Introduction and Overview

BUSS254 Investments

Prof. Ji-Woong Chung

Outline

- Overview of financial markets
- Review: measures of return and risk
- Reading: BKM, Ch. 1 and 5

Overview

Investments - Definition

- Investment involves the acquisition or disposal of assets.
  - Assets: anything that generates cash flows
  - Financial assets vs. Real assets
  - "Investments" vs. "Corporate Finance"
- Allocation decision: Which assets to invest in?
- The goal: Create value
  - Value = Cash flows (returns) in excess of the required amount
  - Required amount = Opportunity cost of capital
  - Alternative investments with similar risk
- Valuation is central: Does it create positive value?
- First step: What is the investment opportunity set?

Financial Markets

- Places where financial assets (stocks, bonds, derivatives) are traded.
- Core functions:
  - Transfer funds (across time/space)
  - Redistribute risks
  - Gather information and discover prices
  - Provide liquidity
- Central institution of modern economies.

Classification of Financial Markets

- Primary vs. Secondary market
- Exchanges vs. Over-the-counter market
- Broker vs. Dealer market
- Direct vs. Indirect market
- Money vs. Capital market
- Cash vs. Futures market

Stylized Fact 1: Financial markets are not perfect

- Institutional constraints
  - Transaction costs, liquidity issues
  - Missing markets
  - Position/trading constraints
- Information asymmetry
  - Between firm stakeholders
  - Between corporate managers and markets
  - Between market participants
- Taxes: Corporate and personal tax effects

- Market imperfections drive financial innovation

Stylized Fact 2: Financial markets are competitive

- Risk-Return Trade-Off: Higher-risk assets offer higher expected returns.
- Efficient Markets: Prices quickly adjust to relevant information.
- Arbitrage is difficult:
  - A strategy is arbitrage if it always generates non-negative cash flows and sometimes generates positive cash flows.
  - Law of One Price (LOOP): Assets with the same payoffs should have the same price.

Investment Strategies

- Depending on your investment philosophy and market outlook, you may choose different strategies.
- Passive Management
  - Holding a diversified portfolio
  - No attempt to find undervalued securities
  - No attempt to time the market
- Active Management
  - Actively seeking mispriced securities
  - Attempting to time the market
  - Using fundamental or technical analysis
  - Higher transaction costs and management fees

We will study ...

- Various kinds of financial markets/assets
- Approaches to valuing financial assets
- Different ways of measuring risk
- Types of investment strategies
- Evaluating performance

Measures of Return and Risk

Simple Return

Suppose a stock price evolves as follows:

```r
P0 <- 100
P1 <- 110
P2 <- 100

R_t <- P1 / P0
r_t <- (P1 - P0) / P0
log_return <- log(P1 / P0)

list(GrossReturn = R_t, NetReturn = r_t, LogReturn = log_return)
```

```
$GrossReturn
[1] 1.1

$NetReturn
[1] 0.1

$LogReturn
[1] 0.09531018
```

Compounded Return

If a bond pays 10% semiannually, its compounded return is:

$$\left(1+\frac{r}{k}\right)^{k}-1$$

Continuously compounded return:

$$\lim_{k\to\infty}\left(1+\frac{r}{k}\right)^{k}-1=e^{r}-1$$

In R:

```r
r <- 0.10
compounded_return <- (1 + r / 2)^2 - 1
continuously_compounded <- exp(r) - 1
list(CompoundedReturn = compounded_return, ContinuousReturn = continuously_compounded)
```

```
$CompoundedReturn
[1] 0.1025

$ContinuousReturn
[1] 0.1051709
```

What is the equivalent c.c. return of Bond XYZ?

$$e^{r}=\left(1+\frac{10\%}{2}\right)^{2}$$

$$r=\ln\left(1+\frac{10\%}{2}\right)^{2}=\ln\frac{P_1}{P_0}=9.758\%$$

Multi-Period Return

Again, suppose $P_0 = 100$, $P_1 = 110$, and $P_2 = 100$. 2-year gross return is

$$R(0,2) = \frac{P_2}{P_0}$$

2-year gross return using 1-year return:

$$R(0,2) = \frac{P_2}{P_0} = \frac{P_1}{P_0}\frac{P_2}{P_1} = R(0,1)R(1,2)$$

cf. Log returns

Annualization

Typically, returns are expressed as an annual return for comparison.

Monthly return 1% for 12 months:

$$r = (1 + 0.01)^{12} - 1$$

A two-year return 10%:

$$\begin{aligned}
&(1 + r_1)(1 + r_2) = 1.1 \\
&\text{Set } r_1 = r_2 = r \\
&(1 + r)^2 = 1.1 \Rightarrow r = (1.1)^{1/2} - 1 = 4.89\%
\end{aligned}$$

In general, an annualized return $= (1 + r_c)^{(365/Days)} - 1$, where r_c is the cumulative (holding-period) return, i.e., $P_t/P_0 - 1$.

Arithmetic vs. Geometric Average Return

- Arithmetic Mean Return:

$$r_{AM} = \frac{(r_1 + r_2 + \cdots + r_T)}{T}$$

- Geometric Mean Return:

$$r_{GM} = \left[(1 + r_1)(1 + r_2) \ldots (1 + r_T)\right]^{1/T} - 1$$

In R:

```r
returns <- c(0.05, 0.10, -0.02, 0.07)
mean_arith <- mean(returns)
mean_geom <- prod(1 + returns)^(1/length(returns)) - 1
list(ArithmeticMean = mean_arith, GeometricMean = mean_geom)
```

```
$ArithmeticMean
[1] 0.05

$GeometricMean
[1] 0.04905428
```

Arithmetic vs. Geometric Average Return

- Fact 1: $r_{AM} \geq r_{GM}$
- Fact 2: The greater the volatility of returns, the greater $r_{AM} - r_{GM}$
- Typically, use r_{AM} as a proxy for the expected return.

Expected Return

- The probability weighted average return
- In population (when we know the probability function),
  - Discrete: $E(r) = \sum_{i=1}^{n} P(r_i) r_i$
  - Continuous: $E(r) = \int_{-\infty}^{+\infty} r f(r) dr$
    * $E(ar_1 + br_2) = aE(r_1) + bE(r_2)$
- In sample (when we only observe history),
  - $\overline{X} = \frac{\sum_{i=1}^{N} x_i}{N}$

Expected Return

- The expected return is the probability-weighted average of all possible returns.
- In practice, the true probability distribution of returns is often unknown (i.e., from the future). Therefore, we estimate it using historical data.
- Typically, we use the arithmetic average of historical returns to estimate the expected return.
  - Law of Large Numbers: If X_i's are independent and identically distributed (i.i.d) with mean μ, then $\frac{1}{N} \sum_{i=1}^{N} X_i \to \mu$ as N approaches infinity.
- The higher the risk, the greater the required rate of return. In equilibrium, the required rate of return should be equal to the expected return.

Risk: Variance/Standard Deviation

- Measures the degree of dispersion of return (around its mean)

In population

- $Var(r) = \sigma^2 = E[(r - E(r))^2] = E(r^2) - E(r)^2$
  - $\sigma(r) = \sqrt{var(r)}$
  - $Var(ar) = a^2 Var(r)$
  - $Var(ar_1 + br_2) = a^2 Var(r_1) + b^2 Var(r_2) + 2abCov(r_1, r_2)$

In sample

- $s^2 = \hat{\sigma}^2 = \frac{\sum_{i=1}^{N}(x_i - \overline{X})^2}{N-1}$

Note

- $Cov(X, Y) = E[(X - E(X))(Y - E(Y))] = E(XY) - E(X)E(Y)$
  - Correlation: $\rho = cov(X, Y)/\sigma(X)\sigma(Y)$, $-1 \leq \rho \leq +1$
  - $Cov(aX + b, eY + f) = aeCov(X, Y)$
  - $Cov(X + Y, Z) = Cov(X, Z) + Cov(Y, Z)$

The Reward-to-Volatility (Sharpe) Ratio

$$\frac{E(r) - r_f}{\sigma}$$

Example in R:

```r
# Load necessary libraries
library(quantmod)
```

```
Loading required package: xts
```

```
Loading required package: zoo
```

```
Attaching package: 'zoo'
```

```
The following objects are masked from 'package:base':

    as.Date, as.Date.numeric
```

```
Loading required package: TTR
```

```
Registered S3 method overwritten by 'quantmod':
  method            from
  as.zoo.data.frame zoo
```

```
library(PerformanceAnalytics)
```

```
Attaching package: 'PerformanceAnalytics'
```

```
The following object is masked from 'package:graphics':

    legend
```

```
# Download adjusted closing prices for all symbols
getSymbols(c("TSLA","BRK-A","^IRX"), from = "2015-01-01", to = "2024-12-31")
```

```
Warning: ^IRX contains missing values. Some functions will not work if objects
contain missing values in the middle of the series. Consider using na.omit(),
na.approx(), na.fill(), etc to remove or replace them.
```

```
[1] "TSLA"  "BRK-A" "IRX"
```

```r
# Extract adjusted closing prices
tsla_prices <- TSLA[, "TSLA.Adjusted"]
brka_prices <- `BRK-A`[, "BRK-A.Adjusted"]
irx_prices <- `IRX`[, "IRX.Adjusted"]

# Merge into a single dataframe
prices <- merge(tsla_prices, brka_prices)

# Calculate daily returns
returns <- na.omit(prices / lag(prices) - 1)
  # or returns <- na.omit(ROC(prices, type = "discrete"))

# Extract individual asset returns
tsla_returns <- returns[, 1]
brka_returns <- returns[, 2]
rf_annual <- irx_prices$IRX.Adjusted / 100  # Convert IRX from percentage to decimal

# Convert annualized risk-free rate to daily rate
rf_daily <- (1 + rf_annual)^(1/252) - 1

# Align risk-free rate with stock returns
rf_daily <- na.omit(rf_daily)
merged_data <- merge(tsla_returns, brka_returns, rf_daily, all = FALSE)

# Calculate excess returns
tsla_excess_returns <- tsla_returns - rf_daily
brka_excess_returns <- brka_returns - rf_daily

# Compute annualized Sharpe Ratios
sharpe_tsla <- mean(tsla_excess_returns) / sd(tsla_excess_returns) * sqrt(252)
sharpe_brka <- mean(brka_excess_returns) / sd(brka_excess_returns) * sqrt(252)

# Print Sharpe Ratios
cat("Sharpe Ratio for TSLA:", sharpe_tsla, "\n")
```

```
Sharpe Ratio for TSLA: 0.8396145
```

```r
cat("Sharpe Ratio for BRK-A:", sharpe_brka, "\n")
```

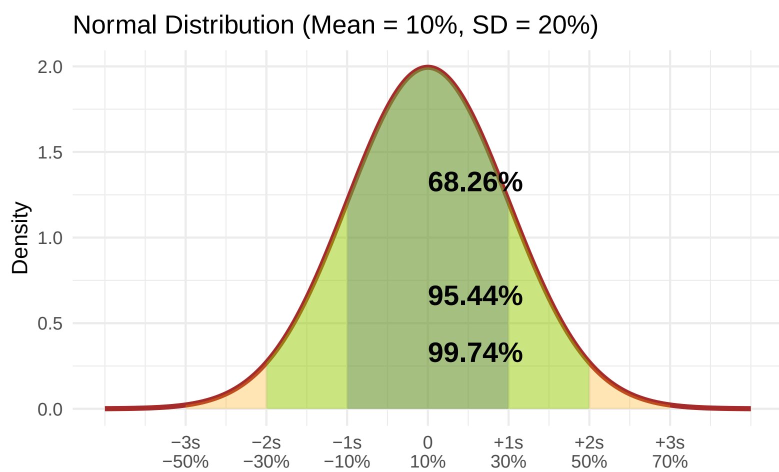
```
Sharpe Ratio for BRK-A: 0.5919776
```

- Risk premium: $E(r) - r_f$ vs. Excess return: $r - r_f$.

Normal Distribution

- We commonly assume that returns are normally distributed, $N(\mu, \sigma^2)$

```
Warning: Using `size` aesthetic for lines was deprecated in ggplot2 3.4.0.
i Please use `linewidth` instead.
```



Normal Distribution

- Probability distribution function, (N(, ^2))

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

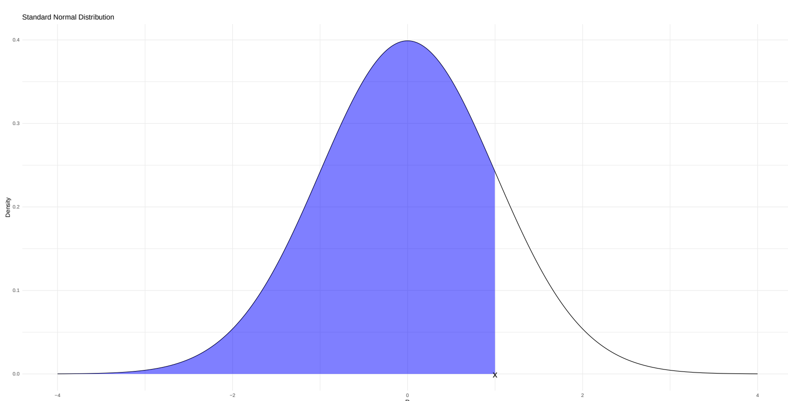
- Standard normal, (N(0,1))

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{r^2}{2}}$$

- If $Z \sim \phi(0,1)$, then $X = \mu + \sigma Z \sim N(\mu, \sigma^2)$
- If $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$, then $X + Y \sim N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2 + 2cov(X,Y))$

Standard Normal Random Variables

- Consider a normal random variable R with $\mu = 0$ and $\sigma = 1$. In other words, $R \sim N(0,1)$. We call it a standard normal random variable.
- Suppose that we want to find the probability that R is lower than x. Graphically, this probability is the shadowed area in the figure below:



Standard Normal Random Variables

- To find this probability, we calculate

$$\text{Prob}(R \leq x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{r^2}{2}} dr \equiv \Phi(x).$$

  $\Phi(x)$ is called the cumulative probability distribution function for a standard normal random variable.

- For any x, the value of $\Phi(x)$ can be found using the excel function, *norm.s.dist*(x, TRUE).

Standard Normal Random Variables

- ***Ex.1*** Suppose that $R_1 \sim \phi(0, 1)$. What is the probability that R_1 is larger than 1?

```
prob_R1 <- 1 - pnorm(1, mean = 0, sd = 1)
cat("P(R1 > 1) =", prob_R1, "\n")
```

```
P(R1 > 1) = 0.1586553
```

- ***Ex.2*** Suppose that $R_2 \sim \phi(0.1, 0.2)$. What is the probability that R_2 is equal to or smaller than 0.5?

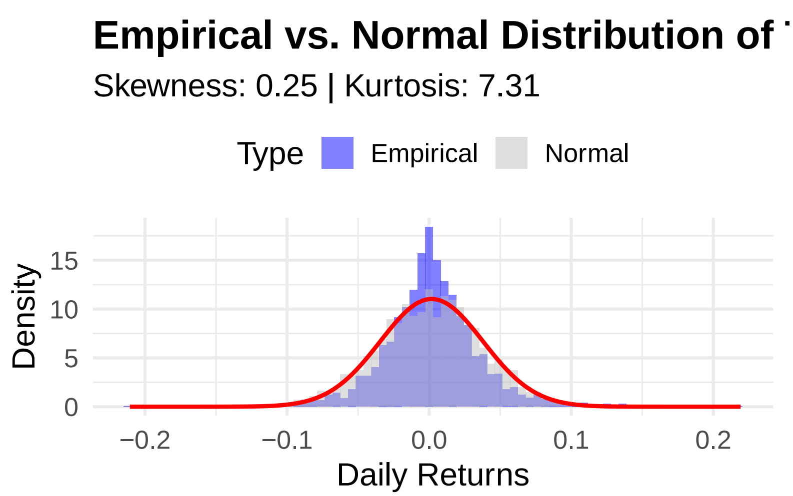
```
prob_R2 <- pnorm(0.5, mean = 0.1, sd = 0.2)
cat("P(R2   0.5) =", prob_R2, "\n")
```

```
P(R2   0.5) = 0.9772499
```

Historic vs. Normal Distribution

- Historical returns often deviate from the normal distribution.
- Empirical distributions can exhibit skewness and kurtosis (fat tails).

```
Attaching package: 'moments'


The following objects are masked from 'package:PerformanceAnalytics':

    kurtosis, skewness


[1] "TSLA"
```



Log-Normal Distribution

- $X \sim LN(\mu, \sigma^2)$ if $\ln(X) \sim N(\mu, \sigma^2)$
- If a rate of return is normally distributed, security prices follow lognormal distribution.
  - log-return $= \ln R_t = \ln \frac{P_t}{P_{t-1}} = \ln P_t - \ln P_{t-1} \sim N(\cdot)$

- By the way,
  - $E(X) = e^{\mu+\sigma^2/2}$. In fact $E(X^n) = e^{n\mu+n^2\sigma^2/2}$
  - $Var(X) = E(X^2) - E(X)^2 = e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2} = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1)$

Risk Measures

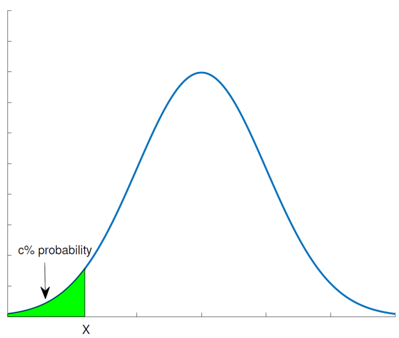
- Companies must assess and manage risks to avoid business failures.
- To understand the risk level of a project or business, we can analyze the probability distribution of possible outcomes.
- Several risk measures are commonly used:
  - Standard Deviation
  - Value at Risk (VaR)
  - Expected Shortfall
  - ...
- Each measure focuses on different aspects of the distribution.

Standard Deviation

- Standard deviation measures the level of uncertainty about the outcomes, or the dispersion of probability distribution.
- The larger standard deviation is, the riskier a project.
- A disadvantage of the standard deviation is that it cannot distinguish between upside and downside movement.

Value at Risk

- Value at Risk (VaR) represents the potential loss in value of a portfolio, given a certain probability over a specific time period.
  - Key question: How much could we lose with a given probability
  - E.g., with a 5% probability, our portfolio may experience a loss greater than the VaR amount.
  - I.e., There is a 95% probability that our loss will not excees the VaR amount.



- That is, we want to find X such that

$$\text{Prob}\left(R \leq X\right) = 0.05$$

Value at Risk

- How can we find X satisfying $\Pr\left(R \leq X\right) = 0.05$, i.e., 95% VaR?
- In a special case when $R \sim \phi(\mu, \sigma)$, we can find X using the Excel function `norm.inv()`.
  - For given $1-p$, `norm.inv(1-p, mean, sigma)` is X that satisfies $\text{Prob}\left(R \leq X\right) = 1 - p$.

* VaR at 5% = `norm.inv(0.05,0,1) = -1.645`
(R: `qnorm(0.05, mean = 0, sd = 1)`)
* VaR at 10% = `norm.inv(0.1,0,1) = -1.282`
(R: `qnorm(0.10, mean = 0, sd = 1)`)

Value at Risk - Example I

- Suppose we own a stock whose return is normally distributed with a mean of 15% and a standard deviation of 30%. What is the 95% Value at Risk (VaR) for this stock?

  Answer: Let X denote the 95% VaR. Then, $\Pr(R \leq X) = \texttt{norm.inv}(0.05, 0.15, 0.30) = -34.3\%$

  Alternatively,

  $$\text{Prob}\,(R \leq X) = \text{Prob}\left(\frac{R - 0.15}{0.3} \leq \frac{X - 0.15}{0.3}\right) = 0.05$$

  Note that $\frac{R-0.15}{0.3} \sim \phi(0,1)$, so we can write

  $$\frac{X - 0.15}{0.3} = \texttt{norm.s.inv}(0.05) = -1.645.$$

  Thus, $X = -34.3\%$.

Value at Risk - Example II

- ***Q.*** A portfolio worth \$10 million has a 1-day standard deviation of \$200,000 and an approximate mean of zero. Assume that the change is normally distributed. What is the 1-day 99% VaR for our portfolio consisting of a \$10 million position in Microsoft? What is the 10-day 99% VaR?

Answer: `norm.s.inv(0.01)` $= -2.326$, meaning that there is a 1% probability that a normally distributed variable will decrease in value by more than 2.326 standard deviations.

Hence, the 1-day 99% VaR is $2.326 \times \$200,000 = \$465,300$.

The 10-day 99% VaR is $2.326 \times (\$200,000 \times \sqrt{10}) = \$1,471,300$.

Value at Risk - Multiple Stocks

- Consider a portfolio consisting of $\boldsymbol{n}$ different stocks.
- The return on the portfolio is

$$R_p = \sum_{i=1}^{n} w_i R_i$$

  where w_i is the fraction of wealth invested in stock i.
- If each stock return is normally distributed, then the portfolio return is also normally distributed.

Value at Risk - Multiple Stocks

- ***Ex.*** Consider a portfolio consisting of stock A and stock B. In the portfolio, \$5 million are invested in each of stock A and stock B. The return on each stock is normally distributed. Stock A has an expected return of 15% and a standard deviation of 30%. Stock B has an expected return of 18% and a standard deviation of 45%. The correlation between stock A and stock B is 0.4. What is the 90% VaR for the portfolio?
  - **Note:** When $X \sim \phi(\mu_x, \sigma_x^2)$ and $Y \sim \phi(\mu_y, \sigma_y^2)$, then $X + Y \sim \phi(\mu_x + \mu_y, \sigma_x^2 + \sigma_y^2 + 2\rho\sigma_x\sigma_y)$

Answer:

– The expected return of the portfolio is:

$$\mu_p = 0.5 \times 0.15 + 0.5 \times 0.18 = 0.165 \text{ or } 16.5\%$$

– The standard deviation of the portfolio is:

$$\sigma_p = \sqrt{(0.5 \times 0.30)^2 + (0.5 \times 0.45)^2 + 2 \times 0.5 \times 0.5 \times 0.4 \times 0.30 \times 0.45} = 0.315 \text{ or } 31.5\%$$

– The 90% VaR for the portfolio is:

$$\text{VaR}_{90\%} = \mu_p + \sigma_p \times \texttt{norm.s.inv}(0.10) = 0.165 + 0.315 \times (-1.282) = -0.239 \text{ or } -23.9\%$$

– Therefore, the 90% VaR for the \$10 million portfolio is:

$$10,000,000 \times 0.239 = \$2,390,000$$

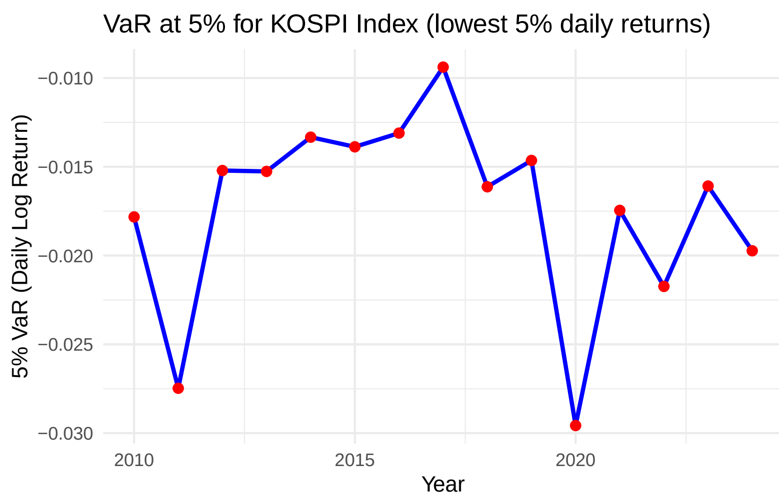
Value at Risk - Historical Data

- We can also calculate the VaR using historical data without assuming a specific distribution.
- For example, let's consider 1-year-long historical data of daily returns for a stock price index.
- We aim to estimate the 5% VaR for the next day's return.
- To do this, we assume that the next day's return will be similar to one of the past year's returns.
- The 5% VaR is then the 5th percentile of these historical returns.

Value at Risk - Some Issues I

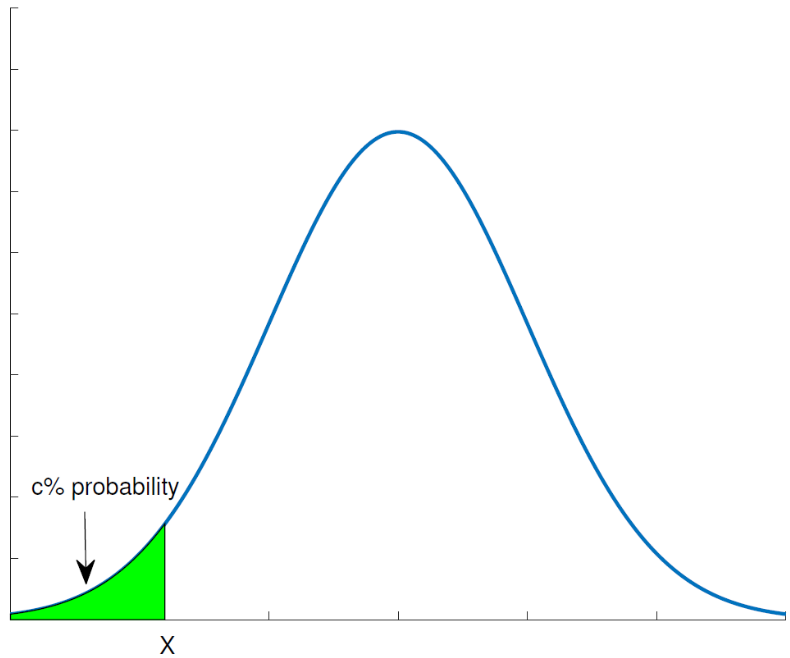
- VaR estimation is based on the assumption that the distribution of future return is the same as (at least similar to) the distribution of past return.
- This assumption may not hold in the real world.

```
######################### Warning from 'xts' package ##########################
#                                                                              #
# The dplyr lag() function breaks how base R's lag() function is supposed to  #
# work, which breaks lag(my_xts). Calls to lag(my_xts) that you type or       #
# source() into this session won't work correctly.                            #
#                                                                              #
# Use stats::lag() to make sure you're not using dplyr::lag(), or you can add #
# conflictRules('dplyr', exclude = 'lag') to your .Rprofile to stop           #
# dplyr from breaking base R's lag() function.                                #
#                                                                              #
# Code in packages is not affected. It's protected by R's namespace mechanism #
# Set `options(xts.warn_dplyr_breaks_lag = FALSE)` to suppress this warning.  #
#                                                                              #
################################################################################


Attaching package: 'dplyr'

The following objects are masked from 'package:xts':

    first, last

The following objects are masked from 'package:stats':

    filter, lag

The following objects are masked from 'package:base':

    intersect, setdiff, setequal, union

Warning: ^KS200 contains missing values. Some functions will not work if
objects contain missing values in the middle of the series. Consider using
na.omit(), na.approx(), na.fill(), etc to remove or replace them.

[1] "KS200"

Warning in to_period(xx, period = on.opts[[period]], ...): missing values
removed from data
```

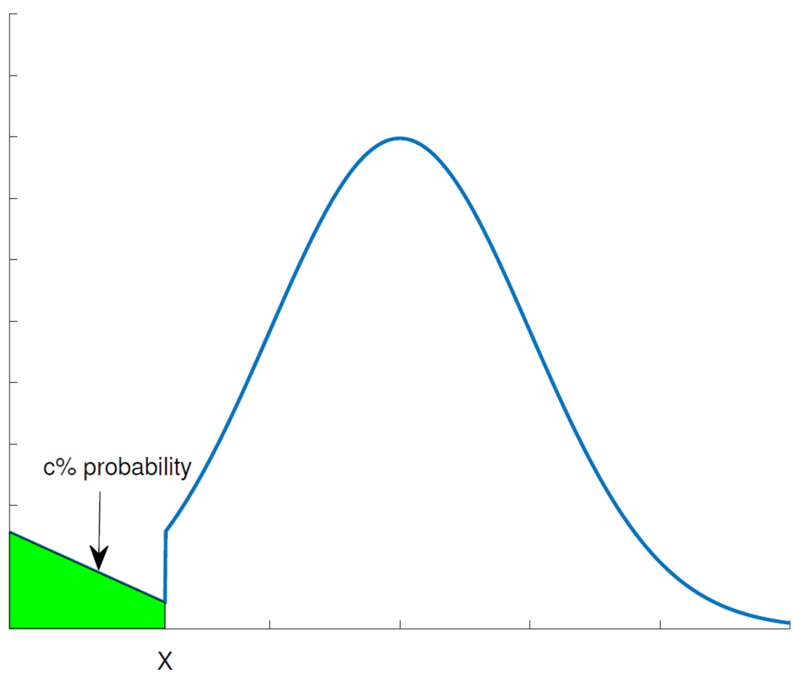


Value at Risk - Some Issues II

- VaR specifies the ***minimum*** loss that will occur with a given probability.
- VaR tells nothing about the expected magnitude of the loss.
- Which is the better between the following two?



c% probability
X



c% probability
X

Expected Shortfall

- ***Expected Shortfall*** is another measure to address the shortcoming of VaR.
  - It asks ***"If things get bad, what is the expected loss?"***
- Suppose that we focus on the loss that will happen with 5% probability. Let $\boldsymbol{V}$ denote the 5% loss (VaR). Then,

$$\text{Expected shortfall} = E\left(R|R \leq V\right)$$

- Also known as Conditional Value at Risk (CVaR)

- Under normal distribution: ES $= \mu - \sigma \frac{\phi((V-\mu)/\sigma)}{\Phi((V-\mu)/\sigma)}$

Expected Shortfall

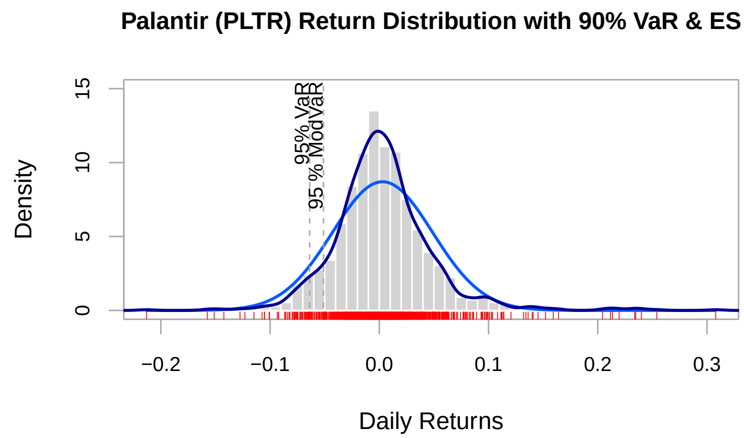
- Once historical data are given, we can compute the expected shortfall.
  - In Excel, use ***"averageif()"***.
- ***Ex.*** Let's use the 1-year-long data of daily returns on a stock index.
  - ***Q1.*** What is the expected shortfall with 5% probability?
  - ***Q2.*** What is the expected shortfall with 10% probability?

Value at Risk and Expected Shortfall

```
[1] "PLTR"
```

```
90% Historical VaR: -0.046151
```

```
90% Expected Shortfall: -0.070268
```



Application: Bank Regulation

- VaR and ES are widely used in the financial industry to measure and manage risk.
- The Basel Committee on Banking Supervision (BCBS) provides global banking regulations.
- Key frameworks include:
  - **1996 Amendment**: Required capital = ***k*** $\times$***VaR (1%, 10 days)***, where ***k*** ***3***.
  - **Basel II (2007)**: Suggested ***VaR(0.1%, 1-year)*** for risk assessment.
  - **Basel IV (2021)**: Recommended 97.5% expected shortfall (ES) for a comprehensive risk view.