Pricing Forwards

BUSS386 Futures and Options

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Lecture Outline

- No Arbitrage Argument
  - What is arbitrage?
  - Short-selling
- Determination of Forward Prices
- Valuing Forward Contracts
- Comparison Between Forward and Futures Prices
- Reading: Ch. 5

Arbitrage

- **Arbitrage** is a trade where investors can make "free lunch" profits.
- For instance, if we see a price difference for the same assets, we can make an arbitrage profit (buy low and sell high).
- *e.g.* Suppose that a stock is traded in both New York Stock Exchange and London Stock Exchange. Its price in New York is \$140, while it is £100 in London. The exchange rate is \$1.43 per pound.
  - Buy a share in New York and sell it in London.
  - Profit = 100×1.43 - 140 = \$3. This profit is risk-free.

Arbitrage - Definition

- Formally, we claim that a trading strategy is an arbitrage if it satisfies the following conditions.

  1. It always generates **non-negative** cash flows, and
  2. It sometimes generates **positive** cash flows.

No Arbitrage Argument

- In the markets, there are numerous investors looking for any arbitrage opportunity.
- Suppose that an arbitrage exists for a certain asset.
- Due to forces of supply and demand, the prices will eventually change. In equilibrium, the prices of one asset will be the same across different markets.
- Generally, arbitrage opportunities quickly disappear.

No Arbitrage Argument

- Also, we can apply the no arbitrage argument to two assets (portfolios) A and B that will generate the same cash flows in the future in every condition.

- The current prices of assets A and B should be the same. Otherwise, an arbitrage exists.

- If current prices are different, we can make an arbitrage through "**buy low and sell high**''.

  $\Rightarrow$ However, an arbitrage should NOT exist in a purely competitive financial market.

Arbitrage - Assumptions

In making an arbitrage strategy, we assume the followings.

- We consider an investor who has nothing in hand at the beginning of the strategy and liquidates all assets at the end.
- We measure profit/loss in terms of **cash flows**.
- The investor can borrow money (sell a bond) or lend money (buy a bond) at the risk-free rate.
  - We can choose any bond amount (face value) as we like.
- *e.g.* If we buy a bond at the rate r,

Action	time 0	time T
buy a bond	-1	e^{rT}

Short Selling

- In constructing an arbitrage, we assume that the market allows short selling.
- **Def.** Short selling is selling an asset that we do not own.
- *e.g.* Suppose that an investor wants to short a stock at time 0 at the current price of \$120.
  - At time 0, the investor borrows the stock, sells immediately, and receives the proceeds of \$120.
  - One year later, stock price falls to \$100. To close the position, the investor buys the stock and return it back to the original owner.

Action	Year 0	Year 1
(Short) Sell a stock	120	-100

Short Selling

- What if the share pays dividend?
- Then, the shorting investor needs to pay the dividend to the original owner.
- *e.g.* An investor shorts a stock at time 0 whose current price is \$120. The stock pays \$5 dividend in six month.
  - Again, by borrowing and selling immediately, the investor receives \$120.
  - In six month, the investor provides the original owner with the \$5 dividend.
  - One year later, stock price falls to \$90. To close the position, the investor buys the stock and return it back to the original owner.

Action	Year 0	Year 0.5	Year 1
(Short) Sell a share	120	-5	-90

Determination of Forward Prices - Basic Idea

- Investors enter a long or short position in forward contract **at zero cost**.
- In other words, the value of forward contract should be **zero** at the time of initiating the contract.
- Conversely, we can determine the forward price, so that the current value of forward contract becomes zero.

Determination of Forward Prices - Setting

- Assumptions
  - No transaction costs.
  - The market participants have the same tax rate on all net trading profits.
  - The market participants can borrow or lend money at the risk-free interest rate.
  - The market participants take advantage of arbitrage opportunities.
- Notation
  - T: delivery date of contract
  - S_0: spot price of the underlying asset today

- S_T: spot price of the underlying asset at time T
- F_0: forward price today
- r: risk-free rate per annum (with continuous compounding)

Determination of Forward Prices

- Consider an underlying asset that pays no dividends. Its current price is S_0.
- What should be the forward price?

Determination of Forward Prices - Derivation 1

- Your goal is to own a stock at T.
  1. long forward with F_0.
  2. borrow S_0, buy a stock, and wait til T.
- At the contract maturity T, the two strategies should have to same cash flow.
  1. $S_T - F_0$
  2. $S_T - S_0 e^{rT}$
- No net cash flow today. Therefore:

$$F_0 = S_0 e^{rT}$$

Determination of Forward Prices - Derivation 2

- Let's consider the following two portfolios:
  1. long forward with F_0 + buy a bond that will pay F_0 at T
  2. buy a stock
- At the contract maturity T, the two portfolios have the same cash flows:
  1. $(S_T - F_0) + F_0$
  2. S_T
- Thus, their current value should be the same:

$$0 + F_0 e^{-rT} = S_0$$

Determination of Forward Prices - Derivation 3

- The payoff from a forward contract at T is $S_T - F_0$
- The present value of the payoff at time 0 is $S_0 - F_0 e^{-rT}$
  - S_0 = PV of $S_T = e^{\alpha T} S_T$, where α is the discount rate accounting for the risk of the stock.
- The value of the forward is zero at 0. Therefore, $0 = S_0 - F_0 e^{-rT}$
- Solving for $F_0 = S_0 e^{rT}$

Determination of Forward Prices - Arbitrage

- What if

$$F_0 \neq S_0 e^{rT}?$$

$\Rightarrow$ An arbitrage exists.

- *e.g.* Consider a 3-month forward contract on a stock whose current price is \$40. The 3-month risk-free interest rate is 5% per annum.

1. What if the forward price is 43 ($> 40e^{0.05\times 3/12}$)?

   $\Rightarrow$ There is an arbitrage:

Action	Cash flow in 0	Cash flow in 3 months
buy stock	-40	S_T
short forward	0	$43 - S_T$
sell bond	40	-42.497
net	0	0.503

Determination of Forward Prices - Arbitrage

1. What if the forward price is 39 ($< 40e^{0.05\times 3/12}$)?

   $\Rightarrow$ There is another arbitrage strategy:

Action	Cash flow in 0	Cash flow in 3 months
sell stock	40	-S_T
buy forward	0	$S_T - 39$
buy bond	-40	41.503
net	0	1.503

Determination of Forward Prices - Example

- Q. Consider a 1-year forward contract on a stock whose current price is \$50. The forward price is \$51, and the risk-free interest rate is 7% per annum. Is there an arbitrage? If so, show the arbitrage strategy.

Determination of Forward Prices - Example

- Q. Consider a 1-year forward contract on a stock whose current price is \$50. The forward price is \$51, and the risk-free interest rate is 7% per annum. Is there an arbitrage? If so, show the arbitrage strategy.

Action	Cash flow in 0	Cash flow in 1 year
buy stock	-50	S_T
short forward	0	$51 - S_T$
sell bond	50	-53.5
net	0	-2.5

Forward and Spot Prices

- Consider a forward contract initiating at time t. Given the maturity date T, the forward price is

$$F_t = S_t e^{r(T-t)}$$

- Thus, the forward and spot prices are usually different. Only at the expiration, they become the same.
- Also, the forward price changes through time.

Forward Prices for Underlying Assets Paying Dividends

Dividend Payment and Forward Prices

- Until now, we have assumed that the underlying assets in forward do not pay any dividends.
- What if the underlying asset will pay dividends in the future? Are there changes in forward prices?
- $\Rightarrow$ Yes, because...

- The current price S_0 of the underlying asset includes the future dividends.
- However, a long/short position in forward will not receive the dividends. Also, the forward payoff is determined by the ex-dividend price.

Determination of Forward Prices - Discrete Dividends

- We consider two different forms of dividend payments.
  1. Discrete dividends: dividends will be paid at certain points in time.
  2. Continuous dividends: dividends will be paid at every instant continuously.
- We first consider the case of discrete dividends.
- Suppose that stock pays dividends until the maturity T. The present value of all future dividends is I.
- The forward price is

$$F_0 = (S_0 - I)e^{rT}$$

Determination of Forward Prices - Discrete Dividends

- Why? Consider the following two portfolios:
  1. long forward with F_0 + buy a bond that will pay $F_0 + Ie^{rT}$ at T
  2. buy a stock
- At the contract maturity T, the two portfolios have the same cash flows:
  1. $(S_T - F_0) + F_0 + Ie^{rT}$
  2. $(S_T + Ie^{rT})$
- The portfolio values are the same at T. Thus, their current values are the same:

$$0 + F_0 e^{-rT} + I = S_0$$

Determination of Forward Prices - Discrete Dividends

- Q1. Consider a 9-month forward contract on a corporate bond. The current price of the corporate bond is \$900, and it will pay \$40 coupon in 4 months. The 4-month and 9-month risk-free rates are 3% and 4%, respectively. If there is no arbitrage, what is the forward price?

  Answer: The forward price is

$$(900 - 40e^{-0.03\times 4/12})e^{0.04\times 9/12} = 886.60$$

Determination of Forward Prices - Discrete Dividends

- Q2. Consider the 9-month forward contract on the corporate bond in Q1. Suppose that the forward price is \$910. Is there an arbitrage? If so, show the arbitrage strategy.

Determination of Forward Prices - Discrete Dividends

- Q2. Consider the 9-month forward contract on the corporate bond in Q1. Suppose that the forward price is \$910. Is there an arbitrage? If so, show the arbitrage strategy.

  Answer: 886.60 < 910. Thus, we can think of the following arbitrage strategy:

Action	Cash flow in 0	Cash flow in 4 months	Cash flow in 9 months
buy corporate bond	-900	40	S_T
short forward	0	0	$910 - S_T$
sell 4-month bond	39.70	-40	0
sell 9-month bond	872.90	0	-910
net	12.60	0	0

Determination of Forward Price - Continuous Dividends

- Some securities pay continuous dividends (e.g, stock index, foreign currency).
  - Once we invest in a stock index, dividends from each individual stock will be paid at different points of time.
  - Having a lot of stocks in the index, we can approximate the index as paying dividends continuously.
- To simplify the argument, we assume that the dividends will be reinvested immediately to buy more shares.

Determination of Forward Price - Continuous Dividends

- Let q denote the dividend yield per annum. Stock price at time 0 is S_0.
  - Let N denote the number of dividend payments in a year.
  - In one period, investor receives dividend $\frac{q}{N}S_t$.
  - Reinvesting the dividend, the investor owns $\frac{q}{N}$ additional shares. Thus, the number of shares increases by factor of $\left(1+\frac{q}{N}\right)$ in one period.
  - When investing for one year, the number of shares increases by factor of $\left(1+\frac{q}{N}\right)^N$. If N becomes infinitely large, the factor becomes e^q.
- If we invest for T years, the number of shares increases by e^{qT}.

Determination of Forward Price - Continuous Dividends

- What if the underlying asset pays continuous dividends with dividend yield q per annum?
- Forward price is

$$F_0 = S_0 e^{(r-q)T}$$

- Why? Consider the two portfolios:
  1. long forward with F_0 + buy a bond that will pay F_0 at T
  2. buy e^{-qT} share of stock
- The two portfolios will have the same cash flows at T:

1. $(S_T - F_0) + F_0$
2. $S_T e^{-qT} e^{qT}$

- Therefore, the two portfolios should have the same present values:

$$0 + F_0 e^{-rT} = S_0 e^{-qT}$$

Determination of Forward Price - Continuous Dividends - Foreign Currency

- If we hold a foreign currency, we receive interests that are paid continuously at the risk-free rate prevailing in the foreign country.
- Thus, foreign currency can be regarded as an asset with continuous dividends.
- Forward price is then

$$F_0 = S_0 e^{(r - r_f)T}$$

where r_f is the foreign risk-free rate.

Determination of Forward Price - Continuous Dividends - Foreign Currency

- Q1. Suppose that the 2-year interest rates in Hong Kong and the United Kingdom are 3% and 1%, respectively, and the spot exchange rate between the British Pound (GBP) and the Hong Kong Dollar (HKD) is 9.65 HKD per GBP. A 2-year forward exchange rate is 11.00. Is there an arbitrage for Hong Kong investors?

Determination of Forward Price - Continuous Dividends - Foreign Currency

- Q1. Suppose that the 2-year interest rates in Hong Kong and the United Kingdom are 3% and 1%, respectively, and the spot exchange rate between the British Pound (GBP) and the Hong Kong Dollar (HKD) is 9.65 HKD per GBP. A 2-year forward exchange rate is 11.00. Is there an arbitrage for Hong Kong investors?

  Answer: $11.00 > 9.65e^{(0.03-0.01)\times 2}$. Thus, there is an arbitrage. We can consider the following strategy:

Action	Cash flow now	Cash flow in 2 years
buy GBP	-9.47	S_T
short forward	0	11.00 - S_T
sell HK bond	10.40	-11.00
net	0.93	0

Determination of Forward Price - Continuous Dividends - Foreign Currency

- Q2. Suppose that the 2-year interest rates in Hong Kong and the United Kingdom are 3% and 1%, respectively, and the spot exchange rate between the British Pound (GBP) and the Hong Kong Dollar (HKD) is 9.65 HKD per GBP. A 2-year forward exchange rate is 9.70. Is there an arbitrage for Hong Kong investors?

Determination of Forward Price - Commodities

- Storing commodities has costs and benefits
- Forward price with proportional storage cost u

$$F_0 = S_0 e^{(r+u)T}$$

- Forward price with convenience yield y

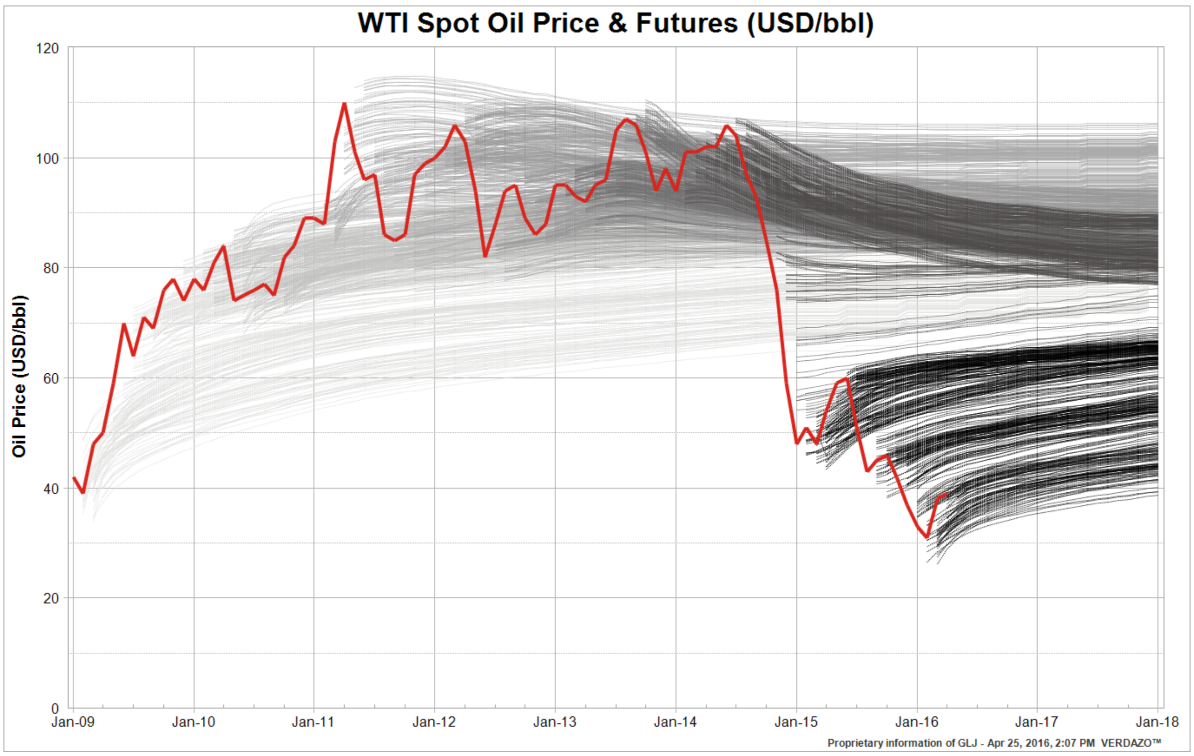
$$F_0 = S_0 e^{(r-y)T}$$

- Together

$$F_0 = S_0 e^{(r-y+u)T}$$

The shape of the forward curve

- Contango is a pattern of forward prices that increases with contract maturity
- Backwardation is a pattern of forward prices over time that decreases with contract maturity



Commodities that cannot be stored

- May be no storage or very limited storage life: electricity, lettuce, strawberries, temperature, rainfall, ...
- For non-storable commodities, forward prices can have information about future spot prices because no-arbitrage conditions don't hold (i.e. not tied to current price)

  - Approach to pricing is to model stochastic future spot prices
  - Also must infer discount rates

Summary

- For stocks, bonds, currencies, metals, stored agricultural commodities, etc., there is no new information in forward prices over what can be learned from spot prices!
- The forward price is tied down by no-arbitrage conditions that depend only on the underlying spot price, interest rates, and associated cash flows between 0 and T (dividends, coupons, storage costs, convenience yield)
- Can we use forward prices to predict future spot price?
- Is the expected future price of a non-dividend paying stock higher or lower than its forward price?

Valuing Forward Contracts

Valuing Forward Contracts

- The value of forward is **zero at the moment we initiate** the contract.
- However, as time passes, its later value can be either negative or positive.
- Suppose that we have a **long position** in a forward with price F_0 that was entered at time 0.
- What is the value f of the forward at time t?

	Value at 0	Value at t
Forward with F_0	0	f

Valuing Forward Contracts

- To find the time-t value of the forward with F_0, we consider another forward that we just start at t.
- Consider the following two portfolios at t:
  1. long forward with F_0 + buy a bond that will pay $F_0 - F_t$ at T
  2. long forward with F_t
- The two portfolios will generate the same cash flows at T:
  1. $(S_T - F_0) + (F_0 - F_t)$
  2. $S_T - F_t$
- Then, the time-t values of the two portfolios should be the same. As a result, the time-t value of the **long position** in forward with F_0 is

$$f + (F_0 - F_t)e^{-r(T-t)} = 0$$

Valuing Forward Contracts

- In a similar way, we can find time-t value of **short position** in forward with F_0 that we started at time 0.
- Consider the two portfolios at t:
  1. short forward with F_0 + buy a bond that will pay $F_t - F_0$ at T
  2. short forward with F_t
- The two portfolios will generate the same cash flows at T:
  1. $(F_0 - S_T) + (F_t - F_0)$
  2. $F_t - S_T$
- Then, the time-t values of the two portfolios should be the same. As a result, the time-t value of the **short position** in forward with F_0 is

$$f = (F_0 - F_t)e^{-r(T-t)}.$$

Valuing Forward Contracts

- We can express the value of forward in a different way by using the forward price F_t

$$F_t = \begin{cases} S_t e^{r(T-t)} & \text{no dividned} \\ (S_t - I)e^{r(T-t)} & \text{discrete divideds} \\ S_t e^{(r-q)(T-t)} & \text{continuous dividends} \end{cases}$$

- As an example, if the underlying asset pays no dividend, the time-t value of the forward is

$$f = S_t - F_0 e^{-r(T-t)}$$

for a long position.

Valuing Forward Contracts

- Q. In August 2020, an investor entered a long position in forward on a stock for delivery in August 2021. At that time, stock price was \$40. Two months later, in October 2020, the stock price becomes \$45. What is the value of the forward? Assume that the risk-free rate of interest is 5%.

Forward vs. Futures Prices

- For the same underlying asset and expiration, the futures and forward prices are very close to each other, but a bit different (due to daily settlement of futures).
- Compare cash flows between forward and futures for a long position:

Day	Forward	Futures
0		
1	0	$F_1 - F_0$
2	0	$F_2 - F_1$
$\vdots$	$\vdots$	$\vdots$
T	$S_T - F_0$	$S_T - F_{T-1}$

- When the risk-free rate is zero, the cumulative gain in futures is the same as the forward payoff. Thus, the forward and futures are the same in cash flows.

  $\Rightarrow$ Futures price = Forward price

Forward vs. Futures Prices

- When the risk-free rate is not zero, the cumulative gain in futures is different from the forward payoff.

Day	Forward	Futures	Interest Factor
0			
1	0	$F_1 - F_0$	$e^{r_1 \cdot (T-1)/365}$
2	0	$F_2 - F_1$	$e^{r_2 \cdot (T-2)/365}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
t	0	$F_t - F_{t-1}$	$e^{r_t \cdot (T-t)/365}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
T	$S_T - F_0$	$S_T - F_{T-1}$	$e^{r_T \cdot 0/365}$

- Whether the cumulative gain in futures is larger/smaller than the forward payoff depends on the **correlation** between risk-free rate and spot price of underlying asset.

Forward vs. Futures Prices

1. What if the price of the underlying asset is **positively** correlated with the interest rate?

   - For a long position, the gain on futures tend to be **larger** than the forward payoff. Why?

     – Suppose that $S_t > S_{t-1}$. Long position is likely to see daily gain ($F_t - F_{t-1} > 0$). This coincides with a larger interest factor due to a higher interest rate.

     – Suppose that $S_t < S_{t-1}$. Long position is likely to see daily loss ($F_t - F_{t-1} < 0$). This coincides with a smaller interest factor due to a lower interest rate.

   - Thus, Futures price > Forward price

Forward vs. Futures Prices

1. What if the price of the underlying asset is **negatively** correlated with the interest rate?

    - For a long position, the gain on futures tend to be **smaller** than the forward payoff. Why?
        - Suppose that $S_t > S_{t-1}$. Long position is likely to see daily gain ($F_t - F_{t-1} > 0$). This coincides with a smaller interest factor due to a lower interest rate.
        - Suppose that $S_t < S_{t-1}$. Long position is likely to see daily loss ($F_t - F_{t-1} < 0$). This coincides with a larger interest factor due to a higher interest rate.
    - Thus, Futures price < Forward price

Forward vs. Futures Prices

- For most contracts, the covariance between futures prices and interest rates is so low that the difference between futures and forward prices will be negligible.
- However, in contracts on long-term fixed-income securities, prices have a high correlation with interest rates, the covariance can be large enough to generate a meaningful spread between forward and futures prices