some securities are correlated beyond the market (e.g. same industry).
  - The index model ignores this correlation and performs portfolio analysis.
  - Compared to the Markowitz algorithm, the index model will under- or overweight the correlated assets because they provide smaller or greater diversification benefit than they actually do.
  - The optimal portfolio from the index model can be significantly different from and inferior to the full-variance Markowitz model.
- We can try to introduce multiple indexes into the model (e.g. industry factor) to alleviate this problem.

Index (Factor) Model

Estimating Index Models

Regression of Amazon's monthly excess returns on market excess returns from June 2013-June 2018

Regression Statistics	
Multiple *R*	0.5351
R-Square	0.2863
Adjusted *R*-Square	0.2742
Standard Error	0.0686
Observations	60

	Coefficients	**Standard Error**	***t*-statistic**	***p*-value**
Intercept	0.0192	0.0093	2.0645	0.0434
Market index	1.5326	0.3150	4.8648	0.0000

- A better index model has higher explanatory power, R^2.
- The intercept, "alpha", is large value but statistically insignificant.
- The coefficient, "beta", is 1.5326: its share price tended to move 1.5326% for every 1% move in the market index.
- The standard deviation of Amazon's residual is $\sigma_{ei} = 6.86\%$ per month, or 23.76% annually.
- The standard deviation of systematic risk is $b\sigma_m = 1.5326 \times 2.81\% = 4.31\%$ per month, smaller than σ_{ei}

Index (Factor) Model

Beta adjustments

- We want to "forecast" future alphas and betas for portfolio construction.
  - Forecasting alphas: Security analysis and/or your genius.
  - Here we used the historical data (sample) to estimate future (population) betas. 1) This involves sampling errors. 2) Also, betas may simply change over time.

- Shrink toward one[2]
  - Blume (1975) find that historical beta tends toward one. He also documents that $b_{i,t+1} = 0.343 + 0.677 b_{i,t}$, based on which we can adjusted beta:

$$\text{Adjusted beta} = \frac{1}{3} + \frac{2}{3}\text{Estimated beta}$$

[2] Other adjustement techniques include Vasicek's (1973) Bayesian adjustment and Rosenberg and Guy's (1976) adjustment using fundamentals.

Index (Factor) Model

Muti-index model

- Multi-index models are an attempt to capture some of the nonmarket influences that cause securities to move together.
- Economic factors accounting for common movement in security prices.

$$R_i = a_i + b_{i1} I_1 + b_{i2} I_2 + ... + b_{iL} I_L + e_i, \text{ where } I_l' s \text{ are independent.}$$

- Alternative representation of index model
  - $R_i - E(R_i) = b_{i1}(I_1 - E(I_1)) + ... + b_{iL}(I_L - E(I_L)) + e_i$
  - The unexpected return is explained by the unexpected shock to factors.
- Example: $R_i = E(R_i) + b_{i,GDP} GDP + b_{i,Int} Intrest + e_i$
  - Utilities: low $b_{i,GDP}$, high $b_{i,Int}$
  - Airlines: high $b_{i,GDP}$, low $b_{i,Int}$
  - The economy expands: GDP is expected to increase, but so are interest rates. For the utility, this is bad news. But for the airline, this is good news.
  - Single-index model cannot capture such differential responses to varying sources of macroeconomic uncertainty.

Index (Factor) Model

Muti-index model: Fama-French (1993)

- Fama and French found that both size (market capitalization) and the ratio of book value of equity to the market value of equity have a strong role in determining the cross section of average return on common stocks.

$$R_i = a_i + b_{iM} R_M + b_{iS} SMB + b_{iH} HML + e_i$$

- They found that a cross section of average returns is negatively related to size and positively related to book to market ratios. That is, small firms and firms with low book to market are riskier than other firms.
- Why? Risk or free lunch?
  - May proxy for hard-to-measure more-fundamental variables.
  - For example, high book-to-market ratios are more likely to be in financial distress, and small stocks may be more sensitive to changes in business conditions
  - Or mispricing?

Index (Factor) Model

Portfolio construction

- Portfolio optimization

  1. We now have estimates for a_i, b_i, σ_{ei}, σ_m, and $E(R_M)$.
  2. Estimate individual securities expected returns, variances, and covariances.
  3. Perform portfolio optimization.

Treynor-Black model

(Optional)

Index (Factor) Model

Implications: Treynor-Black model

$$E(R_i) = a_i + b_i E(R_m)$$

- In the absence of insight generating expectations different from the market consensus, the investor should hold a very large portfolio.
- But what if you (believe to) have special insights?
- The index model provides a conceptual framework to incorporate the subjective analysis into the objective portfolio analysis.
- Procedure:
  - Baseline (no insight/research): hold a market portfolio, e.g., S&P500 index $\rightarrow$ passive portfolio
  - Perform security analysis and identify alpha, i.e., under- or over-valued securities $\rightarrow$ active portfolio
  - Now follow the Markowitz algorithm: Find the tangecy portfolio using the market portfolio and non-zero alpha portfolio.
  - There is a more insightful approach.

Index (Factor) Model

Treynor-Black model

- Suppose we identify non-zero alphas, and attempt to deviate from S&P 500: we may increase not only alpha but also firm-specific risk.
- The optimal risky portfolio is a combination of an active portfolio (A) and the passive portfolio (M)
- Results from Treynor-Black:

  [The derivation: solve $s^* = (1^T \Sigma^{-1} \hat{\mu})^{-1} \Sigma^{-1} \hat{\mu}$]

  - Assume that $\beta_A = 1$: the optimal weight in the active portfolio is proportional to α_A / σ_{eA}^2. The analogous ratio for the index portfolio is $E(R_M)/\sigma_M^2$.
  - The initial position in the active portfolio is

$$w_A^0 = \frac{\alpha_A / \sigma_{eA}^2}{E(R_M)/\sigma_M^2}$$

  - Actual beta may different from 1. Update the weight:

$$w_A^* = \frac{w_A^0}{1 + (1 - \beta) w_A^0}$$

Index (Factor) Model

Treynor-Black model

- The relationship between the Sharpe ratio of an optimal portfolio and that of the index portfolio:

$$S_P^2 = S_M^2 + \left(\frac{\alpha_A}{\sigma_{eA}}\right)^2$$

- Information ratio $\left(\frac{\alpha_A}{\sigma_{eA}}\right)$: The extra return from security analysis compared to the firm-specific risk when deviating from the passive market index.
- The information ratio is maximized if we invest in each security in proportion to its ratio of α_i/σ_{ei}
- Scale this ratio so that the total position in the active portfolio adds up to w_A^*

$$w_i^* = w_A^* \frac{\alpha_i/\sigma_{ei}^2}{\sum \alpha_i/\sigma_{ei}^2}$$

- The contribution of each security to the information ratio of the active portfolio is the its own information ratio (squared):

$$\left(\frac{\alpha_A}{\sigma_{eA}}\right)^2 = \sum_{i=1}^{n} \left(\frac{\alpha_i}{\sigma_{ei}}\right)^2$$

Index (Factor) Model

Treynor-Black model

- Derive: $S_P^2 = S_M^2 + \left(\frac{\alpha_A}{\sigma_{eA}}\right)^2$
- Suppose $\alpha_i > 0$ in $R_i = \alpha_i + \beta_i R_m + e_i \leftrightarrow r_i - r_F = \alpha_i + \beta_i(r_m - r_F) + e_i$.
- Construct a portfolio, P, with \$1.
  - Long \$1 of asset i.
  - $-\$b_i$ unit of the market portfolio.
  - $\$b_i$ unit of the risk-free asset.

$$r_P = r_i - b_i(r_m - r_F) = r_F + \alpha_i + e_i$$

- The tangecy portfolio using M and P: $s^* = (1^T\Sigma^{-1}\hat{\mu})^{-1}\Sigma^{-1}\hat{\mu}$ (Note that P and M are uncorrelated)

$$w_m = \lambda\frac{E(r_m) - r_F}{\sigma_m^2}, \quad w_P = \lambda\frac{E(r_P) - r_F}{\sigma_P^2} = \lambda\frac{\alpha_i}{\sigma_{e_i}^2}$$

$$E(r_T) - r_F = w_m(E(r_m) - r_F) + w_P(E(r_P) - r_F) = \lambda\frac{(E(r_m) - r_F)^2}{\sigma_m^2} + \lambda\frac{\alpha_i^2}{\sigma_{e_i}^2}$$

$$\sigma_T^2 = w_m^2\sigma_m^2 + w_P^2\sigma_P^2 = \lambda^2\frac{(E(r_m) - r_F)^2}{\sigma_m^2} + \lambda^2\frac{\alpha_i^2}{\sigma_{e_i}^2} = \lambda(E(r_T) - r_F)$$

$$SR_T^2 = \frac{(E(r_T) - r_F)^2}{\sigma_m^2} = \frac{1}{\lambda}(E(r_T) - r_F) = \frac{(E(r_m) - r_F)^2}{\sigma_m^2} + \frac{\alpha_i^2}{\sigma_{e_i}^2} = SR_M^2 + SR_P^2$$

Index (Factor) Model

Treynor-Black model

- Procedure:
  1. The initial position of each security in the active portfolio as $w_i^0 = \alpha_i / \sigma_{ei}^2$.
  2. Scale the initial positions to have weights to sum to 1: $w_i = w_i^0 / \sum w_i^0$.
  3. The alpha of the active portfolio: $\alpha_A = \sum w_i \alpha_i$.
  4. The residual variance of the active portfolio: $\sigma_{eA}^2 = \sum w_i^2 \sigma_{ei}^2$.
  5. The initial position in the active portfolio: $w_A^0 = \frac{\alpha_A / \sigma_{eA}^2}{E(R_M) / \sigma_M^2}$
  6. The beta of the active portfolio: $\beta_A = \sum w_i \beta_i$
  7. Adjust the initial position in the active portfolio: $w_A^* = \frac{w_A^0}{1 + (1 - \beta) w_A^0}$.
  8. The optimal risky portfolio: $w_M^* = 1 - w_A^*$ and $w_i^* = w_A^* w_i$.
  9. The risk premium of the optimal portfolio:
     $E(R_P) = (w_M^* + w_A^* \beta_A) E(R_M) + w_A^* \alpha_A$
  10. The variance of the optimal portfolio: $\sigma_P^2 = (w_M^* + w_A^* \beta_A)^2 \sigma_M^2 + [w_A^* \sigma_{eA}]^2$
- Excel spreasheet "`Portfolio Choice using Excel.xlsx`"

Black-Litterman Model
(Optional)

Black-Litterman Model

Overview

- The major challenge to the portfolio problem is the estimation of expected returns for individual securities.
  - Typically, we use historical return, but fraught with estimation error.
- Black and Litterman (1991) attempt to estimate expected return in a more reliable way by incorporating our "subjective" view about future returns.
- The approach is also based on an equilibrium behavior of investors (CAPM - next topic), where a main prediction is that investors should hold the market portfolio which is optimal, i.e., hard to beat.

Black-Litterman Model

Procedure

1. Let investor choose benchmark (e.g., S&P 500 Index)
2. Instead of inputting data and deriving an optimal portfolio, the BL approach assumes that a given portfolio is optimal and derives the expected returns of the benchmark components
   - The implied expected benchmark returns is interpreted as the market's information about the future returns of each asset in the benchmark portfolio
3. Given the expected returns on the benchmark portfolio, let investor express views and re-derive optimal portfolio
   - Because of the correlations between asset returns, an investor's opinion about any particular asset's returns will affect all the other expected returns.

Black-Litterman Model

Solving for expected return

1. Suppose a chosen benchmark portfolio is optimal.
2. From the portfolio optimization technique,

$$s^* = w_B = \frac{\Sigma^{-1}\hat{\mu}}{\lambda} = \frac{\Sigma^{-1}(\mu - r_f)}{\lambda}, \text{ where } \lambda = (1^T\Sigma^{-1}\hat{\mu})$$

3. Therefore, the implied expected returns are

$$\hat{\mu}^{imp} = \lambda\Sigma w_B$$

Black-Litterman Model

Incorporating subjective views

- Updated expected return = the weighted average of the implied expected return and the subjective expected return, where the weight is proportional to the level of conviction.

- For a single security:

$$\hat{\mu}_i^{update} = \frac{\frac{1}{\sigma_{imp}^2}}{\frac{1}{\sigma_{imp}^2} + \frac{1}{\sigma_{sub}^2}} \hat{\mu}_i^{imp} + \frac{\frac{1}{\sigma_{sub}^2}}{\frac{1}{\sigma_{imp}^2} + \frac{1}{\sigma_{sub}^2}} \hat{\mu}_i^{sub}$$

- The greater the variance of the return, the lower the conviction. The inverse of the variance is called the precision.

- Perform portfolio optimization with updated expected returns

Black-Litterman Model

Generalization

- Suppose there are N securities.
- Σ: variance-covariance matrix
- Ω: uncertainty surrounding your views

$$\hat{\mu}^{update} = \frac{\Sigma^{-1}}{\Sigma^{-1} + \Omega^{-1}}\hat{\mu}^{imp} + \frac{\Omega^{-1}}{\Sigma^{-1} + \Omega^{-1}}\hat{\mu}^{sub}$$

- Suppose we only have subjective views on a subset of securities.

$$\hat{\mu}^{update} = \frac{\Sigma^{-1}}{\Sigma^{-1} + P^T\Omega^{-1}P}\hat{\mu}^{imp} + \frac{P^T\Omega^{-1}}{\Sigma^{-1} + P^T\Omega^{-1}P}\hat{\mu}^{sub}$$

- Let's introduce an adjustment scalar τ on Σ

$$\hat{\mu}^{update} = \frac{(\tau\Sigma)^{-1}}{(\tau\Sigma)^{-1} + P^T\Omega^{-1}P}\hat{\mu}^{imp} + \frac{P^T\Omega^{-1}}{(\tau\Sigma)^{-1} + P^T\Omega^{-1}P}\hat{\mu}^{sub}$$

$$\hat{\mu}^{update} = [(\tau\Sigma)^{-1} + P^T\Omega^{-1}P]^{-1}[(\tau\Sigma)^{-1}\hat{\mu}^{imp} + P^T\Omega^{-1}\hat{\mu}^{sub}]$$

Black-Litterman Model

Example

$$\hat{\mu}^{update} = [(\tau\Sigma)^{-1} + P^T\Omega^{-1}P]^{-1}[(\tau\Sigma)^{-1}\hat{\mu}^{imp} + P^T\Omega^{-1}\hat{\mu}^{sub}]$$

To be consistent with BL notations:

$$\hat{\mu}^{BL} = [(\tau\Sigma)^{-1} + P^T\Omega^{-1}P]^{-1}[(\tau\Sigma)^{-1}\Pi + P^T\Omega^{-1}Q]$$

- Consider a one asset case.
  - Implied expected return: $\Pi = 3\%$
  - Variance of the imp. exp. return: $\Sigma = 1.2\%^2$
  - Subjective view on expected return: $Q = 2\%$
  - Uncertainty on the view: $\Omega = 0.25\%^2$
  - $P = 1$. Assume $\tau = 1$, which is an adjustment to the weight on the prior.

$$\begin{aligned}\hat{\mu}^{BL} &= [1(1.2\%^2)^{-1} + 1(0.25\%^2)^{-1}1]^{-1}[(1(1.2\%^2))^{-1}(3\%) + 1(0.25\%^2)^{-1}(2\%)] \\ &= 2.04\%\end{aligned}$$

Black-Litterman Model

Summary

- Advantages
  - Investor's can insert their view
  - Control over the confidence level of views
  - More intuitive interpretation, less extreme shifts in portfolio weights
- Disadvantages
  - Black-Litterman model does not give the best possible portfolio, merely the best portfolio given the views stated
  - As with any model, sensitive to assumptions
  - The choice of parameter values are discretionary

References

- BKM, Chapters 5 through 8 and 10 (Section 1)
- Andrew Lo's lecture notes
- Benninga, Chapter 13