Problem Set: Answer Key

1 Option Greeks

A financial institution has just sold 1,000 7-month European call options on the Japanese yen. Suppose that the spot exchange rate is 0.80 cent per yen, the exercise price is 0.81 cent per yen, the risk-free interest rate in the United States is 8% per annum, the risk-free interest rate in Japan is 5% per annum, and the volatility of the yen is 15% per annum. Calculate the delta, gamma, vega, theta, and rho of the financial institution's position. Interpret each number.

Solution: $S_0 = 0.80$, $K = 0.81$, $r = 0.08$, $\sigma = 0.15$, $T = 0.5833$, and $q = r_f = 0.05$. Hence, the value of one option is:

$$
\begin{aligned}
\Delta &= e^{-r_f T} N(d_1) = e^{-0.05\times 0.5833} \times 0.5405 = 0.5250 \\
\Gamma &= \frac{N'(d_1)\, e^{-r_f T}}{S_0 \sigma \sqrt{T}} = \frac{0.3969 \times 0.9713}{0.80 \times 0.15 \times \sqrt{0.5833}} = 4.206 \\
\nu &= S_0 \sqrt{T} N'(d_1)\, e^{-r_f T} = 0.80\sqrt{0.5833} \times 0.3969 \times 0.9713 = 0.2355 \\
\Theta &= -\frac{S_0 N'(d_1)\, \sigma e^{-r_f T}}{2\sqrt{T}} + r_f S_0 N(d_1)\, e^{-r_f T} - rKe^{-rT} N(d_2) \\
&= -\frac{(0.8)(0.3969)(0.15)(0.9713)}{2\sqrt{0.5833}} + (0.05)(0.8)(0.5405)(0.9713) - (0.08)(0.81)(0.9544)(0.4948) = -0.0399 \\
\rho &= KTe^{-rT} N(d_2) = (0.81)(0.5833)(0.9544)(0.4948) = 0.2231
\end{aligned}
$$

Delta can be interpreted as meaning that, when the spot price increases by a small amount (measured in cents), the value of an option to buy one yen increases by 0.525 times that amount. Gamma can be interpreted as meaning that, when the spot price increases by a small amount (measured in cents), the delta increases by 4.206 times that amount. Vega can be interpreted as meaning that, when the volatility (measured in decimal form) increases by a small amount, the option's value increases by 0.2355 times that amount. When volatility increases by 1% (= 0.01), the option price increases by 0.002355. Theta can be interpreted as meaning that, when a small amount of time (measured in years) passes, the option's value decreases by 0.0399 times that amount. In particular, when one calendar day passes, it decreases by 0.0399/365 = 0.000109. Finally, rho can be interpreted as meaning that, when the interest rate (measured in decimal form) increases by a small amount, the option's value increases by 0.2231 times that amount. When the interest rate increases by 1% (= 0.01), the option's value increases by 0.002231.

2 Portfolio Insurance

A fund manager has a well-diversified portfolio that mirrors the performance of the S&P 500 and is worth \$360 million. The value of the S&P 500 is 1,200, and the portfolio manager would like to buy insurance against a reduction of more than 5% in the value of the portfolio over the next 6 months. The risk-free interest rate is 6% per annum. The dividend yield on both the portfolio and the S&P 500 is 3%, and the volatility of the index is 30% per annum.

1. If the fund manager buys traded European put options, how much would the insurance cost?
2. Create alternative strategies involving traded European call options, and show that they lead to the same result.

Solution:

1. $S_0 = 1200$, $K = 1140$, $r = 0.06$, $\sigma = 0.3$, $T = 0.5$, and $q = 0.03$. Hence, the value of one put option is:

$$Ke^{-rT}N(-d_2) - Se^{-qT}N(-d_1) = 1140e^{-0.06\times0.5} \times 0.4182 - 1200e^{-0.03\times0.5} \times 0.3378 = 63.40$$

The total cost of the insurance is $300,000 \times 63.40 = \$19,020,000$, where $300,000 = \$360\text{m}/1,200$.

2. From the put-call parity:

$$p = c - S_0e^{-qT} + Ke^{-rT}$$

This shows that a put option can be created by buying a call option, shorting the index (e^{-qT}), and investing the remainder at the risk-free rate of interest. Applying this to the situation under consideration, the fund manager should:

(a) Sell $360e^{-0.03\times0.5} = \$354.64$ million worth of shares.

(b) Buy call options on 300,000 times the S&P 500 with exercise price 1140 and maturity in six months. Each call option is \$139.23.

(c) Invest the remaining ($= \$354.64 - \$41.77 = \$311.87$) cash at the risk-free interest rate of 6% per annum.

This strategy gives the same result as buying put options directly (Compare payoffs when the index is below K and above K).

3 Risk Management

A financial institution has the following portfolio of over-the-counter options on sterling:

Type	Position	Delta of option	Gamma of option	Vega of option
Call	$-1,000$	0.50	2.2	1.8
Call	-500	0.80	0.6	0.2
Put	$-2,000$	-0.40	1.3	0.7
Call	-500	0.70	1.8	1.4

A traded option is available with a delta of 0.6, a gamma of 1.5, and a vega of 0.8.

1. What position in the traded option and in sterling (the underlying asset) would make the portfolio both gamma neutral and delta neutral?
2. What position in the traded option and in sterling would make the portfolio both vega neutral and delta neutral?

Solution: The delta of the portfolio is

$$-1000 \times 0.50 - 500 \times 0.80 - 2000 \times (-0.40) - 500 \times 0.80 = -450$$

The gamma of the portfolio is

$$-1000 \times 2.2 - 500 \times 0.6 - 2000 \times 1.3 - 500 \times 1.8 = -6000$$

The vega of the portfolio is

$$-1000 \times 1.8 - 500 \times 0.2 - 2000 \times 0.7 - 500 \times 1.4 = -4000$$

1. A long position in 4,000 traded options will give a gamma-neutral portfolio since the long position has a gamma of $4000 \times 1.5 = 6000$. The delta of the whole portfolio (including traded options) is then:

   $$4000 \times 0.6 - 450 = 1950$$

   Hence, in addition to the 4,000 traded options, a short position of 1,950 in sterling is necessary so that the portfolio is both gamma and delta neutral.

2. A long position in 5,000 traded options will give a vega-neutral portfolio since the long position has a vega of $5000 \times 0.8 = 4000$. The delta of the whole portfolio (including traded options) is then

   $$5000 \times 0.6 - 450 = 2550$$

Hence, in addition to the 5,000 traded options, a short position of 2,550 in sterling is necessary so that the portfolio is both vega and delta neutral.

4 The Black's Model

Consider a futures. Its current price is \$70.00 and its volatility is 16.70% per annum. Suppose the risk-free interest rate is 5.00% per annum (continuous compounding). Use the Black's model to calculate the value of a five-month European put on the futures with a strike price of \$65.00.

Solution: We utilize the Black's model to calculate the option on a futures. In this case, $F_0 = 70$, $\sigma = 16.70\%$, $r = 5\%$, $T = 5/12$, $K = 65$.

$$d_1 = \frac{\ln(F_0/K) + \sigma^2 T/2}{\sigma\sqrt{T}} = 0.74137,$$

and

$$d_2 = \frac{\ln(F_0/K) - \sigma^2 T/2}{\sigma\sqrt{T}} = 0.633572.$$

Hence,

$$p = e^{-rT}\left[KN(-d_2) - F_0 N(-d_1)\right] = 1.038426.$$

5 Implied Volatility

On November 4, 2021, the following options data (379 days until the maturity) was available on the S&P500. Assume that the index is at \$4670.00, that the risk-free rate is 0.25% (continuous compounding), and that the dividend yield is 1.17% (continuous compounding). (Base your calculations on the last trade prices of the put or call).

No.	Strike	Call				Put			
		Last	Net	Bid	Ask	Last	Net	Bid	Ask
1	SPX\$4600.00	\$382.60	+\$12.60	\$379.5	\$382.8	\$344.03	+\$1.08	\$343.3	\$346.2
2	SPX\$4625.00	\$365.98	+\$11.88	\$363.4	\$367.1	\$348.29	−\$3.66	\$352.3	\$355.2
3	SPX\$4650.00	\$321.40	+\$00.00	\$347.7	\$351.4	\$387.59	+\$0.00	\$361.4	\$364.4
4	SPX\$4675.00	\$334.81	+\$11.71	\$332.3	\$336.0	\$366.81	−\$3.94	\$370.9	\$373.8
5	SPX\$4700.00	\$294.80	+\$00.00	\$317.2	\$320.8	\$395.38	+\$0.00	\$380.6	\$383.5
6	SPX\$4725.00	\$266.17	+\$00.00	\$302.4	\$305.9	\$425.23	+\$0.00	\$390.6	\$393.5

1. Calculate the implied volatility for each of the call and put options listed. Use Excel's Goal-Seek function using the BSM formula.

2. For the call and put options with a strike price of $4675.00, estimate the option values when volatility decreases to 0.8 times the implied volatility in part (a), and when it increases to 1.2 times the implied volatility in part (a).

Solution:

1.
   - No.1. Call: 19.799649
   - No.2. Call: 19.530899
   - No.3. Call: 17.755202
   - No.4. Call: 19.064806
   - No.5. Call: 17.508827
   - No.6. Call: 16.544449
   - No.1. Put: 19.098960
   - No.2. Put: 18.615259
   - No.3. Put: 19.992074
   - No.4. Put: 18.143228
   - No.5. Put: 18.915052
   - No.6. Put: 19.740196
2.
   - No.4. Call Price with 80% Implied Volatility: 263.324961
   - No.4. Put Price with 80% Implied Volatility: 298.769108
   - No.4. Call Price with 120% Implied Volatility: 406.204730
   - No.4. Put Price with 120% Implied Volatility: 434.781441

6 Implied Volatility

A European call option on a certain stock has a strike price of $30, a time to maturity of one year, and an implied volatility of 30%. A European put option on the same stock has a strike price of $30, a time to maturity of one year, and an implied volatility of 33%. What is the arbitrage opportunity open to a trader? Does the arbitrage work only when the lognormal assumption underlying Black-Scholes-Merton holds?

Solution: Put-call parity implies that European put and call options have the same implied volatility. If a call option has an implied volatility of 30% and a put option has an implied volatility of 33%, the call is priced too low relative to the put. The correct trading strategy is to buy the call, sell the put, and short the stock. This does not depend on the lognormal assumption underlying Black-Scholes-Merton. Put-call parity is true for any set of assumptions.

7 Option Greeks and Binomial Tree

According to the Black-Scholes-Merton (BSM) model, the delta of a European option on a non-dividend-paying stock has the analytical expressions. We can also compute the delta of an option without invoking the BSM model using the binomial tree model. By varying the price of the underlying stock and recalculating the option price, we can numerically approximate the derivative dC/dS using the formula:

$$\frac{dC}{dS} \approx \frac{\text{New Option Price} - \text{Original Option Price}}{\text{New Stock Price} - \text{Original Stock Price}}$$

Create a binomial tree using the following information: $S_0 = 100$, $K = 100$, $n = 10$, $T = 0.1$, $\sigma = 0.3$, and $r = 0.02$, where n is the number of steps, and find the price of the European call option. Vary the stock price and numerically find delta.
We can numerically approximate gamma as well.

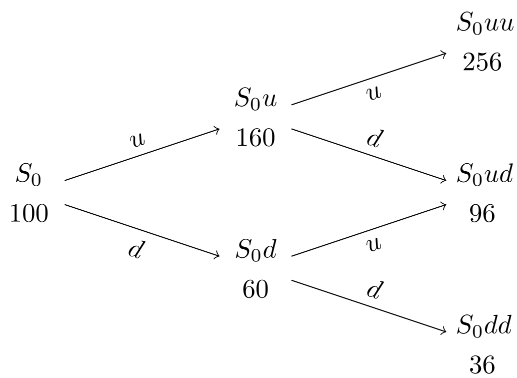
$$\frac{d^2C}{dS^2} \approx \frac{\text{New Delta} - \text{Original Delta}}{\text{New Stock Price} - \text{Original Stock Price}}$$

Find gamma of the option.

Solution: Refer to "MC_Simulation.xlsx".
The price of the European call option is 3.787. In the spreadsheet, we see that the price of the underlying stock in the "up" node at step $i = 1$ is $S_1^u = 103.045$, and the price in the "down" node at $i = 1$ is $S_1^d = 97.045$. The price of the European call option in the "up" node at step $i = 1$ is $c_1^u = 5.381$, and the price in the "down" node at $i = 1$ is $c_1^d = 2.221$. Therefore, the delta is 0.527.
Using the spreadsheet, we can calculate delta for a symmetric 3% increase and decrease in the stock price. For the 3% increase, we plug in $S_0 = 100 \times 1.03 = 103$ and find the delta of the call option to be 0.647. For the 3% decrease, we plug in $S_0 = 100 \times 0.97 = 97$ and find the delta of the call option to be 0.401. The gamma is 0.041.

8 Binomial Tree

Bank XYZ offers an equity option. The expiration date of the option is two periods from now. The option is not initially specified to be a put or a call. Instead, the owner makes this choice after one period. Once the choice is made, the option can be exercised at any time. For example, if after one period the owner chooses for the new exotic option to be a put, it would at that time become identical to an ordinary American put with one period remaining until expiration. A customer has asked for a quote for an option of this type on Stock ABC with a strike price of 100. The current price of Stock ABC is 100 per share, and over each period the stock price evolves as shown on the tree diagram below. The risk-neutral probability of an "up" move is 0.5. The stock does not pay dividends, and the interest rate is 10% per period. What is the lowest price the firm could charge and still break even?



Solution: Let $C(S;n)$ and $P(S;n)$ be the values of American call and put options when the underlying stock price is S and there are n periods remaining until maturity. We begin by finding the payoffs of the call and put options at each node by moving backwards through the tree. Two periods from now, i.e., when $n = 0$, the payoff of a call option will be $256 - 100 = 156$ in the "up-up" node and 0 in either the "up-down" or "down-up" or "down-down" nodes. Similarly, the payoff of a put option will be 0 in the "up-up" node, $100 - 96 = 4$ in the "up-down", "down-up" nodes, and $100 - 36 = 64$ in the "down-down" node.

What would be the values of the call and put options in each node one period from now, i.e., when $n = 1$, given the payoffs in each node at $n = 0$, the risk-neutral probability of an "up" move of 0.5, and an interest rate of 10% per period?

Assume we are in the "up" node at $n = 1$ and the stock price is 160. If we choose to exercise the call option at this node, then its payoff will be $160 - 100 = 60$. Alternatively, if we choose to wait to exercise the call option at $n = 0$, then its value is equal to $(0.5 \cdot 156 + 0.5 \cdot 0)/1.1 = 70.9091$. Thus, the value of the call option in the "up" node at $n = 1$ is $C(160; 1) = \max[60, 70.9091] = 70.9091$. A similar calculation for the put option yields a value of $P(160; 1) = \max[100 - 160, (0.5 \cdot 0 + 0.5 \cdot 4)/1.1] = 1.8182$.

Now, assume we are in the "down" node at $n = 1$ and the stock price is 60. If we choose to exercise the call option at this node, then its payoff will be $60 - 100 = -40$. Alternatively, if we choose to wait to exercise the call option at $n = 0$, then its value is equal to $(0.5 \cdot 0 + 0.5 \cdot 0)/1.1 = 0$. Thus, the value of the call option in the "down" node at $n = 1$ is $C(60; 1) = \max[-40, 0] = 0$. A similar calculation for the put option yields a value of $P(60; 1) = \max[100 - 60, (0.5 \cdot 4 + 0.5 \cdot 64)/1.1] = 40$.

To summarize, the values of the call and put options at $n = 1$ are: $C(160; 1) = 70.9091$, $P(160; 1) = 1.8182$, $C(60; 1) = 0$, $P(60; 1) = 40$. Thus, if the stock price goes down to 60 at $n = 1$, the buyer will choose for the option to be a put, and the put will be exercised immediately. If the stock price instead goes up to 160 at $n = 1$, the buyer will choose for the option to be a call, and the call will be held until its expiration.

Given the optimal exercise policy above, we can find the current value of the option, i.e., at $n = 2$, by discounting the expected payoffs from exercising the put option in the "down" node at $n = 1$ and holding the call option in the "up" node at $n = 1$. The current value of the option is: $[0.5 \cdot C(160; 1) + 0.5 \cdot P(60; 1)]/1.1 = [0.5 \cdot 70.9091 + 0.5 \cdot 40]/1.1 = 50.4132$.

9 Option Greeks

All assumptions of the Black-Scholes-Merton option pricing model hold. Stock XYZ is priced at \$30. It has volatility 25% per year. The annualized continuously-compounded risk-free interest rate is 3.0%.

1. Compute the price of a European call option with strike price \$31, which matures in 6 months.
2. Compute the option Delta at time 0.
3. Suppose that at time 0 the stock price changes instantaneously from \$30 to \$34. Compute the resulting change in the option price.
4. Suppose that at time 0 the stock price changes instantaneously from \$30 to \$34. Compute the resulting change in the value of the replicating portfolio for this option.
5. Suppose that at time 0 the stock price changes instantaneously from \$30 to \$26. Compute the resulting change in the option price.
6. Suppose that at time 0 the stock price changes instantaneously from \$30 to \$26. Compute the resulting change in the value of the replicating portfolio for this option.
7. Does the change in the option price exceed the change in the value of its replicating portfolio? If so, why?

Solution:

1. 1.873
2. $N(d_1) = 0.495$
3. $4.409 - 1.873 = 2.536$ (using the BSM)
4. Delta $\cdot(34 - 30) = 1.98$
5. $0.490 - 1.873 = -1.383$
6. Delta $\cdot(26 - 30) = -1.98$
7. We observe that the change in the option price exceeds the change in the value of its replicating portfolio in both cases. This is due to the positive convexity (Gamma) of the call option.