Derivatives: Part III

Investments

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Lecture Outline

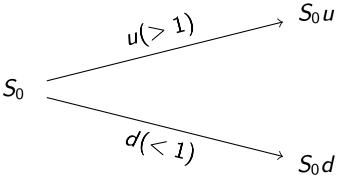
- Financial derivatives: Characteristics
- Forwards and Futures
- Options
  - Pricing: Binomial and Black-Scholes-Merton model

Binomial Tree Model

Binomial Model - Setting

- Assumptions
  - Stock price follows a random walk.

    : In one time step, the stock price can move up or down by a certain amount (only two possible paths).



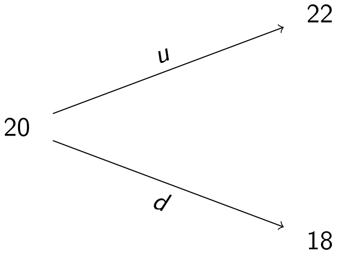
  - There is no arbitrage.

- This may look too simplistic to reflect the reality. Later, we will extend the model to allow multiple steps until the option expiration.

One-Step Binomial Model

e.g. A 3-month European call option has strike price \$21. The risk-free interest rate is 12% per annum.

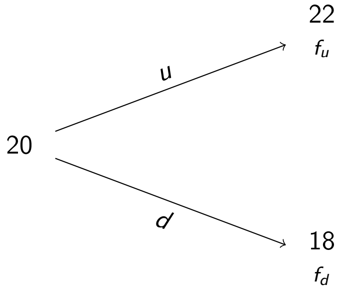
- Current stock price is \$20. The price can move either up to \$22 or down to \$18 during the life of the option.



- What is the price of the call option?

One-Step Binomial Model

1. To price the call, we first determine option payoffs at T:



- f_u(option value when stock price is up)$=\max(22-K,0)=1$
- f_d(option value when stock price is down) $=\max(18-K,0)=0$

One-Step Binomial Model

2. Next, we find a portfolio that replicates the option payoff in every case at T (using stock and bond):

   - Let x denote the number of shares and y the face value of the bond (in dollar) in the replicating portfolio. We want x and y such that

$$\begin{cases} 22x + y = 1 & \text{(at } u\text{)} \\ 18x + y = 0 & \text{(at } d\text{)} \end{cases}$$

   - Solving for the unknowns gives $x = 0.25$, $y = -4.5$. Thus, the replicating portfolio consists of buying 0.25 shares and selling a bond with the face value of -$4.5.

One-Step Binomial Model

3. The option price at time 0 should be equal to the price of the replicating portfolio. Otherwise, an arbitrage exists.

   - The price of the replicating portfolio is

   $$S_0 x + y e^{-rT} = 20 \times 0.25 - 4.5 e^{-0.12 \times 3/12} = 0.633$$

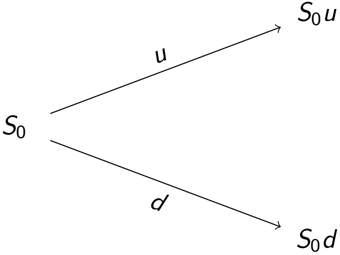
   - Hence, the option price is \$0.633.

   - Note that we do not consider the probabilities of going up or down!

   - In fact, there probabilities are already reflected in the current stock price.

One-Step Binomial Model - General case

- A European call option has strike price K and expiration date T. The risk-free interest rate is r per annum.
- Current stock price is S_0. The price can move either up by u or down by d during the life of the option.



- Notice that we do not assign any probability for up/down movement.

One-Step Binomial Model - General case

1. Find option payoffs at T: $\begin{cases} f_u &= \max(S_0 u - K, 0) \\ f_d &= \max(S_0 d - K, 0) \end{cases}$

2. Find the replicating portfolio (x: number of shares, y: face value of bond).
   - We want to find x and y such that

$$\begin{cases} (S_0 u)x + y = f_u & \text{(at } u\text{)} \\ (S_0 d)x + y = f_d & \text{(at } d\text{)} \end{cases}$$

   - Solving for the unknowns, we obtain

$$x = \frac{f_u - f_d}{S_0 u - S_0 d}, \quad y = \frac{u f_d - d f_u}{u - d}$$

One-Step Binomial Model - General case

3. Calculate the present value (at time 0) of the replicating portfolio.

$$
\begin{aligned}
S_0 x + y e^{-rT} &= \frac{f_u - f_d}{u - d} + e^{-rT}\frac{u f_d - d f_u}{u - d} \\
&= e^{-rT}\left[\frac{e^{rT} f_u - e^{rT} f_d}{u - d} + \frac{u f_d - d f_u}{u - d}\right] \\
&= e^{-rT}\left[\frac{e^{rT} - d}{u - d} f_u + \frac{u - e^{rT}}{u - d} f_d\right] \\
&= e^{-rT}\left[p \times f_u + (1 - p) \times f_d\right]
\end{aligned}
$$

where $p = \frac{e^{rT} - d}{u - d}$.

One-Step Binomial Model

e.g. Go back to the previous example of the European call option with $K = 21$, and $T = 3$ months. The risk-free interest rate is 12% per annum. The current stock price is 20. The price can either move up to 22 or down to 18 during the life of the option.

- We priced the option using the replicating portfolio. Alternatively, we can use the option pricing formula. Here, $u = 22/20 = 1.1$ and $d = 18/20 = 0.9$.

- Then,

$$p = \frac{e^{rT} - d}{u - d} = \frac{e^{0.12 \times 1/4} - 0.9}{1.1 - 0.9} = 0.652$$

- The price of the call option is

$$\begin{aligned} e^{-rT}\left[pf_u + (1-p)f_d\right] &= e^{-0.12 \times 1/4}\left[0.652 \times 1 + (1 - 0.652) \times 0\right] \\ &= \$0.633 \end{aligned}$$

Interpretation: Risk-Neutral Valuation

- The option price $e^{-rT}\left[pf_u + (1-p)f_d\right]$ is similar to the form we would obtain from DCF, when p is interpreted as a probability.
- However, the form is not exactly the same as the DCF.
  - Recall that in DCF, a riskier cash flows is discounted at a higher rate, say r_{call} (e.g. CAPM).
  - However, in the result of option price, the risky option payoff is discounted at the risk-free interest rate.
- Risk-Neutral Valuation
  - The discount rate in the option price is determined **as if** investors do not require a higher return for a riskier investment, that is, as if they are **risk-neutral**.

Risk-Neutral Valuation

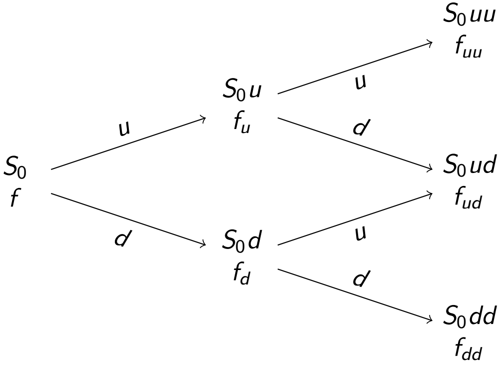
- Risk-neutral valuation is an **interpretation** of the option pricing formula obtained from the replicating portfolio.
- This does not mean that invetors are risk neutral!
- We incorporate risk aversion in two ways:
  - Add risk premium to the cost of capital.
  - Increase the probability of the bad states.
- This interpretation is also useful for multi-step binomial models.

Two-Step Binomial Models

Two-Step Binomial Model

- The current stock price is S_0 and may go up by u or down by d in a time step. Each time step is Δt and the risk-free interest rate is r per annum.

- A European call option has the strike price of K and expires in two steps. What is the option price?



Two-Step Binomial Model

- We start at the option expiration date and find the option payoff at each stock price then.

- At $T = \Delta t$, each price and the following prices can be seen as one-step binomial tree. Thus, we can use the pricing formula of one-step models.

$$\begin{cases} f_u &= e^{-r\Delta t}\left[pf_{uu} + (1-p)f_{ud}\right] \\ f_d &= e^{-r\Delta t}\left[pf_{ud} + (1-p)f_{dd}\right] \end{cases}$$

where $p = \frac{e^{r\Delta t} - d}{u - d}$.

Two-Step Binomial Model

- At $T = 0$,

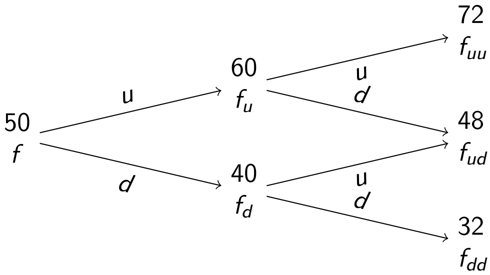
$$\begin{aligned} f_0 &= e^{-r\Delta t}\left[pf_u + (1-p)f_d\right] \\ &= e^{-r\Delta t}\left[p\left(e^{-r\Delta t}\left[pf_{uu} + (1-p)f_{ud}\right]\right) + (1-p)\left(e^{-r\Delta t}\left[pf_{ud} + (1-p)f_{dd}\right]\right)\right] \\ &= e^{-2r\Delta t}\left[p^2 f_{uu} + 2p(1-p)f_{ud} + (1-p)^2 f_{dd}\right] \end{aligned}$$

- This is consistent with the probabilistic interpretation of p.
  - In risk-neutral valuation, p^2, $2p(1-p)$, and $(1-p)^2$ are probabilities of reaching top, middle, and bottom final nodes.

Two-Step Binomial Model - Put Option

- To price a put option, we use put payoffs at the option expiration. The rest of calculation is the same as the call valuation.

Q. Consider a 2-year European put with K=\$52 on a stock with S_0=\$50. Suppose that a time step is 1 year, and in each time step the stock price moves either up or down by 20%. The risk-free interest rate is 5%. What is the price of the put option?



N-Step Binomial Models

N-Step Binomial Model

- Suppose that there are N time steps until the option maturity and each time step is Δt.

- The risk-neutral probability of an increase in stock price during each step is $p(= \frac{e^{r\Delta t} - d}{u-d})$.

- There are $N+1$ nodes at the expiration. Let node j denote the final stock price when the price moves upward j times and downward $N-j$ times. There, the final stock price would be

$$S_0 u^j d^{N-j},$$

where $j = 0, 1, ..., N$.

N-Step Binomial Model

- To determine the price of an European option, we need the probability of reaching each node at the expiration.

- The probability of reaching the node j is

$$\binom{N}{j} p^j (1-p)^{N-j}$$

  - There are multiple paths leading to the node j. The number of the paths is $\binom{N}{j}$, which is j-combinations from a set of N elements.
  - How to calculate $\binom{N}{j}$?
    - In algebra, $\binom{N}{j} = \frac{N!}{j!(N-j)!}$.
    - In Excel, use "combin(N,j)".

N-Step Binomial Model

- For each node, the probability and option payoff is as follows:

No. of up	No. of down	Probability	Stock price at T	Option payoff
0	N	$p^0(1-p)^N$	$S_0 u^0 d^N$	f_0
1	$N-1$	$\binom{N}{1} p^1(1-p)^{N-1}$	$S_0 u^1 d^{N-1}$	f_1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
j	$N-j$	$\binom{N}{j} p^j(1-p)^{N-j}$	$S_0 u^j d^{N-j}$	f_j
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
N	0	$p^N(1-p)^0$	$S_0 u^N d^0$	f_N

- The price of European option is then

$$e^{-r(N\Delta t)} \sum_{j=0}^{N} \binom{N}{j} p^j (1-p)^{N-j} f_j$$

where f_j is the option payoff at node j.

N-Step Binomial Model

Q. Consider a 3-year European call with K=\$30 on a stock with S_0=\$30. Suppose that a time step is 1 year, and in each time step the stock price moves either up or down by 10%. The risk-free interest rate is 5%. What is the price of the call option?

Answer: First, the option payoffs at each of 4 nodes are

$$
\begin{aligned}
f_0 &= \max(30(1.1)^0(0.9)^3 - 30, 0) = 0 \\
f_1 &= \max(30(1.1)^1(0.9)^2 - 30, 0) = 0 \\
f_2 &= \max(30(1.1)^2(0.9)^1 - 30, 0) = 2.67 \\
f_3 &= \max(30(1.1)^3(0.9)^0 - 30, 0) = 9.93.
\end{aligned}
$$

The risk-neutral probability is $p = \frac{e^{0.05\times 1} - 0.9}{1.1-0.9} = 0.756$. Then, the option price is

$$e^{-0.05\times 3} \sum_{j=0}^{3} \binom{N}{j} (0.756)^j (1-0.756)^{N-j} f_j = 4.01$$

Black-Scholes-Merton Model

Binomial vs. BSM Model

- In the binomial model, we assume that the price can change discretely at a constant interval.
- In contrast, actual stock prices change almost every instant. Thus, the assumption in the binomial model may over-simplify the reality.
- The Black-Scholes-Merton model recognizes the fact that stock prices change continuously over time. Based on the recognition, the model provides the option prices.
- Still, the BSM model and the binomial model are closely connected.
  - When we make the time step in the binomial model infinitesimally small, we can obtain the analytic expression of BSM option prices.

Black-Scholes-Merton Model

- The prices of European call and put options on non-dividend-paying stock are

$$c_0 = S_0 N(d_1) - Ke^{-rT} N(d_2)$$

$$p_0 = Ke^{-rT} N(-d_2) - S_0 N(-d_1)$$

where

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$
$$d_2 = \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T},$$

and $N(x)$ is the cumulative probability distribution function for a standard normal random variable.

Black-Scholes-Merton Model

- The BSM model provides an analytic form that determines the option price as a function of the followings:
  - Current stock price S_0
  - Strike price K
  - Time to expiration T
  - Risk-free interest rate r
  - Volatility of underlying asset σ
- Through the BSM model, we can find the option price by simply inputting numbers into the option-pricing formula.

BSM formula: Interpretation

- The BSM expresses the option as a portfolio of stocks and bonds.

- $N(d_1)$ is the fraction of share we hold in the replicating portfolio at t. In fact, we can show that:

  $\Delta_c = \frac{\partial C}{\partial S} = N(d_1) > 0$

  $\Delta_p = \frac{\partial P}{\partial S} = -N(-d_1) < 0$

- For a call, $Ke^{-rTN(d_2)}$ is the amount of initial borrowing in the replicating portfolio.

- The value of the call is the cost of the replicating portfolio.

$$\begin{aligned} c_0 &= \Delta_c \times S - B \\ &= S_0 e^{-\delta T} N(d_1) - Ke^{-rT} N(d_2) \end{aligned}$$

Black-Scholes-Merton Model - Example

Q. There is a 6-month European call option on a stock whose current price is \$42. The strike price is \$40, and the risk-free interest rate is 10% per annum. The stock volatility is 20% per annum. What is the price of the option?

Answer:

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} = \frac{\ln(42/40) + (0.1 + 0.2^2/2)(0.5)}{0.2\sqrt{0.5}} = 0.7693$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.6278$$

$$\begin{aligned} c &= S_0 N(d_1) - Ke^{-rT} N(d_2) \\ &= 42 \times N(0.7693) - 40e^{-0.1\times0.5} \times N(0.6278) \\ &= 42 \times \text{norm.s.dist}(0.7693, \text{TRUE}) - 40e^{-0.1\times0.5} \times \text{norm.s.dist}(0.6278, \text{TRUE}) \\ &= \$4.759. \end{aligned}$$

Pricing Options on Dividend-Paying Stocks

- The basic BSM formula prices European put and call options on an underlying security that pays no dividends and whose price is log-normally distributed
- When there is a known dividend or dividend yield the formula can be easily adjusted:
- Options with known dividend
  - Use $S^* = S_0 - PV(D)$, instead of S_0.
- Options with known dividend yield
  - Use $S^* = Se^{-\delta T}$
  - $d_1 = \frac{\ln(S_0/K)+(r-\delta+\sigma^2/2)T}{\sigma\sqrt{T}}$
- Refer to Hull 13.11, 15.12, 17.3, 21.3 more on these variations.

Implied Volatility

- Implied volatility (IV) is the value of the volatility input to an option pricing model that makes the model value equal the option price observed in the market.
    - IV is the market's forecast of volatility. Higher uncertainty $\rightarrow$ Higher option price $\rightarrow$ Higher IV
    - There is a one-to-one correspondence between implied volatility and option price. In some markets, options are quoted in terms of implied volatility, rather than price.
    - Note that implied volatility depends on the option pricing model used to calculate it. IV as commonly reported is always computed from the Black-Scholes model.

Implied Volatility

	A	B	C	D	E	F	G	H	I	J	K
1	INPUTS			OUTPUTS			FORMULA FOR OUTPUT IN COLUMN E				
2	Standard deviation (annual)	.2783		d1	0.0029		(LN(B5/B6)+(B4–B7+.5*B2^2)*B3)/(B2*SQRT(B3))				
3	Expiration (in years)	.5		d2	–0.1939		E2–B2*SQRT(B3)				
4	Risk-free rate (annual)	.06		N(d1)	0.5012		NORMSDIST(E2)				
5	Stock price	100		N(d2)	0.4231		NORMSDIST(E3)				
6	Exercise price	105		B/S call value	7.0000		B5*EXP(–B7*B3)*E4–B6*EXP(–B4*B3)*E5				
7	Dividend yield (annual)	0		B/S put value	8.8968		B6*EXP(–B4*B3)*(1–E5)–B5*EXP(–B7*B3)*(1–E4)				
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The VIX Index

- The CBOE publishes indices of implied volatility. The most popular index, the SPX VIX, is an index of the implied volatility of 30-day options on the S&P 500 calculated from a wide range of calls and puts.
  - One contract is on 1,000 times the index.
  - Futures on the VIX started in 2004 and options on the VIX in 2006.
- Example: Suppose that a trader buys an April futures contract on the VIX when the futures price is 18.5 (corresponding to a 30-day S&P 500 volatility of 18.5%) and closes out the contract when the futures price is 19.3 (corresponding to an S&P 500 volatility of 19.3%). The trader makes a gain of $800.

The VIX Index

Delta Hedging

- Delta (hedge ratio): the change in option price for a $1 increase in the stock price.
    - Call option: positive
    - Put option: negative

- Delta is simply the slope of the option value-asset price curve.

- In BSM model, the delta for a call is $N(d_1)$ and for a put is $N(d_1) - 1$.

- Example: An investor is long one call option on a stock with a delta of 0.75. She could delta hedge the call option by shorting 0.75 shares of the underlying stocks.

Dynamic Hedging

- The challenge with synthetic put (call) positions is that deltas constantly change.
- As the stock price falls, the absolute value of the appropriate hedge ratio increases (decreases).
- Therefore, market declines require updating hedging,
- Additional conversion of equity (cash) into cash (equity).
- This constant updating of the hedge ratio is called dynamic hedging

Does the Model Explains the Actual Prices?

- BSM model generates values fairly close to actual prices of traded options
- Biggest concern is volatility
  - The implied volatility of all options on a given stock with the same expiration date should be equal
  - Empirical tests show that implied volatility actually falls as exercise price increases.
    - Market thinks that a sudden price drop is more likely than the normal distribution predicts.
    - Another possible explanation is that in-the-money calls have become popular alternatives to outright stock purchases as they offer leverage and hence increased ROI.

Volatility Smile

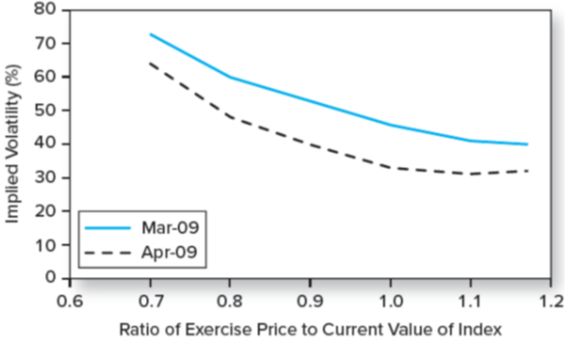

Figure 21.15 The S&P 500 option smirk on two dates

Source: *The CBOE Skew Index*, Chicago Board Options Exchange, 2010.

References

- BKM, Chapters 20 through 23
- Hull, Chapters 1 through 5, 11 through 13
- Prof. Deborah Lucas' lecture notes