Exotic Options

BUSS386. Futures and Options

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Lecture Outline

- Exotic options
  - Descriptions and uses
  - Pricing with Monte Carlo simulations and binomial trees

- Chapters 26 and 27

Exotic Options

Exotic Options

- Nonstandard options, often constructed by tweaking ordinary options
- Exotic options solve specific business problems that ordinary options cannot
- Typically created and sold by investment banks and professional money managers, who in turn hedge the positions and earn a commission
- Goal is not to memorize or derive formulas
- The relevant questions are:
    1. What is the rationale for the use of an exotic option?
    2. Can the exotic option be approximated by a portfolio of ordinary options?
        - Such links can sometimes show us how to modify BSM to price exotics
    3. Is the exotic option cheap or expensive relative to a standard option that achieves a similar goal?
    4. What's the general approach for pricing them when there isn't a formula?

Binary options

Path-Independent Exotic Options

- Cash-or-nothing:
  - Call: pays 1 if $S_T > K$, zero otherwise: $e^{-r(T-t)}N(d_2)$
  - Put: pays 1 if $S_T < K$, zero otherwise: $e^{-r(T-t)}N(-d_2)$
- Asset-or-nothing:
  - Call: pays stock price if $S_T > K$, zero otherwise: $Se^{-q(T-t)}N(d_1)$
  - Put: pays stock price if $S_T < K$, zero otherwise: $Se^{-q(T-t)}N(-d_1)$
- Look familiar? What is the value of a portfolio that is
  - long an asset-or-nothing call option with strike K, and
  - short K cash-or-nothing call options with strike K

Compound options

Path-Independent Exotic Options

- An option to buy or sell an option
  - Call on call
  - Put on call
  - Call on put
  - Put on put
- Often priced by backward induction on a binomial tree
- Example:
  - A company bidding for a large project requires $200 million in financing for 2 years if they win.
  - They face the risk of higher rates between bidding and contract award.
  - To manage this, they could buy a two-year interest rate cap starting at the contract award date, but this is costly if they don't win.
  - Alternatively, they can purchase a call option on the interest rate cap. If they win, they exercise the option to secure the cap at a set premium; if not, they let the option expire. This approach lowers upfront costs and reduces risk.

Gap options

Path-Independent Exotic Options

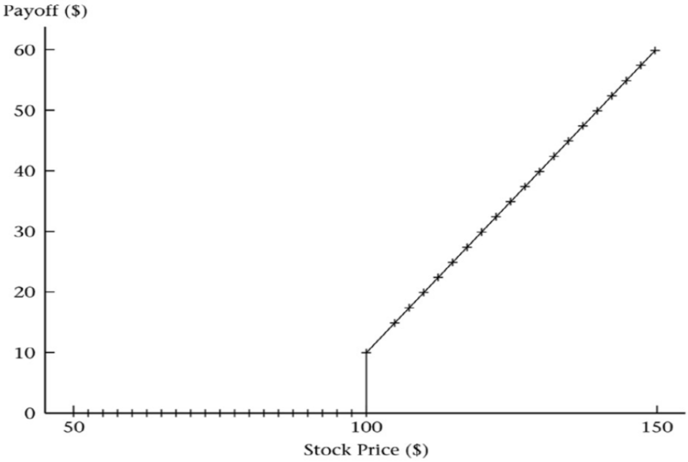
- A gap call options pays $S - K_1$ when $S > K_2$
- The value of a gap call

$$Se^{-qT}N(d_1) - K_1e^{-rT}N(d_2),$$

$$d_1 = \frac{\ln\left(\frac{S}{K_2}\right) + (r - q + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$
$$d_2 = d_1 - \sigma\sqrt{T}$$

Illustration

- Pays $S – K_1$ when $S > K_2$. $K_1 = 90$ and $K_2 = 100$
- Does this option cost more or less without the gap with $K = K_1$?



Barrier options

Path-Dependent Exotic Options with Path-Independent Valuation

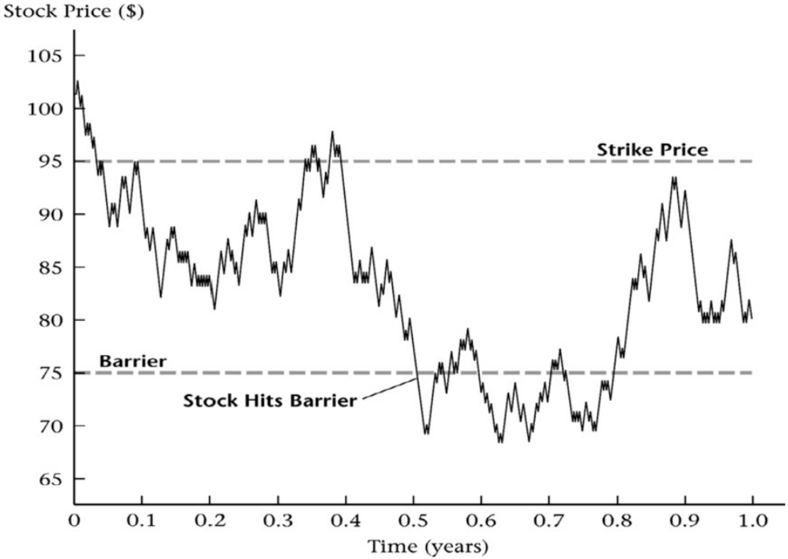
- Payoff depends on whether over its life the underlying price hits a certain barrier
  - Implies payoff is path dependent

- Barrier puts and calls
  - Knock-out options: go out of existence if the underlying price
    - Down-and-out: falls below a barrier
    - Up-and-out: rises above a barrier
  - Knock-in options: come into existence if the underlying price
    - Down-and-in: falls below a barrier
    - Up-and-in: rises above a barrier
  - Rebate options: make a fixed payment if the underlying price
    - Down rebates: falls below a barrier
    - Up rebates: rises above a barrier

- What is worth more, a barrier option or an otherwise identical option?

Barrier options

Illustration: Down-and-in option



Barrier options

Pricing

- Parity relations
  - $c = c_{UI} + c_{UO}$
  - $c = c_{DI} + c_{DO}$
  - $p = p_{UI} + p_{UO}$
  - $p = p_{DI} + p_{DO}$

- Can price on binomial tree (but complicated by path dependence) or use Monte Carlo simulation
  - Example: Down-and-in and up-and-out currency put options
  - Standard put: $x_0 = 0.9, \sigma = 0.1, r_\$ = 0.06, r_€ = 0.03, t = 0.5$

	Standard	Down-and-In Barrier ($)		Up-and-Out Barrier ($)		
Strike ($)	($)	0.8000	0.8500	0.9500	1.0000	1.0500
$K = 0.8$	0.0007	0.0007	0.0007	0.0007	0.0007	0.0007
$K = 0.9$	0.0188	0.0066	0.0167	0.0174	0.0188	0.0188
$K = 1.0$	0.0870	0.0134	0.0501	0.0633	0.0847	0.0869

Lookback options

Path-Dependent Exotic Options with Path-Dependent Valuation

- (Strike) Floating lookback call
  - Buyers can buy at lowest observed price in some time interval, $S_T - S_{min}$
- (Strike) Floating lookback put
  - Buyers can sell at highest observed price in some time interval, $S_{max} - S_T$
- Fixed lookback call
  - $\max(S_{max} - K, 0)$
- Fixed lookback put
  - $\max(K - S_{min}, 0)$
- Comments
  - Relatively expensive
  - Closed form solutions assume continuous looks and lognormal process
  - Related to "shout" options[1]

[1] A European option where the holder can "shout" to the writer at one time during its life. The option holder receives either the usual payoff from a European option or the intrinsic value at the time of the shout.

Non-standard American options

Path-Dependent Exotic Options with Path-Dependent Valuation

- Bermudan option (between American and European)
  - Can be exercised on certain pre-specified dates prior to expiration
  - Strike price may change over the life of the option
  - Generally can be priced like American options on binomial tree
- Examples include some callable bonds, corporate warrants, employee stock options that are reset when they become far out-of-the-money
  - In a 7-year warrant, exercise might be possible on particular dates during years 3 to 7, with the strike price being \$30 during years 3 and 4, \$32 during the next 2 years, and \$33 during the final year.

Asian options

Path-Dependent Exotic Options with Path-Dependent Valuation

- The payoff on an Asian option is based on the average price over some period of time
  - They are path dependent
- Examples of when Asian options are useful
  - Profit depends on average price (e.g., of exchange rates, oil, electricity) over a period of time
  - There is concern that the price at a single point in time might be subject to manipulation
  - When price swings are frequent due to thin or illiquid markets
  - Convertible bonds have an embedded Asian option. Typically the exercise of the conversion option is based on the stock price over a 20-day period at the end of the bond's life
- What feature of an Asian call option tends to make it less valuable than an otherwise identical European call option? Volatility!

Basic Types of Asian options

- Average can be based on geometric or arithmetic mean
  - Suppose we record stock price every h periods from $t = 0$ to T.
  - Arithmatic average: $A(T) = \frac{1}{N}\sum_{i=1}^{N} S_{ih}$
  - Geometric average: $G(T) = (S_h \times S_{2h} \times \cdots \times S_{Nh})^{1/N}$
- Average can be used for the asset price or strike price: average price option and average strike option. For European:

	Arithmatic	Geometric
Average price call	$\max[0, A(T) - K]$	$\max[0, G(T) - K]$
Average price put	$\max[0, K - A(T)]$	$\max[0, K - G(T)]$
Average strike call	$\max[0, S_T - A(T)]$	$\max[0, S_T - G(T)]$
Average strike call	$\max[0, A(T) - S_T]$	$\max[0, G(T) - S_T]$

Example

Hedging currency exposure

- XYZ has monthly revenue of €100m, and cost in dollars, x: spot dollar price of a euro. In one year, the converted amount in dollar:

$$€100 \sum_{i=1}^{12} x_i \times e^{r(12-i)/12}$$

- Ignoring interest, the amount of euro exposure that needs to be hedged is

$$\sum_{i=1}^{12} x_i = 12 \times \frac{1}{12} \sum_{i=1}^{12} x_i$$

- An arithmatic average price put option that puts a floor K, on the average exchange rate received:

$$\max \left(0, K - \frac{1}{12} \sum_{i=1}^{12} x_i \right)$$

Example

Hedging currency exposure

- Alternative strategies?
- Currency options can be valued using BSM formula with constant dividend yield, recognizing that currency earns risk-free interest rate
- Example: Assume the current exchange rate is \$0.9/EUR, strike $K = 0.9$, $r = 6\%$, $r_{euro} = 3\%$, dollar/euro volatility $\sigma = 10\%$.

a. 12 European puts expiring in 1 year	0.2753
b. A basket of 12 monthly options	0.2178
c. 12 Geometric average puts	0.1796
d. 12 Arithmetic average puts	0.1764
e. Currency swap	?

Pricing Asian options

- Closed form solution for geometric case
  - Uses Black's Model and log-normal approximation for the average mean and variance.[2] (See, e.g., Chapter 26.13 in Hull)

- For arithmatic case:
  - Binomial tree (but path dependence is a problem)
  - Monte Carlo simulation

[2] When the asset price follows geometric Brownian motion, the geometric average of the price is exactly lognormal. However, the arithmetic average of lognormal variables is not lognormal.

Exchange options

Multivariate Options

- Pays off only if the underlying asset outperforms some other asset (the benchmark asset)

$$\max(0, S_T - N_T)$$

- The value of a European exchange call

$$Se^{-q_S T} N(d_1) - Ne^{-q_N T} N(d_2),$$

$$d_1 = \frac{\ln\left(\frac{Se^{-q_S T}}{Ne^{-q_N T}}\right) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} = \frac{\ln\left(\frac{S}{N}\right) + (q_N - q_S + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

$$\sigma = \sqrt{\sigma_S^2 + \sigma_N^2 - 2\rho\sigma_S\sigma_N}$$

- We can estimate market implied correlations between assets using the market prices of exchange options.

Quantos

Multivariate Options

- A quanto is a contract that allows an investor in one currency to hold an asset denominated in another currency without exchange rate risk
- Example: Nikkei put warrants traded on the American Stock Exchange
- Payoff and premium are in dollars, but directly scaled by the yen price of the Nikkei index relative to a yen strike price
  - Quantos are attractive because they shield the purchaser from exchange rate fluctuations.
  - If a US investor were to invest directly in the Japanese stocks that comprise the Nikkei, he would be exposed to both fluctuations in the Nikkei index and fluctuations in the USD/JPY exchange rate.
  - Essentially, a quanto has an embedded currency forward with a variable notional amount.
  - Forward/futures won't perfectly hedge the risk because the notional amount to hedge changes over time.
  - It is that variable notional amount that give quantos their name - "quanto" is short for "quantity adjusting option."

Pricing Exotic Options

Pricing Exotics

- Multiple approaches to pricing
    - Modified Black-Scholes-Merton (See, e.g., Chapter 26 in Hull)
    - Binomial trees
    - Monte Carlo simulation

Valuation

- Path-independent exotics
  - These can be valued easily using the standard methodology, such as the Binomial model. Some have closed form valuation equations (Binary, Compound, Packages)
- Path-dependent exotics with path-independent valuation
  - Like American options, these securities have payoffs that depend on the path followed by the underlying asset, but they can still be priced with standard methods (American, Barrier, Lookback)
- Path-dependent exotics with path-dependent valuation
  - For these, the path followed by the underlying determines the payoff in a way that requires different valuation techniques, such as Monte Carlo simulation (Asian, MBS)
- Multivariate options: The payoff is a function of more than one random variable (Exchange, Quantos): Monte Carlo simulation

Binomial (Risk Neutral) Trees

- Recall the one-step tree
- Assume $S_0 = 100, K = 100, T = 1, r = 2\%, \sigma = 30\%$
- Given $u = e^{\sigma\sqrt{T}} = 1.34986$, the price of any derivative security with payoff $V(S_1)$ can be computed as

$$V_0 = E^Q[e^{-rT} f(S_1)] = e^{-rT}[pf(S_{1,u}) + (1-p)f(S_{1,d})]$$

- For instance, a call option has price given by

$$\begin{tikzcd}
 & \begin{array}{l} S_{1,d} = 74.082 \\ f_{1,d} = 0 \end{array} \\
\begin{array}{l} S_0 = 100 \\ q = 0.4597 \\ c_0 = e^{-rT} p c_{1,u} = 15.731 \end{array} \arrow[ur] \arrow[dr] & \\
 & \begin{array}{l} S_{1,u} = 134.986 \\ f_{1,u} = 34.986 \end{array}
\end{tikzcd}$$

Monte Carolo Simulation on Risk Neutral Trees

- Alternative way of computing the expected future payoff is to simulate up and down movement with a computer.
- In Excel, $RAND()$ simulates a uniform between $[0, 1]$

1. Generate $RAND()$ many times, say N:
   - Whenever the realization $RAND() > q$, we say we went down the tree;
   - Whenever the realization $RAND() < q$, we say we went up the tree;
2. The stock price at $T = 1$ will be $S_{1,u}$ or $S_{1,d}$, depending on the outcome of Step 1. Let S_1^j denote the realization of S_1 in simulation run j.
3. Compute the payoff of the security at $T = 1$ for each simulation run, e.g., $V(S_1^j) = \max(S_1^j - K, 0)$
4. The value of the security is the average of the many realizations.

$$\hat{V}_0 = \frac{1}{N} \sum_{j=1}^{N} e^{-rT} V(S_1^j)$$

Monte Carolo Simulation on Risk Neutral Trees

- For instance, given $p = 0.4587$, we obtain the following table.

RAND()	Move on Tree	Price at T	Payoff	discounted
0.457335	up	134.986	34.986	34.293
0.393937	up	134.986	34.986	34.293
0.090053	up	134.986	34.986	34.293
0.878148	down	74.082	0	0
0.658659	down	74.082	0	0
0.759579	down	74.082	0	0
0.798027	down	74.082	0	0
0.061689	up	134.986	34.986	34.293
0.969222	down	74.082	0	0
0.392675	up	134.986	34.986	34.293
			Average	17.147
			st. error	5.715

- With $N = 10$, $\hat{V}_0 = 17.147$ is quite different from $V_0 = 14.731$.

- As N increases, the value gets more precise.

Monte Carolo Simulation on Risk Neutral Trees

- How many simulations?
  - N should be large enough to obtain a small standard error for our estimate of the option price.
  - This is computed as the standard deviation of the discounted payoffs from the simulations, divided by $\sqrt{N}$.
  - In the previous example, $s.e. = 5.715$.
    - This implies that with 95% probability, the true value of the security is between $[\hat{V}_0 - 2 \times s.e., \hat{V}_0 + 2 \times s.e.] = [5.715, 28, 577]$
    - Given $N = 10$, we are 95% confident that the true value is between 5.715 and 28.477.
  - Increasing $N = 1000$, we obtain $\hat{V}_0 = 15.725$ with $s.e. = 0.52$
    - The confidence intervalue is $[14.644, 16.866]$, much tigher than before.

Multi-step Trees

- As the nubmer of steps increases, the estimate becomes more precise.

Binomial Tree Model											
	Stock Assumption			Option Assumption			Tree			Risk Neutral Prob	
	mu	0.1		T	1		n	10		q*	0.486836
	sigma	0.3		K	100		h	0.1			
	r	0.02		Call or Put	1	(=1 for call	u	1.099514		Price Binomial	3.787
	div yield	0					d	0.909493		Delta Binomial	0.166
	S0	100					p	0.52919			

time ==>	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
i==>	0	1	2	3	4	5	6	7	8	9	10
0	100.000	109.951	120.893	132.924	146.151	160.696	176.687	194.270	213.603	234.859	258.231
1		90.949	100.000	109.951	120.893	132.924	146.151	160.696	176.687	194.270	213.603
2			82.718	90.949	100.000	109.951	120.893	132.924	146.151	160.696	176.687
3				75.231	82.718	90.949	100.000	109.951	120.893	132.924	146.151
4					68.422	75.231	82.718	90.949	100.000	109.951	120.893
5						62.229	68.422	75.231	82.718	90.949	100.000
6							56.597	62.229	68.422	75.231	82.718
7								51.475	56.597	62.229	68.422
8									46.816	51.475	56.597
9										42.579	46.816
10											38.725

Option Pricing Tree

time ==>	0.000	0.100	0.200	0.300	0.400	0.500	0.600	0.700	0.800	0.900	1.000
i==>	0.0	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0
0	12.530	18.247	25.881	35.666	47.656	61.691	77.484	94.868	114.002	135.059	158.231
1		7.156	11.075	16.699	24.431	34.528	46.948	61.294	77.086	94.470	113.603
2			3.465	5.782	9.429	14.947	22.879	33.522	46.551	60.895	76.687
3				1.280	2.345	4.231	7.481	12.871	21.292	33.123	46.151
4					0.275	0.566	1.164	2.396	4.932	10.151	20.893
5						0.000	0.000	0.000	0.000	0.000	0.000
6							0.000	0.000	0.000	0.000	0.000
7								0.000	0.000	0.000	0.000
8									0.000	0.000	0.000
9										0.000	0.000
10											0.000

Multi-step Trees: Monte Carlo Simulation

- Generate stock prices: When $RAND() > p$, go down. Otherwise, go up
- 1,000 simulations of stock prices. Get S_T, compute $f_T e^{-rT}$, take the average.

Option prices by simulation												
	Simulated Call Price									Price Binomial	12.530	
	13.196									Price Black Scholes	12.822	
	0.685											
				Simulation of Risk Neutral Stock Prices								
time ==>	Discounted	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
i==>	Call Payoff	0	1	2	3	4	5	6	7	8	9	10
1	20.479	100	90.949	82.718	75.231	82.718	90.949	100.000	90.949	100.000	109.951	120.893
2	0.000	100	90.949	82.718	75.231	68.422	75.231	68.422	62.229	56.597	62.229	68.422
3	0.000	100	90.949	82.718	90.949	82.718	90.949	82.718	75.231	68.422	62.229	56.597
4	20.479	100	109.951	120.893	109.951	120.893	132.924	120.893	109.951	120.893	109.951	120.893
5	0.000	100	109.951	100.000	90.949	82.718	75.231	68.422	75.231	82.718	75.231	68.422
6	75.169	100	109.951	100.000	90.949	100.000	109.951	120.893	132.924	146.151	160.696	176.687
7	45.238	100	90.949	100.000	109.951	100.000	109.951	100.000	109.951	120.893	132.924	146.151
8	0.000	100	109.951	120.893	109.951	100.000	90.949	82.718	75.231	68.422	75.231	82.718
9	0.000	100	90.949	82.718	75.231	68.422	62.229	56.597	62.229	68.422	75.231	68.422
10	45.238	100	109.951	100.000	90.949	100.000	90.949	100.000	109.951	120.893	132.924	146.151

Why Monte Carlo simulations?

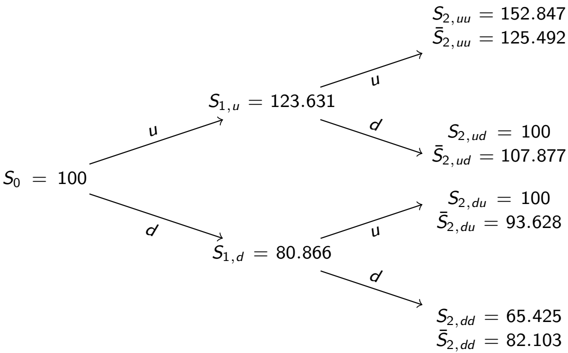
- Why do we need Monte Carlo simulations when we have the tree itself?
  - Monte Carlo Simulations may be useful to price derivative securities with path dependent payoff
  - That is, the value at maturity depends on the path taken by the stock during the life of the security

- For instance, recall that an Asian call option has a payoff given by

$$\max\left(\frac{1}{T}\sum_{t=0}^{T} S_t - K, 0\right)$$

- These options are very hard to price on a tree without simulations

Why Monte Carlo simulations?



- Even though $S_{2,ud} = S_{2,du} = 100$, $\bar{S}_{2,ud} \neq \bar{S}_{2,du} \Rightarrow$ Non-combining tree
- When the number of steps gets large, path dependent options become much more difficult to price without Monte Carlo
- However, for American type options, we use Monte Carolo binomial trees to infer the optimal exercise policy.

Monte Carlo simulations without trees

- There is no reason to limit MC simulations to trees
- The main requirement to be able to price by MC simulations is to satisfy conditions for risk neutral pricing to be valid
  - That is, dynamic replication can be achieved
- These no arbitrage conditions are naturally satisfied on the trees we have constructed
- However, once we decide we can use risk neutral pricing, we can simulate out of any distribution
  - For example, MC can generate prices based on the lognormal model, as in BSM
  - MC can be used to incorporate time-varying volatility, for instance by using the Heston Model.

Monte Carlo simulations under log-normality

- With the lognormal model, one way to simulate stock price is to use the following algorithm:
  - For given Δt,

$$S_{t+\Delta t} = S_t e^{(r-\frac{\sigma^2}{2})\Delta t + \sigma \epsilon_t \sqrt{\Delta t}}$$

    where $\epsilon_t \sim \phi(0,1)$

$$\ln \frac{S_{t+\Delta t}}{S_t} = (r - \frac{\sigma^2}{2})\Delta t + \sigma \epsilon_t \sqrt{\Delta t}$$

  - Therefore, $\ln \frac{S_{t+h}}{S_t} \sim \phi((r - \frac{\sigma^2}{2})\Delta t, \sigma^2 \Delta t)$

Monte Carlo simulations with multiple factors

- Consider an option that pays the maximum betwen the return on Google and Apple from 0 to T.
- The payoff at T is:

$$\max\left(\frac{S_T}{S_0}, \frac{N_T}{N_0}\right)$$

- How much is the premium of this security?
  - The risk neutral stock price processes are:

$$S_{t+\Delta t} = S_t e^{(r-\frac{\sigma^2}{2})\Delta t + \sigma\epsilon_{1,t}\sqrt{\Delta t}}$$

$$N_{t+\Delta t} = N_t e^{(r-\frac{\sigma^2}{2})\Delta t + \sigma\epsilon_{2,t}\sqrt{\Delta t}}$$

  - Very likely that the returns of Google and Apple is correlated. Where $v \sim \phi(0,1)$,

$$\epsilon_{2,t} = \rho\epsilon_{1,t} + \sqrt{1-\rho^2}v$$

Monte Carlo simulations with multiple factors

- For each simulation i, compute the discounted payoff

$$V^i = e^{-rT} \max\left(\frac{S_T^i}{S_0}, \frac{N_T^i}{N_0}\right)$$

- The price of the security is then:

$$\hat{V}_0 = \frac{1}{n} \sum_{i=1}^{n} V^i$$

- Assuming $\sigma_S = \sigma_N = 0.3, r = 0.02$ and $\rho = 0.7$, $\hat{V}_0 = 1.134$.

- Consider another option with the payoff:

$$\max\left(\frac{S_T}{S_0} - \frac{N_T}{N_0}, 0\right)$$

- It pays the difference in the returns.

- In this case, $\hat{V}_0 = 0.1$. Make sense?

Monte Carlo simulations with multiple factors

Stochastic volatility and the Heston model

- Heston model for stochastic volatility is an example with non-traded factor.
- Assume that under the risk neutral probability,

$$S_{t+\Delta t} - S_t = S_t(r\Delta t + \epsilon_{s,t}\sqrt{v_t \Delta t})$$
$$v_{t+\Delta t} - v_t = \kappa(\bar{v} - v_t)\Delta t + \epsilon_{v,t}\xi\sqrt{v_t \Delta t}$$

where κ and ξ are constants, and $\epsilon_{s,t}$ and $\epsilon_{v,t}$ have a correlation ρ.

Summary

- Tools for pricing include modified BSM, binomial trees, and Monte Carlo
- Binomial trees
  - Generally need to use binomial trees for American-style options where a decision has to be made about when to exercise
  - Most useful when working backwards and seeing ordered outcomes is essential
- Monte Carlo simulation
  - One of the main tools used by practitioners to price complex securities under fairly general assumptions about the underlying stochastic processes
    1. Simulate many paths of underlying stochastic variables under the risk neutral probabilities
    2. For each path, compute the discounted simulated payoff of the derivative security
    3. Estimate the derivative price as the average of discounted payoffs across paths

Summary

- MCS are especially useful to value path dependent securities, or securities that depend on the value of multiple underlying variables.
  - Barrier options, Asian options, Lookback options
  - Options on maximu, options on the relative returns across securities
- MCS are also useful to value securities under general processes for underlying stochastic variables such as
  - Stochastic volatility
  - Stochastic interest rate
  - Jumps
- The ever increasing gains in the computer speed makes MCS methodology increasingly attractive.