Option Greeks

BUSS386. Futures and Options

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Lecture Outline

- The Greeks
- Applications
  - Capital protection products
  - Risk management
- Disclamer: the discussion is based on the BSM model. The empirical (actual) values of the Greeks are usually different from what the BSM model predicts!

“The Greeks”

- The sensitivity of option value to various factors
- They are known as “The Greeks.”
  1. Delta
  2. Gamma
  3. Theta
  4. Vega
  5. Rho
- They are used for risk management as well as trading.
- For options that can be priced using BSM, they often take a simple form.

Delta, Δ

1. Delta: Sensitivity of option to changes in the underlying price.

$$\Delta = \frac{\partial V}{\partial S} = N(d_1) \text{ for Calls}$$

    - For dividend paying underlyings: $e^{-q(T-t)}N(d_1)$
    - For put: $N(d_1) - 1$. With dividends: $e^{-q(T-t)}(N(d_1) - 1)$

- It tells how many units of the underlying asset one should trade in order to hedge the market risk exposure of the option.
    - For example, if $\Delta = 0.50$ for a given call option, the position that is long one call and short 0.50 shares of stock will be hedged against a (small) change in the stock price up or down (Delta neutral hedge)
    - Delta measures **market risk**.

- Approximately the probability that an option finishes in the money (in a risk neutral world).
    - $\text{Prob}\,(S_T \geq K) = N(d_2) = N(d_1 - \sigma\sqrt{T}) \propto N(d_1)$

Delta, Δ (cont'd)

- Consider a call option on a non-dividend paying stock, where $S_0 = 49, K = 50, r = 0.05, \sigma = 0.20$, $T = 0.3846$.

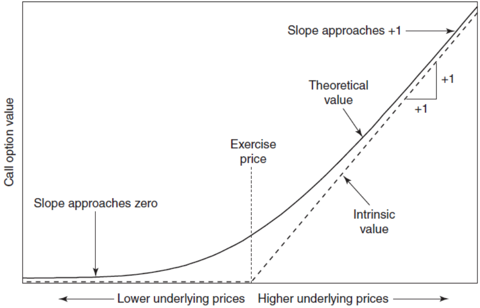
$$d_1 = \frac{ln(49/50) + (0.05 + 0.2^2/2)0.3846}{0.2 \times \sqrt{0.3846}} = 0.0542$$

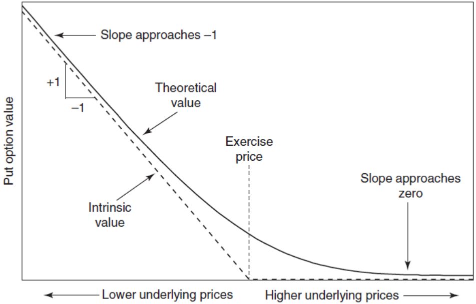
- Delta is $N(d_1) = 0.522$. When the stock price changes by ΔS, the option price changes by $0.522\Delta S$.

Delta, Δ (cont'd)

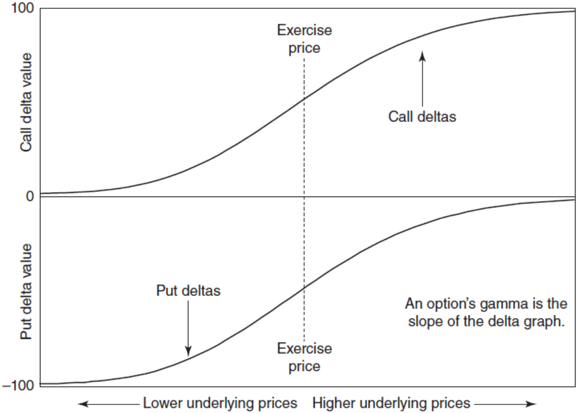
- The delta of is positive for calls and negative for puts.
- The delta is close to ± 1 for deep in the money options.
- The delta of far out of the money option is close to 0.
- At the money option has delta of about ± 0.50.

Delta, Δ (cont'd)





Delta, Δ (cont'd)



Gamma, Γ

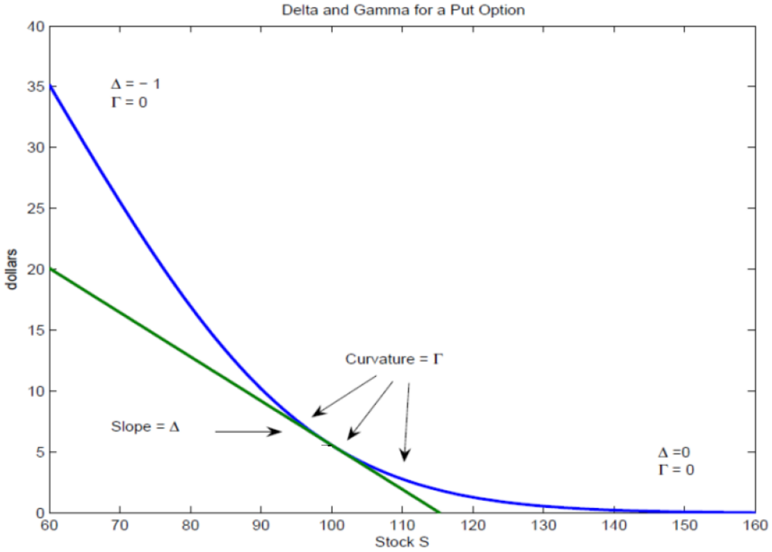
2. Gamma: Sensitivity of Delta to changes in the underlying price.

$$\Gamma = \frac{\partial \Delta}{\partial S} = \frac{N'(d_1)}{S\sigma\sqrt{T}}$$

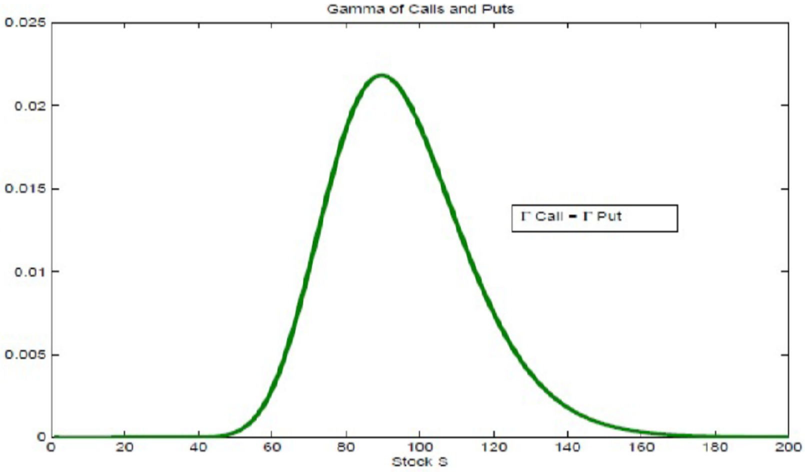
where $N'(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$, a PDF for a standard normal distribution.

- Identical for both calls and puts. For dividend paying underlyings?
- Gamma measures risk for a delta neutral hedge.
- Gamma is related to the curvature of the option value function.
    - For a long position, always positive.
    - Gamma is the largest at the money.
    - Gamma is small in the deep in or out of the money.

Gamma, Γ (cont'd)

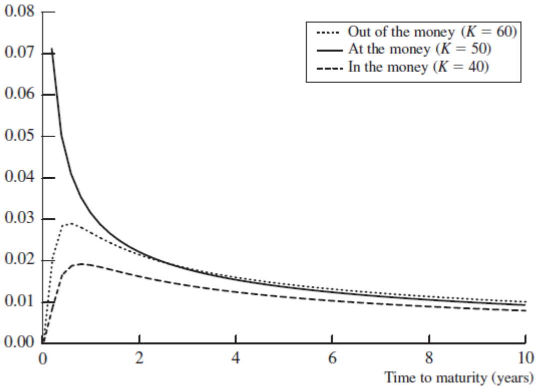


Gamma, Γ (cont'd)



Gamma, Γ (cont'd)

Variation of gamma with time to maturity for a stock option ($S = 50$, $r = 0$, $\sigma = 25\%$).



For an at-the-money option, gamma increases as the time to maturity decreases. Short-life at-the-money options have very high gammas, which means that the value of the option holder's position is highly sensitive to jumps in the stock price.

Gamma, Γ (cont'd)

- Consider a call option on a non-dividend paying stock, where $S_0 = 49, K = 50, r = 0.05, \sigma = 0.20$, $T = 0.3846$.

$$\frac{N'(d_1)}{S\sigma\sqrt{T}} = 0.066$$

- When the stock price changes by ΔS, the delta of the option changes by $0.066\Delta S$.

Gamma, Γ (cont'd)

- A call has a Delta of 0.54 and Gamma of 0.04.
  - Stock goes up \$1: Delta will become more positive by the Gamma amount.
  - New Delta value: 0.58

- Another call has a Delta of 0.75 and Gamma of 0.03
  - Stock is down \$1: Delta will become less positive by Gamma amount.
  - New Delta value: 0.72

- XYZ: $S = \$50, K = \$50, C = \$2, \Delta = 0.50, \Gamma = 0.06$
  - Should XYZ go up to \$51, the 50 strike call will be worth around \$2.50 when using delta only.
  - Using gamma as well: $c(51) - c(50) = \Delta(51 - 50) + \frac{1}{2}\Gamma(51 - 50)^2$
  - Delta = (Dollar) duration and Gamma = (Dollar) convexity

Theta, Θ

3. Theta: Sensitivity of option to passage of time, t.

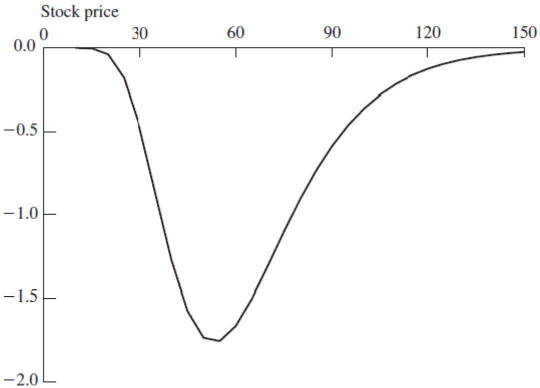
$$\Theta = \frac{\partial V}{\partial t} = \begin{cases} -\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}N(d_2) & \text{for Calls} \\ -\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} + rKe^{-rT}N(-d_2) & \text{for Puts} \end{cases}$$

where $N'(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$, a PDF for a standard normal distribution.

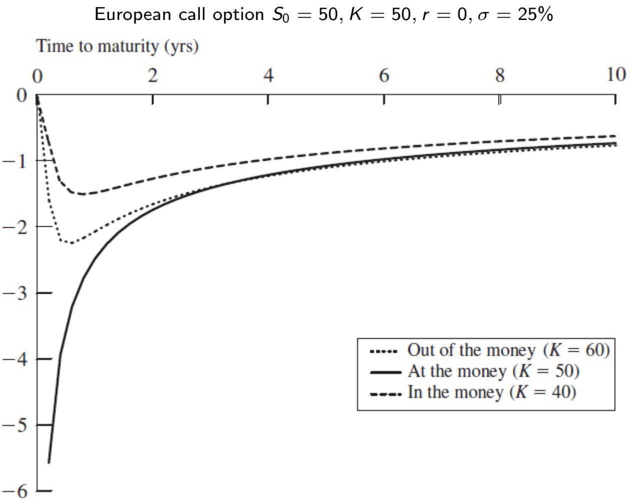
- Theta measures **Time Decay**.
- Theta decreases (more negative) when the option closer to expiration and at the money.

Theta, Θ (cont'd)

European call option $K = 50, r = 0, \sigma = 25\%, T = 2$



Theta, Θ (cont'd)



Theta, Θ (cont'd)

(Assume long position)

- $\Theta < 0$ for a call option on non-dividend paying stock.
  - As time elapses (keeping S fixed), the variance of the final price decreases and the exercise price (your debt) is less heavily discounted.

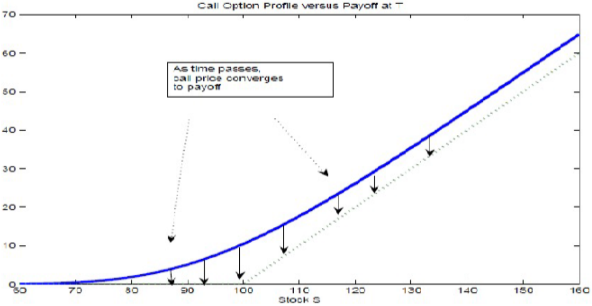
- Θ can be positive for a call on dividend paying stock.
  - As time passes, the call option holder is more likely to avoid a price drop due to dividend payout.
  - Deep out of the money call will not be affected much.

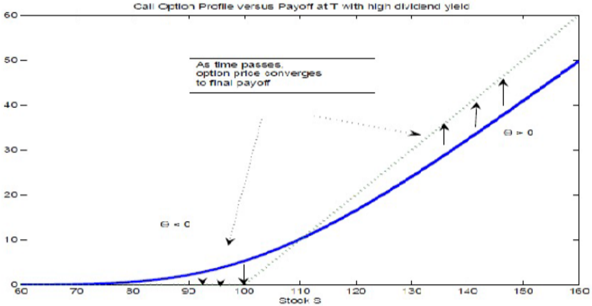
- For a put option, $\Theta > 0$ for low S and $\Theta < 0$ for high S
  - When S is high, payoff is zero, but put price is positive. $\rightarrow$ As time passes (keeping S fixed), the put price must decline.

  - When put is deep in-the-money (e.g., $S = 0$), the payoff at T is K. Put price, $p = Ke^{-r(T-t)}$, which increases with t.

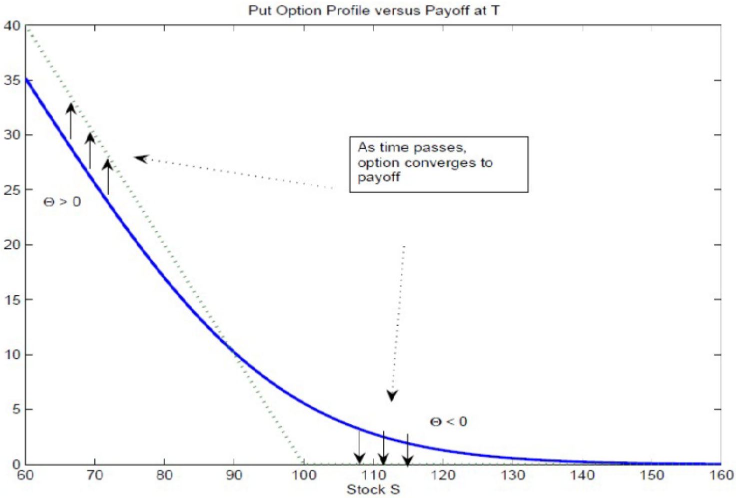
- $\Theta < 0$ for American options.

Theta, Θ (cont'd)





Theta, Θ (cont'd)



Theta, Θ (cont'd)

- Consider a call option on a non-dividend paying stock, where $S_0 = 49, K = 50, r = 0.05, \sigma = 0.20$, $T = 0.3846$.

$$-\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}N(d_2) = -4.31$$

- The theta is $-4.31/365 = -0.0118$ per calendar day, or $-4.31/252 = -0.0171$ per trading day.

Vega, ν

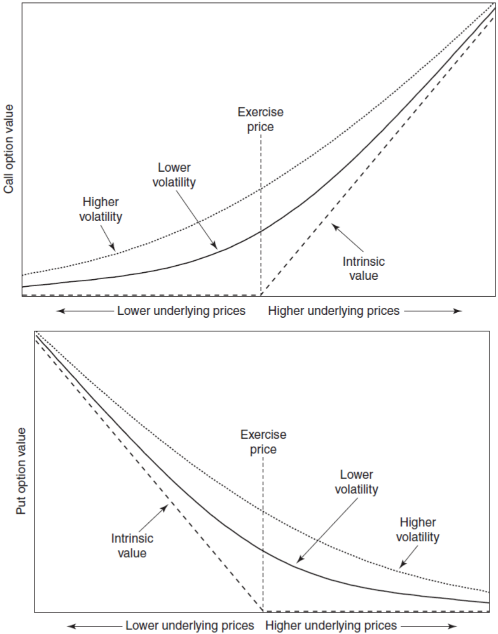
4. Vega: Sensitivity of option to a change in volatility σ.

$$\nu = \frac{\partial V}{\partial \sigma} = S\sqrt{T}N'(d_1) > 0$$

where $N'(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$, a PDF for a standard normal distribution.

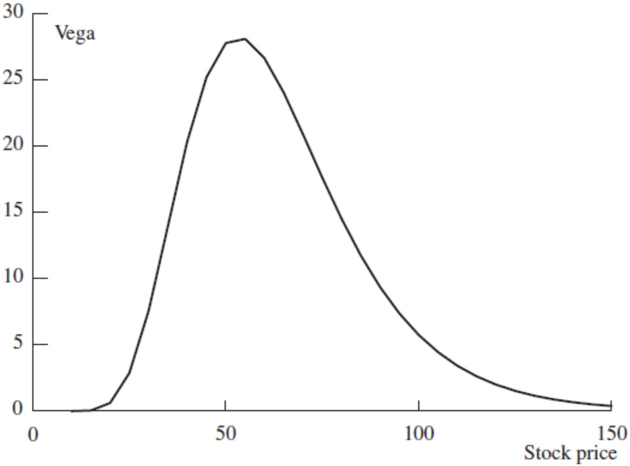
- Vega measures exposure to **Volatility Risk**
- The vega of European and American calls and puts is positive.
- For very deep OTM or ITM options, the vega is close to zero.
- The vega of a call or put peaks near the money.
- Buying a portfolio with positive vega is "buying volatility". Typically we do this by buying a call and a put — a straddle.

Vega, ν (cont'd)



Vega, ν (cont'd)

Variation of vega with stock price for an option $K = 50$, $r = 0$, $\sigma = 25\%$, $T = 2$



Vega, ν (cont'd)

- Consider a call option on a non-dividend paying stock, where $S_0 = 49, K = 50, r = 0.05, \sigma = 0.20$, $T = 0.3846$.

$$S\sqrt{T}N'(d_1) = 12.1$$

- Thus a 1% (0.01) increase in the implied volatility from (20% to 21%) increases the value of the option by approximately $0.01 \times 12.1 = 0.121$.

Rho, ρ

5. Rho: Sensitivity of option to a change in the interest rate.

$$\rho = \frac{\partial V}{\partial r} = \begin{cases} KTe^{-rT}N(d_2) > 0 & \text{for Calls} \\ -KTe^{-rT}N(-d_2) < 0 & \text{for Puts} \end{cases}$$

- Rho measures exposure to Interest Rate Risk.
- It depends on whether the option holder will pay K (call) or receive K (put). The PV of K declines as r increases, making the payment made smaller for the long call and payment received smaller for the long put.

Rho, ρ (cont'd)

- Consider a call option on a non-dividend paying stock, where $S_0 = 49, K = 50, r = 0.05, \sigma = 0.20, T = 0.3846$.

$$KTe^{-rT}N(d_2) = 8.91$$

- This means that a 1% (0.01) increase in the risk-free rate (from 5% to 6%) increases the value of the option by approximately $0.01 \times 8.91 = 0.0891$.

Summary

Table 19.6 Greek letters for European options on an asset providing a yield at rate q.

Greek letter	*Call option*	*Put option*
Delta	$e^{-qT}N(d_1)$	$e^{-qT}[N(d_1)-1]$
Gamma	$\dfrac{N'(d_1)e^{-qT}}{S_0\sigma\sqrt{T}}$	$\dfrac{N'(d_1)e^{-qT}}{S_0\sigma\sqrt{T}}$
Theta	$-S_0N'(d_1)\sigma e^{-qT}/(2\sqrt{T})$ $+\,qS_0N(d_1)e^{-qT}-rKe^{-rT}N(d_2)$	$-S_0N'(d_1)\sigma e^{-qT}/(2\sqrt{T})$ $-\,qS_0N(-d_1)e^{-qT}+rKe^{-rT}N(-d_2)$
Vega	$S_0\sqrt{T}N'(d_1)e^{-qT}$	$S_0\sqrt{T}N'(d_1)e^{-qT}$
Rho	$KTe^{-rT}N(d_2)$	$-KTe^{-rT}N(-d_2)$

Exercise

- Here are the current market prices for XYZ stock and two XYZ options. The Greek letter risk exposures come from the Black-Scholes model. The interest rate is 8% and the implied volatility is 0.25.

	Market price	delta	gamma	vega	theta
XYZ Stock	100	1	0	0	0
XYZ Call 105 strike, 1 month	1.25	0.29	0.047	.099	-.044
XYZ Put 95 strike, 1 month	0.83	-0.21	0.039	.084	-.030

- You are long 105-strike calls on 100,000 shares. (That is, you have 100,000 call options, each covering one share.)
  1. How would you set up a delta hedge for this position?
  2. What would the overall hedged position be worth? (What is the net cost to set it up?)
  3. What are the Greek letter exposures for the overall position?

Exercise (cont'd)

	Market price	delta	gamma	vega	theta
XYZ Stock	100	1	0	0	0
XYZ Call 105 strike, 1 month	1.25	0.29	0.047	.099	-.044
XYZ Put 95 strike, 1 month	0.83	-0.21	0.039	.084	-.030

1. Position delta is $100,000 \times 0.29 = 29,000$. Hedge by shorting $29,000$ shares.
2.
$$
\begin{aligned}
Calls &= 100,000 \times 1.25 = 125,000 \\
Stocks &= -29,000 \times 100 = -2,900,000 \\
Total &= -2,775,000
\end{aligned}
$$
3.
$$
\begin{aligned}
Delta &= 100,000 \times 0.29 + (-29,000) \times 1 = 0 \\
Gamma &= 100,000 \times 0.047 + (-29,000) \times 0 = 4,700 \\
Vega &= 100,000 \times 0.099 + (-29,000) \times 0 = 9,900 \\
Theta &= 100,000 \times -.044 + (-29,000) \times 0 = -4,400
\end{aligned}
$$

Exercise (cont'd)

- Tomorrow, XYZ stock opens at 95. Here is the new set of option prices and Greek letters.

	Market Price	delta	gamma	vega	theta
XYZ Stock	95	1.0	0	0	0
XYZ Call 105 strike, 1 month	0.30	0.10	0.025	.047	-.021
XYZ Put 95 strike, 1 month	3.35	-0.46	0.044	.108	-.052

4. If you liquidate right now, what would the profit or loss on the hedged position be?
5. If you don't liquidate, what stock trade will you need to do to become delta neutral again?

Exercise (cont'd)

	Market Price	delta	gamma	vega	theta
XYZ Stock	95	1.0	0	0	0
XYZ Call 105 strike, 1 month	0.30	0.10	0.025	.047	-.021
XYZ Put 95 strike, 1 month	3.35	-0.46	0.044	.108	-.052

4. If you unwind at the new prices your profit is:

$$\begin{aligned} Calls &= 100,000 \times (0.30 - 1.25) = -95,000 \\ Stocks &= -29,000 \times (95 - 100) = +145,000 \\ Total &= +50,000 \end{aligned}$$

5. If you wanted to rehedge, with the new delta, you should only be short

$$100,000 \times 0.10 = 10,000.$$

You have to buy back 19,000 of the shares you shorted.

Who cares about the Greeks?

- They are very important for market makers (MM).
- When a MM trades an option, s/he immediately trades stocks to cover delta risk.
    - MM is not betting on direction, but volatility.
- MM has a portfolio of different options, strikes, maturities and constantly monitoring the overall Delta, Gamma, Vega, and Theta portfolio risk.

Protected Principal Note

- Remember that this is an investment strategy where investors do not lose any of principal (initial investment) and sometimes earn additional profits. Also called "capital protected note".
- Investment banks often offer such securities, and hedge the short position with options or dynamic trading strategies.

Protected Principal Note: Example

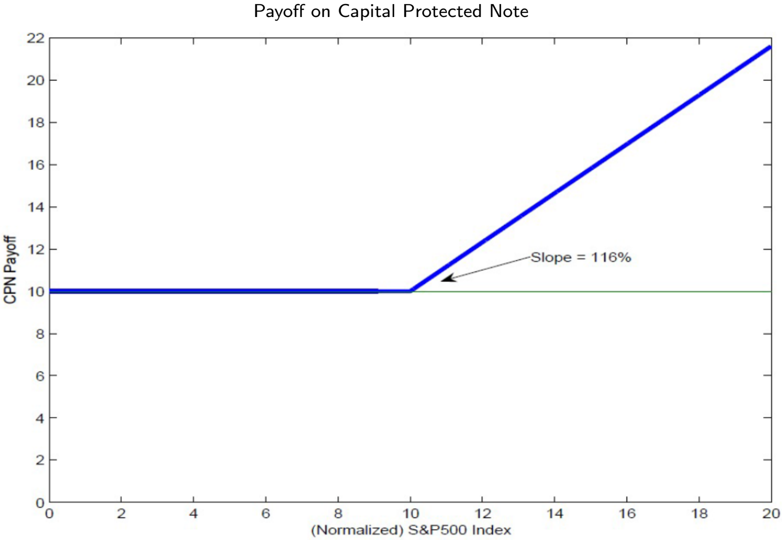
- On Feb 22, 2008, you as an MM sold a Capital Protected Note with:
  - Maturity: February 20, 2015
  - Issue price: \$10
  - Principal: \$10
  - Interest: 0%
  - Principal protection: 100%
  - Payoff at maturity = principal + Supplemental Redemption Amount (SRA) if positive

$$SRA = \$10 \times 116\% \times \frac{\text{Final Index Value} - \text{Initial Index Value}}{\text{Initial Index Value}}$$

  - Index is S&P 500 normalized to have Initial Index Value = \$10

- You want to protect your short position against increases in the stock price index.

Protected Principal Note: Example (cont'd)



Protected Principal Note: Example (cont'd)

- The payoff on the note can be decomposed into:
  - A zero coupon bond with principal \$10 and maturity $T = 7$.
  - 1.16 at-the-money call options on the normalized S&P 500 with maturity $T = 7$.
    - The reference index is normalized so that $S_0 = \beta \times S\&P500 = \$10$
    - On 2/28/08, $S\&P500 = 1353.1 \rightarrow \beta = 10/1353.1$

- Other data on 2/28/08
  - Interest rate, $r = 3.23\%$ (continuously compounded)
  - Dividend yield on S&P 500, $q = 2\%$
  - Forecast of market volatility over the 7 years, $\sigma = 15\%$

- The value of the security using BSM for dividend-paying stock is:

$$e^{-rT}(\$10) + (1.16)Call(S_0, K, r, \delta, \sigma, T)$$
$$= \$7.9764 + (1.16)\$1.7 = \$9.9483$$

- Investors give up interest on principal in exchange for a call option.

Protected Principal Note: Example (cont'd)

- At $t = 0$, you have a short position in the Capital Protected Note

- Hedge with an offsetting long position:
  - Buy a zero coupon bond for \$7.9764 to hedge the bond component.
  - Buy 1.16 units of the replicating portfolio for the embedded call option.

- Setting up the replicating portfolio for each call:
  - We can calculate the call $\Delta = e^{-qT}N(d_1) = 0.5747$.
  - Then the bond position $= Call(S_0, K, r, \delta, \sigma, T) - \Delta \times S_0 = 1.7 - 0.5747 \times 10 = -4.047$
  - In sum, for each call option, invest $0.5747 \times \$10 = \$5.747$ in the S&P 500 and borrow \$4.047

- Value of replicating portfolio $= \$5.747 - \$4.047 = 1.7$

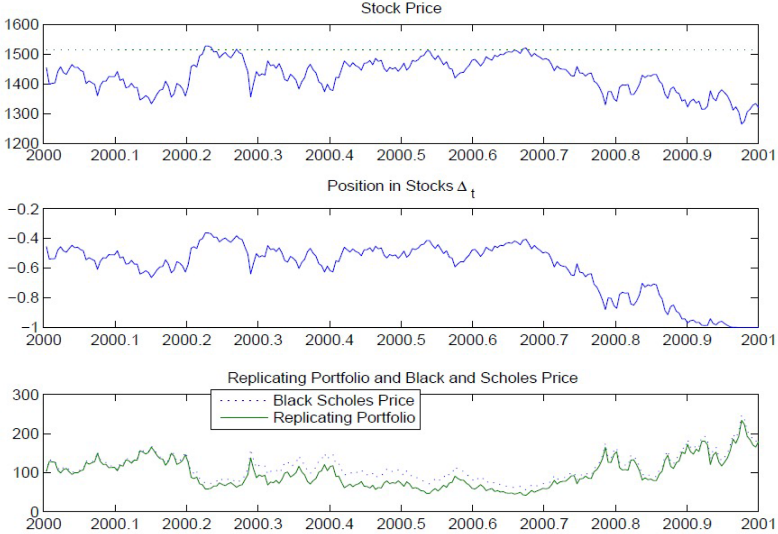
- Multiply both positions by 1.16 to scale up to the replicating portfolio for the Capital Protected Bond

Dynamic Delta Hedging

- Theoretically we need to frequently rebalance the portfolio as the Δ changes.
  - It will change with the stock price.
  - It will also change the passage of time, and any changes in r and σ.
- Recalculate Δ and new value of call.
- Adjust holdings of stocks and bonds in replicating portfolio to match new option value.
- The effectiveness of dynamic hedging depends on:
  - Frequency of rebalancing
  - Stability and accuracy of parameters (e.g., volatility)
  - Whether jumps in stock prices

How well does dynamic replication work in practice?

Replicating a put option on S&P 500 index, $T = 1, \sigma =$ std.dev in 1999.



Portfolio Insurance

- In 1981 UC Berkeley Profs Hayne Leland and Mark Rubinstein joined forces with John O'Brien and formed Leland, O'Brien and Rubinstein Associates, Incorporated (LOR).
  - Business Idea: Use dynamic replication to offer investors insurance on their portfolios.
  - For example, a pension plan fully invested in equity could purchase insurance to insure the portfolio against losses of value
  - LOR did not directly sell insurance. Rather they advised clients on the dynamic asset allocation that would insure against a drop in value of the portfolio.
  - Large potential demand from both mutual funds and pension plans. Business started slow but took off in 1984-86. By 1987 an estimated $100 billion in assets were covered by portfolio insurance products

Portfolio Insurance: Example (cont'd)

- A portfolio is worth \$90 million. To protect against market downturns:
  1. Buy a 6-month European put option on the portfolio.

$$S_0 = \$90, K = \$87, r = 0.09, q = 0.03, \sigma = 0.25, T = 0.5$$

  2. Create the put option synthetically.

$$d_1 = \frac{ln(90/87) + (0.09 - 0.03 + 0.25^2/2)0.5}{0.25\sqrt{0.5}} = 0.4499$$

  The delta is $e^{-qT}(N(d_1) - 1) = -0.3215$.

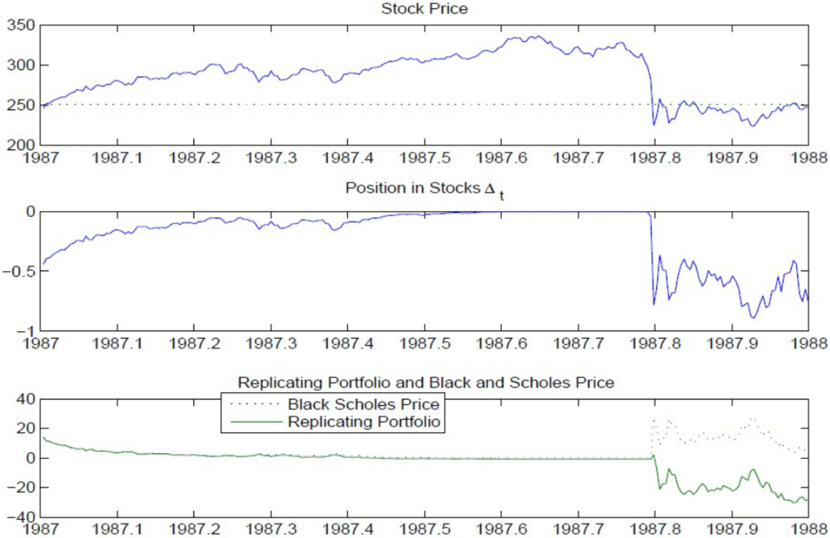
  - This shows that 32.15% of the portfolio should be sold initially and invested in risk-free assets.
  - If the value of the original portfolio reduces to \$88 million after 1 day, the delta of the required option changes to 0.3679 and a further 4.64% of the original portfolio should be sold and invested in risk-free assets.

Portfolio Insurance (cont'd)

- Various forms of portfolio insurance were developed.
- The most significant development in this period was "perpetual" insurance
  - In a standard fixed term contract, the investor was covered only for some period of time (typically, 3 years)
  - Most pension funds have long term liabilities, limiting the value of term insurance
  - Perpetual insurance allowed the investor (insured) to decide to exercise its option at any time in the future
- But then came the 1987 Crash ...

How well does dynamic replication work in practice?

This example is for put options around the time of the 1987 market crash.



How well does dynamic replication work in practice?

- This example illustrates in practice what we learned in theory: delta-hedging strategies only work well when stock price movement are fairly smooth

- With large jumps in stock prices, it is impossible to rebalance fast enough to eliminate all risk

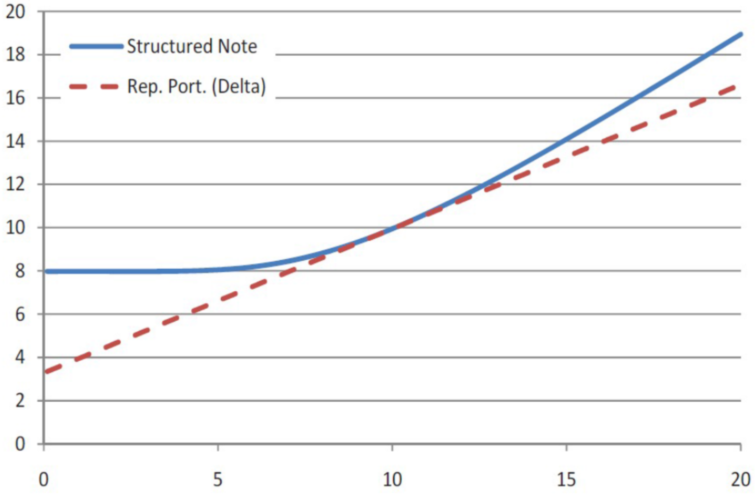
Delta-gamma hedging

- We have seen that there are some issues with delta hedging
  - We need to rebalance the portfolio frequently, which is expensive with transactions costs
  - The hedge can break down when there are large changes in stock prices
- The problems can be alleviated by "delta-gamma" hedging.
- This involves adding to the hedge portfolio a security with a positive gamma.
- Consider a portfolio Π which is short the T-dated call $Call(S, T)$ (like the one embedded in the Capital Protected Note), long N stocks, and long N^C of T_1-dated calls, $Call(S, T_1)$.

$$\Pi = -Call(S, T) + N \times S + N^C \times Call(S, T_1)$$

- We want to hedge both the sensitivity of Π to changes in the stock price ($\frac{\partial \Pi}{\partial S} = 0$) and the change in that sensitivity to changes in the stock price, i.e., the convexity, so that ($\frac{\partial^2 \Pi}{\partial S^2} = 0$).

Delta hedging: Capital Protected Note



Delta-gamma hedging (cont'd)

- The delta-gamma hedging requires:

$$\frac{\partial \Pi}{\partial S} = 0 \rightarrow -\frac{\partial C}{\partial S} + N + N^C \frac{\partial C_1}{\partial S} = 0 \qquad \text{(Delta hedging)}$$

$$\frac{\partial^2 \Pi}{\partial S^2} = 0 \rightarrow -\frac{\partial^2 C}{\partial S^2} + N^C \frac{\partial^2 C_1}{\partial S^2} = 0 \qquad \text{(Gamma hedging)}$$

- Solving the two equations with two unknowns:

$$N^C = \frac{\Gamma(S, T)}{\Gamma(S, T_1)}$$

$$N = \Delta(S, T) - N^C \Delta(S, T_1)$$

- Note that the position in stocks is smaller (if $N^C > 0$) than in the case of only Delta-hedging, as we now have to hedge the position in the short-term call option, which is used to hedge against Gamma.

Delta-gamma hedging (cont'd)

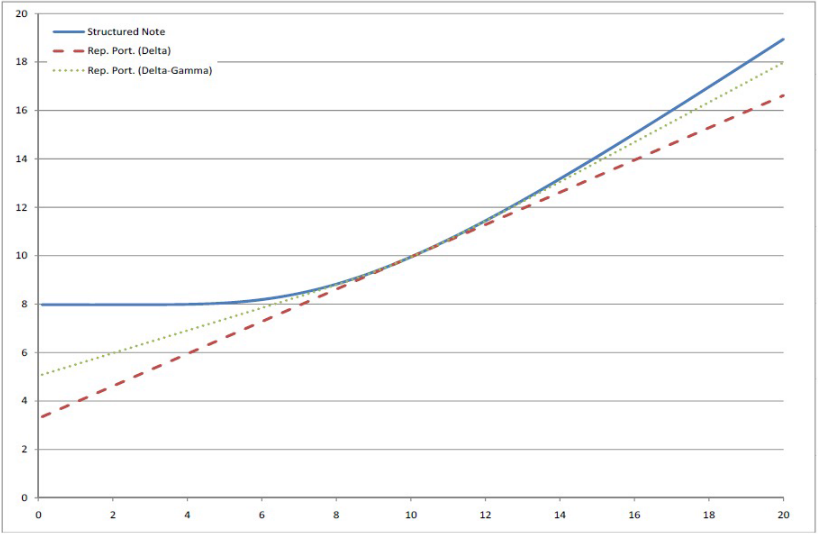
- For example, using a one-year option to hedge the Capital Protected Note, we have:

$$\begin{array}{lll} Call(S,T) = 1.7, & \Gamma(S,T) = 0.0801, & \Delta(S,T) = 0.5747 \\ Call(S,T_1) = 0.6443, & \Gamma(S,T_1) = 0.2575, & \Delta(S,T_1) = 0.5512 \end{array}$$

- Then we obtain:

$$\begin{aligned} \text{Short-term call} &= N^C = 0.3113 \\ \text{Stocks} &= N = 0.4031 \\ \text{Bonds} &= 1.7 - N \times S - N^C \times Call(S,T_1) = -2.5315 \end{aligned}$$

Delta-gamma hedging (cont'd)



Delta-gamma hedging (cont'd)

- The Delta-Gamma hedging allows for larger swings in the stock price before calling for rebalancing.
- Less frequent rebalancing implies lower transaction costs.
  - But we have more transaction costs from rebalancing T_1-dated options.
  - We need to use very liquid, exchange-traded options to minimize transaction costs.
- Additional benefit is that large sudden changes in stock prices are now better hedged.

Vega-hedging

- Again, consider a portfolio Π which is short the T-dated call $Call(S, T)$, long N stocks, and long N^C of T_1-dated calls, $Call(S, T_1)$.

$$\Pi = -Call(S, T) + N \times S + N^C \times Call(S, T_1)$$

- We want to hedge the sensitivity of Π to changes in the volatility ($\frac{\partial \Pi}{\partial \sigma} = 0$).

- The vega hedging requires:

$$\frac{\partial \Pi}{\partial \sigma} = 0 \rightarrow -\frac{\partial C}{\partial \sigma} + N^C \frac{\partial C_1}{\partial \sigma} = 0 \qquad \text{(Vega hedging)}$$

- Hence,

$$N^C = \frac{\nu(S, T)}{\nu(S, T_1)}$$

Vega-hedging (cont'd)

- If we want this portfolio to be Gamma-neutral as well, we need at least two traded options.

$$\Pi = -Call(S,T) + N \times S + N_1^C \times Call(S,T_1) + N_2^C \times Call(S,T_2)$$

- The Vega-Gamma hedging requires:

$$\frac{\partial \Pi}{\partial \sigma} = 0 \rightarrow -\frac{\partial C}{\partial \sigma} + N_1^C \frac{\partial C_1}{\partial \sigma} + N_2^C \frac{\partial C_2}{\partial \sigma} = 0 \qquad \text{(Vega hedging)}$$
$$\frac{\partial^2 \Pi}{\partial S^2} = 0 \rightarrow -\frac{\partial^2 C}{\partial S^2} + N_1^C \frac{\partial^2 C_1}{\partial S^2} + N_2^C \frac{\partial^2 C_2}{\partial S^2} = 0 \qquad \text{(Gamma hedging)}$$

- Hence,

$$N_2^C = \frac{\nu \Gamma_1 - \nu_1 \Gamma}{\nu_2 \Gamma_1 - \nu_1 \Gamma_2}$$
$$N_1^C = \frac{\nu - N_2^C \nu_2}{\nu_1} = \frac{\Gamma - N_2^C \Gamma_2}{\Gamma_1}$$

Vega-hedging (cont'd)

- Consider a portfolio that is delta neutral, with a gamma of $-5,000$ and a vega of $-8,000$.
- The options shown in the following table can be traded.

	Delta	*Gamma*	*Vega*
Portfolio	0	–5000	–8000
Option 1	0.6	0.5	2.0
Option 2	0.5	0.8	1.2

- Unlike the formula we derived where we hedge short call, here assume that we try to hedge long calls.

Vega-hedging (cont'd)

- To make the portfolio gamma and vega neutral, both Option 1 and Option 2 can be used. If w_1 and w_2 are the quantities of Option 1 and Option 2 that are added to the portfolio, we require that

$$-5,000 + 0.5w_1 + 0.8w_2 = 0$$
$$-8,000 + 2.0w_1 + 1.2w_2 = 0$$

- The solution to these equations is $w_1 = 400, w_2 = 6,000$.

- The portfolio can therefore be made gamma and vega neutral by including 400 of Option 1 and $6,000$ of Option 2.

- The delta of the portfolio, after the addition of the positions in the two traded options, is $400 \times 0.6 + 6,000 \times 0.5 = 3,240$. Hence, $3,240$ units of the asset would have to be sold to maintain delta neutrality. This doesn't affect Gamma and Vega.

General Approach to Risk Management

- The Greek letter risk exposures "add up." The total risk exposure of a position is the sum of the exposures of the component securities.
- In general you need at least one hedge instrument per type of risk.
  - For example, to hedge both delta and gamma, you need a minimum of two hedge instruments.
- Not every instrument can be used for every type of risk.
  - For example, the bond can't hedge Delta, and neither the bond nor the stock can hedge Gamma.
  - Two options may have the same value for two different Greek letters (for example, a European call and put with the same maturity and strike have the same Gamma and the same Vega). In that case, you can't use the two options to hedge those two risks separately

General Approach to Risk Management (cont'd)

- If there are more hedge instruments than risks to be hedged, the solution is not unique. This allows optimizing on other aspects of the hedge. Things to optimize on include
  - minimizing the overall cost
  - maximizing the (theoretical) expected profit from selling overvalued options and buying undervalued ones
  - minimizing the amount of future rebalancing that will be required, etc.

Appendix: Relation between Delta, Gamma, and Theta

Relation between Delta, Gamma, and Theta

- Black and Scholes derived a partial differential equation (pde) and solved it to obtain the BSM pricing equation.
- The Black–Scholes pde for a portfolio of options is:

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV$$

- It can be rewritten using the Greeks:

$$\Theta + rS\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma = rV$$

- For a delta-neutral portfolio, $\Delta = 0$:

$$\Theta + \frac{1}{2}\sigma^2 S^2 \Gamma = rV$$

- When Theta is large and positive, Gamma tends to be large and negative, and vice versa. Theta can be regarded as a proxy for Gamma in a delta-neutral portfolio.