Volatility

BUSS386. Futures and Options

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Lecture Outline

- Volatility
  - Does the BSM predict market option price?
  - Volatility Smile/Smirk
  - Volatility Surface

Volatility

Volatility

- Volatility is the only unobservable input in the BSM model.

$$C(S, K, T, r, \sigma)$$

  - In the BSM model, volatility is assumed to be a constant known parameter.

- However in the real world it is neither constant nor known.

- So naturally there is a lot of dispersion around the average volatility expectation in the market.

Stylized Facts about Volatility

- Volatility is not constant; it varies substantially over time
- Variation is not random; there are extended periods of high volatility and periods of low volatility ("volatility clusters")
- There appears to be "mean reversion" in volatility; periods of unusually high or low volatility tend to be followed by a reversion to more normal behavior
- In equity markets, volatility increases when stock prices fall, and (may) decrease when prices rise (this is often called the "leverage effect")

Stylized Facts about Volatility (cont'd)

[We will talk about the following today]

- Implied volatility has a regular structure across options with different strike prices, known as the "smile" or the "skew"
- Implied volatility also shows systematic "term structure" effects for options with different maturities

Estimating/Forecasting Volatility

Historical volatility

- Compute K log returns from past prices: $R_{t-k} = \ln(S_{t-k}/S_{t-k-1})$, for $k = 1, ..., K$
- Volatility estimate = annualized standard deviation of R

$$\sqrt{h \frac{\sum_{k=0}^{K} R_{t-k}^2}{K}}$$

	A	B	C	D
1	**Date**	**Price**	**log return**	**Squared**
2	t	102	0.985%	9.707E-05
3	t-1	101	4.041%	0.0016329
4	t-2	97	-2.041%	0.0004165
5	t-3	99	-1.005%	0.000101
6	t-4	100		
7		**average**	0.495%	0.0005619
8		**annualize (*255)**	126.242%	0.14328
9				
10			**volatility**	37.852%

Estimating/Forecasting Volatility

Exponentially weighted moving average

- Compute log returns
- Downweight data as it ages, by multiplying each squared deviation by w_k, for some weight $w < 1.0$.
- Volatility estimate = annualized value of

$$\sqrt{h \frac{\sum_{k=0}^{K} w^k R_{t-k}^2}{\sum_{k=0}^{K} w^k}}$$

	A	B	C	D	E	F
1	Date	Price	log return	Squared	weight (0.9)	w*Squared
2	t	102	0.985%	9.707E-05	1	9.70677E-05
3	t-1	101	4.041%	0.0016329	0.9	0.001469638
4	t-2	97	-2.041%	0.0004165	0.81	0.000337383
5	t-3	99	-1.005%	0.000101	0.729	7.36357E-05
6	t-4	100				
7		average	0.495%	sum	3.439	0.001977724
8		annualize (*255)	126.242%		weighted average	0.000575087
9					annualize (*255)	0.146647175
10						
11					volatility	38.295%

Estimating/Forecasting Volatility

Generalized Autoregressive Conditional Heteroskedasticity (GARCH)

- It models variance at date t as a combination of last period's variance, σ_{t-1}^2, and last period's squared random price shock ϵ_{t-1}^2.
- The simplest GARCH model has two equations:
    - Return equation: $r_t = \mu + \epsilon_t$
    - Variance equation: $\sigma_t^2 = \omega + \alpha\epsilon_{t-1}^2 + \beta\sigma_{t-1}^2$
      (Typical values: $0.05 \leq \alpha \leq 0.08$ and $0.90 \leq \beta \leq 0.95$, $\alpha + \beta < 1.0$)[1]

	A	B	C	D	E	F
1	Date	Price	log return	Squared*255	lagged	Conditional variance
2	t	102	0.985%	0.024752275	0.41639735	2.676%
3	t-1	101	4.041%	0.416397351	0.10621312	1.103%
4	t-2	97	-2.041%	0.106213121	0.02575736	0.856%
5	t-3	99	-1.005%	0.025757359		2.576%
6	t-4	100				

[1]GARCH has a great advantage over more complicated stochastic volatility (SV) models because the same ϵ shocks drive both the return and the volatility.

Practical Issues in Estimating Volatility from Past Prices

- If the price of the underlying asset did follow a lognormal diffusion with constant mean and volatility, as assumed by Black and Scholes, you would get the best volatility estimate by using as much past data, sampled at as fine an interval, as you could get. But with real world prices, several practical issues arise:
  - What observation interval to use (daily? monthly? intraday?)
    - Higher frequency is better, so use daily data
  - Whether to estimate the mean
    - No!
  - How much past data to include
    - As much as possible, but not from much different economic environments
  - How to deal with "outliers," i.e., events like October 19, 1987
    - There is no perfect answer; use judgment and consider how sensitive the results are to the outliers

Forecasting Volatility: Suggestions

- Here are several suggestions[2]
  - Different methods should be evaluated based on their out-of-sample forecasting performance.
  - The overall accuracy of all methods is relatively low.
  - Assuming a mean of zero is preferable to using deviations from the sample mean, which often serves as a poor estimator.
  - Simpler models, such as historical volatility measured over a long sample period, tend to perform comparably to more complex models while offering greater robustness.
  - Volatility forecasts are generally more reliable for long horizons compared to short horizons.
  - GARCH models perform well for very short horizons but require substantial data to produce accurate parameter estimates.

[2] by Stephen Figlewski, former NYU professor

Implied Volatility

- Implied volatility (IV) is the value of the volatility input to an option pricing model that makes the model value equal the option price observed in the market.
    - IV is regarded the best volatility estimate possible, because the market has access to much more information than any model can incorporate.
    - There is a one-to-one correspondence between implied volatility and option price. In some markets, options are quoted in terms of implied volatility, rather than price.
        - Higher uncertainty $\rightarrow$ Higher option price $\rightarrow$ Higher IV

- IV of a European call option is the same as IV of a European put option for the same underlying asset, with the same strike price and time to maturity.
    - $p(\sigma_p) + S_0 = c(\sigma_c) + e^{-rT}K$ with the BSM, i.e., $\sigma_p = \sigma_c$.
    - At market prices of put/call, $p_{mkt} + S_0 = c_{mkt} + e^{-rT}K$.
    - $p(\sigma_p) - p_{mkt} = c(\sigma_c) - c_{mkt}$
    - Set $\sigma_p = IV$. Then LHS $= 0 =$ RHS, in which case $c(\sigma_c) = c_{mkt}$. Hence, $\sigma_p = \sigma_c = IV$.

The VIX Index

Trading volatility

- The CBOE publishes indices of implied volatility. The most popular index, the SPX VIX, is an index of the implied volatility of 30-day options on the S&P 500 calculated from a wide range of calls and puts.
  - Futures on the VIX started in 2004
    - One contract is on 1,000 times the index, with minimum tick size of 0.01 (1 volatility basis point).
  - Options on the VIX in 2006.
    - (Call) Payoff = $\$100 \times \max(VIX_T - K, 0)$, where T is option maturity and K is the strike level.
  - The VIX is like a temperature: you can observe it but you can't store it over time. So there is no cost of carry pricing model for VIX products. Prices are based on expectations.

- Example: Suppose that a trader buys an April futures contract on the VIX when the futures price is 18.5 (corresponding to a 30-day S&P 500 volatility of 18.5%) and closes out the contract when the futures price is 19.3 (corresponding to an S&P 500 volatility of 19.3%). The trader makes a gain of $800.

The VIX Index (Optional)

How the VIX is computed

- The original formula for computing the VIX was changed in 2003. The "Old VIX," (still computed and published as the VXO), was based on 8 at-the-money calls and puts on the OEX index and used the Black-Scholes model to extract the implied volatilities. These IVs were combined into a weighted average 30-day at-the-money implied volatility. (There was also a technical problem in annualizing the index, which made it significantly biased upward.)

- The new VIX uses all out of the money SPX index calls and puts expiring just before and just after 30 days and extracts, not individual IVs, but the whole risk neutral probability distribution (without using Black-Scholes or any other pricing model). The implied volatility is the standard deviation from this implied distribution.

The VIX Index

Trading volatility

- Variance swaps: a forward contract on the realized variance of returns over some period ending at a future date with a strike variance level that is set at the beginning.
  - Example: A trader who expects high volatility in the stock market until the end of the year 2016 could "buy" a variance swap with a strike variance rate of $(0.20)^2$ and maturity Dec. 31. The quote is often done in terms of the volatility implied by the given variance, which is 20% in this case.
  - Realized variance is calculated as the annualized mean of R_t^2 , where R_t is the log return, for dates t from now through Dec. 31. Mean return is assumed to be zero. The difference between realized variance and the strike is multiplied by a notional principal to get the payoff in dollars.

- A volatility swap is essentially the same thing, except the payoff is in terms of the difference between realized volatility and the strike.
  - A volatility swap seems to make more sense, since volatility is what investors care about, but valuation is trickier mathematically, meaning it is a lot harder to hedge than a variance swap, so vol swaps are less popular.

The VIX Index

BSM option price = Market option price?

- Let's compare BSM options prices to market prices at a point in time
  - The data is from May 3, 2007
  - The S&P 500 index was at $S = 1502.39$
  - The one-month risk-free rate was at $r = 4.713\%(c.c.)$
  - The dividend yield on the S&P 500 was about $q = 1.91\%$
  - Using the previous 3 months of returns:
    $\sigma = \sqrt{\frac{1}{63}\sum_{i=1}^{63}(r_{t-i} - \bar{r})} \times \sqrt{252} = 12.35\%$

Comparing BSM predictions to market prices

SPX (S&P 500 INDEX)	Today	cc rate	div yield	volatility
1502.39	5/3/2007	0.04713	0.0191	0.1236

				CALLS			PUTS		
Maturity	Time to T	Strike	Moneyness K/S	Mkt Price	B/S	BSC/Mkt	Mkt Price	B/S	BSP/Mkt
6/15/2007	0.12	1430	0.952	83.9	80.12	0.955	6.2	3.19	0.514
6/15/2007	0.12	1435	0.955	79.4	75.74	0.954	6.7	3.78	0.564
6/15/2007	0.12	1440	0.958	75	71.44	0.953	7.3	4.46	0.610
6/15/2007	0.12	1445	0.962	70.6	67.24	0.952	7.9	5.23	0.662
6/15/2007	0.12	1450	0.965	66.3	63.14	0.952	8.7	6.10	0.701
6/15/2007	0.12	1455	0.968	62.1	59.15	0.952	9.3	7.08	0.761
6/15/2007	0.12	1460	0.972	57.9	55.27	0.955	10.1	8.17	0.809
6/15/2007	0.12	1465	0.975	53.8	51.52	0.958	10.9	9.39	0.862
6/15/2007	0.12	1470	0.978	49.8	47.89	0.962	11.9	10.74	0.902
6/15/2007	0.12	1475	0.982	45.9	44.40	0.967	12.6	12.22	0.970
6/15/2007	0.12	1480	0.985	42.1	41.05	0.975	14.1	13.84	0.982
6/15/2007	0.12	1485	0.988	38.4	37.84	0.986	15.4	15.61	1.014
6/15/2007	0.12	1490	0.992	34.8	34.79	1.000	17.05	17.52	1.028
6/15/2007	0.12	1495	0.995	31.4	31.88	1.015	18.55	19.59	1.056
6/15/2007	0.12	1500	0.998	28.05	29.13	1.039	20.35	21.82	1.072
6/15/2007	0.12	1505	1.002	24.55	26.54	1.081	21.95	24.19	1.102
6/15/2007	0.12	1510	1.005	22	24.10	1.095	24	26.73	1.114
6/15/2007	0.12	1515	1.008	19.3	21.81	1.130	26.2	29.41	1.123
6/15/2007	0.12	1520	1.012	16.6	19.68	1.186	28.6	32.25	1.128
6/15/2007	0.12	1525	1.015	14.8	17.70	1.196	31.2	35.24	1.130
6/15/2007	0.12	1530	1.018	12.3	15.86	1.290	34	38.38	1.129
6/15/2007	0.12	1535	1.022	10.3	14.17	1.376	37	41.66	1.126
6/15/2007	0.12	1540	1.025	8.6	12.61	1.467	40.3	45.07	1.118
6/15/2007	0.12	1545	1.028	7.05	11.19	1.587	43.7	48.62	1.113
6/15/2007	0.12	1550	1.032	5.95	9.89	1.663	47.4	52.30	1.103
6/15/2007	0.12	1555	1.035	4.5	8.72	1.937	51.2	56.09	1.096
6/15/2007	0.12	1560	1.038	3.7	7.65	2.068	55.2	60.00	1.087
6/15/2007	0.12	1565	1.042	2.9	6.69	2.308	59.4	64.01	1.078
6/15/2007	0.12	1570	1.045	2.325	5.83	2.509	63.7	68.13	1.070
6/15/2007	0.12	1575	1.048	1.9	5.07	2.667	68.2	72.33	1.061

When K/S is low: OTM Puts & ITM Calls

Volatility Smile/Smirk

- Note that the BSM assumes that volatility is constant.
- "Smirk" or "Skew": However, the volatility used to price a low-strike-price option (i.e., a OTM put or a ITM call) is significantly higher than that used to price a high-strike-price option (i.e., a ITM put or a OTM call).
- "Smile": The implied volatility is relatively low for at-the money options. It becomes progressively higher as an option moves either into the money or out of the money (usually for currency options).

What could explain the discrepancies?

- A negative correlation between equity prices and volatility.
  1. Leverage: Price $\downarrow \Rightarrow$ Leverage $\uparrow \Rightarrow$ Volatility $\uparrow$
  2. Volatility feedback effect: Volatility $\uparrow \Rightarrow$ Required return $\uparrow \Rightarrow$ Price $\downarrow$

     A price drop begets a price drop.

$\Rightarrow$ Extra demand for OTM puts: Volatility skew appeared after the crash of 1987 (Black Monday: DSIA dropped more than 22%)

- What about ITM calls? ITM calls have become popular alternatives to outright stock purchases as they offer leverage and hence increased ROI? Δ of ITM call ≈ 1.

Volatility Prediction: Implied Volatility

- The volatility smile shows that the basic Black-Scholes option pricing model does not fully explain how options are priced in the market.
  - I.e., implied volatility contains information about the volatility that will occur in the future, but it is biased as a forecast.
- More sophisticated option pricing models can be constructed that are consistent with the existence of a volatility smile, but are they the true explanation for it? (see Appendix)
- Despite all of these issues, market makers prefer to use implied volatility in their models, because they want the model values to match the prices they are seeing in the market. This leads to the use of "practitioner Black-Scholes," which is the Black-Scholes equation, but with a different volatility input for each option.

A Simple Practitioners' Approach

- Record the shape of the "smile" from recent past observations.
- Assume that the same smile will persist in the future.
- Obtain the IV of the ATM option, via BSM. (Relatively accurate)
- Use the graph to adjust for the bias of the BSM equation, given K/S. Plug the adjusted volatility in the BSM equation.
- We may use the adjusted BSM equation to calculate the greeks Δ, Γ, Θ.
- Market makers typically fit a smooth curve (for example, a quadratic or exponential curve) through the graph of IV.
    - They judge options with IV above the curve as overpriced and options with IV below the curve as underpriced. They make a market in these options based on this information.

A More Sophisticated Practitioners' Approach

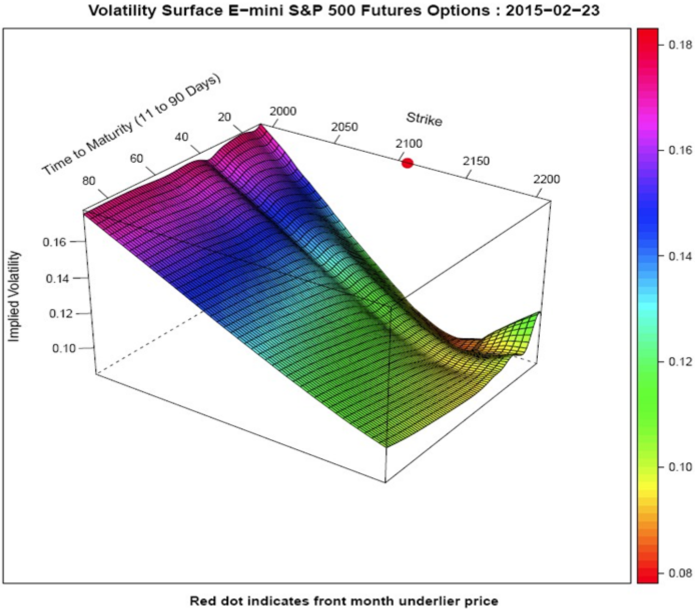
- Evaluate the relative price of different options.
- It is hard to compare the value of options with different strike prices or maturities.
  - Different strike prices $\rightarrow$ different intrinsic values
  - Different maturities $\rightarrow$ different time value of money and uncertainty

Table 20.2 Volatility surface.

	K/S_0				
	0.90	*0.95*	*1.00*	*1.05*	*1.10*
1 month	14.2	13.0	12.0	13.1	14.5
3 month	14.0	13.0	12.0	13.1	14.2
6 month	14.1	13.3	12.5	13.4	14.3
1 year	14.7	14.0	13.5	14.0	14.8
2 year	15.0	14.4	14.0	14.5	15.1
5 year	14.8	14.6	14.4	14.7	15.0

A More Sophisticated Practitioners' Approach

Volatility Surface



A More Sophisticated Practitioners' Approach

Example

- When a new option has to be valued, we look up the appropriate volatility in the table.
- For example, when valuing a 9-month option with a K/S_0 ratio of 1.05, we would interpolate between 13.4 and 14.0 in Table 20.2 to obtain a volatility of 13.7%.
- When valuing a 1.5-year option with a K/S_0 ratio of 0.925, a two-dimensional (bilinear) interpolation would be used to give an implied volatility of 14.525%.
- This is the volatility that would be used in the Black–Scholes–Merton formula or a binomial tree.
- We can also consider estimating $\hat{\sigma}(K/S, T)$.
    - From the prices of several exchange-traded options, regress $\hat{\sigma}$ on $(K/S), T, (K/S)^2, (K/S) \times T,$ and T^2

Summary

- Despite its inaccuracies BSM serves as a useful benchmark.
  - Gives decent approximation to prices close to the money.
- It also works reasonably well to hedge options positions against changes in stock prices using delta or delta-gamma hedging.
- Models have been proposed to correct some of the shortcomings.
  - Stochastic volatility
  - Jumps
  - Fat tails
- All of these models are consistent with the idea that OTM puts are expensive relative to BSM prices because investors seeking protection from large losses (e.g., jumps down) must pay a higher (insurance) premium

Appendix: Volatility Models
(Chapter 27)

Local Volatility Models

- Assume that σ depends on S itself.

$$dS_t = \mu S_t dt + \sigma(S_t, t) dW_t$$

- The Constant Elasticity of Variance model (Cox, 1975):

$$dS_t = \mu S_t dt + \sigma S_t^{\gamma} dW_t$$

where $dW_t = \epsilon\sqrt{dt}, \epsilon \sim \phi(0,1)$.

  - When $\gamma = 1$, the BSM model.
  - If $\gamma < 1$, a lower S_t implies a higher volatility $\Rightarrow$ volatility smirk.[3]

[3] Obviously, when $\gamma > 1$, volatility increases with S_t; sometimes observed in options on futures.

Stochastic Volatility Models

- Assume that σ_t is moving over time.
- The Heston model:

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_t$$
$$dv_t = \theta(\omega - v_t)dt + \xi\sqrt{v_t} dB_t$$

  - ω: the mean long-term variance,
  - θ: the rate at which the variance reverts toward its long-term mean,
  - ξ: the volatility of the variance process,
  - dW_t and dB_t are correlated with the constant correlation value ρ.

- If $\rho < 0$, i.e., volatility is negatively related to stock returns, OTM put options become relatively more expensive $\Rightarrow$ volatility smirk
- Intuition: A decline in price $\rightarrow$ higher volatility $\rightarrow$ higher probability of even larger declines $\rightarrow$ higher price of insurance against downturns

Jumps in stock prices

- We know that periodically there are large jumps in stock prices (e.g., 1987 and 2020)

$$dS_t = \mu S_t dt + \sigma S_t dW_t + J(Q) S_t dP_t$$

  - $P_t = 0$ most of time, but $P_t = 1$ with small probability.
  - $J(Q)$ can be random or constant (jump size).

- If $J(Q) < 0$, then OTM put options are relatively more expensive.
  - If $J(Q) < 0$, it become more likely that negative outcomes occur.
  - Investors are willing to pay high premium to insure against such negative events.

- Pricing with "jumps" are more complicated.

Implied tree models

- Previously we saw how to find option prices given a stock price tree
- With implied trees we start with some observed options prices and calibrate the stock price tree to be consistent with those prices
  - We can then use the tree to price other options
- Suppose $S_0 = 1502.39, K = 1500, \sigma = 12.36\%, r = 4.713\%, \delta = 1.91\%$, and $T = 0.12$.
  - $u = e^{\sigma\sqrt{T}} = 1.0437$ and $d = 1/u = 0.9581$.
  - Risk neutral probability, $p = \frac{e^{(r-\delta)T} - d}{u-d} = 0.5286$
  - $c = 28.394$, but market price $c^{mkt} = 20.35$
  - So, adjust σ (meaning adjust u and S_1) so that $c = c^{mkt}$.
  - Turns out: $\sigma = 8.24\%$, i.e., $p = 0.5446$, makes $c = c^{mkt}$.
  - Using the new values, we can price other options.