PROBLEM SET: ANSWER KEY

1 Delta

What does it mean to assert that the delta of a call option is 0.7? How can a short position in 1,000 options be made delta neutral when the delta of each option is 0.7?

Solution: A delta of 0.7 means that, when the price of the stock increases by a small amount, the price of the option increases by 70% of this amount. Similarly, when the price of the stock decreases by a small amount, the price of the option decreases by 70% of this amount. A short position in 1,000 options has a delta of -700 and can be made delta neutral with the purchase of 700 shares.

2 Delta

Calculate the delta of an at-the-money six-month European call option on a non-dividend-paying stock when the risk-free interest rate is 10% per annum and the stock price volatility is 25% per annum. Use the BSM model.

Solution: In this case, $S_0 = K$, $r = 0.1$, $\sigma = 0.25$, and $T = 0.5$. Also,

$$d_1 = \frac{\ln(S_0/K) + \left(0.1 + 0.25^2/2\right)0.5}{0.25\sqrt{0.5}} = 0.3712$$

The delta of the option is $N(d_1)$ or 0.64.

3 Portfolio Insurance

Why did portfolio insurance not work well on October 19, 1987?

Solution: Portfolio insurance involves creating a put option synthetically. It assumes that as soon as a portfolio's value declines by a small amount, the portfolio manager's position is rebalanced by either (a) selling part of the portfolio, or (b) selling index futures. On October 19, 1987, the market declined so quickly that the sort of rebalancing anticipated in portfolio insurance schemes could not be accomplished.

4 Delta of Futures Option

What is the delta of a short position in 1,000 European call options on silver futures? The options mature in eight months, and the futures contract underlying the option matures in nine months. The current nine-month futures price is \$8 per ounce, the exercise price of the options is \$8, the risk-free interest rate is 12% per annum, and the volatility of silver futures prices is 18% per annum. Hint: Use Black's model.

Solution: The delta of a European futures call option is usually defined as the rate of change of the option price with respect to the futures price (not the spot price). It is

$$e^{-rT}N\left(d_1\right)$$

In this case, $F_0 = 8$, $K = 8$, $r = 0.12$, $\sigma = 0.18$, $T = 0.6667$

$$d_1 = \frac{\ln(8/8) + \left(0.18^2/2\right) \times 0.6667}{0.18\sqrt{0.6667}} = 0.0735$$

$N\left(d_1\right) = 0.5293$ and the delta of the option is

$$e^{-0.12 \cdot 0.6667} \times 0.5293 = 0.4886$$

The delta of a short position in 1,000 futures options is therefore -488.6. Hence, a long position in nine-month futures on 488.6 ounces is necessary to hedge the option position.

Extra: If you want to hedge the option position using the underlying asset (i.e., silver), how many units of silver do you need? (Assume no storage cost for silver)
The delta of a nine-month futures contract is $e^{0.12 \cdot 0.75} = 1.094$ assuming no storage costs. (This is because silver can be treated in the same way as a non-dividend-paying stock when there are no storage costs. $F_0 = S_0 e^{rT}$). Hence, the delta of the option with respect to silver price change is (the delta of the option with respect to futures price change) times (the delta of the futures with respect to silver price changes), i.e., $\frac{\mathrm{d}c}{\mathrm{d}P} = \frac{\mathrm{d}c}{\mathrm{d}F} \cdot \frac{\mathrm{d}F}{\mathrm{d}P}$. Therefore, we need a long position in $-488.6 \times 1.094 = -534.6$ ounces of silver.

5 Delta and Gamma

A company uses delta hedging to hedge a portfolio of long positions in put and call options on a currency. Which of the following would give the most favorable result?

1. A virtually constant spot rate
2. Wild movements in the spot rate

Explain your answer.

Solution: A long position in either a put or a call option has a positive gamma. When gamma is positive, the hedger gains from a large change in the stock price and loses from a small change in the stock price. Hence, the hedger will fare better in case (b).

6 Gamma and Vega

Under what circumstances is it possible to make a European option on a stock index both gamma neutral and vega neutral by adding a position in one other European option with the same underlying asset but different K and T? Use the BSM model.

Solution: Assume that S_0, K, r, σ, T, q are the parameters for the option held and $S_0, K^*, r, \sigma, T^*, q$ are the parameters for another option. Suppose that d_1 has its usual meaning and is calculated on the basis of the first set of parameters while d_1^* is the value of d_1 calculated on the basis of the second set of parameters. Suppose further that w of the second option are held for each of the first option held. The gamma of the portfolio is:

$$\alpha\left[\frac{N'(d_1)\, e^{-qT}}{S_0\sigma\sqrt{T}} + w\frac{N'(d_1^*)\, e^{-qT^*}}{S_0\sigma\sqrt{T^*}}\right]$$

where α is the number of the first option held.
Since we require gamma to be zero:

$$w = -\frac{N'(d_1)\, e^{-q(T-T^*)}}{N'(d_1^*)}\sqrt{\frac{T^*}{T}}$$

The vega of the portfolio is:

$$\alpha\left[S_0\sqrt{T}N'(d_1)\, e^{-qT} + wS_0\sqrt{T^*}N'(d_1^*)\, e^{-qT^*}\right]$$

Since we require vega to be zero:

$$w = -\sqrt{\frac{T}{T^*}}\frac{N'(d_1)\, e^{-q(T-T^*)}}{N'(d_1^*)}$$

Equating the two expressions for w

$$T^* = T$$

Hence, the maturity of the option held must equal the maturity of the option used for hedging.

7 The Put-Call Parity and Greeks

Use the put-call parity relationship to derive, for a non-dividend-paying stock, the relationship between:

1. The delta of a European call and the delta of a European put.

2. The gamma of a European call and the gamma of a European put.
3. The vega of a European call and the vega of a European put.
4. The theta of a European call and the theta of a European put.

Solution:

1. For a non-dividend paying stock, put-call parity gives at a general time t:

$$p + S = c + Ke^{-r(T-t)}$$

Differentiating with respect to S:

$$\frac{\partial p}{\partial S} + 1 = \frac{\partial c}{\partial S}$$

or

$$\frac{\partial p}{\partial S} = \frac{\partial c}{\partial S} - 1$$

This shows that the delta of a European put equals the delta of the corresponding European call less 1.0.

2. Differentiating with respect to S again

$$\frac{\partial^2 p}{\partial S^2} = \frac{\partial^2 c}{\partial S^2}$$

Hence, the gamma of a European put equals the gamma of a European call.

3. Differentiating the put-call parity relationship with respect to σ

$$\frac{\partial p}{\partial \sigma} = \frac{\partial c}{\partial \sigma}$$

showing that the vega of a European put equals the vega of a European call.

4. Differentiating the put-call parity relationship with respect to t

$$\frac{\partial p}{\partial t} = rKe^{-r(T-t)} + \frac{\partial c}{\partial t}$$

This is in agreement with the thetas of European calls and puts given in the lecture note since $N(d_2) = 1 - N(-d_2)$.

8 Hedging using Greeks

A financial institution has the following portfolio of over-the-counter options on sterling:

Type	Position	Delta of Option	Gamma of Option	Vega of Option
Call	$-1,000$	0.5	2.2	1.8
Call	-500	0.8	0.6	0.2
Put	$-2,000$	-0.40	1.3	0.7
Call	-500	0.70	1.8	1.4

A traded option is available with a delta of 0.6, a gamma of 1.5, and a vega of 0.8.

1. What position in the traded option and in sterling would make the portfolio both gamma neutral and delta neutral?
2. What position in the traded option and in sterling would make the portfolio both vega neutral and delta neutral?

Solution: The delta of the portfolio is

$$-1,000 \times 0.50 - 500 \times 0.80 - 2,000 \times (-0.40) - 500 \times 0.70 = -450$$

The gamma of the portfolio is

$$-1,000 \times 2.2 - 500 \times 0.6 - 2,000 \times 1.3 - 500 \times 1.8 = -6,000$$

The vega of the portfolio is

$$-1,000 \times 1.8 - 500 \times 0.2 - 2,000 \times 0.7 - 500 \times 1.4 = -4,000$$

1. A long position in 4,000 traded options will give a gamma-neutral portfolio since the long position has a gamma of $4,000 \times 1.5 = +6,000$. The delta of the whole portfolio (including traded options) is then:

   $$4,000 \times 0.6 - 450 = 1,950$$

   Hence, in addition to the 4,000 traded options, a short position of 1,950 in sterling is necessary so that the portfolio is both gamma and delta neutral.

2. A long position in 5,000 traded options will give a vega-neutral portfolio since the long position has a vega of $5,000 \times 0.8 = +4,000$. The delta of the whole portfolio (including traded options) is then

   $$5,000 \times 0.6 - 450 = 2,550$$

Hence, in addition to the 5,000 traded options, a short position of 2,550 in sterling is necessary so that the portfolio is both vega and delta neutral.

9 Portfolio Insurance

Suppose that \$70 billion of equity assets are the subject of portfolio insurance schemes. Assume that the schemes are designed to provide insurance against the value of the assets declining by more than 5% within one year. Suppose $S_0 = 70$, $K = 66.5$, $T = 1$. Other parameters are estimated as $r = 0.06$, $\sigma = 0.25$, and $q = 0.03$. Calculate the value of the stock or futures contracts that the administrators of the portfolio insurance schemes will attempt to sell if the market falls by 23% in a single day.

Solution: We can regard the position of all portfolio insurers taken together as a single put option. Compute the value of the option:

$$\begin{aligned} d_1 &= \frac{\ln(70/66.5) + \left(0.06 - 0.03 + 0.25^2/2\right)}{0.25} = 0.4502 \\ N(d_1) &= 0.6737 \end{aligned}$$

The delta of the option is

$$\begin{aligned} & e^{-qT}\left[N(d_1) - 1\right] \\ &= e^{-0.03}(0.6737 - 1) \\ &= -0.3167 \end{aligned}$$

This shows that 31.67% or \$22.17 billion of assets should have been sold before the decline.
After the decline, $S_0 = 53.9$, $K = 66.5$, $T = 1$, $r = 0.06$, $\sigma = 0.25$, and $q = 0.03$.

$$\begin{aligned} d_1 &= \frac{\ln(53.9/66.5) + \left(0.06 - 0.03 + 0.25^2/2\right)}{0.25} = -0.5953 \\ N(d_1) &= 0.2758 \end{aligned}$$

The delta of the option has dropped to

$$e^{-0.03\cdot 1}(0.2758 - 1) = -0.7028$$

This shows that cumulatively 70.28% of the assets originally held should be sold. An additional 38.61% of the original portfolio should be sold. The sales measured at pre-crash prices are about \$27.0 billion. At post-crash prices, they are about \$20.8 billion.

10 Hedging with Delta and Gamma

A bank's position in options on the dollar-euro exchange rate has a delta of 30,000 and a gamma of $-80,000$. Explain how these numbers can be interpreted. The exchange rate (dollars per euro) is 0.90. What position would you take to make the position delta neutral? After a short period of time, the exchange rate moves to 0.93. Estimate the new delta. What additional trade is necessary to keep the position delta neutral? Assuming the bank did set up a delta-neutral position originally, has it gained or lost money from the exchange-rate movement?

Solution: The delta indicates that when the value of the euro exchange rate increases by \$0.01, the value of the bank's position increases by $0.01 \times 30,000 = \$300$. The gamma indicates that when the euro exchange rate increases by \$0.01, the delta of the portfolio decreases by $0.01 \times 80,000 = 800$. For delta neutrality, 30,000 euros should be shorted. When the exchange rate moves up to 0.93, we expect the delta of the portfolio to decrease by $(0.93 - 0.90) \times 80,000 = 2,400$, so that it becomes 27,600. To maintain delta neutrality, it is therefore necessary for the bank to unwind its short position by 2,400 euros so that a net 27,600 have been shorted. When a portfolio is delta neutral and has a negative gamma, a loss is experienced when there is a large movement in the underlying asset price. (The short position experienced a loss since the price has gone up.) We can conclude that the bank is likely to have lost money.

11 Implied Volatility

A stock price is currently \$50 and the risk-free interest rate is 5%. Compute implied volatilities, assuming no dividends. Are the option prices consistent with the assumptions underlying Black-Scholes-Merton?

Stock Price	Maturity = 3 months	Maturity = 6 months	Maturity = 12 months
45	7.00	8.30	10.50
50	3.50	5.20	7.50
55	1.60	2.90	5.10

Solution: Implied volatilities are:

Stock Price	Maturity = 3 months	Maturity = 6 months	Maturity = 12 months
45	37.78	34.99	34.02
50	34.15	32.78	32.03
55	31.98	30.77	30.45

The option prices are not exactly consistent with Black-Scholes-Merton. If they were, the implied volatilities would be all the same. We usually find in practice that low strike price options on a stock have significantly higher implied volatilities than high strike price options on the same stock.

12 Implied Volatilities of Calls and Puts

A European call and put option have the same strike price and time to maturity. The call has an implied volatility of 30% and the put has an implied volatility of 25%. What trades would you do?

Solution: The put has a price that is too low relative to the call's price. The correct trading strategy is to buy the put, buy the stock, and sell the call.

13 Implied Volatilities of Calls and Puts

The market price of a European call is $3.00 and its price given by Black-Scholes-Merton model with a volatility of 30% is $3.50. The price given by this Black-Scholes-Merton model for a European put option with the same strike price and time to maturity is $1.00. What should the market price of the put option be? Explain the reasons for your answer.

Solution: With the notation in the text

$$c_{\text{bs}} + Ke^{-rT} = p_{\text{bs}} + Se^{-qT}$$
$$c_{\text{mkt}} + Ke^{-rT} = p_{\text{mkt}} + Se^{-qT}$$

It follows that

$$c_{\text{bs}} - c_{\text{mkt}} = p_{\text{bs}} - p_{\text{mkt}}$$

In this case, $c_{\text{mkt}} = 3.00$, $c_{\text{bs}} = 3.50$, and $p_{\text{bs}} = 1.00$. It follows that p_{mkt} should be 0.50.

14 OTM Options and Volatility

Option traders sometimes refer to deep-out-of-the-money options as being options on volatility. Why do you think they do this?

Solution: A deep-out-of-the-money option has a low value. Decreases in its volatility reduce its value. However, this reduction is small because the value can never go below zero. Increases in its volatility, on the other hand, can lead to significant percentage increases in the value of the option. The option does, therefore, have some of the same attributes as an option on volatility.

15 Practitioners' Approach

Using the table below, calculate the implied volatility a trader would use for an 8-month option with $K/S_0 = 1.04$.

	K/S_0				
	0.90	0.95	1.00	1.05	1.10
1 month	14.2	13.0	12.0	13.1	14.5
3 month	14.0	13.0	12.0	13.1	14.2
6 month	14.1	13.3	12.5	13.4	14.3
1 year	14.7	14.0	13.5	14.0	14.8
2 year	15.0	14.4	14.0	14.5	15.1
5 year	14.8	14.6	14.4	14.7	15.0

Solution: The implied volatility is 13.45%. We get the same answer by (a) interpolating between strike prices of 1.00 and 1.05 and then between maturities six months and one year and (b) interpolating between maturities of six months and one year and then between strike prices of 1.00 and 1.05.