Problem Set: Answer Key

1 Volatility

The volatility of a stock price is 30% per annum. What is the standard deviation of the percentage price change in one trading day?

Solution: The standard deviation of the percentage price change in time Δt is $\sigma\sqrt{\Delta t}$ where σ is the volatility. In this problem $\sigma = 0.3$ and, assuming 252 trading days in one year, $\Delta t = 1/252 = 0.004$ so that $\sigma\sqrt{\Delta t} = 0.3\sqrt{0.004} = 0.019$ or 1.9%.

2 BSM Assumption

What does the Black-Scholes-Merton stock option pricing model assume about the probability distribution of the stock price in one year? What does it assume about the probability distribution of the continuously compounded rate of return on the stock during the year?

Solution: The Black-Scholes-Merton option pricing model assumes that the probability distribution of the stock price in 1 year (or at any other future time) is lognormal. It assumes that the continuously compounded rate of return on the stock during the year is normally distributed. ($\ln\left(\frac{S_t}{S_0}\right)$ is normally distributed)

3 Using BSM Model

What is the price of a European call option on a non-dividend-paying stock when the stock price is \$52, the strike price is \$50, the risk-free interest rate is 12% per annum, the volatility is 30% per annum, and the time to maturity is three months?

Solution: In this case $S_0 = 52$, $K = 50$, $r = 0.12$, $\sigma = 0.30$, and $T = 0.25$.

$$d_1 = \frac{\ln(52/50) + \left(0.12 + 0.3^2/2\right) 0.25}{0.30\sqrt{0.25}} = 0.5365 \tag{1}$$

$$d_2 = d_1 - 0.30\sqrt{0.25} = 0.3865 \tag{2}$$

The price of the European call is

$$
\begin{aligned}
& 52N(0.5365) - 50e^{-0.12 \cdot 0.25}N(0.3865) && (3)\\
&= 52 \times 0.7042 - 50e^{-0.03} \times 0.6504 && (4)\\
&= 5.06 && (5)
\end{aligned}
$$

or \$5.06.

4 Using BSM Model: No Dividend

Calculate the price of a three-month European put option on a non-dividend-paying stock with a strike price of \$50 when the current stock price is \$50, the risk-free interest rate is 10% per annum, and the volatility is 30% per annum.

Solution: In this case $S_0 = 50$, $K = 50$, $r = 0.1$, $\sigma = 0.3$, $T = 0.25$, and

$$
\begin{aligned}
d_1 &= \frac{\ln(50/50) + (0.1 + 0.09/2)0.25}{0.3\sqrt{0.25}} = 0.2417 && (6)\\
d_2 &= d_1 - 0.3\sqrt{0.25} = 0.0917 && (7)
\end{aligned}
$$

The European put price is

$$
\begin{aligned}
& 50N(-0.0917)e^{-0.1 \cdot 0.25} - 50N(-0.2417) && (8)\\
&= 50 \times 0.4634e^{-0.1 \cdot 0.25} - 50 \times 0.4045 = 2.37 && (9)
\end{aligned}
$$

or \$2.37.

5 Using BSM Model: With Continuous Dividend

What difference does it make to your calculations in the previous problem if a dividend of \$1.50 is expected in two months?

Solution: In this case we must subtract the present value of the dividend from the stock price before using Black-Scholes-Merton. Hence the appropriate value of S_0 is

$$S_0 = 50 - 1.50e^{-0.1667 \cdot 0.1} = 48.52$$

As before $K = 50$, $r = 0.1$, $\sigma = 0.3$, and $T = 0.25$. In this case

$$d_1 = \frac{\ln(48.52/50) + (0.1 + 0.09/2)0.25}{0.3\sqrt{0.25}} = 0.0414 \quad (10)$$

$$d_2 = d_1 - 0.3\sqrt{0.25} = -0.1086 \quad (11)$$

The European put price is

$$50N(0.1086)e^{-0.1 \cdot 0.25} - 48.52N(-0.0414) \quad (12)$$

$$= 50 \times 0.5432e^{-0.1 \cdot 0.25} - 48.52 \times 0.4835 = 3.03 \quad (13)$$

or \$3.03.

6 BSM Assumption

A stock price follows log-normal distribution (i.e., geometric Brownian motion) with an expected return of 16% and a volatility of 35%. The current price is \$38.

a What is the probability that a European call option on the stock with an exercise price of \$40 and a maturity date in six months will be exercised?

b What is the probability that a European put option on the stock with the same exercise price and maturity will be exercised?

Solution:

a The required probability is the probability of the stock price being above \$40 in six months time. Suppose that the stock price in six months is S_T

$$\ln S_T \sim \varphi\left[\ln 38 + \left(0.16 - \frac{0.35^2}{2}\right)0.5, 0.35^2 \times 0.5\right]$$

i.e.,

$$\ln S_T \sim \varphi\left(3.687, 0.247^2\right)$$

Since $\ln 40 = 3.689$, we require the probability of $\ln(S_T) > 3.689$. This is

$$1 - N\left(\frac{3.689 - 3.687}{0.247}\right) = 1 - N(0.008)$$

Since $N(0.008) = 0.5032$, the required probability is 0.4968.

b In this case the required probability is the probability of the stock price being less than \$40 in six months time. It is

$$1 - 0.4968 = 0.5032$$

7 BSM Application

Assume that a non-dividend-paying stock has an expected return of μ and a volatility of σ. An innovative financial institution has just announced that it will trade a derivative that pays off a dollar amount equal to $\ln S_T$ at time T where S_T denotes the values of the stock price at time T.

a Use risk-neutral valuation to calculate the price of the derivative at time t in term of the stock price, S, at time t.

b (Optional) Confirm that your price satisfies the differential equation

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf$$

Solution:

a At time t, the expected value of $\ln S_T$ is

$$\ln S + \left(\mu - \sigma^2/2\right)(T - t)$$

In a risk-neutral world the expected value of $\ln S_T$ is therefore

$$\ln S + \left(r - \sigma^2/2\right)(T - t)$$

Using risk-neutral valuation the value of the derivative at time t is

$$e^{-r(T-t)}\left[\ln S + \left(r - \sigma^2/2\right)(T - t)\right]$$

b If

$$f = e^{-r(T-t)}\left[\ln S + \left(r - \sigma^2/2\right)(T - t)\right]$$

then

$$\frac{\partial f}{\partial t} = re^{-r(T-t)}\left[\ln S + \left(r - \sigma^2/2\right)(T-t)\right] - e^{-r(T-t)}\left(r - \sigma^2/2\right) \tag{14}$$
$$\frac{\partial f}{\partial S} = \frac{e^{-r(T-t)}}{S} \tag{15}$$
$$\frac{\partial^2 f}{\partial S^2} = -\frac{e^{-r(T-t)}}{S^2} \tag{16}$$

The left-hand side of the Black-Scholes-Merton differential equation is

$$e^{-r(T-t)}\left[r\ln S + r\left(r - \sigma^2/2\right)(T-t) - \left(r - \sigma^2/2\right) + r - \sigma^2/2\right] \tag{17}$$
$$= e^{-r(T-t)}\left[r\ln S + r\left(r - \sigma^2/2\right)(T-t)\right] \tag{18}$$
$$= rf \tag{19}$$

Hence the differential equation is satisfied.

8 (Optional) BSM Application

Consider a derivative that pays off S_T^n at time T where S_T is the stock price at that time. When the stock pays no dividends and its price follows geometric Brownian motion, it can be shown that its price at time $t (t \leq T)$ has the form

$$h(t,T)S^n$$

where S is the stock price at time t and h is a function only of t and T.

a By substituting into the Black-Scholes-Merton partial differential equation derive an ordinary differential equation satisfied by $h(t,T)$.

b What is the boundary condition for the differential equation for $h(t,T)$?

c Show that

$$h(t,T) = e^{\left[0.5\sigma^2 n(n-1) + r(n-1)\right](T-t)}$$

Solution: If $G(S,t) = h(t,T)S^n$ then $\partial G/\partial t = h_t S^n$, $\partial G/\partial S = hnS^{n-1}$, and $\partial^2 G/\partial S^2 = hn(n-1)S^{n-2}$ where $h_t = \partial h/\partial t$. Substituting into the Black-Scholes-Merton differential equation we obtain

$$h_t + rhn + \frac{1}{2}\sigma^2 hn(n-1) = rh$$

The derivative is worth S^n when $t = T$. The boundary condition for this differential equation is therefore $h(T,T) = 1$.
The equation

$$h(t,T) = e^{\left[0.5\sigma^2 n(n-1)+r(n-1)\right](T-t)}$$

satisfies the boundary condition since it collapses to $h = 1$ when $t = T$. It can also be shown that it satisfies the differential equation in (a). Alternatively we can solve the differential equation in (a) directly. The differential equation can be written

$$\frac{h_t}{h} = -r(n-1) - \frac{1}{2}\sigma^2 n(n-1)$$

The RHS is constant. The LHS is $\frac{\mathrm{d}}{\mathrm{dt}}\ln(h)$. The solution to this is

$$\ln h = \left[r(n-1) + \frac{1}{2}\sigma^2 n(n-1)\right](T-t)$$

or $h(t,T) = e^{\left[0.5\sigma^2 n(n-1)+r(n-1)\right](T-t)}$.

9 (Optional) BSM Differential Equation

With the notation used in the class

a What is $N'(x)$?

b Show that $SN'(d_1) = Ke^{-r(T-t)}N'(d_2)$, where S is the stock price at time t

$$d_1 = \frac{\ln(S/K) + \left(r + \sigma^2/2\right)(T-t)}{\sigma\sqrt{T-t}} \tag{20}$$

$$d_2 = \frac{\ln(S/K) + \left(r - \sigma^2/2\right)(T-t)}{\sigma\sqrt{T-t}} \tag{21}$$

c Calculate $\partial d_1/\partial S$ and $\partial d_2/\partial S$.

d Show that when

$$c = SN(d_1) - Ke^{-r(T-t)}N(d_2) \tag{22}$$

$$\frac{\partial c}{\partial t} = -rKe^{-r(T-t)}N(d_2) - SN'(d_1)\frac{\sigma}{2\sqrt{T-t}} \tag{23}$$

where c is the price of a call option on a non-dividend-paying stock.

e Show that $\partial c/\partial S = N(d_1)$.

f Show that the c satisfies the Black-Scholes-Merton differential equation.

g Show that c satisfies the boundary condition for a European call option, i.e., that $c = \max(S-K, 0)$ as t tends to T.

Solution:

a Since $N(x)$ is the cumulative probability that a variable with a standardized normal distribution will be less than x, $N'(x)$ is the probability density function for a standardized normal distribution, that is,

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

b

$$\begin{aligned} N'(d_1) &= N'\left(d_2 + \sigma\sqrt{T-t}\right) && (24) \\ &= \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{d_2^2}{2} - \sigma d_2\sqrt{T-t} - \frac{1}{2}\sigma^2(T-t)\right] && (25) \\ &= N'(d_2) \exp\left[-\sigma d_2\sqrt{T-t} - \frac{1}{2}\sigma^2(T-t)\right] && (26) \end{aligned}$$

Because

$$d_2 = \frac{\ln(S/K) + \left(r - \sigma^2/2\right)(T-t)}{\sigma\sqrt{T-t}}$$

it follows that

$$\exp\left[-\sigma d_2\sqrt{T-t} - \frac{1}{2}\sigma^2(T-t)\right] = \frac{Ke^{-r(T-t)}}{S}$$

As a result

$$SN'(d_1) = Ke^{-r(T-t)}N'(d_2)$$

which is the required result.

c

$$\begin{aligned} d_1 &= \frac{\ln\frac{S}{K} + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} && (27) \\ &= \frac{\ln S - \ln K + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} && (28) \end{aligned}$$

Hence

$$\frac{\partial d_1}{\partial S} = \frac{1}{S\sigma\sqrt{T-t}}$$

Similarly

$$d_2 = \frac{\ln S - \ln K + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

and

$$\frac{\partial d_2}{\partial S} = \frac{1}{S\sigma\sqrt{T-t}}$$

Therefore:

$$\frac{\partial d_1}{\partial S} = \frac{\partial d_2}{\partial S}$$

d

$$c = SN(d_1) - Ke^{-r(T-t)}N(d_2) \tag{29}$$

$$\frac{\partial c}{\partial t} = SN'(d_1)\frac{\partial d_1}{\partial t} - rKe^{-r(T-t)}N(d_2) - Ke^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial t} \tag{30}$$

From (b):

$$SN'(d_1) = Ke^{-r(T-t)}N'(d_2)$$

Hence

$$\frac{\partial c}{\partial t} = -rKe^{-r(T-t)}N(d_2) + SN'(d_1)\left(\frac{\partial d_1}{\partial t} - \frac{\partial d_2}{\partial t}\right)$$

Since

$$d_1 - d_2 = \sigma\sqrt{T-t} \tag{31}$$

$$\frac{\partial d_1}{\partial t} - \frac{\partial d_2}{\partial t} = \frac{\partial}{\partial t}(\sigma\sqrt{T-t}) \tag{32}$$

$$= -\frac{\sigma}{2\sqrt{T-t}} \tag{33}$$

Hence

$$\frac{\partial c}{\partial t} = -rKe^{-r(T-t)}N(d_2) - SN'(d_1)\frac{\sigma}{2\sqrt{T-t}}$$

e From differentiating the Black-Scholes-Merton formula for a call price we obtain

$$\frac{\partial c}{\partial S} = N(d_1) + SN'(d_1)\frac{\partial d_1}{\partial S} - Ke^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial S}$$

From the results in (b) and (c) it follows that

$$\frac{\partial c}{\partial S} = N(d_1)$$

f Differentiating the result in (e) and using the result in (c), we obtain

$$\frac{\partial^2 c}{\partial S^2} = N'(d_1)\frac{\partial d_1}{\partial S} \tag{34}$$

$$= N'(d_1)\frac{1}{S\sigma\sqrt{T-t}} \tag{35}$$

From the results in (d) and (e)

$$\frac{\partial c}{\partial t} + rS\frac{\partial c}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 c}{\partial S^2} = -rKe^{-r(T-t)}N(d_2) - SN'(d_1)\frac{\sigma}{2\sqrt{T-t}} \tag{36}$$

$$+ rSN(d_1) + \frac{1}{2}\sigma^2 S^2 N'(d_1)\frac{1}{S\sigma\sqrt{T-t}} \tag{37}$$

$$= r\left[SN(d_1) - Ke^{-r(T-t)}N(d_2)\right] \tag{38}$$

$$= rc \tag{39}$$

This shows that the Black-Scholes-Merton formula for a call option does indeed satisfy the Black-Scholes-Merton differential equation.

g Consider what happens in the formula for c in part (d) as t approaches T. If $S > K$, d_1 and d_2 tend to infinity and $N(d_1)$ and $N(d_2)$ tend to 1. If $S < K$, d_1 and d_2 tend to zero. It follows that the formula for c tends to $\max(S - K, 0)$.

10 Black-Scholes-Merton Model

Consider a 3-month European put option on a non-dividend-paying stock. The current stock price is \$52, the strike price is \$50, the risk-free interest rate is 12% per annum, and the volatility is 30% per annum. What is the price of the put according to the Black-Scholes-Merton model?

Solution:

$$d_1 = \frac{\ln\left(S_0/K\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = \frac{\ln(52/50) + \left(0.12 + \frac{0.3^2}{2}\right)(3/12)}{0.3\sqrt{3/12}} = 0.5365 \tag{40}$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.3865 \tag{41}$$

Thus, the put price is

$$p_0 = Ke^{-rT}N\left(-d_2\right) - S_0 N\left(-d_1\right) = 50e^{-0.12\times(3/12)}N(-0.3865) - 52N(-0.5365) = \$1.580$$

11 Valuation of a Derivative

Consider a derivative on a stock with the time to expiration T and the following payoff:

$$\begin{cases} 0 & \text{if } S_T < K_1 \\ K_1 & \text{if } K_1 \leq S_T < K_2 \\ 0 & \text{if } K_2 \leq S_T \end{cases}$$

where $K_2 > K_1$. Assume that stock price follows log-normal distribution. What is the present value of the derivative? Provide an analytic expression of the price using $N()$, the cumulative probability distribution function of a standard normal random variable.

Solution: Let V denote $\ln S_T$. Then, V is normally distributed, i.e., $V \sim \phi(m, w)$. Let $g(V)$ denote the probability density function of V. To find the present value of the derivative, we first compute the expected option payoff:

$$
\begin{aligned}
E[\text{Payoff}(V)] &= \int_{-\infty}^{\infty} \text{Payoff}(V) \cdot g(V)\, dV & (42)\\
&= \int_{-\infty}^{\ln K_1} \text{Payoff}(V) \cdot g(V)\, dV + \int_{\ln K_1}^{\ln K_2} \text{Payoff}(V) \cdot g(V)\, dV + \int_{\ln K_2}^{\infty} \text{Payoff}(V) \cdot g(V)\, dV & (43)\\
&= \int_{-\infty}^{\ln K_1} 0 \cdot g(V)\, dV + \int_{\ln K_1}^{\ln K_2} K_1 \cdot g(V)\, dV + \int_{\ln K_2}^{\infty} 0 \cdot g(V)\, dV & (44)\\
&= K_1 \int_{\ln K_1}^{\ln K_2} g(V)\, dV & (45)\\
&= K_1 \cdot \text{Prob}\left(\ln K_1 \le V \le \ln K_2\right) & (46)\\
&= K_1 \cdot \text{Prob}\left(K_1 \le S_T \le K_2\right) & (47)\\
&= K_1 \cdot \left[\text{Prob}\left(K_1 \le S_T\right) - \text{Prob}\left(K_2 \le S_T\right)\right] & (48)\\
&= K_1 \cdot \left[N\left(\frac{\ln\left(S_0/K_1\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) - N\left(\frac{\ln\left(S_0/K_2\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)\right] & (49)
\end{aligned}
$$

Next, multiplying by the discount factor, we obtain the present value as follows:

$$
f_0 = e^{-rT} K_1 \cdot \left[N\left(\frac{\ln\left(S_0/K_1\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) - N\left(\frac{\ln\left(S_0/K_2\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)\right]
$$

12 One-Step Binomial Model

Consider a 2-year European put with strike price \$100. The current stock price is \$100 and the stock price can move either up or down by 20% once during the life of the option. The risk-free interest rate is 5% per annum.

a Find the option payoff at each of two stock prices at the expiration.

b Suppose that we are in time 0. Find the replicating portfolio of the option.

c What is the current price of the option?

Solution:

a The option payoff in year 2 is

$$
f_u = \max(100 - (100)(1.2), 0) = 0 f_d = \max(100 - (100)(0.8), 0) = 20
$$

b Let x denote the number of shares and y denote the face value of bond in the replicating portfolio. Then, x and y should satisfy

$$120x + y = 080x + y = 20$$

Solving for unknowns, we obtain $x = -0.5$ and $y = 60$.

c The present value of the replicating portfolio is $(-0.5)(100) + 60e^{-0.05\times 2} = 4.290$. Because the replicating portfolio will have the same value as the put option two years later in every case, their present values should be the same. Thus, the option price is \$4.290.

13 Option Pricing in Discounted Cash Flow

Let's revisit the European put in Question 1. This time, we want to use the DCF approach to determine the option price. Risk-averse investors require the return on stock to be 7% per annum.

a What is the real probability p^* of an increase in the stock price?

b What is the discount rate for the put option in DCF? Find the annual rate r_{put} in continuous compounding.

c What is the current price of the put from DCF? Is this price the same as the price from Question 1 (c)?

Solution:

a The real probability is

$$p^* = \frac{e^{\alpha T} - d}{u - d} = \frac{e^{0.07\times 2} - 0.8}{1.2 - 0.8} = 0.876$$

b First, we calculate the expected return on the replicating portfolio. The expected gross return over two years is

$$\frac{(-0.5)(100)}{(-0.5)(100) + 60e^{-0.05\times 2}} \times e^{0.07\times 2} + \frac{60e^{-0.05\times 2}}{(-0.5)(100) + 60e^{-0.05\times 2}} \times e^{0.05\times 2} = 0.579526$$

This should be equal to the required return on the put, so

$$e^{r_{\text{put}}\times 2} = 0.576526$$

Thus, $r_{\text{put}} = -27.28\%$. (Can this be true?)

c The put price is

$$e^{-r_{\text{put}}\times 2}\left[p^* \times 0 + (1 - p^*) \times 20\right] = e^{-(-0.2728)\times 2}[(0.876)(0) + (1 - 0.876)(20)] = 4.290$$

Strike Price	Price	Strike Price	Price
130	20.00	150	5.50
135	15.60	155	3.50
140	11.60	160	2.10
145	8.20	165	1.20

This price is the same as the result from the risk-neutral valuation in Q1(c).

14 Making Sense of Option Price in the Market

On 12 November 2020, the market prices of call option on Johnson & Johnson stock is as follows. The expiration date of these option is 16 April 2021.

The stock price on 12 November 2020 was \$147.80. The risk-free interest rate is 0.5% per annum. Suppose that a call option with strike price \$130 is considered. The time to expiration is 5/12 years. Suppose that a binomial tree with 10 steps is used to estimate the option price. The volatility is estimated to be 30% per annum.

a Calculate u, d, and risk-neutral probability p.

b Construct the table showing the possible stock prices and option payoff at the option maturity and their probabilities.

c What is the option price?

d Compare the theoretical price to the market price. Are they close to each other? What can we do to better estimate the option price?

If the alternative estimate of volatility is used,

$$u = e^{0.299781\sqrt{(5/12)/10}}, \quad d = e^{-0.299781\sqrt{(5/12)/10}}$$

Solution:

a The volatility is $\sigma = 0.3$, time to expiration is $T = 5/12$ years, and each step in the 10-step binomial tree is $\Delta t = (5/12)/10 = 1/24$ years. Thus,

$$u = e^{\sigma\sqrt{\Delta t}} = e^{0.3\sqrt{1/24}} = 1.0645 \tag{50}$$

$$d = e^{-\sigma\sqrt{\Delta t}} = e^{-0.3\sqrt{1/24}} = 0.9394 \tag{51}$$

$$p = \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{0.005\cdot 1/24} - 0.9394}{1.0645 - 0.9394} = 0.4838 \tag{52}$$

Number of up	Number of down	Stock Price at Maturity	Probability	Payoff
10	0	308.682	0.00069	178.682
9	1	266.406	0.00740	136.406
8	2	229.920	0.03563	99.920
7	3	198.431	0.10170	68.431
6	4	171.254	0.19050	41.254
5	5	147.800	0.24468	17.800
4	6	127.558	0.21824	0.000
3	7	110.088	0.13348	0.000
2	8	95.011	0.05358	0.000
1	9	81.998	0.01274	0.000
0	10	70.768	0.00136	0.000

Number of up	Number of down	Stock Price at Maturity	Probability	Payoff
10	0	272.540	0.00074	142.540
9	1	241.145	0.00783	111.145
8	2	213.367	0.03719	83.367
7	3	188.789	0.10472	58.789
6	4	167.042	0.19351	37.042
5	5	147.800	0.24519	17.800
4	6	130.775	0.21574	0.775
3	7	115.710	0.13017	0.000
2	8	102.381	0.05154	0.000
1	9	90.588	0.01209	0.000
0	10	80.153	0.00128	0.000

If the alternative estimate of volatility is used,

$$u = e^{0.299781\sqrt{(5/12)/10}} = 1.0631, \quad d = e^{-0.299781\sqrt{(5/12)/10}} = 0.9406$$

b Construct the table showing the possible stock prices and option payoff at the option maturity and their probabilities:

If the alternative estimate of volatility is used,

c The option price today is

$$e^{-0.005\times(5/12)} \sum_{j=0} p_j f_j = \$23.817$$

If the alternative estimate of volatility is used, the option price is \$21.886.

d The estimate from the binomial tree, \$23.817, is fairly close to the market price of \$20.00. To better estimate the option price, we can consider the following:

- We may increase the number of steps in the binomial tree.

- In obtaining the volatility, we may use the number of trading days per year (approximately 252 days) instead of calendar days.
- We may consider the stock's dividend payment. This will change the ex-dividend prices in the binomial tree.

15 Application to Corporate Finance

A company's current value of assets is \$120 million and the volatility of the asset value is 10% per annum. The company has issued a debt whose face value is \$100 million and it needs to repay the debt in two years. The risk-free interest rate is 5% per annum. Use a two-step binomial tree of the asset value in the following questions.

a What is the current value of the debt?

b What is the risk-neutral probability of the company's default on the debt?

Solution:

a The increasing/decreasing factor in the binomial tree is $u = e^{\sigma\sqrt{\Delta t}} = e^{0.1\sqrt{1}}$, and $d = e^{-\sigma\sqrt{\Delta t}} = e^{-0.1\sqrt{1}}$. The payoff of the debt two years later is as follows:

$$f_{uu} = \min\left(120e^{0.2}, 100\right) = 100 f_{ud} = \min\left(120e^{0.1}e^{-0.1}, 100\right) = 100 f_{dd} = \min\left(120e^{-0.2}, 100\right) = 98.248$$

The risk-neutral probability is $p = \frac{e^{0.05} - e^{-0.1}}{e^{0.1} - e^{-0.1}} = 0.731$. Then, the current value of the debt is

$$f_0 = e^{-0.05\times 2}\left[(0.731)^2(100) + 2(0.731)(1-0.731)(100) + (1-0.731)^2(98.248)\right] = \$90.369 \text{ million}$$

b The default occurs when the asset value decreases twice in the binomial tree. The probability of this path is $(1 - 0.731)^2 = 0.0724$.

16 Distribution of Stock Prices

Consider a stock whose current stock price is \$250 and its volatility is 20%. The risk-free interest rate is 3% per annum. The future stock price is log-normally distributed. Risk-averse investors require the stock return to be 10% per annum.

a What is the risk-neutral probability that stock price in year 2 is higher than \$300?

b What is the real probability that stock price in year 2 is higher than \$300?

Solution:

a The stock price is log-normally distributed,

$$\ln S_T \sim \phi\left(\ln S_0 + \left(r - \frac{\sigma^2}{2}\right)T, \sigma\sqrt{T}\right)$$

Thus, the risk-neutral probability is

$$\begin{aligned}
\text{Prob}\left(S_T > 300\right) &= \text{Prob}\left(\ln S_T > \ln 300\right) && (53)\\
&= \text{Prob}\left(\frac{\ln S_T - \ln S_0 - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} > \frac{\ln 300 - \ln S_0 - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) && (54)\\
&= \text{Prob}\left(\frac{\ln S_T - \ln S_0 - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} > \frac{\ln 300 - \ln 250 - \left(0.03 - \frac{0.2^2}{2}\right)2}{0.2\sqrt{2}}\right) && (55)\\
&= 1 - N(0.5739) && (56)\\
&= 0.283 && (57)
\end{aligned}$$

b The stock price is log-normally distributed,

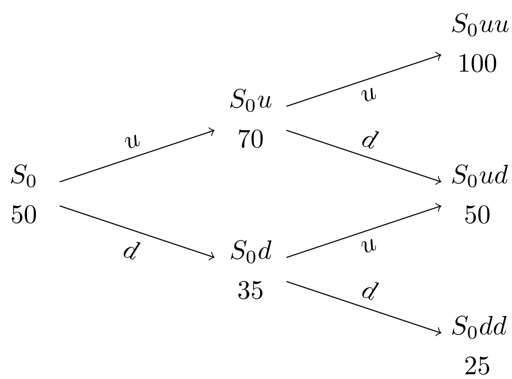
$$\ln S_T \sim \phi\left(\ln S_0 + \left(\alpha - \frac{\sigma^2}{2}\right)T, \sigma\sqrt{T}\right)$$

Thus, the real probability is

$$\begin{aligned}
\text{Prob}\left(S_T > 300\right) &= \text{Prob}\left(\ln S_T > \ln 300\right) && (58)\\
&= \text{Prob}\left(\frac{\ln S_T - \ln S_0 - \left(\alpha - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} > \frac{\ln 300 - \ln S_0 - \left(\alpha - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) && (59)\\
&= \text{Prob}\left(\frac{\ln S_T - \ln S_0 - \left(\alpha - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} > \frac{\ln 300 - \ln 250 - \left(0.1 - \frac{0.2^2}{2}\right)2}{0.2\sqrt{2}}\right) && (60)\\
&= 1 - N(0.0789) && (61)\\
&= 0.469 && (62)
\end{aligned}$$

17 Dynamic Replication

There are two main conceptual frameworks for pricing derivatives: (i) replicating portfolios and (ii) risk-neutral pricing. Risk-neutral pricing states that the price of any derivative is equal to the expectation of its discounted future payoffs, where the expectation is computed using risk-neutral probabilities and discounting is at the risk-free rate: Price $= e^{-rT}E[\text{Payoff}]$, where $E[\cdot]$ denotes the risk-neutral expectation. The replicating portfolio approach to pricing an option should imply the same price as the risk-neutral pricing approach. In a multi-period setting, however, constructing the replicating portfolio may be cumbersome. As an example, consider a stock price that evolves according to the binomial tree below (Assume one-step is 1 year.):



Assume that the risk-free rate is 11%. Using the replicating portfolio approach, we can price a call option with maturity $T = 2$ and strike price $K = 50$ in five steps.

1. Determine the payoff from the call option at each node of the tree at the final node $t = 2$.

2. Find the position "delta" to invest in stocks and the amount of risk-free bonds in the replicating portfolio at each node of the tree at $t = 1$.

3. Find the value of the replicating portfolio at each node of the tree at $t = 1$. By no arbitrage, this is also the value of the call option at each node.

4. Repeat the step for the $t = 0$ node.

Solution:

1. Since the strike price of the call option is 50, $c_{2;uu} = 50$, $c_{2;ud} = c_{2;du} = 0$, and $c_{2;dd} = 0$.

2. At $t = 1$, $\Delta_{1,u} = \frac{c_{2,uu} - c_{2;ud}}{S_0uu - S_0ud} = \frac{50-0}{100-50} = 1$ in the "up" node, and $\Delta_{1,d} = \frac{c_{2,ud} - c_{2,dd}}{S_0ud - S_0dd} = \frac{0-0}{50-25} = 0$ in the "down" node. The amount of bonds is $B_{1,u} = -e^{-r}\left(-c_{2;uu} + \Delta_{1,u}S_0uu\right) = -44.7917$ and $B_{1,d} = -e^{-r}\left(-c_{2;dd} + \Delta_{1,d}S_0dd\right) = 0$.

3. The value of the replicating portfolio, i.e., call option, at $t = 1$ is $c_{1;u} = \Delta_{1,u} S_0 u + B_{1,u} = 25.2083$ in the "up" node and $c_{1;d} = \Delta_{1,d} S_0 d + B_{1,d} = 0$ in the "down" node.

4. $\Delta_0 = (25.2083 - 0)/(70 - 35) = 0.7202$, $B_0 = -e^{-r}\left(-c_{1,u} + \Delta_0 S_{1,u}\right) = -22.5824$, and $c_0 = \Delta_0 S_1 + B_0 = 13.4294$.