Problem Set 7 - Answer Key

1. Consider the following three stocks.

	Expected return	Standard deviation
A	5%	10%
B	20%	40%
C	20%	40%

The return on stock C is not correlated with A and B. Correlation of returns between A and B is -0.5.

a Suppose that an investor allocates 40% of her wealth to stock A, 60% to stock C, and nothing to stock B. What is the standard deviation and expected return of her portfolio?

b Suppose that an investor allocates 40% of her wealth to stock A, 60% to stock B, and nothing to stock C. What is the standard deviation and expected return of her portfolio?

c Suppose that an investor constructs an equally weighted portfolio out of stocks A, B, and C. What is the standard deviation and expected return of her portfolio?

Solution:

	Expected return	Standard deviation
a	14%	24.33%
b	14%	22.27%
c	15%	17.95%

Portfolio (a) has the same return as portfolio (b). However, because the correlation between stocks A and B is lower than that between stocks A and C, portfolio (b) has a lower volatility. The equally weighted portfolio (c) has higher expected return and lower volatility because it takes greater advantage of low correlation between the three stocks and, as a result, achieves a better risk-return trade-off.

2. Suppose there are 2 stocks in the market.

	Expected return	Standard deviation
A	5%	10%
C	20%	40%

The returns on stocks A and C are not correlated.

a Build a portfolio with an expected return of 10%

b Suppose that the maximum risk you can afford to take is 20%. Build a portfolio that achieves this target. What is the expected return of this portfolio?

c Suppose that there is an investor with low tolerance for risk. Construct a minimum variance portfolio (MPV) for this investor. What is the expected return of this minimum variance portfolio?

Solution:

a. Target return is 10%. Suppose you invest w in stock A and $(1-w)$ in stock C. Then: $wr_A+(1-w)r_B = 10\%$. Hence $w = 2/3$.

b. Target volatility is 20%. Suppose you invest w in stock A and $(1-w)$ in stock C. Then: $w^2\sigma_A^2 + (1-w)^2\sigma_C^2 + w(1-w)\sigma_{A,C} = 0.2^2$. Hence $w = 1.365$ or 0.517. The expected return is -0.48%, when $w = 1.365$, and 12.25% when $w = 0.517$.

c. Solve the following problem: $\min_w w^2(0.1)^2 + (1-w)^2(0.4)^2$. The solution is $w = 94.12\%$. So the minimum variance portfolio is 94.12% invested in A and 5.88% invested in C. The portfolio achieves: Standard deviation of 9.7% and expected return of 5.88%. You can also use the GMVP formula we derived in class, $w = (1^T\Sigma^{-1}1)^{-1}\Sigma^{-1}1$.

3. Use the same data from 2. In addition, there is a riskless asset with guaranteed return of 3%. Suppose there are two investors, X and Z, who maximize risk-return tradeoff and have mean-variance preferences. Investor X can only invest in stocks A and C. Investor Z, in addition to being able to invest in stocks A and C, can also invest in the riskless bond. Investor X targets standard deviation of 20% and investor Z targets standard deviation of 12%.

a What is the maximum Sharpe ratio investor X can achieve?

b What is the maximum Sharpe ratio investor Z can achieve?

c Describe the portfolio of investor Z. What is the expected return of this portfolio?

d If investor X had access to the risk-free asset, could her return have been improved? If yes, compute the expected return for a 20% target level of risk.

Solution:

a. Recall from question 2 part b that the expected return on X's portfolio is 12.25%. The Sharpe ratio of X's portfolio is: $SR = \frac{12.25\% - 3\%}{20\%} = 0.462$.

b. We need to find the portfolio of risky assets, A and C, which has the highest Sharpe ratio (i.e. tangency portfolio). Suppose we invest w in A and $(1-w)$ in C. The tangency portfolio is $w = (1^T\Sigma^{-1}\hat{\mu})^{-1}\Sigma^{-1}\hat{\mu}$, where $\hat{\mu}$ is an excess return. Then the tangency portfolio is the portfolio with 65.31% invested in A and 34.69% invested in C. The Shart ratio is 0.47. Alternatively, you can solve the following problem:

$$\max_{w} \frac{wr_A + (1-w)r_B - r_f}{\sqrt{w^2\sigma_A^2 + (1-w)^2\sigma_C^2}}$$

You can solve this problem algebraically or numerically (using Excel's Solver).

c. Since investor Z targets a 12% portfolio volatility, she needs to mix the risk-free asset with the tangency portfolio. Suppose she invests w in the tangency portfolio and $(1-w)$ in the risk-free asset. Then: $w^2\sigma_T^2 + (1-w)^2\sigma_f^2 = 0.12^2$. Hence $w = \frac{0.12}{0.1524} = 78.24\%$. So investor Z needs to invest 78.24% in the tangency portfolio and 21.76% in the riskfree asset to achieve the maximimum Sharpe ratio while targeting 12% volatility. Z's portfolio consists of:
- 21.76% risk-free asset
- 51.10% stock A (= 78.24% $\times$ 65.31%)
- 27.14% stock C (= 78.24% $\times$ 34.69%)

The expected return on Z's portfolio is: 8.64%.

d. Suppose X invests w in the tangency portfolio and $(1-w)$ in the risk-free asset. Then: $w^2\sigma_T^2 + (1-w)^2\sigma_f^2 = 0.2^2$. Hence $w = \frac{0.2}{0.1524} = 130.24\%$. So investor X needs to invest 130.40% into the tangency

portfolio and -30.40% into the risk-free asset. Recall that the tangency portfolio is 65.31% A and 34.69% C. So X's portfolio is:
- -30:40% risk-free asset
- 85.16% stock A (= 130.40% $\times$ 65.31%)
- 45.24% stock C (= 130.40% $\times$ 34.69%)

The expected return of X's portfolio is: 12.39%.

5. Suppose that the expected return on the market portfolio is 14% and the standard deviation of its returns is 25%. The risk-free rate is 6%. Consider a portfolio with expected return of 16% and assume that it is located on the Capital Market Line (CML).

a What is the beta of this portfolio?

b What is the standard deviation of this portfolio?

c What is its correlation with the market?

Solution:

a. Using the CAPM, $E(r_p) = 16\% = r_f + \beta_p[E(r_m) - r_f] = 6\% + \beta_p[14\% - 6\%]$. Therefore $\beta_p = 1.25$.

b. One approach is to consider the fact that the portfolio (P) is located on the CML, in which case standard deviation of a portfolio is proportional to $\frac{\sigma_p}{\sigma_m}$, i.e., $E(r_p) = r_f + \frac{\sigma_p}{\sigma_m}[E(r_m) - r_f]$. Hence, $\sigma_p = \sigma_m \times 1.25 = 31.25\%$.

Another approach is to create a portfolio that replicates the portfolio p using the market portfolio and the risk-free asset. We invest w in the risk-free asset and $1 - w$ in the market portfolio. We want this portfolio to have an expected return of 16%. $16\% = w6\% + (1 - w)14\%$. Hence $= -0.25$. The variance of the replicating portfolio is $Var(r_R) = Var(wr_f + (1 - w)r_m) = (1 - w)^2\sigma_m$. Therefore, $\sigma_R = (1 - w)\sigma_m = 1.25 \times 25\% = 31.25\%$. The standard deviation of the replicating portfolio is the same as the standard deviation of the portfolio in question

c. Use beta definition. $\beta_p = \frac{\sigma_{p,m}}{\sigma_m^2} = \frac{\rho_{p,m}\sigma_p\sigma_m}{\sigma_m^2}$. Hence, $\rho_{p,m} = 1$.

6. Suppose an endowment fund is currently investing 20% of its portfolio in a risk-free asset and investing the remaining 80% with a private equity fund, YZW LLC. The endowment fund analyzed holdings of YZW's portfolio and found that its expected returns over the next year will be 10%, with a standard deviation of 25%.

Research conducted by the endowment fund indicates that the market portfolio will have expected return of 10% and standard deviation of 20% during the next year. The correlation of returns between YZW's portfolio and the market portfolio is 0.8. Suppose that the risk-free asset will yield 5% return over the next year.

a Compute the CAPM beta of YZW's portfolio.

b Compute the CAPM beta and alpha of the endowment fund's portfolio.

c Suppose that you were retained as an advisor to the endowment fund. Would you recommend divesting part of the holdings in YZW and reallocating them to the market portfolio? (Hint: compare Sharpe ratios)

Solution:

a. $\beta_{YZW} = \frac{\rho_{YZW,m}\sigma_Y ZW \sigma_m}{\sigma_m^2} = 1$.

b. The endowment fund invests 20% into the risk-free asset and 80% into Eternal Growth. So, $\beta_{EF} = 20\%\beta_f + 80\%\beta_{YZW} = 0.8$. To compute the alpha, we first find the endowment fund's expected return: $r_{EF} = 20\% r_f + 80\% r_{YZW} = 20\%5\% + 80\%10\% = 9\%$. Then, the CAPM alpha is the difference between the expected return that the endowment fund expects to generate (9%) and the expected return under the CAPM. The expected return of the endowment fund under CAPM is: $E[r_{EF}] = r_f + \beta_{EF}(E[r_m] - r_f) = 5\% + 0.8(10\% - 5\%) = 9\%$. $E[r_{EF}]$ under CAPM is the same as the endowment fund's expected return. So the CAPM alpha is 0.

c. We can compare the Sharpe ratio of the endowment fund to the Sharpe ratio of the market portfolio. The endowment fund Sharpe ratio is $SR_{EF} = \frac{r_{EF} - r_f}{\sigma_{EF}} = 0.2$. Note that $\sigma_{EF} = 0.8\sigma_{YZW} = 20\%$. The Sharpe ratio of the market portfolio is $SR_m = \frac{r_m - r_f}{\sigma_m} = 0.25$. Since the market portfolio Sharpe ratio is greater than the endowment fund Sharpe ratio, the endowment fund is not efficient. Its Sharpe ratio can be improved by reallocating assets from YZW fund to the market portfolio.

7. Suppose company XYZ is a publicly traded firm. Its current share price is \$22.5. The total number of outstanding shares is 20 million. XYZ has 525,000 bonds outstanding, each with a face value of \$100, maturing one year from now. The bonds do not pay a coupon and are considered risk-free. XYZ's stock has a market beta of 0.8. The expected market risk premium is 6%, and the risk-free rate is 5%.

a What is the current market value of the firm?

b What is the market beta of the firm?

c Suppose that the beta of the firm's assets and the expected interest rates are constant over the next two years. What is the expected price of the stock in two years if it does not pay dividends? Assume that the performance of the company in Year 1 does not predict its performance in Year 2. (Hint: the firm pays off its current debt after one year.)

d If the firm issues 551,250 2-year zero-coupon bonds with face value \$100 to buy back its stock, will it change the equity beta of the firm? If yes, compute the new equity beta. Suppose that investors consider these new bonds risk-free. (Hint: the firm's asset beta is not affected by capital structure)

Solution:

a. The current market value of equity is: $E = \$22.5 \times 20,000,000 = \$450,000,000$ The current market value of debt is: $D = \frac{1}{1+5\%} \times \$100 \times 525,000 = \$50,000,000$. So the market value of the firm's assets is: $A = E + D = \$450,000,000 + \$50,000,000 = \$500,000,000$.

b. $\beta_A = \frac{E}{E+D}\beta_E + \frac{D}{E+D}\beta_D = \frac{\$450m}{\$500m} \times 0.8 = 0.72$.

c. The stock price in two years is $P_2 = P_0(1+r_1)(1+r_2)$. Taking the expectation of both sides gives us: $E(P_2) = P_0E((1+r_1)(1+r_2))$. Next, note that we assume the stock performance in year 1 is independent of its performance in year 2. This allows us to separate the $(1+r_1)and(1+r_2)$ when taking the expectation of the product: $E(P_2) = P_0(1+E(r_1))(1+E(r_2))$. To determine the expected stock price in year 2, we need to determine the expected returns in year 1 and year 2. We will use the CAPM to do this. Note that in year 2, the equity beta is 0.72 (equal to the asset beta found in part b) since the debt matures at the end of year 1, making the firm all equity-financed. We have:

$$E(r_1) = r_f + \beta_E(E(r_m) - r_f) = 5\% + 0.8(6\%) = 9.8\% E(r_2) = r_f + \beta_E(E(r_m) - r_f) = 5\% + 0.72(6\%) = 9.32\%$$

Hence, the expected share price in Year 2: $E(P_2) = \$22.5(1 + 9.8\%)(1 + 9.32\%) = \$27.01$.

d. Again, $\beta_A = \frac{E}{E+D}\beta_E + \frac{D}{E+D}\beta_D$. Since both the old and new tranches of debt are risk-free,

$D = 0$. So, $\beta_A = \frac{E}{E+D}\beta_E$. Rearranging give us: $\beta_E = \frac{E+D}{E}\beta_A$. The value of the firm's assets does not change as a result of this new debt issuance. To find the new equity value, we need to find the new value of debt: $D_{new} = \frac{1}{1+5\%}(551,250)(\$100) = \$50,000,000$. The total value of debt $D = D_{old} + D_{new} = \$50mm + \$50mm = \$100mm$. The new value of equity after the bond issuance and share buy-back is $E = A - D = \$500mm - \$100mm$. So the new equity beta is:$\beta_E = \frac{\$500mm}{\$400mm}(0.72) = 0.9$.

8. Consider a two-factor model: $r_i = E[r_i] + \beta_{i,1}f_1 + \beta_{i,2}f_2 + \epsilon_i$ and $E[f_1] = 0, E[f_2] = 0, Cov(f_1, f_2) = 0$ Assume that the standard deviation of Factor 1 is 7% and that of Factor 2 is 5%. There are two stocks with the following properties (loadings mean betas):

	Expected return	Load on Factor 1	Load on Factor 2	$\sigma(\epsilon_i)$
A	10%	1	1.5	25%
B	10%	2	3	10%

Assume that the idiosyncratic component of returns of stocks A and B is uncorrelated with each other, as well as with both factor returns.

a Compute standard deviation of returns on Stocks A and B.

b Compute the correlation between returns on Stock A and Stock B.

Solution:

a. The return on stock A is given by the factor model: $r_A = E[r_A] + \beta_{A,1}f_1 + \beta_{A,2}f_2 + \epsilon_A$. (Note that $E[r_A] = \alpha$, the notation we used in class.) Therefore, $Var[r_A] = Var[\beta_{A,1}f_1] + Var[\beta_{A,2}f_2] + Var[\epsilon_A] + 2Cov[\beta_{A,1}f_1, \beta_{A,2}f_2] + 2Cov[\beta_{A,1}f_1, \epsilon_A] + 2Cov[\beta_{A,2}f_2, \epsilon_A] = \beta_{A,1}^2 Var[f_1] + \beta_{A,2}^2 Var[f_2] + Var[\epsilon_A] + 2\beta_{A,1}\beta_{A,2}Cov[f_1, f_2] + 2\beta_{A,1}Cov[f_1, \epsilon_A] + 2\beta_{A,2}Cov[f_2, \epsilon_A]$. Since this is a factor model, by its definition, all the covariances are zeros. Hence,$Var[r_A] = \beta_{A,1}^2 Var[f_1] + \beta_{A,2}^2 Var[f_2] + Var[\epsilon_A]$. Plug in the numbers from the table, we get $Var[r_A] = 0.073, \sigma = 27.02\%$ and $Var[r_B] = 0.0521, \sigma = 22.83\%$.

b. $Cov(r_A, r_B) = Cov(E[r_A] + \beta_{A,1}f_1 + \beta_{A,2}f_2 + \epsilon_A, E[r_B] + \beta_{B,1}f_1 + \beta_{B,2}f_2 + \epsilon_B)$. The covariance between a constant ($E[r_A]$ or $[r_B]$) and a random variable is zero. Also, $Cov(\epsilon_A, \epsilon_B) = 0$ and $Cov(\epsilon_i, f_j) = 0$ for $i = A, B$ and $j = A, B$. Hence, the equation becomes: $Cov[\beta_{A,1}f1, \beta_{B,1}f1] + Cov[\beta_{A,2}f2, \beta_{B,1}f1] + Cov[\beta_{A,1}f1, \beta_{B,2}f2] + Cov[\beta_{A,2}f2, \beta_{B,2}f2] = \beta_{A,1}\beta_{B,1}Cov[f1, f1] + \beta_{A,2}\beta_{B,1}Cov[f2, f1] + \beta_{A,1}\beta_{B,2}Cov[f1, f2] + \beta_{A,2}\beta_{B,2}Cov[f2, f2]$. Use the fact that $Cov[f_1, f_2] = 0$, and plug in the numbers. Then the covariance is 0.02105, and the correlation is $0.02105/(27.02\%22.83\%) = 34.13\%$.

9. Consider a two-factor model: $r_i = E[r_i] + \beta_{i,1} f_1 + \beta_{i,2} f_2 + \epsilon_i$. There are three well-diversified portfolios with the following properties:

	Expected return	Load on Factor 1	Load on Factor 2
A	8%	0.95	1.15
B	6%	0.85	0.70
C	10.5%	1.20	1.50

Assume that the current risk-free rate is 1.5%. Construct an arbitrage strategy that generates $1,000 today and zero payoff in the future. (Hint: Use the Arbitrage Pricing Theory. You don't want to take any factor risk.)

Solution:

Let us start building our arbitrage strategy by assuming that we buy (go long) portfolios A, B, and C and the risk-free bond in the following (dollar) amounts: x_A, x_B, x_C, x_f . We will allow these amounts to be negative, which would indicate that we are shorting these securities. We want to construct a trading strategy that gives us $1,000 today, i.e., $-x_A - x_B - x_C - x_f = 1,000$. Notice the negative signs. This is because we assumed that we are buying x of each security, which represents cash outflow. According to the APT, no arbitrage implices:1) No exposure to systematic risk, 2) No idiosyncratic risk, 3) Zero expected return.

Let us start with zero expected return: $x_A(1+E[r_A]) + x_B(1+E[r_B]) + x_C(1+E[r_C]) + x_f(1+E[r_f])) = 0$. Plugging the numbers from the table, we get $x_A(1+0.08) + x_B(1+0.06) + x_C(1+0.105) + x_f(1+0.015)) = 0$.

The next assumption is that there is no exposure to systematic risk factor 1. Therefore, $\frac{x_A}{V_P}\beta_{A,1} + \frac{x_B}{V_P}\beta_{B,1} + \frac{x_C}{V_P}\beta_{C,1} = 0$, where $V_P = x_A + x_B + x_C + x_f$. Plugging the numbers we get $0.95x_A + 0.85x_B + 1.20x_C = 0$. Similarly, the exposure of the portfolio to factor 2 must be zero: $\frac{x_A}{V_P}\beta_{A,2} + \frac{x_B}{V_P}\beta_{B,2} + \frac{x_C}{V_P}\beta_{C,2} = 0$, so $1.15x_A + 0.70x_B + 1.50x_C = 0$.

Now we have to solve a system of four equations. You can for the problem using matrix and solve it in Excel. The solution is $x_A = -235,480$, $x_B = 24,360$, $x_C = 169,167$, and $x_f = 40,953$. Looking at our solution, we can conclude that we short portfolio A and go long into other portfolios. This means that if portfolios B and C are correctly priced by the factors, then portfolio A offered too low return given its systematic risk. Alternatively put, portfolio A is too expensive.

10. Consider a three-factor model: $r_i = E[r_i] + \beta_{i,1} f_1 + \beta_{i,2} f_2 + \beta_{i,3} f_3 + \epsilon_i$. There are four well-diversified portfolios with the following properties:

	Expected return	Load on Factor 1	Load on Factor 2	Load on Factor 2
A	21.65%	0.95	1.20	0.85
B	19.23%	1.00	0.75	1.35
C	21.48%	1.10	1.05	0.95
D	23.15%	1.25	1.05	1.15

Compute factor risk premia (also known as factor risk prices) as well as the risk-free rate.

Solution:

The APT pricing equation states that the expected excess return on a portfolio is the sum of products of factor loadings with the risk-premia of the factors: $E[r_P] - r_f = \lambda_1 \beta_{P,1} + \lambda_2 \beta_{P,2} + \lambda_3 \beta_{P,3}$. Hence, the APT pricing relationship implies restrictions on the factors riskpremia and the risk-free rate:

$21.65\% - r_f = \lambda_1 0.95 + \lambda_2 1.20 + \lambda_3 0.85$

$19.23\% - r_f = \lambda_1 1.00 + \lambda_2 0.75 + \lambda_3 1.35$

$21.48\% - r_f = \lambda_1 1.10 + \lambda_2 1.05 + \lambda_3 0.95$

$23.15\% - r_f = \lambda_1 1.25 + \lambda_2 1.05 + \lambda_3 1.15$

The solution is $(\lambda_1, \lambda_1, \lambda_1, r_f) = 6.51\%, 9.95\%, 3.47\%, 0.57\%)$.

11. Download the data contained in the file posted on course webpage. The data contains historical monthly returns on eight firms from January 2005 through December 2019. Four of these firms are from the gold mining industry. And the other four are from the technology industry.

We will consider a two-factor model that has the market and the price of gold as the risk. To proxy those factors, we will use factor-mimicking portfolios: For the market factor, we will use the returns on the market portfolio, and for the price of gold, we will use the returns on the Spider Gold Shares ETF. The data with the returns of those factors and the risk-free rates are also in the file.

The following equation describes the two-factor model:

$$r_i - r_f = \alpha_i + \beta^i_{mkt}(r_{mkt} - r_f) + \beta^i_{gld}(r_{gld} - r_f)$$

a Using the OLS regression over the full sample, estimate factor loadings, β_{mkt}, β_{gld}, and the alphas for each of the eight stocks.

Report, separately, the arithmetic average of the factor loadings and alphas for the gold mining stocks and for the technology stocks.

b Assume that the returns on all stocks satisfy the exact 2 -factor APT model with the stock market and the gold price as the two factors. Take your estimated factor loadings as exact. Suppose you are given a forecast for the expected excess monthly returns going forward: 0.75% for the stock market portfolio, and 0.15% for the gold price. Using the average factor loadings you've computed in part (a), compute the expected excess return for the gold Mining sector, and the technology sector. Report both numbers as monthly, in %.

(Solution is provided in the file.)