Black-Scholes-Merton Model

BUSS386. Futures and Options

Professor Ji-Woong Chung

Lecture Outline

- Black-Scholes-Merton Model
  - Log-Normal Property of Stock Prices
  - Derivation
  - Iterpretation

BSM Model

Binomial vs. BSM Model

- In the binomial model, we assume that the price can change discretely at a constant interval.
- In contrast, actual stock prices change almost every instant. Thus, the assumption in the binomial model may over-simplify the reality.
- The Black-Scholes-Merton model recognizes the fact that stock prices change continuously over time. Based on the recognition, the model provides the option prices.
- Still, the BSM model and the binomial model are closely connected.
  - When we make the time step in the binomial model infinitesimally small, we can obtain the analytic expression of BSM option prices.

BSM Model - Distribution of Future Stock Price

- One of assumptions in the BSM model is that stock price follows a **log-normal** distribution:

$$\ln(S_T) \sim \phi(m, s)$$

  where $\phi(m, s)$ denotes a normal distribution with mean m and the standard deviation s.[1]

- In fact, we can prove this log-normality from the binomial model.
  - Assuming that the tree has infinitely many steps until option expiration (or unit time steps becomes very short), the stock price becomes log-normally distributed.

- Let's prove the log-normality.

[1] In Alternative Derivation II, we show that when stock price follows a geometric Brownian motion, $ln(S)$ follows a generalized Wiener process (using Ito's lemma). And $ln(S)$ is normal, i.e., S is log-normal.

Log-Normal Property of Stock Prices

- Consider a binomial tree where the stock price can go up by u or down by d in each step.

- The expiration date of option is T and there are n time steps until the maturity.
  - The length of each step is $\Delta t \left(= \frac{T}{n}\right)$.

- The stock price at the expiration T is

$$S_T(j) = S_0 u^j d^{n-j}$$

when there are $j \ (= 0, 1, 2, \cdots, n)$ upward movements.

Log-Normal Property of Stock Prices

- The number of upward movement j is a random variable with the following probability:

$$\binom{n}{j} p^j (1-p)^{n-j}$$

where p is the risk-neutral probability of upward movement.

- In statistics, we call this a binomial random variable.

$$j \sim B(n, p)$$

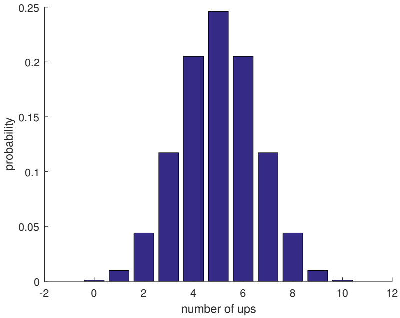
- The binomial random variable has the following property:
  - The mean is np.
  - The standard deviation is $\sqrt{np(1-p)}$.

Log-Normal Property of Stock Prices

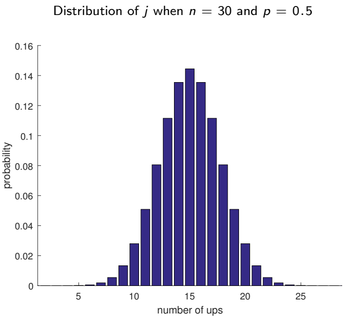
- What will happen if n becomes infinitely large? (This would be equivalent to making each step infinitesimally small).

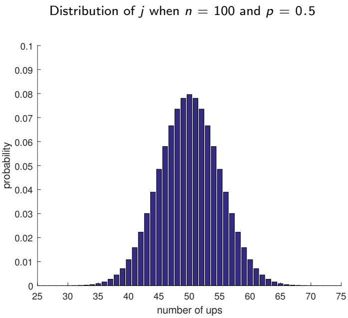
- To see this, let's increase the number of steps n.

Distribution of j when $n = 10$ and $p = 0.5$



Log-Normal Property of Stock Prices





- We find that as n goes to infinity, a binomial distribution approaches a normal distribution. $\rightarrow$ The Central Limit Theorem

- Hence, as n approaches infinity, the number of upward movement will be normally distributed

$$j \sim \phi(np, \sqrt{np(1-p)}).$$

Log-Normal Property of Stock Prices

- We now know the distribution of j. Next, let's find the distribution of $S_T(j)$.

- Using $u = e^{\sigma\sqrt{\Delta t}}$ and $d = e^{-\sigma\sqrt{\Delta t}}$,

$$\begin{aligned} S_T(j) &= S_0 u^j d^{n-j} \\ &= S_0 e^{(\sigma\sqrt{\Delta t})j} e^{(-\sigma\sqrt{\Delta t})(n-j)} \\ &= S_0 e^{(2\sigma\sqrt{\Delta t})j - n\sigma\sqrt{\Delta t}} \end{aligned}$$

- The log of stock price is

$$\ln S_T(j) = \ln S_0 + (2\sigma\sqrt{\Delta t})j - n\sigma\sqrt{\Delta t}$$

- As j is normally distributed, $\ln S_T$ is also normally distributed. Hence, $S_T(j)$ is log-normally distributed.

Log-Normal Property of Stock Prices

- To further identify the distribution, let's find the mean and the standard deviation of $\ln S_T$.

- The mean of $\ln S_T$ is

$$
\begin{aligned}
E(\ln S_T) &= \ln S_0 + 2\sigma\sqrt{\Delta t}E(j) - n\sigma\sqrt{\Delta t} \\
&= \ln S_0 + 2\sigma\sqrt{\Delta t}(np) - n\sigma\sqrt{\Delta t}
\end{aligned}
$$

Log-Normal Property of Stock Prices

- To proceed, we use $p = \frac{e^{r\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}$ (i.e., risk-neutral probability). Here we use the Tylor series of e^x and also the fact $\Delta t \to 0$ as $n \to \infty$.[2]

$$p \approx \frac{1 + r\Delta t - (1 - \sigma\sqrt{\Delta t} + \sigma^2 \Delta t/2)}{(1 + \sigma\sqrt{\Delta t} + \sigma^2\Delta t/2) - (1 - \sigma\sqrt{\Delta t} + \sigma^2\Delta t/2)}$$
$$= \left(\frac{1}{2} + \frac{(r - \sigma^2/2)\sqrt{\Delta t}}{2\sigma}\right)$$

- Plugging this p into $E(\ln S_T)$ in the previous page, the mean becomes

$$E(\ln S_T) = \ln S_0 + \left(r - \frac{\sigma^2}{2}\right) T$$

where we use the fact $\Delta t = \frac{T}{n}$.

[2] $e^{r\Delta t} \approx 1 + r\Delta t + \frac{1}{2}r^2\Delta t^2$ (set Δt as x), $e^{-\sigma\sqrt{\Delta t}} \approx 1 - \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t$ (set $\sqrt{\Delta t}$ as x).

Log-Normal Property of Stock Prices

- The standard deviation of $\ln S_T$ is

$$\begin{aligned}\text{Std.Dev.}(\ln S_T) &= 2\sigma\sqrt{\Delta t} \times \sqrt{np(1-p)} \\ &= 2\sigma\sqrt{Tp(1-p)}.\end{aligned}$$

- Next, let's simplify the standard deviation. We find

$$\begin{aligned}p(1-p) &= \left(\frac{1}{2} + \frac{(r-\sigma^2/2)\sqrt{\Delta t}}{2\sigma}\right)\left(\frac{1}{2} - \frac{(r-\sigma^2/2)\sqrt{\Delta t}}{2\sigma}\right) \\ &= \frac{1}{4} - \frac{\left(r-\sigma^2/2\right)^2}{4\sigma^2}\Delta t \approx \frac{1}{4}\end{aligned}$$

- Thus, the standard deviation of $\ln S_T$ becomes

$$\text{Std.Dev.}(\ln S_T) = \sigma\sqrt{T}.$$

Log-Normal Property of Stock Prices

- Combining the mean and the standard deviation, we conclude

$$\ln S_T \sim \phi\left(\ln S_0 + \left(r - \frac{\sigma^2}{2}\right) T, \sigma\sqrt{T}\right)$$

  in the risk-neutral world.

Log-Normal Property of Stock Prices - Real probability

- Consider the real world where investors require the return α per annum on stock. Then, we can use the real probability p^* instead of p.

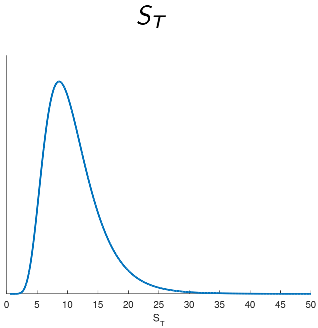
$$p^* = \frac{e^{\alpha \Delta t} - d}{u - d}$$

- Following the same logic as in the risk-neutral world, the real world distribution of stock price is

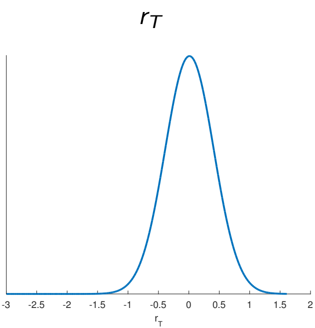
$$\ln S_T \sim \phi\left(\ln S_0 + \left(\alpha - \frac{\sigma^2}{2}\right) T, \sigma\sqrt{T}\right)$$

Log-Normal Property of Stock Prices - Example

- Suppose that $S_0 = 10$, $r = 0.09$, $\sigma = 0.4$, and $T = 1$. Below are the probability density functions of S_T and r_T $(= \ln(S_T/S_0))$.



log-normal distribution



normal distribution

Probability of Option Exercise

- Using the distribution of future stock price, we can determine the probability of option exercise.
- Consider a European call with strike price K and expiration date T.
- What is the probability of option exercise,

$$\text{Prob}\left(S_T \geq K\right)$$

  when S_T is log-normally distributed?

Probability of Option Exercise

- The probability is ...

$$
\begin{aligned}
&\mathrm{Prob}\left(S_T \geq K\right) = \mathrm{Prob}\left(\ln S_T \geq \ln K\right) \\
=&\mathrm{Prob}\left(\frac{\ln S_T - \ln S_0 - (r - \sigma^2/2)T}{\sigma\sqrt{T}} \geq \frac{\ln K - \ln S_0 - (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right) \\
=&1 - \mathrm{Prob}\left(\underbrace{\frac{\ln S_T - \ln S_0 - (r - \sigma^2/2)T}{\sigma\sqrt{T}}}_{\sim\phi(0,1)} < \frac{\ln K - \ln S_0 - (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right) \\
=&1 - N\left(\frac{\ln K - \ln S_0 - (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right) \\
=&N\left(\frac{-\ln K + \ln S_0 + (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right) \equiv N(d_2)
\end{aligned}
$$

where $\phi(0,1)$ is a standard normal random variable, and $N(x)$ is the cumulative distribution function of the standard normal.

Next

- Using the log-normal distribution of stock price, we can calculate the expected payoff of an option. This will lead us to the Black-Scholes-Merton formula.

- The exercise probability, $N(d_2)$, will be a part of the BSM result.

Math Review

Derivation of the BSM formula - Math Review

- In the derivation of the BSM formula, we need to compute the expected value of a function of random variable.
- This requires the understanding of a normal random variable and its probability density function.
- In addition, the calculation requires us to change variable in integration. This technique will be reviewed in the next slide.

Math Review - Normal Distribution

- Recall that to define $N(x)$, we consider a standard normal random variable Z.
- For a certain value x, $N(x)$ is the probability that Z is lower than or equal to x.

$$N(x) \equiv \text{Prob}(Z \leq x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz.$$

- Graphically, $N(x)$ is the shadowed area in the below figure.

Figure 14.3 Shaded area represents $N(x)$.

- In Excel, we can use the function "norm.s.dist(x, TRUE)" to compute $N(x)$.

Math Review - Change of variable in integration

- Suppose that we integrate function $f(y)$ with respect to y:

$$\int f(y)dy.$$

- In addition, y is a function of another variable x, $y = g(x)$.

- Then, we can rewrite the above integration with respect to x

$$\int f(y)dy = \int f(g(x))g'(x)dx.$$

- Intuitively, we change dy to $g'(x)dx$ based on the derivative

$$\frac{dy}{dx} = g'(x)$$

Math Review - Change of variable in integration

e.g. Y is a normal random variable with mean m and the standard deviation w. $f(Y)$ is a function of the variable. Then, the expectation of $f(Y)$ is

$$E[f(Y)] = \int_{-\infty}^{\infty} f(y) \frac{1}{\sqrt{2\pi w^2}} e^{-\frac{(y-m)^2}{2w^2}} dy$$

- Consider a new variable $z = \frac{y-m}{w}$. Then, $y = m + wz$ and $(dy) = w(dz)$. We can rewrite the above integration in terms of z:

$$\begin{aligned}
\int_{-\infty}^{\infty} f(y) \frac{1}{\sqrt{2\pi w^2}} e^{-\frac{(y-m)^2}{2w^2}} dy &= \int_{-\infty}^{\infty} f(m+wz) \frac{1}{\sqrt{2\pi w^2}} e^{-\frac{z^2}{2}} w(dz) \\
&= \int_{-\infty}^{\infty} f(m+wz) \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz
\end{aligned}$$

Derivation of Black-Scholes-Merton Model

Black-Scholes-Merton Model - Assumptions

- The derivation of BSM option price is based on following assumptions.
  - The stock price follows a log-normal distribution.
  - The risk-free rate, r, is constant and the same for all maturities.
  - There are no dividends during the life of the derivative.
  - There are no transaction costs or taxes.
  - There are no arbitrage opportunities.

Black-Scholes-Merton Model - Derivation

- Using the present-value approach, the call price is

$$c_0 = e^{-rT} E\left[\max(S_T - K, 0)\right].$$

  when we compute the expected payoff under the risk-neutral probability $\Rightarrow$ Risk-neutral valuation

- Utilizing the log-normal distribution of S_T, we can compute the expected option payoff. Then, by discounting as above, we obtain the option price.

Black-Scholes-Merton Model - Derivation

- First, let's calculate $E\left[\max(S_T - K, 0)\right]$
- Note that S_T is log-normally distributed in the risk-neutral world.

$$\ln S_T \sim \phi\left(\underbrace{\ln S_0 + \left(r - \frac{\sigma^2}{2}\right)T}_{\equiv m}, \underbrace{\sigma\sqrt{T}}_{\equiv w}\right)$$

- To simplify the notation, let V denote $\ln S_T$. So, $V \sim \phi(m, w)$. Then, the probability density function of V is

$$g(V) = \frac{1}{\sqrt{2\pi w^2}} e^{-\frac{(V-m)^2}{2w^2}}$$

- Let's use $g(V)$ to compute the expected payoff of the call.

Black-Scholes-Merton Model - Derivation

- The expected payoff is

$$\begin{aligned}
E\left[\max(S_T - K, 0)\right] &= E\left[\max(e^V - K, 0)\right] \\
&= \int_{-\infty}^{\infty} \max(e^V - K, 0) g(V) dV \\
&= \int_{-\infty}^{\ln K} \underbrace{\max(e^V - K, 0)}_{=0} g(V) dV + \int_{\ln K}^{\infty} \underbrace{\max(e^V - K, 0)}_{=e^V - K} g(V) dV \\
&= \int_{\ln K}^{\infty} (e^V - K) g(V) dV \\
&= \underbrace{\int_{\ln K}^{\infty} e^V \cdot g(V) dV}_{\equiv \mathbb{A}} - \underbrace{\int_{\ln K}^{\infty} K \cdot g(V) dV}_{\equiv \mathbb{B}}
\end{aligned}$$

- Let's calculate $\mathbb{A}$ and $\mathbb{B}$ separately and combine later.

Black-Scholes-Merton Model - Derivation

- Let's find $\mathbb{B}$ first.

$$\begin{aligned}
\mathbb{B} &= \int_{\ln K}^{\infty} K \cdot g(V) dV \\
&= K \int_{\ln K}^{\infty} g(V) dV \\
&= K \cdot \text{Prob}\left(V \geq \ln K\right) \\
&= K \cdot \text{Prob}\left(\underbrace{e^{V}}_{=S_T} \geq \underbrace{e^{\ln K}}_{=K}\right) \\
&= K \cdot N(d_2)
\end{aligned}$$

where $d_2 = \frac{\ln(S_0/K)+(r-\sigma^2/2)T}{\sigma\sqrt{T}}$.

Black-Scholes-Merton Model - Derivation

- Next, let's find $\mathbb{A}$.

$$\mathbb{A} = \int_{\ln K}^{\infty} e^V \cdot g(V) dV = \int_{\ln K}^{\infty} e^V \cdot \frac{1}{\sqrt{2\pi w^2}} e^{-\frac{(V-m)^2}{2w^2}} dV$$

- To simplify the calculation, define a new variable $Q = \frac{V-m}{w}$. Then, $V = m + wQ$, and $(dV) = w(dQ)$ in the change of variable in the integration.

$$\begin{aligned}
\mathbb{A} &= \int_{\frac{\ln K - m}{w}}^{\infty} e^{m+wQ} \frac{1}{\sqrt{2\pi w^2}} e^{-\frac{Q^2}{2}} w \times dQ \\
&= \int_{\frac{\ln K - m}{w}}^{\infty} e^{m+wQ} \frac{1}{\sqrt{2\pi}} e^{-\frac{Q^2}{2}} dQ \\
&= \int_{\frac{\ln K - m}{w}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{Q^2}{2}+wQ+m} dQ \\
&= \ldots
\end{aligned}$$

Black-Scholes-Merton Model - Derivation

$$
\begin{aligned}
\mathbb{A} &= \int_{\frac{\ln K - m}{w}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{Q^2}{2} + wQ - \frac{w^2}{2} + \frac{w^2}{2} + m} dQ \\
&= e^{m + \frac{w^2}{2}} \int_{\frac{\ln K - m}{w}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{Q^2}{2} + wQ - \frac{w^2}{2}} dQ \\
&= e^{m + \frac{w^2}{2}} \int_{\frac{\ln K - m}{w}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(Q-w)^2}{2}} dQ
\end{aligned}
$$

- To simplify, define a new variable $Y = Q - w$. Then, $Q = Y + w$ and $(dQ) = (dY)$ in the change of variable in the integration.

$$
\begin{aligned}
\mathbb{A} &= e^{m + \frac{w^2}{2}} \int_{\frac{\ln K - m - w^2}{w}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{Y^2}{2}} dY \\
&= e^{m + \frac{w^2}{2}} \times \text{Prob}\left(Y \geq \frac{\ln K - m - w^2}{w} \right) \\
&= \ldots
\end{aligned}
$$

Black-Scholes-Merton Model - Derivation

$$
\begin{aligned}
\mathbb{A} &= e^{m+\frac{w^2}{2}} \times \left[1 - \text{Prob}\left(Y < \frac{\ln K - m - w^2}{w}\right)\right] \\
&= e^{m+\frac{w^2}{2}} \times \left[1 - N\left(\frac{\ln K - m - w^2}{w}\right)\right] \\
&= e^{m+\frac{w^2}{2}} \times N\left(\frac{-\ln K + m + w^2}{w}\right)
\end{aligned}
$$

Black-Scholes-Merton Model - Derivation

- In $\mathbb{A}$,

$$m + \frac{w^2}{2} = \ln S_0 + \left(r - \frac{\sigma^2}{2}\right) T + \frac{\sigma^2 T}{2} = \ln S_0 + rT$$

$$\frac{-\ln K + m + w^2}{w} = \frac{\ln S_0 - \ln K + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}}$$

- Thus,

$$\begin{aligned}
\mathbb{A} &= e^{m + \frac{w^2}{2}} \times N\left(\frac{-\ln K + m + w^2}{w}\right) \\
&= S_0 e^{rT} \times N\left(\underbrace{\frac{\ln S_0 - \ln K + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}}}_{\equiv d_1}\right) \\
&= S_0 e^{rT} \times N(d_1)
\end{aligned}$$

Black-Scholes-Merton Model - Derivation

- Now, let's combine $\mathbb{A}$ and $\mathbb{B}$.

$$\begin{aligned} E\left[\max(S_T - K, 0)\right] &= \mathbb{A} - \mathbb{B} \\ &= S_0 e^{rT} \times N(d_1) - K \times N(d_2) \end{aligned}$$

- The current price of the call is

$$\begin{aligned} c_0 &= e^{-rT} E\left[\max(S_T - K, 0)\right] \\ &= e^{-rT}\left[S_0 e^{rT} \times N(d_1) - K \times N(d_2)\right] \\ &= S_0 N(d_1) - K e^{-rT} N(d_2) \end{aligned}$$

Black-Scholes-Merton Model - Derivation

- Once the call option is obtained, we can easily drive the put price using the put-call parity.

$$
\begin{aligned}
p_0 &= c_0 + Ke^{-rT} - S_0 \\
&= S_0 N(d_1) - Ke^{-rT} N(d_2) + Ke^{-rT} - S_0 \\
&= -S_0 \left[1 - N(d_1)\right] + Ke^{-rT} \left[1 - N(d_2)\right] \\
&= -S_0 N(-d_1) + Ke^{-rT} N(-d_2)
\end{aligned}
$$

Black-Scholes-Merton Model

- The BSM model provides an analytic form that determines the option price as a function of the followings:
  - Current stock price S_0
  - Strike price K
  - Time to expiration T
  - Risk-free interest rate r
  - Volatility of underlying asset σ
- Through the BSM model, we can find the option price by simply inputting numbers into the option-pricing formula.

Black-Scholes-Merton Model - Result

- The prices of European call and put options on non-dividend-paying stock are

$$c_0 = S_0 N(d_1) - Ke^{-rT} N(d_2)$$

$$p_0 = Ke^{-rT} N(-d_2) - S_0 N(-d_1)$$

where

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T},$$

and $N(x)$ is the cumulative probability distribution function for a standard normal random variable.

Black-Scholes-Merton Model - Example

Q. There is a 6-month European call option on a stock whose current price is \$42. The strike price is \$40, and the risk-free interest rate is 10% per annum. The stock volatility is 20% per annum. What is the price of the option?

Black-Scholes-Merton Model - Example

Q. There is a 6-month European call option on a stock whose current price is \$42. The strike price is \$40, and the risk-free interest rate is 10% per annum. The stock volatility is 20% per annum. What is the price of the option?

Answer:

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} = \frac{\ln(42/40) + (0.1 + 0.2^2/2)(0.5)}{0.2\sqrt{0.5}} = 0.7693$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.6278$$

$$\begin{aligned} c &= S_0 N(d_1) - K e^{-rT} N(d_2) \\ &= 42 \times N(0.7693) - 40e^{-0.1\times 0.5} \times N(0.6278) \\ &= 42 \times \text{norm.s.dist}(0.7693, \text{TRUE}) - 40e^{-0.1\times 0.5} \times \text{norm.s.dist}(0.6278, \text{TRUE}) \\ &= \$4.759. \end{aligned}$$

Black-Scholes-Merton Model - Example

- What if we use the binomial model for the previous question?
- Let's start with 10-step binomial model and increases the number of steps.

number of steps	option price
10	4.800
20	4.768
50	4.762
⋮	⋮
500	4.759
BSM price	4.759

- As the number of steps increases, the binomial price converges to the BSM price.

Black-Scholes-Merton Model - Another Example

Q. Consider a derivative on a stock with the time to expiration T and the following payoff:

$$\begin{cases} 0 & \text{if } S_T < K_1 \\ K_1 & \text{if } K_1 \leq S_T < K_2 \\ 0 & \text{if } K_2 \leq S_T \end{cases}$$

where $K_2 > K_1$. What is the present value of the derivative? Provide an analytic expression of the price using $N(\cdot)$, the cumulative probability distribution function of a standard normal random variable.

Black-Scholes-Merton Model - Another Example

Answer: Let V denote $\ln S_T$. Then, V is normally distributed, i.e., $V \sim \phi(m, w)$. Let $g(V)$ denote the probability density function of V. To find the present value of the derivative, we first compute the expected option payoff:

$$
\begin{aligned}
E\left[\text{Payoff}\right] &= \int_{-\infty}^{\infty} \text{Payoff} \cdot g(V)dV \\
&= \int_{-\infty}^{\ln K_1} \text{Payoff} \cdot g(V)dV + \int_{\ln K_1}^{\ln K_2} \text{Payoff} \cdot g(V)dV \\
&\qquad + \int_{\ln K_2}^{\infty} \text{Payoff} \cdot g(V)dV \\
&= \int_{-\infty}^{\ln K_1} 0 \cdot g(V)dV + \int_{\ln K_1}^{\ln K_2} K_1 \cdot g(V)dV + \int_{\ln K_2}^{\infty} 0 \cdot g(V)dV \\
&= K_1 \int_{\ln K_1}^{\ln K_2} g(V)dV \\
&= K_1 \cdot \text{Prob}\left(\ln K_1 \leq V \leq \ln K_2\right) \\
&= K_1 \cdot \text{Prob}\left(K_1 \leq S_T \leq K_2\right) \\
&= K_1 \cdot \left[\text{Prob}\left(K_1 \leq S_T\right) - \text{Prob}\left(K_2 \leq S_T\right)\right]
\end{aligned}
$$

Black-Scholes-Merton Model - Another Example

Answer (cont'd):

$$= K_1 \cdot \left[N\left(\frac{\ln(S_0/K_1) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \right) - N\left(\frac{\ln(S_0/K_2) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \right) \right].$$

Next, multiplying by the discount factor, we obtain the present value as follows:

$$f_0 = e^{-rT} K_1 \cdot \left[N\left(\frac{\ln(S_0/K_1) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \right) - N\left(\frac{\ln(S_0/K_2) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \right) \right].$$

BSM formula: Interpretation

- The BSM expresses the option as a portfolio of stocks and bonds.

- $N(d_1)$ is the fraction of share we hold in the replicating portfolio at t. In fact, we can show that:

  $\Delta_c = \frac{\partial C}{\partial S} = N(d_1) > 0$

  $\Delta_p = \frac{\partial P}{\partial S} = -N(-d_1) < 0$

- For a call, $Ke^{-rT}N(d_2)$ is the amount of initial borrowing in the replicating portfolio.

- The value of the call is the cost of the replicating portfolio.

$$\begin{aligned} c_0 &= \Delta_c \times S - B \\ &= S_0 e^{-\delta T} N(d_1) - Ke^{-rT} N(d_2) \end{aligned}$$

Extending the BSM model

The BSM for dividend payout

- Suppose the underlying pays continuous dividend q .
  - Dividend should, for the purposes of option valuation, be defined as the reduction in the stock price.

- Replace the stock price S in the formula by Se^{-qT}

$$c = S_0 e^{-qT} N(d_1) - Ke^{-rT} N(d_2)$$

  , where $d_1 = \frac{\ln(S_0/K)+(r-q+\sigma^2/2)T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$. (called Merton model)

- Delta $= e^{-qT} N(d_1)$

- Put-Call parity: $p + S_0 e^{-qT} = c + Ke^{-rT}$
  - Given the price of puts and calls, we can solve this for the "implied dividend yield q".

- For proof, go to the slide, "The BSM for dividend payout: Derivation".

The BSM for option on futures

- Futures call option
  - Right to enter into a long futures contract at a pre-specified futures price
  - If exercised, holder gets long position in futures contract plus cash difference between most recent settlement price on futures and strike price on futures option
  - Effective payoff is $max(F - K, 0)$, where F is the current price of the futures.

- Futures put option
  - Right to enter into a short futures contract at a pre-specified futures price
  - If exercised, holder gets short position in futures contract and receives cash difference between strike price and most recent settlement price
  - Effective payoff is $max(K - F, 0)$, where F is the current price of the futures

The BSM for option on futures (cont'd)

- Example: On August 15 and a trader has one September futures call option contract on copper with a strike price of 320 cents per pound. One futures contract is on $25,000$ pounds of copper.

- The futures price of copper for delivery in September is currently 331 cents, and at the close of trading on August 14 it was 330 cents.

  - If the option is exercised, the trader receives a cash amount of $25,000 \times (330 - 320)$ cents $= +2,500$, plus a long position in a futures contract to buy $25,000$ pounds of copper in September.
  - If the position in the futures contract is closed out immediately. The trader gets the \$2,500 cash payoff plus an amount $25,000 \times (331 - 330)$ cents $= +250$, reflecting the change in the futures price since the last settlement.
  - The total payoff from exercising the option on August 15 is \$2,750, which equals \$25,000$(F - K)$, where F is the futures price at the time of exercise and K is the strike price.

The BSM for option on futures (cont'd)

- Futures options have potential advantages over spot options
  - Futures contracts may be easier to trade and more liquid than the underlying asset.
  - Exercise of option does not lead to delivery of underlying asset.
  - Futures options and futures usually trade on same exchange (reduced margin requirement).
- European futures options and European spot options are equivalent when futures contract matures at the same time as the option ($F_T = S_T$).
- Popular contracts include agricultural commodities, energy, gold, VIX, and interest rates
- Most futures options are American-style.

The BSM for option on futures (cont'd)

- The underlying is a futures contract, so S in the equation is the futures price, call it F.
  - Remember $F_0 = S_0 e^{rT}$. As time passes, e^{rT} shrinks at the rate of r like dividend yield q.
  - Let's assume Futures = Forward here.

- Replace the stock price S in the formula by the discounted value of the futures price F: Fe^{-rT}

$$c = Fe^{-rT}N(d_1) - Ke^{-rT}N(d_2) = e^{-rT}[FN(d_1) - KN(d_2)]$$

, where $d_1 = \frac{\ln(F/K)+(\sigma^2/2)T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$

- Delta $= e^{-rT}N(d_1)$

- Put-Call parity: $p + Fe^{-rT} = c + Ke^{-rT}$

The BSM for option on futures (cont'd)

- Fischer Black derived this model in 1976.
  - Avoids need to calculate convenience yield or income on underlying asset (already incorporated in futures)
  - The underlying is a forward rather than a futures price.
    - When interest rates are assumed to be deterministic, forward and futures prices are equal and so this is valid.
  - Forward price follows the log-normal distribution.
  - Very useful in applications beyond futures options (e.g., options on bonds)
  - Note the formula is for European, not for American.
    - Binomal tree model can be used.

The BSM for currency option

- The price of the underlying is the exchange rate (in \$ per unit of FX). The underlying pays interest at the foreign riskless rate, so set $q = r_F$. The riskless rate r is the domestic rate (Garman-Kohlhagen Model).

- Replace the stock price S in the formula by $Se^{-r_F T}$

$$c = S_0 e^{-r_F T} N(d_1) - Ke^{-rT} N(d_2)$$

, where $d_1 = \frac{\ln(S_0/K) + (r - r_F + \sigma^2/2)T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$

- Delta $= e^{-r_F T} N(d_1)$

- Put-Call parity: $p + S_0 e^{-r_F T} = c + Ke^{-rT}$

- Using the Black's model: $c = e^{-rT}[FN(d_1) - KN(d_2)]$, where F is the futures price on currency.

Alternative Derivation I

Review

- This derivation is also based on the Binomal Tree model in the risk-neutral world.
  - The final stock price: $S_0 u^j d^{n-j}$.
  - The payoff from a European call option: $\max(S_0 u^j d^{n-j} - K, 0)$
  - The probability of j upward and $n-j$ downward steps: $\frac{n!}{j!(n-j)!} p^j (1-p)^{n-j}$
  - The expected payoff: $\sum_{j=0}^{n} \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j} \max(S_0 u^j d^{n-j} - K, 0)$
  - The option value: $c = e^{-rT} \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j} \max(S_0 u^j d^{n-j} - K, 0)$

Alternative Formulation of Call Price

The Payoff is positive if $S_0 u^j d^{n-j} > K \Rightarrow \ln(S_0/K) > -j\ln(u) - (n-j)\ln(d)$

Plug in $u = e^{\sigma\sqrt{T/n}}, u = e^{-\sigma\sqrt{T/n}}$

$\Rightarrow \ln(S_0/K) > n\sigma\sqrt{T/n} + 2j\sigma\sqrt{T/n}$

$\Rightarrow j > \dfrac{n}{2} - \dfrac{\ln(S_0/K)}{2\sigma\sqrt{T/n}}$

Therefore, $c = e^{-rT}\sum_{j>\alpha} \dfrac{n!}{j!(n-j)!} p^j (1-p)^{n-j} \max(S_0 u^j d^{n-j} - K, 0)$,

$$\text{where } \alpha = \frac{n}{2} - \frac{\ln(S_0/K)}{2\sigma\sqrt{T/n}}$$

Write $c = e^{-rT}(S_0 U_1 - K U_2)$,

$$\text{where } U_1 = \sum_{j>\alpha} \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j} u^j d^{n-j} \text{ and}$$

$$U_2 = \sum_{j>\alpha} \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j}$$

Increase the Number of Steps in a Binomial Tree

Fact: as $n \to \infty, j \sim B(n,p) \longrightarrow \phi(np, \sqrt{np(1-p)})$

U_2 is $\Pr(j > \alpha)$, therefore, $U_2 = \Pr(\frac{j-np}{\sqrt{np(1-p)}} > \frac{\alpha - np}{\sqrt{np(1-p)}}) = N\left(\frac{np-\alpha}{\sqrt{np(1-p)}}\right)$

$$\Rightarrow U_2 = N\left(\frac{\ln(S_0/K)}{2\sigma\sqrt{Tp(1-p)}} + \frac{\sqrt{n}(p-1/2)}{\sqrt{p(1-p)}}\right)$$

Remember $p = \dfrac{e^{rT/n} - e^{-\sigma\sqrt{T/n}}}{e^{\sigma\sqrt{T/n}} - e^{-\sigma\sqrt{T/n}}}$

Applying the Taylor expansion: $e^{rT/n} \approx 1 + r(T/n)$

$$e^{\sigma\sqrt{T/n}} \approx 1 + \sigma\sqrt{T} + \frac{1}{2}\sigma^2(T/n)$$

$$e^{-\sigma\sqrt{T/n}} \approx 1 - \sigma\sqrt{T} + \frac{1}{2}\sigma^2(T/n)$$

Hence, as $n \to \infty$,

$$p(1-p) \to 1/4 \text{ and } \sqrt{n}(p-1/2) \to \frac{(r-\sigma^2/2)\sqrt{T}}{2\sigma}$$

$$\Rightarrow U_2 = N\left(\frac{\ln(S_0/K) + (r-\sigma^2/2)T}{\sigma\sqrt{T}}\right)$$

Increase the Number of Steps in a Binomial Tree

$$U_1 = \sum_{j>\alpha} \frac{n!}{j!(n-j)!}(up)^j (d(1-p))^{n-j}$$

Let $p^* = \dfrac{pu}{pu + (1-p)d}$

Then $1 - p^* = \dfrac{(1-p)d}{pu + (1-p)d}$

$$\Rightarrow U_1 = (pu + (1-p)d)^n \sum_{j>\alpha} \frac{n!}{j!(n-j)!}(p^*)^j (1-p^*)^{n-j}$$

$$U_1 = e^{rT} \sum_{j>\alpha} \frac{n!}{j!(n-j)!}(p^*)^j (1-p^*)^{n-j}$$

Therefore, following the same step, $U_1 = e^{rT} N\left(\dfrac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}\right)$

$$\Rightarrow c = e^{-rT}(S_0 U_1 - K U_2) = c = S_0 N(d_1) - K e^{-rT} N(d_2))$$

Alternative Derivation II

Overview

- This derivation is not based on the Binomial Tree model.
    - The Binomial Model set up a framework in which the underlying asset and the riskless bond could be combined to create a position that exactly replicates the payoff on the option.
    - The Black-Scholes model is derived in a similar way: The option and the stock are combined to create a hedged position that is like a riskless bond.
    - The riskless option-stock hedged position must return the riskless rate of interest. This leads to a fair price for the option.

Underlying Assumptions of the BSM Model

- Options are European
- "Perfect" markets – no transactions costs, no taxes, no constraints on short selling with full use of the proceeds, no indivisibilities, etc.
- No limits on borrowing or lending at a known risk free rate of interest
- The price of the underlying asset follows a "lognormal diffusion" process
- The return volatility of the underlying asset is known
- No dividends or cash payouts from the underlying asset prior to option maturity

The Asset Price Process

- The BSM model assumes the price of the underlying asset follows a "lognormal diffusion" process:

$$dS_t = \mu S_t dt + \sigma S_t dz_t$$
$$\Rightarrow \frac{dS_t}{S_t} = \mu dt + \sigma dz_t$$

  - dS_t = the change in stock price over the next instant
  - μ = the "drift," that is, the average rate of capital gains as a continuously compounded annualized figure
  - dt = an "instant"
  - σ = the volatility, expressed as an annual rate
  - dz_t = "Brownian motion," a very small random shock to the price over the next instant.

Definitions

- A variable z_t follows a Brownian motion (Wiener process) if
  1. $dz_t = \epsilon\sqrt{dt}$, where $\epsilon \sim \phi(0,1)$.
     - $E[dz_t] = 0$ and $\sigma(dz_t) = \sqrt{dt}$
  2. dz_t for any two different short intervals of time are independent.
  3. $z_0 = 0$ and z is continuous in t.
- $dS_t = \mu dt + \sigma dz_t$: Generalized Wiener process
  - μ and σ are constant.
- $dS_t = \mu(S_t, t)dt + \sigma(S_t, t)dz$: Ito process
  - μ and σ are functions of S_t and time t.

The Process for a Stock Price

- $dS_t = \mu dt$?
  - There is no uncertainty.
  - $S_t = \mu t$, i.e., stock price grows by μ. $\Rightarrow$ Not realistic!
- $dS_t = \mu dt + \sigma dz$?
  - There is uncertainty, dz.
  - But stock price can take a negative value!
- $dS_t/S_t = \mu dt + \sigma dz$
  - The most widely used model of stock price behavior.
  - For a risk-free asset, $\mu = r$ and $\sigma = 0$. Hence, $S_t = e^{rt}$.
  - Ito process, log-normal diffusion process, geometric Brownian motion

Ito's Lemma

- Suppose x follows an Ito process: $dx = a(x,t)dt + b(x,t)dz$
- Ito shows that a function of x and t, $G(x,t)$ follows another Ito process:

$$dG = \left(\frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2 \right) dt + \frac{\partial G}{\partial x} b dz$$

  - Apply a Taylor series expansion on $G(x,t)$:

$$dG \approx \frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial t} dt + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} dx^2 + \frac{1}{2} \frac{\partial^2 G}{\partial t^2} dt^2 + \frac{\partial^2 G}{\partial x \partial t} dxdt$$

    - $dx^2 \approx b^2 dz^2 = b^2 \epsilon^2 dt^3$
    - $E(b^2 \epsilon^2 dt) = b^2 dt$ and $Var(\epsilon^2 dt) = 2dt^2 \approx 0$ $(\because Var(\epsilon^2) = E(\epsilon^4) - E(\epsilon^2)^2 = 3 - 1 = 2)$.
    - Ignore higher order terms (e.g. $dt^{1.5}, dt^2$).

$$dG \approx \frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial t} dt + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2 dt$$

  - Plug in $dx = a(x,t)dt + b(x,t)dz$.

[3] $dtdz = 0$ and $(dz)^2 = dt$

Apply Ito's Lemma

- Apply Ito's lemma on $dS_t = \mu S_t dt + \sigma S_t dz_t$

$$dG = \left(\frac{\partial G}{\partial S} \mu S + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial G}{\partial S} \sigma S dz$$

- Now consier $G = \ln S_t$.

$$\frac{\partial G}{\partial S} = \frac{1}{S_t}, \quad \frac{\partial^2 G}{\partial S^2} = -\frac{1}{S_t^2}, \quad \frac{\partial G}{\partial t} = 0$$

- Therefore,

$$dG = \left(\mu - \frac{\sigma^2}{2} \right) dt + \sigma dz$$

- It follows a generalized Wiener process.
    - $G_T - G_0 = \ln S_T - \ln S_0 \sim \phi \left(\left(\mu - \frac{\sigma^2}{2} \right) T, \sigma^2 T \right)$
    - That is, $\ln S_T \sim \phi \left(\ln S_0 + \left(\mu - \frac{\sigma^2}{2} \right) T, \sigma^2 T \right)$

Apply Ito's Lemma

- The stochastic process usually assumed for a stock price is geometric Brownian motion.
- Under this process the return to the holder of the stock in a small period of time is normally distributed and the returns in two nonoverlapping periods are independent.
- The value of the stock price at a future time has a lognormal distribution.
- The Wiener process dz underlying the stochastic process for S is exactly the same as the Wiener process underlying the stochastic process for G.
- Both are subject to the same underlying source of uncertainty.

The BSM Differential Equation

- The stock price process, $dS_t = \mu S_t dt + \sigma S dz_t$.
- V is the price of a call option, a function of S and t.

$$dV = \left(\frac{\partial V}{\partial S} \mu S + \frac{\partial V}{\partial t} + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial V}{\partial S} \sigma S dz$$

- Construct a portfolio: Long 1 unit of the call option and short $\frac{\partial V}{\partial S}$ number of shares. The value of the portfolio is:

$$\begin{aligned} \Pi &= V - \frac{\partial V}{\partial S} S \\ d\Pi &= dV - \frac{\partial V}{\partial S} dS \\ &= \left(\frac{\partial V}{\partial t} + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \sigma^2 S^2 \right) dt \end{aligned}$$

The BSM Differential Equation

- Because this equation does not involve dz, the portfolio must be riskless during time dt. Therefore,

$$\Pi = e^{rdt}$$
$$d\Pi = r\Pi dt$$
$$\left(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2\right) dt = r\left(V - \frac{\partial V}{\partial S}S\right) dt$$
$$\Rightarrow \frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 = rV$$

- This is called the Black–Scholes–Merton differential equation.
- Solving the differential equation with the boundary conditions, e.g., $V = \max(S - K, 0)$ when $= T$, gives a formula for a European call option.
  - Unfortunately, no exact analytic formula for the value of an American put option on a non-dividend-paying stock has been produced.

NB There is no μ, the expected return!

The BSM Differential Equation

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 = rV$$

- If $V(S, T) = S_T$, i.e., the stock itself, $V(S, t) = S_t$
- If $V(S, T) = K$, i.e., constant, then $V(S, t) = Ke^{-r(T-t)}$
- If $V(S, T) = S_T - K$, i.e., forward, then $V(S, t) = S_t - Ke^{-r(T-t)}$
- Does $V(S, 0) = S_0 N(d_1) - Ke^{-rT} N(d_2)$ satisfy the equation?
- The PDE above is so general that it can solve (mostly numerially) for V depending on the boundary conditions.

The BSM for dividend payout: Derivation

- In time dt the holder of the portfolio earns capital gains equal to $d\Pi$ and dividends on the stock position equal to

$$dD = qS\frac{\partial V}{\partial S}dt$$

- The change in the wealth of the portfolio holder in time Δt is the sum of $\Delta\Pi$ and ΔD:

$$\Delta W_t = \left(-\frac{\partial V}{\partial t} - \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + qS\frac{\partial V}{\partial S}\right)\Delta t$$

- The portfolio is instantaneously riskless.

$$\Delta W_t = r\Pi\Delta t$$

$$\left(-\frac{\partial V}{\partial t} - \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + qS\frac{\partial V}{\partial S}\right)\Delta t = r\left(-V + \frac{\partial V}{\partial S}S\right)\Delta t$$

$$\frac{\partial V}{\partial t} + (r-q)S\frac{\partial V}{\partial S} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 = rf$$

The BSM for dividend payout: Derivation

- The PDE does not involve any variable affected by risk preferences, μ. We can again apply the risk-neutral valuation, i.e., $c = e^{-rT}E[(\max(S_T - K, 0)]$ with risk-neutral probabilities.

- The expected growth rate in the stock price is $r - q$.

$$dS = (r - q)Sdt + \sigma S dz$$

- The expected stock price at T is $S_0 e^{(r-q)T}$. Going through the same steps, we get:

$$E[(\max(S_T - K, 0)] = S_0 e^{r-q} N(d_1) - KN(d_2)$$

and

$$c = S_0 e^{-q} N(d_1) - Ke^{-rT} N(d_2)$$