Fixed Income Securities

BUSS254 Investments

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Lecture Outline

- Coupon bonds vs. Zero (coupon) bonds
- Zero prices and zero (spot) rates
- Forward rates
- Yield to Maturity
- Swap rates
- Interest rate risk: measurement and management
- Term structure of interest rates
- Reading: BKM, Ch. 14, 15, and 16, Tuckman and Serrat, Ch. 1 - 4

Fixed Income: Basics

Fixed Income Securities

- A promise to deliver future known cash flows.
  - Examples: money market instruments, treasury bonds, notes, and bills, corporate bonds etc.
- Terminologies
  - Indenture: contract between issuer and holders specifying interest, principal, and other items (covenants).
  - Par (Face) value: the amount of money paid at maturity.
  - Maturity: the end date of the security's life.
  - Coupon payments: interest payments made periodically through the life of the security.

    - Coupon rate = Coupon/Par value
    - Current yield = Coupon/Price
  - Yield to Maturity: the internal rate of return (IRR) of a bond investment

Types of Bonds

- Callable bonds
  - The issuer can repurchase the bond at a specified call price before the maturity.
  - Deferred callable bonds: come with a period of call protection (not-callable period).
  - Puttable bonds: gives the issuer the option
- Convertible bonds
  - Give bondholders an option to exchange each bond for a specified number of shares of common stock (conversion ratio)
  - Deferred callable bonds: come with a period of call protection (not-callable period).
- Floating-Rate Bonds
  - Coupon rate varies inversely with a benchmark interest rate.
- International bonds
  - Foreign bonds: foreign issuer issues in domestic currency
    - Yankee, Samurai, Bulldog, Dim-sum bonds etc.
  - Eurobonds: issue in foreign country with domestic currency
    - Eurodollar, Euroyen, Euro-euro etc.

Types of Bonds (cont'd)

- Asset-Backed Bonds
  - The income from a specified group of assets is used to service the debt.
  - Business revenue: movie sales, real estate etc.
- Catastrophe Bonds
  - Payments halted/cancelled if earthquake, terror, pandemic etc.
- Indexed Bonds
  - Payments tied to an index
  - Treasury Inflation Protected Securities (TIPS).

Coupon Bonds

- The quoted coupon rate is annualized.
  - If the quoted coupon rate is c, and bond maturity is time T, then each \$1 par value (quantity) of the bonds pays out cash flows (number of payments per year: N)

0

0.5

c/N

1

c/N

...

T

1 + c/N

- Institutionally speaking, the prices of government bonds form the basis for the pricing in fixed income markets.
- All other fixed income instruments, including derivatives, are priced in relation to the prices of these benchmark bonds

Zero-Coupon Bonds

- It is convenient to unpack the original coupon bonds into individual zero-coupon bonds, or zeroes - bonds with a single cash flow equal to face value at maturity.
  - Treasury zeroes called STRIPS (Separate Trading of Registered Interest and Principal of Securities) are actually traded in a secondary market (since 2007 in Korea, 1960s in US).
  - The process of detaching the interest payments from the bond is called coupon stripping
    * E.g., a 10-year bond with a $40,000 face value and a 5%, semi-annual coupon payments: 21 zero-coupon bonds can be created (=20 semi-annual coupon + the bond itself). Each stripped coupon has a $1,000 face value. All 21 securities are distinct and are traded separately in the market.

Zero Prices or "Discount Factors''

- Let d_t denote the price today of the t-year zero that pays off $1 in t years.
  - This is also called the t-year "discount factor.''
- A Coupon Bond as a Portfolio of Zeroes
  - $10,000 par of a one and a half year, 8.5% bond makes the following payments:

0

0.5

$425

1

$425

1.5

$10,425

- This is the same as a portfolio of three different zeroes:
  - $425 par of a 6-month zero
  - $425 par of a 1-year zero
  - $10425 par of a 1 1/2-year zero
- Also same as an annuity + a zero

No Arbitrage and The Law of One Price

- The Law of One Price: assets which offer the same cash flows must sell for the same price.
  - Why? If not, then one could buy the cheaper asset and sell the more expensive, making a profit today with no cost in the future.
  - This would be an arbitrage opportunity, which could not persist in equilibrium in a frictionless market
  - An arbitrage trading strategy: always generates non-negative cash flows and sometimes strictly positive cash flows.
- However, when there are "limits to arbitrage'' such as transaction costs, capital constraints, or barriers to trading across markets, then violations of the law of one price can persist.
  - For example, prices of Treasury STRIPS and Treasury bonds don't fit the pricing relationship exactly

Example

Years to maturity	Discount factor	Bond cash flow	Value
0.5	0.9730	$425	$414
1	0.9476	$425	$403
1.5	0.9222	$10,425	$9,614
			$10,430

- The value of $10,000 par of a 1.5-year 8.5% Treasury coupon bond based on the zero prices (discount factors) in the table is $10,430.
  - If the 1.5-year 8.5% coupon bond deviates significantly from this value, it'd create an arbitrage opportunity.
- The LOOP implies that a coupon bond price is (per $1 face value):

$$P(c, T) = (c/2) \times d_{0.5} + (c/2) \times d_1 + ... + (1 + c/2) \times d_T$$

Deriving Zero Prices from Coupon Bond Prices

- Consider three coupon bonds below. Find discount factors.

Bonds	Years to maturity	Coupon	Prices
A	0.5	4.250%	99.40625
B	1	4.375%	98.96875
C	1.5	8.500%	104.31250

1. $(1 + 0.0425/2)d_{0.5} = 0.9940625 \Rightarrow d_{0.5} = 0.973$
2. $(0.04375/2)d_{0.5} + (1 + 0.04375/2)d_1 = 0.9896875 \Rightarrow d_1 = 0.948$
3. $(0.085/2)d_{0.5} + (0.085/2)d_1 + (1 + 0.085/2)d_{1.5} = 1.043125 \Rightarrow d_{1.5} = 0.922$, which closely match the actual STRIPS prices

- You just found a term structure.
- This process is called the bootstrap method.

Bootstrapping

$$P(T) = \sum_{t=1}^{2T} \$c \frac{1}{(1 + r_t/2)^t} + \$M \frac{1}{(1 + r_T/2)^{2T}}$$

$$\Rightarrow r_T = \left(\frac{c + M}{P(T) - \sum_{t=1}^{2T-1} \frac{c}{(1+r_t/2)^t}} \right)^{\frac{1}{2T}} - 1$$

- Using matrices: Suppose we have T coupon bonds and the maximum maturity among these bond is T years.

$$\begin{bmatrix} P(T_{0.5}) \\ \cdots \\ P(T_T) \end{bmatrix} = \begin{bmatrix} CF_{0.5}^1 & \cdots & CF_T^1 \\ \cdots & \ddots & \cdots \\ CF_{0.5}^T & \cdots & CF_T^T \end{bmatrix} \begin{bmatrix} d_{0.5} \\ \cdots \\ d_T \end{bmatrix}$$

$$\Rightarrow P = CF \times D$$

$$\Rightarrow D = CF^{-1} P$$

Bootstrapping (cont'd)

$$P = \begin{bmatrix} 101.50 \\ 99.80 \end{bmatrix} \text{ and } CF = \begin{bmatrix} 105.25 & 0 \\ 4.43 & 104.43 \end{bmatrix}$$

$$D = CF^{-1}P$$

- Use `mmult` and `minverse` functions in Excel.

Regression Approach

- The bootstrap method is only feasible when the number of coupon bonds = the number of time periods.

$$\text{Pricing error: } P - CF \times D \equiv \epsilon$$

$$\text{Solve } \min_D \sum_{i=1}^{N} \epsilon_i^2 = \sum_{i=1}^{N} \left(P(T_i) - \sum_{t=1}^{T} CF_t^i D_t^i \right)$$

$$\text{The solution: } D = (CF'CF)^{-1}(CF'P)$$

- Instead of finding the precise values for D, we try to find the best estimate of D that predicts P.

Parametric Yield Curve Models

- The resulting estimated yield curve is usually not smooth
- Interpolation is often used to calculate the discount factor/spot rate for maturity that we do not have.
- Parametric function to model the discount factor or the spot curve (or the forward curve) can give you the desired "smoothness' ".
  - At the potential expense of higher pricing error.
- Example: The Polynominal Yield Curve

$$r_t = \alpha + \beta_1 t + \beta_2 t^2 + \beta_3 t^3$$

- Popular models: Nelson-Seigel model, Svennson model etc.

Excercise

- Consider the following three bonds

Bond	t=1	t=2	t=3	Price
A	100	0	0	96.0
B	5	105	0	99.3
C	10	10	110	108.8

- There is bond D whose price is \$84 and pays 100 in t=3. Is there an arbitrage opportunity?

$$\begin{bmatrix} 96.0 \\ 99.3 \\ 108.8 \end{bmatrix} = \begin{bmatrix} 100 & 0 & 0 \\ 5 & 105 & 0 \\ 10 & 10 & 110 \end{bmatrix} \begin{bmatrix} DF_1 \\ \cdots \\ DF_T \end{bmatrix}$$

- $DF_1 = 0.96, DF_2 = 0.90, DF_3 = 0.82.$
- Hence, no-arbitrage price, $Price(D) = 0.82 \times 100 = 82$

Excercise (cont'd)

- Can you find an abitrage strategy (portfolio)?

$$\begin{bmatrix} -96 & -99.3 & -108.8 & -84 \\ 100 & 5 & 10 & 0 \\ 0 & 105 & 10 & 0 \\ 0 & 0 & 110 & 100 \end{bmatrix} \begin{bmatrix} x_A \\ x_B \\ x_C \\ x_D \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Solve for vector X. The choice of the payoff on the RHS is arbitrary.

Zero (Spot) Rates

- People prefer to quote interest rate than prices because it is easier to understand.
- If you buy a t-year zero and hold it to maturity, you lend at rate r_t where r_t is defined by

$$d_t \times \left(1 + \frac{r_t}{2}\right)^{2t} = 1,$$

or

$$r_t = 2 \times \left(d_t^{-\frac{1}{2t}} - 1 \right)$$

- Call r_t the t-year zero rate or t-year discount rate.
  - Spot rate: a spot rate is the rate on a spot loan, directly observable from, e.g., STRIPS price
  - Zero rate: extracted from, e.g., Treasury (coupon) bonds. Proxy for spot rate.
  - Obviously, we can use either d_t or r_t to value bonds.

Example

- Suppose STRIPS rates are as follows:
  - 0.5-year rate: 5.54%
  - 1-year rate: 5.45%
  - 1.5-year rate: 5.47%
- What is 0.5-year and 1-year zero prices?

$$d_t = \frac{1}{(1 + \frac{r_t}{2})^{2t}}.$$

Therefore,

$$d_{0.5} = \frac{1}{\left(1 + \frac{0.0554}{2}\right)^{2 \times 0.5}} = 0.973,$$

and

$$d_1 = \frac{1}{\left(1 + \frac{0.0545}{2}\right)^{2 \times 1}} = 0.9476.$$

and

$$d_{1.5} = ?$$

Yield to Maturity

- Yield to maturity is a convenient way to express price in terms of a single rate of interest (IRR).
- It is the single y that solves:

$$\frac{c/2}{(1+r_{0.5}/2)^1} + \frac{c/2}{(1+r_1/2)^2} + ... + \frac{c/2+M}{(1+r_T/2)^{2T}}$$

$$= \frac{c/2}{(1+y/2)^1} + \frac{c/2}{(1+y/2)^2} + ... + \frac{c/2+M}{(1+y/2)^{2T}}$$

- Realized return can be different from the YTM.
  - Purchase price, coupon, sales price, and reinvestment rate of coupon
  - Ex-post return is equal to its initial yield if all of the coupons are reinvested at the initial yield

Yield to Maturity (cont'd)

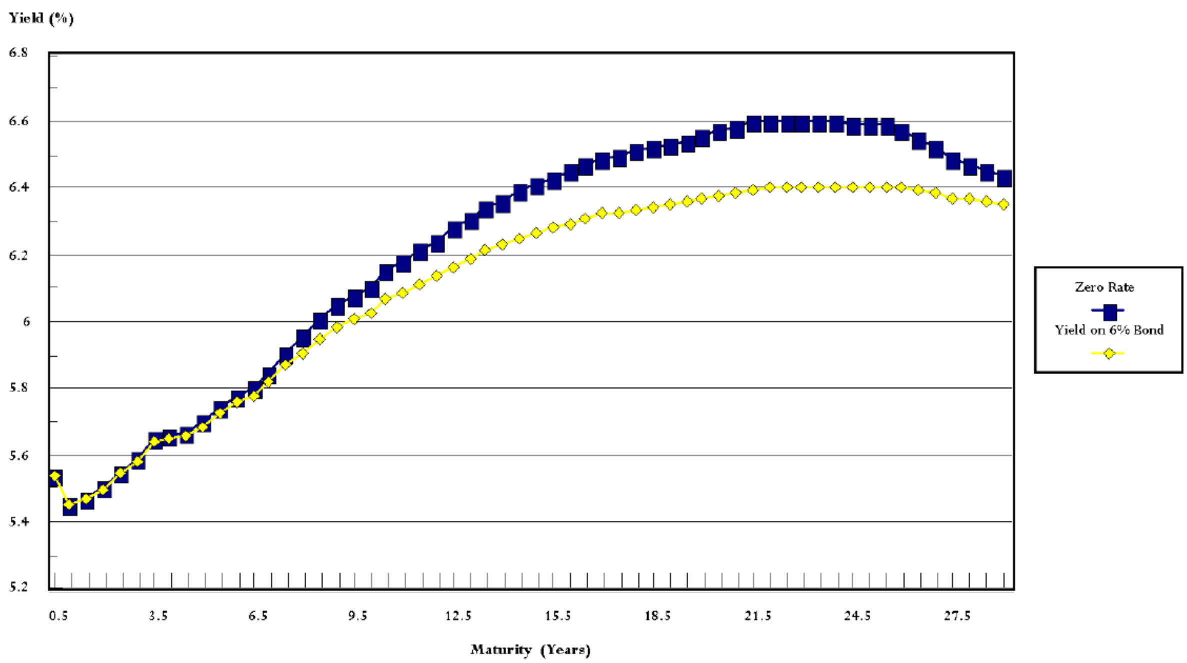
- Example: The 1.5-year, 8.5%-coupon bond
- Using the zero rates 5.54%, 5.45%, and 5.47%, the bond price is 1.043066 per dollar par value.
- This implies a yield of 5.4704%:

$$\begin{aligned} 1.043066 &= \frac{0.0425}{(1+0.0554/2)^1} + \frac{0.0425}{(1+0.0545/2)^2} + \frac{1.0425}{(1+0.0547/2)^3} \\ &= \frac{0.0425}{(1+0.54704/2)^1} + \frac{0.0425}{(1+0.54704/2)^2} + \frac{1.0425}{(1+0.54704/2)^3} \end{aligned}$$

- YTM $\approx$ weighted average of the t-zero rates, where the weight is proportional to the relative PV of cash flow at t. (Will see this later)
- YTM (and IRR) assumes that coupons are reinvested at the YTM.

Yield Curves for Zeroes and 6% Bonds

- A yield curve plots yields or zero rates for different maturities of bonds. It depicts the term structure of interest rates.
  - A yield curve should be based on a group of homogeneous bonds (e.g. credit and liquidity)
- Why does the coupon bond yield curve lie below the zero curve?



Source: Prof. Jennifer Carpenter, 2017

The Coupon Effect

- Yields vary across bonds for different reasons – credit, liquidity, maturity. It turns out they also vary with coupon.

1. If the yield curve is not flat, then bonds with the same maturity but different coupons will have different yields.
2. If the yield curve is upward-sloping, then for any given maturity, higher coupon bonds will have lower yields.

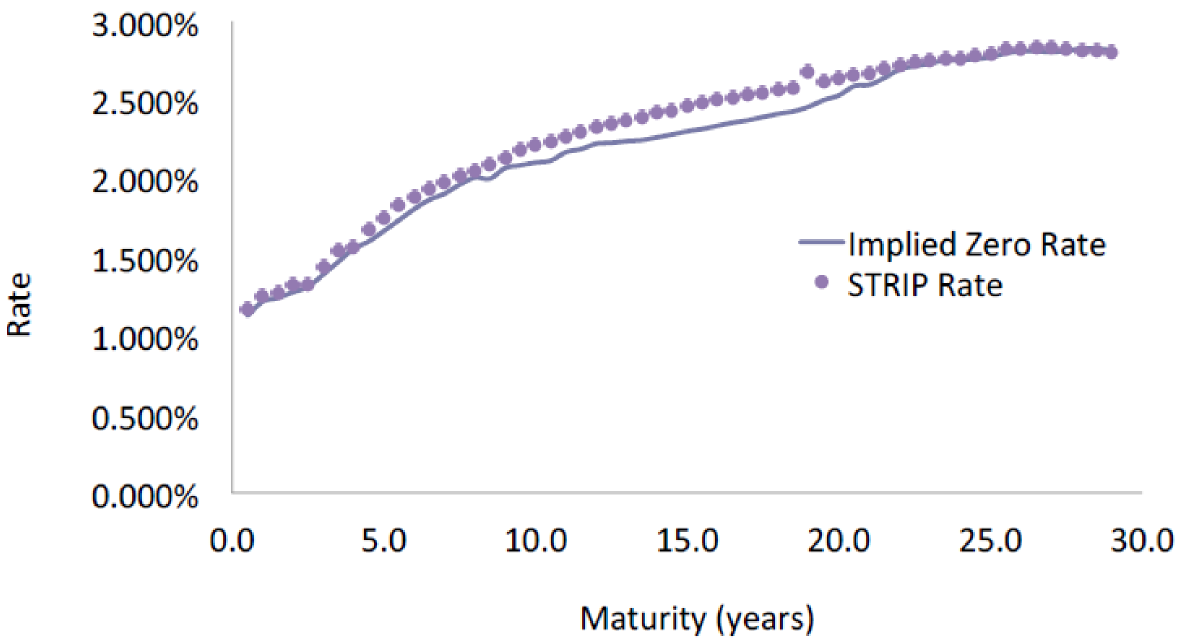
- PV of c and the associated zero rates before maturity contributes more to the YTM when c is larger. As zero rates are increasing in time, the weighted average becomes smaller, i.e., the yield is smaller.

3. If the yield curve is downward-sloping, then for any given maturity, higher coupon bonds will have higher yields.

- Therefore, it is not necessarily true that the bond with a higher yield is a superior investment.

Yield Curve of US Treasury Zero Rates

- This graph plots the zero rates implied by Treasury coupon bond prices (line), and the actual traded Treasury STRIPS rates (dots).
- It shows that the Law of One Price holds very closely across the US Treasury coupon bond and STRIPS markets.



Prof. Jennifer Carpenter, 2017

Yield and Price

- Present value of a coupon bond with \$1 par value

$$P = \frac{c/2}{y/2}\left(1 - \frac{1}{(1+y/2)^{2T}}\right) + \frac{1}{(1+y/2)^{2T}}$$

(Bond = Annuity + Zero)

- If $c = y, P = 1$ (the bond is priced at par)
- If $c > y, P > 1$ (the bond is priced at a premium to par)
- If $c < y, P < 1$ (the bond is priced at a discount)
- The yield on a zero is the zero rate: $c = 0; y = r_T$

Price and Time

- Bond price should converge to the par value at maturity.
  - For a premium bond, each coupon adds to the current bond price.
  - For a discount bond, each coupon erodes the current bond price.
  - As time passes, we lose each coupon.
  - Hence, bond price decreases for a premium bond and increases for a discount bond.

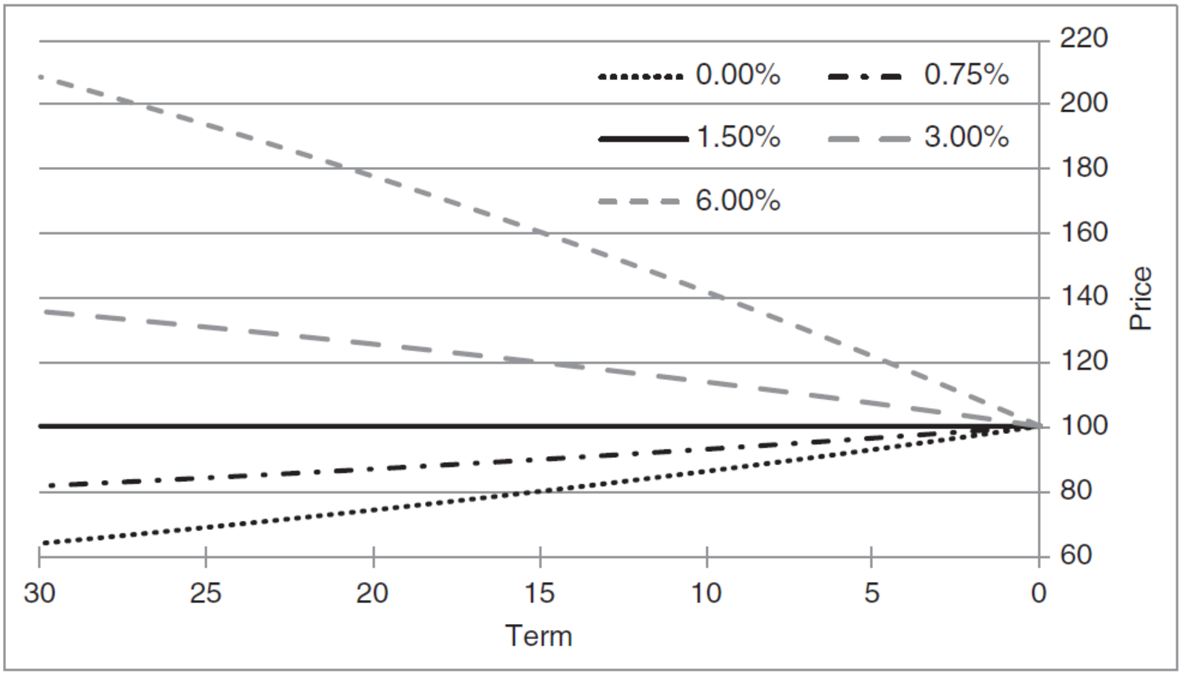


FIGURE 3.1 Prices of Bonds with Different Coupons and Maturities. All Yields Equal 1.5%.

Forward Rates

- The rate on a forward loan, which is an agreement today to lend money at some time in the future and to be repaid some time after that.
- Forward rates are related to spot rates by the no-arbitrage condition.

$$\left(1+\frac{r_t}{2}\right)^{2t} = \left(1+\frac{r_{t-0.5}}{2}\right)^{2t-1}\left(1+\frac{f_{t-0.5,t}}{2}\right)$$

- Denote $ f_{0,0.5} = f_{0.5}$. By definition, $ f_{0.5} = r_{0.5}$

$$\left(1+\frac{r_t}{2}\right)^{2t} = \left(1+\frac{f_{0.5}}{2}\right)\left(1+\frac{f_1}{2}\right)\ldots\left(1+\frac{f_t}{2}\right)$$

- Forward rates can also be expressed in terms of discount factors

$$1+\frac{f_{t-0.5,t}}{2} = \frac{d_{t-0.5}}{d_t}$$

- Multiple-period forward rate, $f_{t-\tau,t}$?

Interest Rate Swaps

- Let's digress...
- Swaps and bonds together comprise a significant portion of fixed income markets, and swaps, because they are relatively liquid, have become benchmarks against which to evaluate other fixed income instruments.
- In an interest rate swap, two parties agree to exchange a series of interest payments.

Interest Rate Swaps (cont'd)

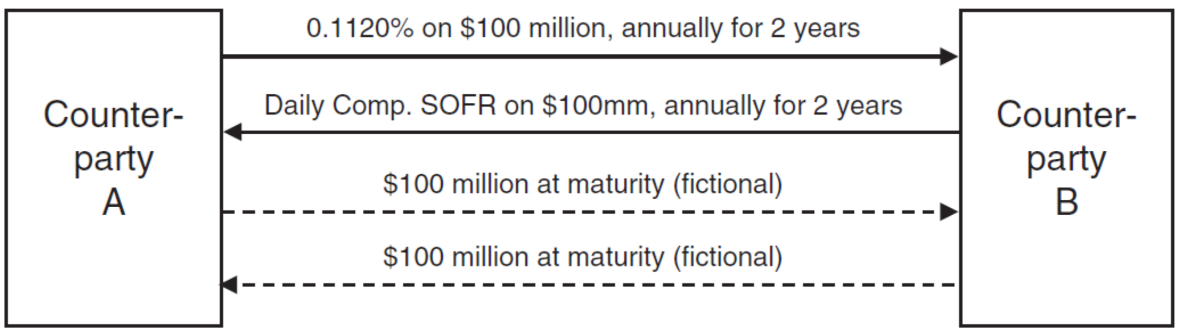


FIGURE 2.1 A SOFR Swap.

- A pays B the fixed *swap rate* of 0.1120% annually for two years on a *notional amount* of $100 million.
- B pays A daily compounded SOFR annually for two years on this same notional amount.
- No cash is exchanged on the trade or settlement date.
- The notional amount is never paid or received by either counterparty
- Daily SOFR represents the rate on extremely safe, overnight loans made that day (replacing LIBOR in 2021)

Interest Rate Swaps (cont'd)

- A SOFR swap follow the actual/360 day-count
  - Payment on fixed lag:

$$\$100,000,000 \times 0.1120\% \times \frac{365}{360} = \$113,556$$

  - Payment on floating lag. Assume SOFR was 0.10% for five days; 0.50% for 170 days; and 0.01% for 190 days:

$$\$100,000,000 \left(1 + \frac{0.1\%}{360}\right)^{5} \left(1 + \frac{0.5\%}{360}\right)^{170} \left(1 + \frac{0.01\%}{360}\right)^{190}$$

$$= \$100,243,071$$

Term Structure of SOFR Rates

The SOFR curve, for example, gives the fixed rates that can be exchanged for SOFR for various terms.

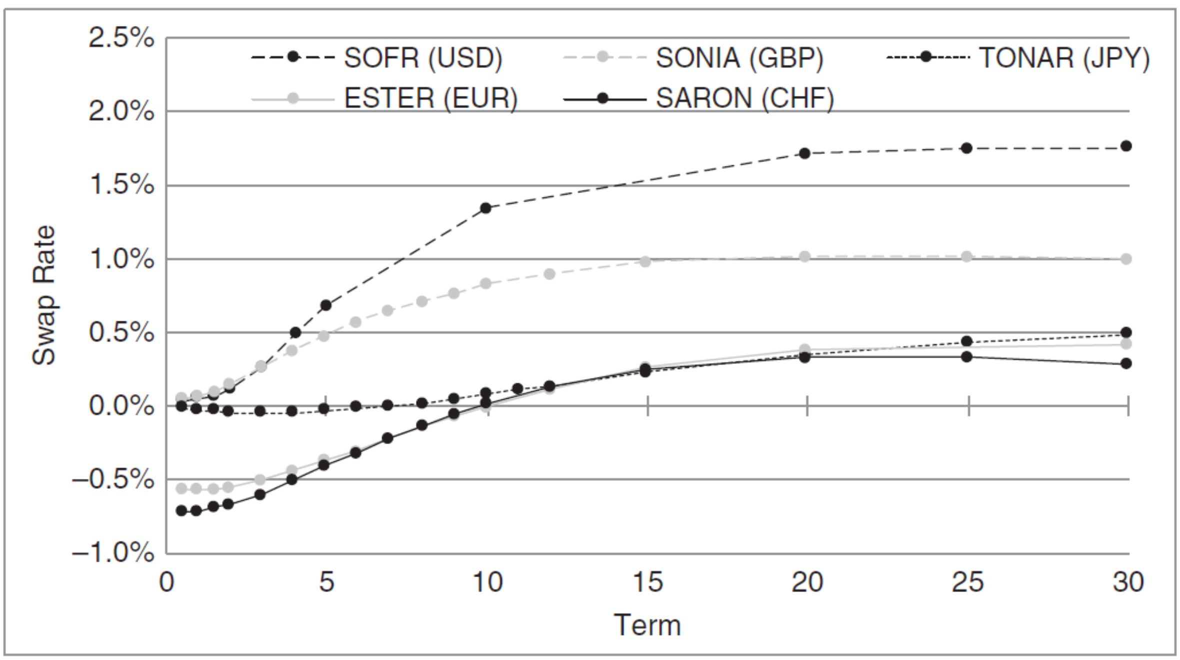


FIGURE 2.2 Term Structures in Different Currencies, as of May 14, 2021.

Pricing Interest Rate Swaps

- A: Long in floating-rate bond $ $ B: Long in fixed-rate bond
  - The value of a floating-rate bond that always pays the fair market rate is worth par, or face amount, today.
  - $P_0 = (1+c)M/(1+r_{0.5}) = (1+r_{0.5})M/(1+r_{0.5}) = M$
- The swap has no cash flow at initiation and at maturity.
  - B buys a fixed-rate bond by borrowing at the floating rate (leveraged long position)
  - Hence, PV(fixed-rate) = PV(floating-rate) = Par value
  - The swap rate is determined so that PV(fixed-rate) = Par value.

Swap Rates

- Swap rate = Par rate
  - Par rate (yield): the rate paid on an investment that costs par and promises to repay par at maturity.
  - Measures the coupon rate where a coupon bond is traded at par.
    * If the yield curve is flat, then the par yield = YTM
    * The par-yield is often used as a reference for pricing new issues: Can determine the coupon rate that a new bond with a given maturity will pay in order to sell at par today.
- Given a (spot) yield curve, we can extract par rate from a coupon bond (M: par value):

$$M = \sum_{t=1}^{2T} \frac{(c^*/2)M}{(1+r_{t/2}/2)^{2t}} + \frac{M}{(1+r_T/2)^{2T}}$$

$$c^* = 2 \times \frac{1 - \frac{1}{(1+r_T/2)^{2T}}}{\sum_{t=1}^{2T} \frac{1}{(1+r_{t/2}/2)^{2t}}} = 2 \times \frac{1-d_T}{\sum_{t=1}^{2T} d_{t/2}}$$

- Note, here, we assume cash flows (coupon) occurs exactly every half year.
- Swap rate is approaximately the average of spot rates.

Extracting discount factors from SOFR swap rates

TABLE 2.1 Swap Rates, Spot Rates, and Forward Rates Implied by USD SOFR Swaps, as of May 14, 2021. Rates Are in Percent.

Term	Swap Rate	Spot Rate	Forward Rate	Discount Factor
0.5	0.0340	0.0348	0.0348	0.999826
1.0	0.0460	0.0466	0.0585	0.999534
1.5	0.0670	0.0681	0.1111	0.998979
2.0	0.1120	0.1136	0.2500	0.997732

SOFR swaps ≤ 1 year make one payment at maturity. > 1 year typically make one stub payment followed by annual payments to maturity. A 1.5-year swap makes a stub payment after six months and another payment one year later.

$$
\begin{aligned}
100\left(1+0.034\%\frac{184}{360}\right)d_{0.5} &= 100 \\
100\left(1+0.046\%\frac{365}{360}\right)d_{1} &= 100 \\
\$0.067\frac{184}{360}d_{0.5}+100\left(1+0.067\%\frac{365}{360}\right)d_{1.5} &= 100 \\
\$0.112\frac{184}{360}d_{1}+100\left(1+0.112\%\frac{365}{360}\right)d_{2} &= 100
\end{aligned}
$$

Solving for d's (can use matrix), you can use discount factors in the table. Can create a yield curve.

Swap, Spot, and Forward Rates

- The spot rate approximately equals the average of the forward rates.

$$
\ln\left(1+\frac{r_t}{2}\right)^{2t} = \ln\left(1+\frac{f_{0.5}}{2}\right)\left(1+\frac{f_1}{2}\right)\ldots\left(1+\frac{f_t}{2}\right)
$$

$$
r_t \approx \frac{f_{0.5}+f_1+\ldots+f_t}{2t}
$$

- Spot rates increase when forward rates are greater than spot rates.
- When spot rates are increasing, swap (par) rates are below spot rates.

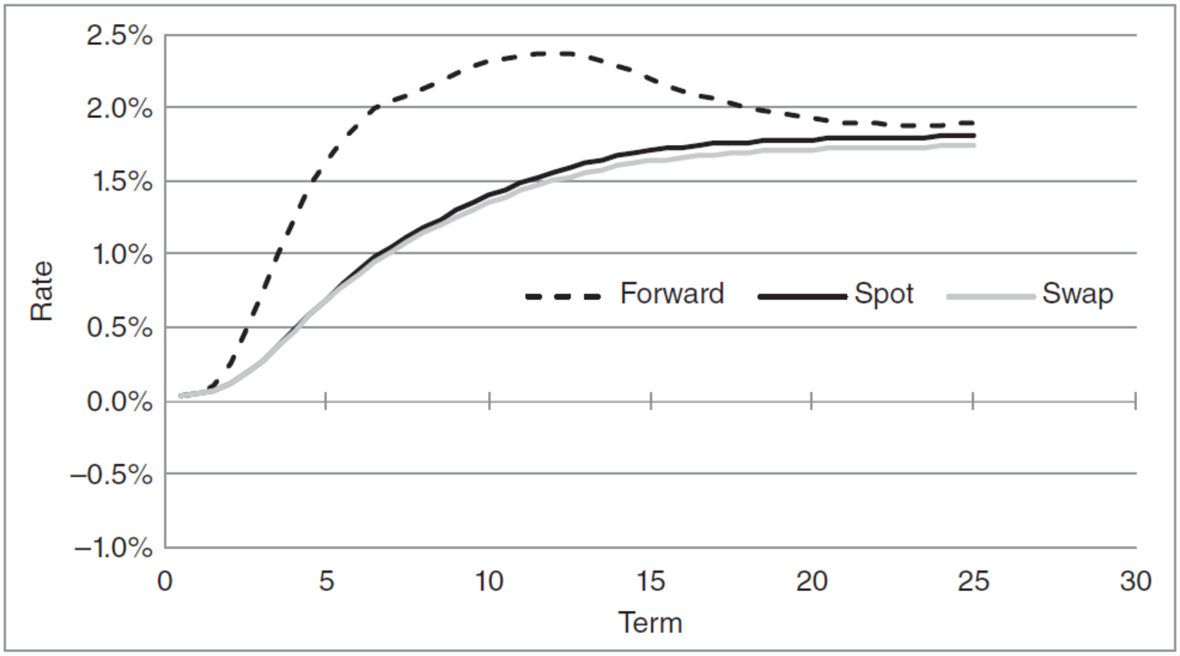


FIGURE 2.3 SOFR Rate Curves, as of May 14, 2021.

Bond Pricing Between Coupon Payment Dates

- Bond price = Quoted price + Accrued interest
  - Bond price: full/dirty price
  - Quoted price: flat/clean price
  - Accrued interest: the portion of the next coupon that the seller has earned but will not receive.
- Suppose you purchase $10,000 face amount of the US Treasury 0.625s of 08/15/2030, for settlement on May 17, 2021. The flat/quoted price is 91.78125.
  - Last coupon payment of $\$10,000 \times 0.625\%/2 = \$31.25$ on February 15, 2021, and makes its next coupon payment of $31.25 on August 15, 2021.
  - You pay the seller the accrued interest: $\$0.625\%/2 \times 91/181 = 0.15711\%$ \$, using the ACT/ACT day-count.
  - Hence, the full price is $\$10,000 \times (91.78125\% + 0.15711\%) = \$9,193.836$

Why not just quote bonds using the full price?

- Market convention!
  - full price changes dramatically over time
  - When trading bonds day to day, it is more intuitive to follow flat prices and negotiate transactions in those terms

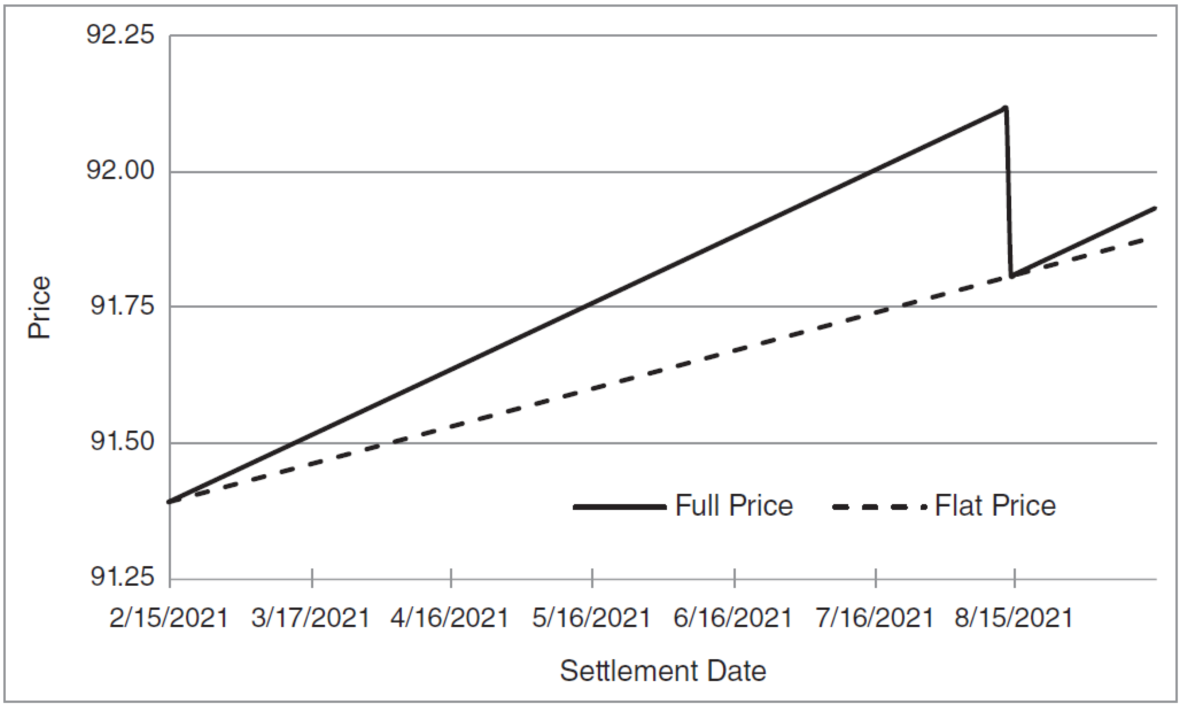


FIGURE 1.4 Full and Flat Prices for the 0.625s of 08/15/2030, Assuming Constant Interest Rates.

Interest Rate Risk

Interest Rate Risk

- Bond investing faces several types of risk: interest rate risk, credit risk, liquidity risk etc.
- Our focus: interest rate risk, the change in bond price due to changes in the interest rates in the market.
- How to measure the exposure to interest rate risk?

  - Senario analysis: compute how much bond price would change if interest rate change by $x\%$.
  - In the old days without sufficient computing power, this was not easy.
  - People came up with "simpler'' and approximate measure of interest rate risk.
  - These measures prove to be useful in modern days for commmunication and risk management purpose.

Which Interest Rate?

- We want to measure the bond price sensitivity of interest rate.
- But which interest rate?
  - Spot rate, zero rate, swap rate, forward rate, yield etc. and their term structures (1-, 2-yr rate etc.).
- They tend to move together, but not exactly.
  - We will first study the measures based on yield-to-maturity, i.e., the sensitivity of bond price to a change in yield. That is, set $r_t = y$ in the bond pricing formula.
  - They are simpler and widely used in practice.

Dollar Duration and DV01

NB Beware of units (decimal or percentage). It's confusing!

- Dollar duration: $D_d = -dP/dy$
  - The change in the dollar price of a bond for a 1 unit (i.e. 100%) change in interest rates (per \$100 face)
- $DV01$ (Dollar Value of 0.01%): $DV01 = D_d/10,000, (100\% = 10,000bps)$
  - The change in the price of a bond for 1 bps of change in interest rates (per \$100 face)
  - a.k.a Dollar Value of a Basis Point (DVBP), Present Value of a Basis Point (PVBP).

Dollar Duration and DV01 (cont'd)

Par value: 100, Coupon rate: c

$$P(y) = \frac{100c}{2} \sum_{t=1}^{2T} \frac{1}{(1+\frac{y}{2})^t} + \frac{100}{(1+\frac{y}{2})^{2T}}$$

$$P(y) = \frac{100c}{y} \left(1 - \frac{1}{(1+\frac{y}{2})^{2T}}\right) + \frac{1}{(1+\frac{y}{2})^{2T}}$$

Then $DV01 = \frac{1}{10{,}000} D_d = \frac{1}{10{,}000} dP/dy$.

$$DV01 = \frac{1}{10,000} \frac{1}{1+\frac{y}{2}} \left[\frac{100c}{2} \sum_{t=1}^{2T} \frac{t}{2} \frac{1}{(1+\frac{y}{2})^t} + T \frac{100}{(1+\frac{y}{2})^{2T}} \right]$$

$$DV01 = \frac{1}{10,000} \left[\frac{100c}{y^2} \left(1 - \frac{1}{(1+\frac{y}{2})^{2T}}\right) + T \left(1 - \frac{c}{y}\right) \frac{100}{(1+\frac{y}{2})^{2T+1}} \right]$$

Duration

- (Modified or Adjusted) Duration: $D = -\frac{(dP/P)}{dy} = D_d/P$
  - The *percent* change in price for a 100 basis point change in yield.

$$D = \frac{1}{P} \frac{1}{1+\frac{y}{2}} \left[\frac{100c}{2} \sum_{t=1}^{2T} \frac{t}{2} \frac{1}{(1+\frac{y}{2})^t} + T \frac{100}{(1+\frac{y}{2})^{2T}} \right]$$

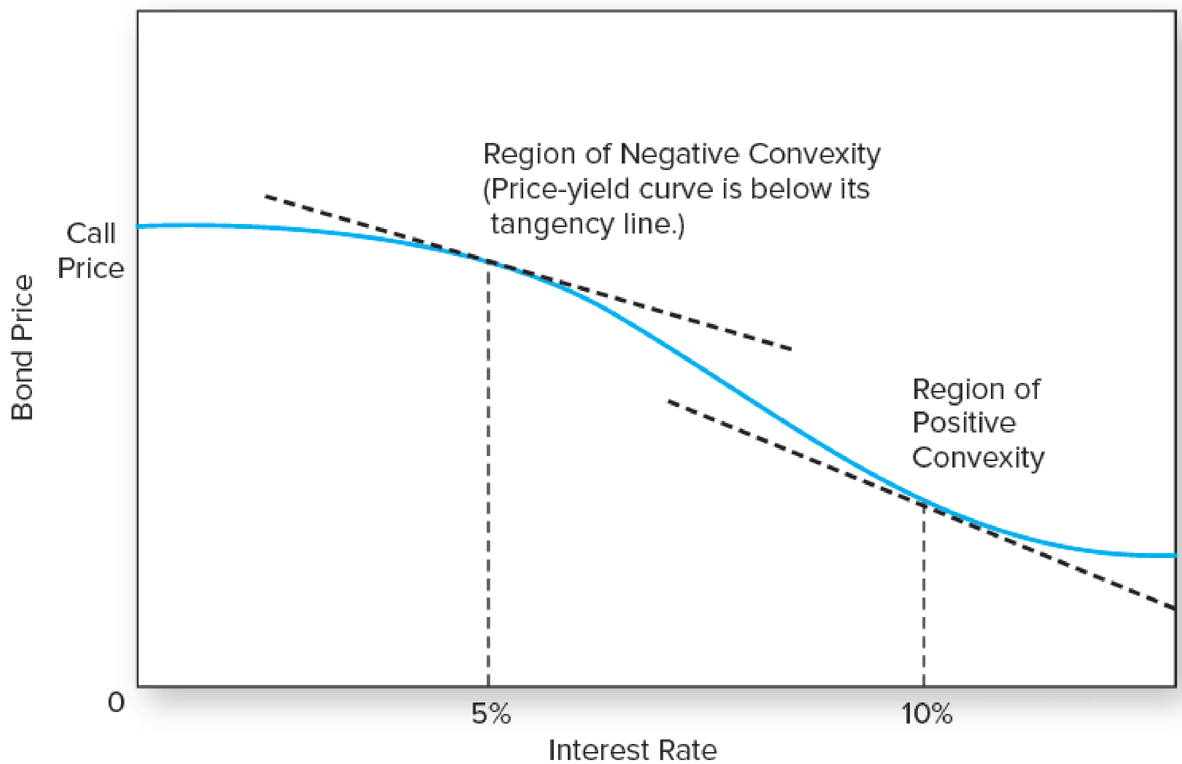
$$D = \frac{1}{P} \left[\frac{100c}{y^2} \left(1 - \frac{1}{(1+\frac{y}{2})^{2T}}\right) + T \left(1 - \frac{c}{y}\right) \frac{100}{(1+\frac{y}{2})^{2T+1}} \right]$$

Interpretations

- The first derivative of price function, $P(y)$. The slope (tangent line) of $P(y)$ at y.
- The terms inside the brackets in (3) $= \sum_{t=1}^{2T} \frac{t}{2} W_t$
  - The weighted sum of the times at which cash flows are received, with $W_t = $ the present value of the cash flow received at that time.
- The terms inside the brackets in (5) $= \sum_{t=1}^{2T} \frac{t}{2} \omega_t$
  - $\omega_t = W_t/P$, the present value of the cash flow received at that time divided by the bond price.
- Macaulay Duration = Duration $\times(1 + y/2)$
  - The first measure of duration was developed by Frederick Macaulay in 1938.
  - The average maturity of a bond's cash flows
  - It approximates minus the percent change in price per 100 bp change in the continuously compounded yield.

Effective Duration

- An *empirical* change in bond price w.r.t change in yield.
  - Simply replace d with Δ. To be more realistic, replace y with r_t.
  - Instead of taking derivative, perturbe y by x bp and see how much P changes.
  - $D_d = -dP/dy \Longrightarrow -\Delta P/\Delta y$
  - $D = -\frac{(dP/P)}{dy} \Longrightarrow -\frac{(\Delta P/P)}{\Delta y}$
- Does not assume parallel shifts in bond yields (talk later).
- Useful for bonds with embedded options (e.g. callable bonds, mortgages) and portfolios of bonds, where the price-yield relationship is "peculiar.' '

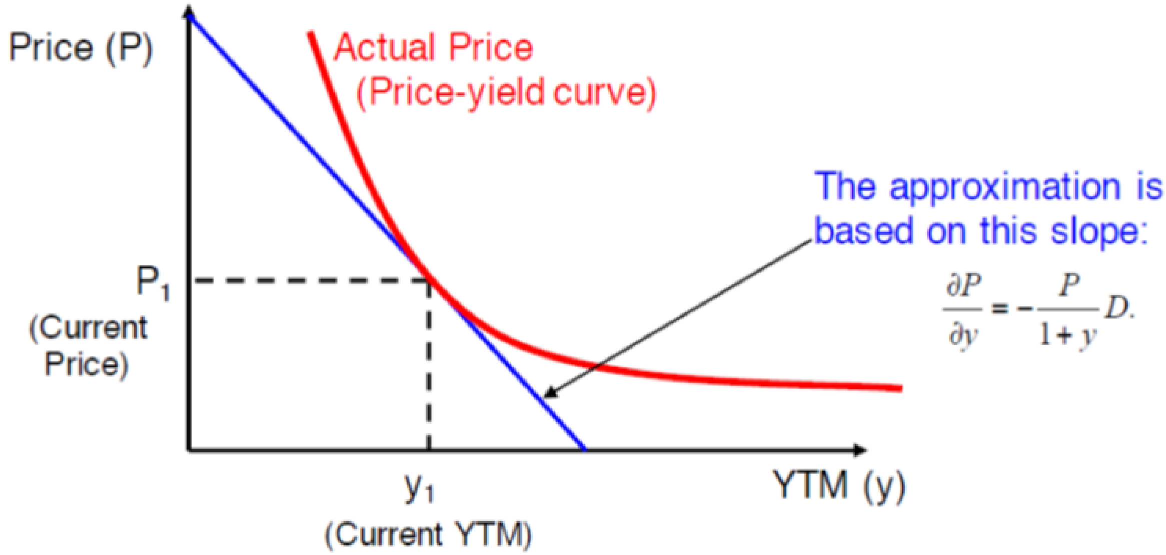

Region of Negative Convexity
(Price-yield curve is below its
tangency line.)
Call
Price
Bond Price
Region of
Positive
Convexity
0
5%
10%
Interest Rate

Duration: Example

	H	I	J	K	L	M
1	Par	$100		Coupon	4.00%	
2				Yield	5.00%	
3						
4						
5	Time	CF	PV	W	Time x W	
6	0.5	$2	$1.951	2.01%	0.0100	
7	1	$2	$1.904	1.96%	0.0196	
8	1.5	$2	$1.857	1.91%	0.0286	
9	2	$2	$1.812	1.86%	0.0373	
10	2.5	$2	$1.768	1.82%	0.0454	
11	3	$102	$87.954	90.45%	2.7134	
12		Total:	$97.246	100.00%	2.8543	
13						
14	Yield Chg	Price				
15	4.80%	$97.789		**Mac Duration**	2.8543	
16	5.00%	$97.246		**Modified Duration**	2.7847	=L15/(1+$L$2/2)
17	5.20%	$96.706		**Effective Duration**	2.7847	=-(I15-I17)/(H15-H17)/I16
18				**DV01**	0.0271	=L16*I16/10000

See Excel spreadsheet, *Fixed_Income.xlsx*

Duration: Graphical Representation



- The approximation using duration can be understood as a tangent line approximation.

$DV01$ and Duration for Zero-coupon and Par bond

- Zero coupon bonds: set $c = 0$ in (4) and (6). $P = 100/(1+y/2)^{2T}$

$$DV01_{c=0} = \frac{T}{100(1+\frac{y}{2})^{2T+1}}$$

$$D_{c=0} = \frac{T}{(1+\frac{y}{2})}$$

{$\$Dur = 100T/(1+y/2)^{2T+1}$,Macaulay Duration $= T$}

- Par bonds: set \$ c = y\$ in (4) and (6). $P = 100$.

$$DV01_{c=y} = \frac{1}{100y}\left(1 - \frac{1}{(1+\frac{y}{2})^{2T}}\right)$$

$$D_{c=y} = \frac{1}{y}\left(1 - \frac{1}{(1+\frac{y}{2})^{2T}}\right)$$

Properties of Duration

- The average of the time to the bond's promised cash flows.
- A measure of the price sensitivity (semi-elasticity) to a change in YTM.

1. $D \approx T$ ($D_{Mac} = T$) for zero-coupon bonds.
2. Higher maturity, higher D
3. Higher coupon, lower D
4. Higher yield (market rate), lower D

Convexity

- The interest rate sensitivity of a bond falls as rates increase
- Convexity is a measure of the curvature of the price-rate curve.
  - All else equal, the greater, the better.

$$C = \frac{1}{P}\frac{d^2P}{dy^2}$$

$$C = \frac{1}{P(1+\frac{y}{2})^2}\left[\frac{100c}{2}\sum_{t=1}^{2T}\frac{t}{2}\left(\frac{t}{2}+0.5\right)\frac{1}{(1+\frac{y}{2})^t} + T(T+0.5)\frac{100}{(1+\frac{y}{2})^{2T}}\right]$$

Involves quadratic functions of times; convexity increases much faster with maturity than duration.

Convexity: Example

	H	I	J	K	L	M	N
1	Par	$100		Coupon	4.00%		
2				Yield	5.00%		
3							
4							
5	Time	CF	PV	W	Time x W	t*(t+0.5)*W	
6	0.5	$2	$1.951	2.01%	0.0100	0.0100	
7	1	$2	$1.904	1.96%	0.0196	0.0294	
8	1.5	$2	$1.857	1.91%	0.0286	0.0573	
9	2	$2	$1.812	1.86%	0.0373	0.0932	
10	2.5	$2	$1.768	1.82%	0.0454	0.1363	
11	3	$102	$87.954	90.45%	2.7134	9.4967	
12		Total:	$97.246	100.00%	2.8543	9.8229	
13							
14	Yield Chg	Price					
15	4.80%	$97.789		**Mac Duration**	2.8543		
16	5.00%	$97.246		**Modified Duration**	2.7847	=L15/(1+$L$2/2)	
17	5.20%	$96.706		**Effective Duration**	2.7847	=-(I15-I17)/(H15-H17)/I16	
18				**DV01**	0.0271	=L16*I16/10000	
19				**Mac Convexity**	9.8229		
20				**Modified Convexity**	9.3496	=L19/(1+L2/2)^2	

See Excel spreadsheet, *Fixed_Income.xlsx*

Approximation Using Convexity

- The Tayloar series:

$$f(x) - f(x_0) \approx f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2$$

$$\Rightarrow P(y) - P(y_0) \approx P'(y_0)(y - y_0) + \frac{1}{2} P''(y_0)(y - y_0)^2$$

- $P'(y_0)$: $-$Dollar duration, $P''(y_0)$: Dollar convexity

$$\Rightarrow \frac{P(y) - P(y_0)}{P(y)} \approx \frac{P'(y_0)}{P(y)}(y - y_0) + \frac{1}{2} \frac{P''(y_0)}{P(y)}(y - y_0)^2$$

- Percent change in price:

$$\frac{\Delta P}{P} \approx -D\Delta y + \frac{1}{2}C\Delta y^2$$

See Excel spreadsheet, *Fixed_Income.xlsx*

Duration and Convexity of Portfolios

- The D_d ($DV01$) of a portfolio is the **sum** of its component D_d ($DV01$). Let \$ P = P_i\$.

$$\frac{dP}{dy} = \sum \frac{dP_i}{dy}$$

- The duration of a portfolio is the value-weighted **average** of its component durations.

$$-\frac{1}{P}\frac{dP}{dy} = \sum \frac{1}{P}\frac{dP_i}{dy} = \sum \frac{P_i}{P}\frac{1}{P_i}\frac{dP_i}{dy}$$

$$D_P = \sum \frac{P_i}{P} D_i$$

- Similarly, the convexity of a portfolio the value-weighted sum of its component convexities.

Portfolio Yield

- The yield of the portfolio, y, is the single discount rate that gives the portfolio the same price as the individual security yields (or zero rates), call them $y_1, y_2, ..., y_n$.

$$P = \sum P_i(y_i) = \sum P_i(y)$$

- Therefore,

$$0 = \sum P_i(y) - P_i(y_i) \approx \sum P_i'(y_i)(y - y_i)$$

$$\Rightarrow y = \frac{\sum P_i'(y_i) y_i}{\sum P_i'(y_i)}$$

- The portfolio yield is approximately the dollar-duration weighted average of the individual security (or zero) yields.
  - Example: A coupon bond = a portfolio of zeros. The yield of the coupon bond = the weighted average of zeros' yields.

Assumptions About Term Structure

- Note that the interest rate risk sensitivities we consider here is with respect to the change in "yield to maturity' '.
  - That is, they examine what happens if the yield changes.
  - Yield is affected by market interest rates (e.g., zero rates). And we want to measure the sensitivity of bond price to market interest rates.
  - If the yield curve is flat, i.e., $r_t = r$, then $r_t = r = y$.
  - If the yield curve shifts in parallel by x bp, then yield changes (approximately) by x bp.
  - In this case, our measures are reasonable proxy for interest rate sensitivity. But this is only when the shift is small.
  - Besides, if the market interest rates move in non-parallel manner, the approximation does a poor job.
  - Further discussion along this line can be found, e.g., Tuckman and Serrat, Chapter 4, 5, and 6.

Yield vs. Rates

	A	B	C	D	E	F	G	H	I	J	K
1	Par	$100		Coupon	4.00%		Par	$100		Coupon	4.00%
2											
3							Curve shift:	1.00%			
4								.			
5	zero rates	Time	CF	PV	CFs		Yield Curve	Time	CF	PV	CFs
6					($94.902)						($92.320)
7	1%	0.5	$2	$1.990	$2.000		2.00%	0.5	$2	$1.980	$2.000
8	2%	1	$2	$1.961	$2.000		3.00%	1	$2	$1.941	$2.000
9	3%	1.5	$2	$1.913	$2.000		4.00%	1.5	$2	$1.885	$2.000
10	4%	2	$2	$1.848	$2.000		5.00%	2	$2	$1.812	$2.000
11	5%	2.5	$2	$1.768	$2.000		6.00%	2.5	$2	$1.725	$2.000
12	6%	3	$102	$85.423	$102.000		7.00%	3	$102	$82.977	$102.000
13			Total:	$94.902					Total:	$92.320	
14											
15				Yield:	5.8783%					Yield:	6.8766%
16											
17										Yield Diff:	0.9983%

See Excel spreadsheet, *Fixed_Income.xlsx*

Credit Risk

- Measured by rating agencies such as Moody's Investor Services, Standard & Poor's Corporation, and Fitch Investors

Bond Ratings

	Very High Quality		High Quality		Speculative		Very Poor	
Standard & Poor's	AAA	AA	A	BBB	BB	B	CCC	D
Moody's	Aaa	Aa	A	Baa	Ba	B	Caa	C

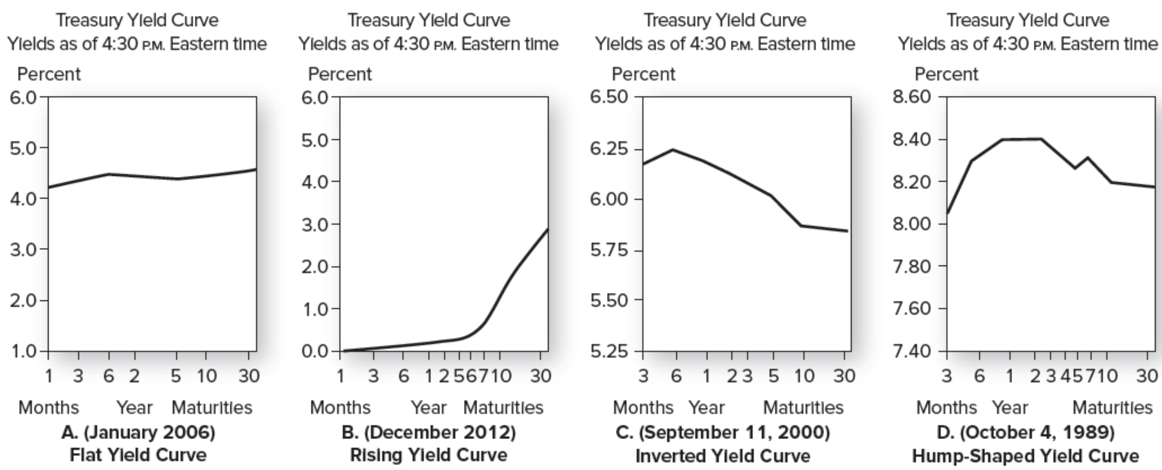
At times both Moody's and Standard & Poor's have used adjustments to these ratings: S&P uses plus and minus signs: A+ is the strongest A rating and A– the weakest. Moody's uses a 1, 2, or 3 designation, with 1 indicating the strongest.

- Credit Default Swap (CDS)
  - An insurance policy on the default risk of a bond or loan: paying premium for default protection
- Collateralized debt obligations (CDO)
  - Can be viewed as a mechanism to reallocate credit risk in the fixed-income markets.

Term Structure of Interest Rates

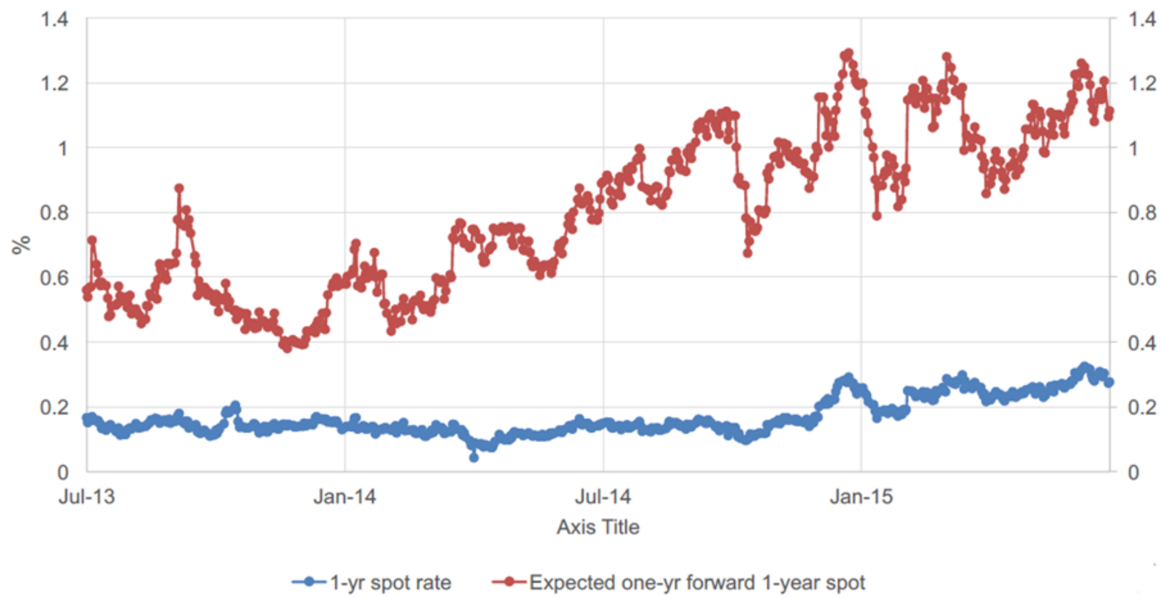
What is the term structure of interest rate?

- The relationship between yield to maturity and time to maturity.
- The yield curve plots the term structure of interest rate.
    - Provides useful insight about how the market thinks about future interest rate movements.
    - Allows central bankers to gauge market expectation about inflation, growth and risks
    - Allows you to price assets



Theories of the Term Structure

- The Pure Expectation Hypothesis
    - Forward rates reflect today's expectation of what spot rates will be in the future.
    - No-arbitrage implies $(1+r_2)^2 = (1+r_1)(1+f_{1,2})$.
    - The PEH suggests that $(1+r_1)(1+f_{1,2}) = (1+r_1)(1+E[r_{1,2}])$
    - Also, $(1+r_2)^2 = (1+r_1)(1+E[r_{1,2}])$. Hence, $r_2 \approx \frac{r_1+E[r_{1,2}]}{2}$
    - If $r_2 > r_1$, then $ E[r_{1,2}]>r_1$



Theories of the Term Structure (cont'd)

- The Liquidity Preference Hypothesis
  - Investors require premium to hold a bond with a particular maturity over another maturity.
  - Suppose $r_2 = \frac{r_1 + E[r_{1,2}] + tp_{1,2}}{2}$.
  - Even when $r_1 = E[r_{1,2}]$, i.e., the market does not expect rates to rise, $r_2 > r_1$ if $tp_{1,2} > 0$.
  - Alternatively, the no-arbitrage condition suggests that we can define it as $tp_{1,2} = f_{1,2} - E[r_{1,2}]$
  - That is, the excess of the forward rate over pure expectation of future interest rate
  - The term-premium can then be decomposed into two components.
    * Price risk premium (increasing with the maturity; First-order): Higher the duration, more sensitive is the price to a change in interest rates;
    * Convexity premium (decreasing with the maturity, Second-order):
  - The term-premium can be positive or negative depending on which component dominates

Theories of the Term Structure (cont'd)

- Market Segmentation (Preferred Habitat) Hypothesis
  - Bonds of a given maturity are mainly traded by a particular group of investors.
  - The supply and demand conditions of a bond with a given maturity are independent on the supply and demand conditions of bonds of other maturity.
  - Arbitrage opportunity across maturities is missing.
  - This explanation is less popular nowadays.

Interpreting the Yield Curve

- The yield curve contains a mixture of
  - information about expected returns (risk) on bonds of different maturities and
  - forecasts of future yield changes
- In general, it is difficult to disentangle these two without a model of expected returns or interest rate forecasts

Applications: Risk Management

Asset-Liability Management

- Suppose you have liabilities or obligations consisting of a stream of fixed cash flows you must pay in the future.
  - Pension liabilities
  - Insurance liabilities
- How can you structure an asset portfolio to fund these liabilities?

Immunization

- The liabilities have a certain market value. It changes as time passes and as interest rates change.
- Construct an asset portfolio with
  - the same market value and
  - the same duration as the liabilities
  - so that the asset value tracks the liability value over time.
- Can include derivatives as well as fixed cash flows.

Duration/Market Value Matching

- Change in value \$ -\$D $\times$\$ change in rates
- Matching the dollar duration of assets and liabilities means matching their changes in value if all rates change by the same amount – hedges against parallel yield curve shifts.
- Matching market value means liabilities are fully funded.
- Hedging against parallel shifts is really just a first step.

Structuring an Asset Portfolio

- Suppose your liabilities have market value of \$100M and duration of 6.
- You want to structure an asset portfolio with the same market value and duration.
- Construct an asset portfolio with just two securities:
  - A bond with price \$100 and duration 8
  - A CMO with price \$70 and duration 4
- What are the number of units of the bond and CMO in the immunizing asset portfolio?

$$
\begin{aligned}
& x \times \$100 + y \times \$70 = \$100M \\
& \frac{x \times \$100}{\$100M} \times 8 + \frac{y \times \$70}{\$100M} \times 4 = 6 \\
& \Rightarrow x = 454,545, y = 714,286
\end{aligned}
$$

Simply Dollar Duration Matching

- Suppose your liabilities have dollar duration of 100M and your assets have dollar duration 500M
- You want to leave your existing assets in place and close the gap by selling interest rate swap contracts.
- Suppose each swap contract has present value zero and dollar duration of 10M.
- How many contracts must you sell to give your net position zero dollar duration?

$$
\begin{aligned}
&\$500M + x \times \$10M = \$100M \\
&\Rightarrow x = -40
\end{aligned}
$$

Example: Bank's Immunization Strategy

Many financial institutions hold short-term liabilities (checking and savings accounts, certificates of deposit, etc.) and long-term assets (car loans, home mortgages, etc.).

Assets	\$300m (D=5 years)	Liab.	\$285m (D=3 years)
		Equity	\$15m (D=D_E years)

- Duration of a portfolio is the average of the durations of the portfolio's components: $5 = (285/300)3 + (15/300)D_E $
  - $D_E = 43$, the bank (and regulators) may worry about this level of duration mismatch.
  - If the yield increase by 25 bp, equity reduces by 11%!

Example: Bank's Immunization Strategy (cont'd)

The bank wants to reduce the duration of its equity from 43 to 0. That is, it wants to immunize its portfolio.

- Its assets include \$80.625 mortgages whose duration is 8 years and other assets \$219.38 with duration of 3.9.
- If it exchanges all these mortgages for cash, then is the portfolio immunized?
  - $(80.63/300)0 + (219.38/300)3.9 = (285/300)3 + (15/300)D_E$
  - $D_E \approx 0$

- However, the bank gave up the returns from mortgages.
- "Zero risk' ' is not (usually) optimal – bank business is to achieve returns by taking some risks.

Limitations of Immunization

- As interest rate changes, the duration of the portfolio changes.
  - Maintaining an immunized portfolio requires continuous adjusting as the rate changes.
  - Even if the rate doesn't change, the duration changes over time.
- A duration-neutral portfolio is protected when the rates for all maturities change by the same percentage point (i.e., when there is a parallel shift in the yield curve).
- Immunization is costly.
  - In our example, exchanging mortgages for cash entails giving up future revenue

Example: Market Maker's Risk Management

- A dealer in corporate bonds finds herself with an inventory of \$1mm in a 5 year 6.9% bonds (semiannual payments) at the end of the trading day, priced at par.
- The bonds are illiquid, so selling them would entail a loss. Holding them overnight is risky, since their price might fall if rates rise.
- An alternative to selling the corporate bonds is to short more liquid Treasury bonds. The following bonds are available:
  - 10 yr, 8% Treasury, $p = \$1,109.0$ per \$1,000 face
  - 3 yr, 6.3% Treasury, $p = \$1,008.1$ per \$1,000 face
- How much of the 10 year bond would she need to short to hedge? How much of the 3 year bond?
- If yields rise by 1% overnight on all the bonds, show the result of the transactions the next day when the short position is closed out.

Example (cont'd)

1. Find modified duration of the bond to be hedged
   - For 5 year 6.9% bond: $y = 6.9\%, D = 4.1688$
2. Find modified duration of the bonds to be shorted
   - For 10 year 8% bond: $y = 6.5\%, D = 7.005$
   - For 3 year 6.3% bond: $y = 6.00\%, D = 2.700$
3. Find x and y in:
   - $x(7.005) = \$1mm(4.1688) \Rightarrow x = \$593,861.5$
   - $y(2.7) = \$1mm(4.1688) \Rightarrow y = \$1,540,720$
4. If yields rise by 1% overnight on all the bonds:
   - For 5 yr, yield to 7.9%,: $P = \$959.423/1,000 = 0.959423$, Loss: $(\$1,000,000)(1 - 0.959423) = \$40,577$
   - For 10 year yield to 7.5% $P = \$1034.81/1109 = 0.93311$, Loss: $(\$595,167)(1 - 0.93311) = \$39,810$
   - For 3 year yield to 7%: $P = \$981.42/\$1,008.1 = 0.97353$, Loss: $(\$1,543,947)(1 - .97353) = \$40,861$

Example (cont'd)

- What if the dealer wants the added protection of doing a gamma neutral hedge?
- Investment must be both delta neutral and gamma neutral. This requires matching deltas and gammas, and requires investments in both bonds.
  - $P1 = \$1m, D1 = 4.1688, C1 = 21.038$
  - $P2 = ?, D2 = 7.005, C2 = 62.98$
  - $P3 = ?, D3 = 2.700, C3 = 8.939$
- Match hedge ratios:
  - $\$1m(4.1688) = P2(7.005) + P3(2.700)$
- Match gamma:
  - $$1m(21.038) = P2(62.98) + P3(8.939) $

- 2 linear equations in two unknowns. Solve for P2 and P3.

Barbells and Bullets

- Consider a bullet portfolio invested 100% in a 20-year zero.
  - Its duration ≈ 20, convexity $\approx 20^2 = 400$
- Consider a barbell portfolio invested 50% in a 10-year zero and 50% in a 30-year zero.
  - Its duration $\approx 0.5(10) + 0.5(30) = 20$.
  - Its convexity $\approx 0.5(10)^2 + 0.5(30)^2 = 500$
- More generally, for a given duration, the more disperse the cash flows are the greater the convexity. Example: a coupon bond has greater convexity than the same-duration zero.
  - A Barbell strategy takes advantage of increased convexity by duplicating the duration of an existing bond using a portfolio with one shorter-term bond and one longer-term bond.

Barbells and Bullets: Example

- Suppose an investor holds \$1-million value of the 5-year Treasury note.
- This could be sold and used to purchase a portfolio of \$$x$-million value of the 2-year note and \$$(1 - x)$-million value of the 10-year note.

Issue	YTM(%)	Duration	Convexity
2-year	0.55	1.91	0.05
5-year	1.40	4.73	0.25
10-year	2.00	8.82	0.87

- Find x so that $1.91x + 8.8(1 - x) = 4.73$. $x = 0.592$.
- The convexity of the portfolio is $0.05(0.592) + 0.87(1 - 0.592) = 0.38$

Does the Barbell Always Outperform the Bullet?

- Should we always prefer a higher-convexity portfolio?
  - Markets recognize the contribution of convexity as rates change and generate lower yield for higher convexity portfolios.
    * If there is an immediate parallel shift in interest rates either up or down (high volatility), the barbell will outperform the bullet.
    * If interest rates stay the same, the bullet will outperform the barbell in terms of yield.
    * If this shift is not parallel, anything could happen.
- Tread-off: yield vs. convexity!