Ministério da Educação
UNIVERSIDADE FEDERAL DO PARANÁ
Setor de Ciências Exatas

AN 2

Nota:

Disciplina:

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Curso:

Professor:

Aluno:

Turma:

Data: / /

① Coord. esféricas

$$\begin{cases} x = \rho \operatorname{sen}\phi \cos\theta \\ y = \rho \operatorname{sen}\phi \operatorname{sen}\theta \\ z = \rho \cos\phi \end{cases}$$

Domínio de Integração

$1 \leq \rho \leq 2$

$0 \leq \phi \leq \pi/4$

$0 \leq \theta \leq 2\pi$

$\Rightarrow dV = \rho^2 \operatorname{sen}\phi \, d\rho \, d\phi \, d\theta$

$$\int_0^{2\pi}\int_0^{\pi/4}\int_1^2 \frac{\rho^2\cos^2\phi}{(\rho^2)^{3/2}} \rho^2 \operatorname{sen}\phi \, d\rho \, d\phi \, d\theta$$

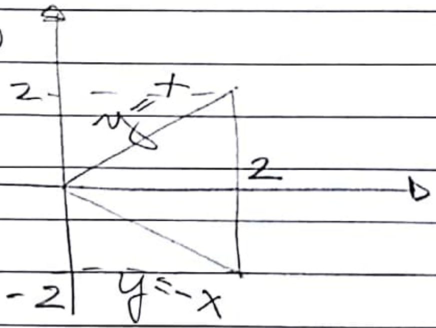
$$= \int_0^{2\pi}\int_0^{\pi/4}\int_1^2 \rho \cos^2\phi \operatorname{sen}\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_1^2 \rho \, d\rho \int_0^{\pi/4} \underbrace{\cos^2\phi}_{u^2} \underbrace{\operatorname{sen}\phi \, d\phi}_{-du} \int_0^{2\pi} d\theta = \left[\frac{\rho^2}{2}\Big|_1^2\right] \cdot \int_1^{\sqrt{2}/2} -u^2 \, du \cdot [2\pi]$$

$$= \left(\frac{4}{2} - \frac{1}{2}\right)\left(-\frac{u^3}{3}\Big|_1^{\sqrt{2}/2}\right)(2\pi) = \frac{3}{2} \cdot \left[-\frac{1}{3}\left(\frac{\cancel{2}\sqrt{2}}{4\,\cancel{2}} - 1\right)\right] \cdot 2\pi$$

$$= \frac{\cancel{3}}{\cancel{2}}\left(-\frac{1}{\cancel{3}}\right)\left(\frac{\sqrt{2}}{4} - 1\right)(\cancel{2}\pi) = \pi\left(1 - \frac{\sqrt{2}}{4}\right)$$

②



Teorema de Green:

$$\int_C F \cdot dr = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Assim, $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = \dfrac{\partial}{\partial x}(x^2 e^y) - \dfrac{\partial}{\partial y}(0) = 2xe^y$

Teorema de Green:

$$\iint_S 2xe^y \, dA = \int_0^2 \int_{-x}^{x} 2xe^y \, dy\, dx = 2\int_0^2 \int_{-x}^{x} xe^y \, dy\, dx$$

$$= 2\int_0^2 xe^y \Big|_{-x}^{x} dx = 2\int_0^2 x(e^x - e^{-x})\, dx = 2\left[\int_0^2 xe^x dx - \int_0^2 xe^{-x} dx\right]$$

$u = x \quad dv = e^x dx \quad | \quad u = x \quad dv = e^{-x}$

$du = dx \quad v = e^x \quad | \quad du = dx \quad v = -e^{-x}$

$$= 2\left[\left(xe^x - \int_0^2 e^x dx\right) - \left(-xe^{-x} + \int e^{-x} dx\right)\right] =$$

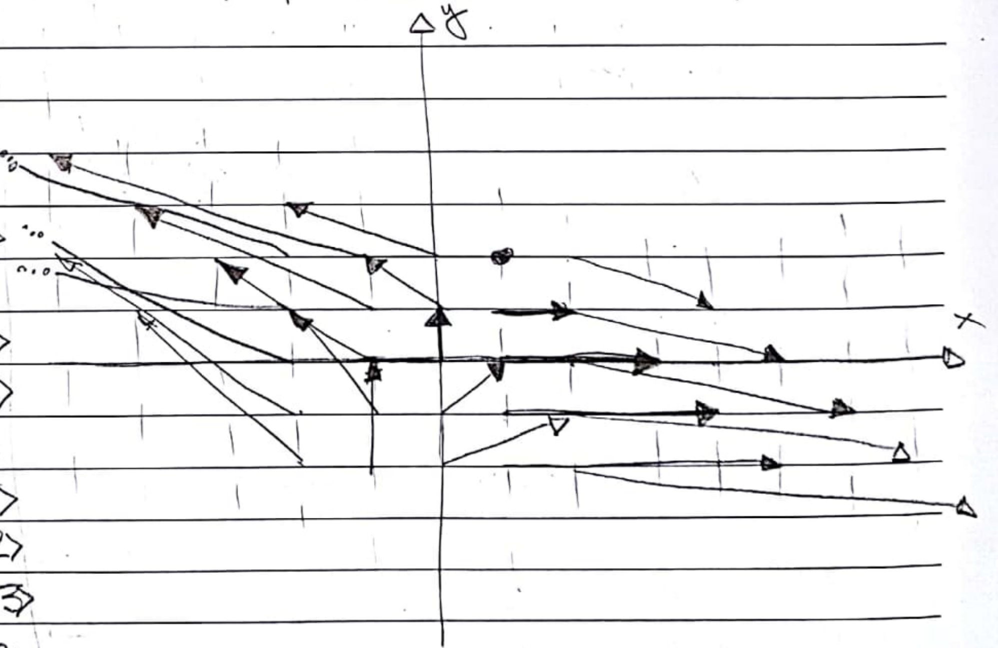
$$= 2\left(xe^x - e^x + xe^{-x} - e^{-x}\right)\Big|_0^2 = 2\left[\left(2e^2 - e^2 + 2e^{-2} - e^{-2}\right) - (0 - 1 + 0 - 1)\right]$$

$$= 2\left(e^2 + e^{-2} + 2\right)$$

③

a)

(x, y)	F	(x, y)	F	(x, y)	F
(0, 0)	⟨0, 1⟩	(−2, 0)	⟨−4, 3⟩	(0, −2)	⟨2, 1⟩
(1, 0)	⟨2, 0⟩	(−2, −1)	⟨−3, 3⟩	(1, −2)	⟨4, 0⟩
(1, 1)	⟨1, 0⟩	(−2, −2)	⟨−2, 3⟩	(2, −2)	⟨6, −1⟩
(0, 1)	⟨−1, 1⟩	(−1, −2)	⟨0, 2⟩	(2, −1)	⟨5, −1⟩
(−1, 1)	⟨−3, 2⟩				
(−1, 0)	⟨−2, 2⟩				
(−1, −1)	⟨−1, 2⟩				
(0, −1)	⟨1, 1⟩				
(1, −1)	⟨3, 0⟩				
(2, 0)	⟨4, −1⟩				
(2, 1)	⟨3, −1⟩				
(2, 2)	⟨2, −1⟩				
(1, 2)	⟨0, 0⟩				
(0, 2)	⟨−2, 1⟩				
(−1, 2)	⟨−4, 2⟩				
(−2, 2)	⟨−6, 3⟩				
(−2, 1)	⟨−5, 3⟩				



b) Note que $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -1-(-1) = 0 \Rightarrow F$ é conservativo

$f_x = 2x - y \Rightarrow f = \int (2x-y)\,dx = x^2 - yx + g(y)$

$f_y = -x + g'(y) = 1 - x \Rightarrow g'(y) = 1 \Rightarrow g(y) = \int dy = y + C$

$\therefore f = x^2 - yx + y + C$

TFIL: $\int_C F \cdot dr = \int_C \nabla f \cdot dr = f\left(\begin{smallmatrix}\text{pto}\\ \text{final}\end{smallmatrix}\right) - f\left(\begin{smallmatrix}\text{pto}\\ \text{inicial}\end{smallmatrix}\right)$

$f\left(\begin{smallmatrix}\text{pto}\\ \text{final}\end{smallmatrix}\right) = f(r(1)) = f(\langle 1, e\rangle) = 1 - e + e + C$

$f\left(\begin{smallmatrix}\text{pto}\\ \text{inicial}\end{smallmatrix}\right) = f(r(0)) = f(\langle 0, 1\rangle) = 0 - 0 + 1 + C$

TFIL: $\int_C F \cdot dr = (1+C) - (1+C) = 0$ //

c) Como F é conservativo e a curva C dada pela circunferência $(x-3)^2 + (y-2)^2 = 2^2$ é uma curva fechada, então pelo TFIL,

$\int_C F \cdot dr = 0$ //

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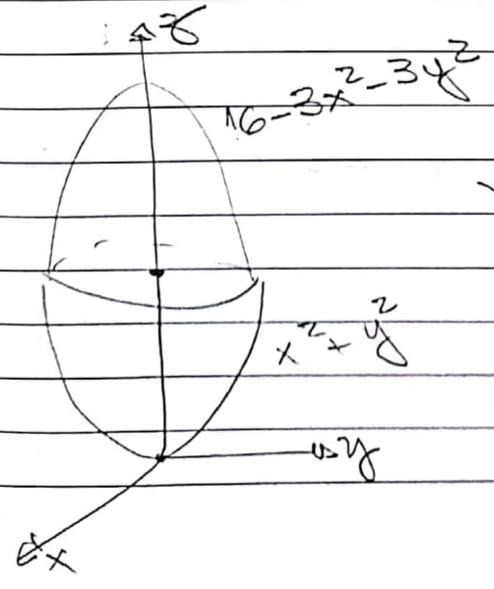


$$\iint_S \int_{x^2+y^2}^{16-3x^2-3y^2} 1 \cdot dz\,dA =$$

$$= \iint_S (\underbrace{16-3x^2-3y^2-x^2-y^2}_{16-4x^2-4y^2})\,dA$$

A intersecção dos dois paraboloides é:

$16-3x^2-3y^2 = x^2+y^2 \Rightarrow 16 = 4x^2+4y^2$

$\Rightarrow \underbrace{x^2+y^2=4}_{S} \Rightarrow$ circunf. de centro $(0,0)$, $r=2$

$$\iint_S 16-4(x^2+y^2)\,dA = \int_0^{\pi/2}\int_0^2 (16-4r^2)\,r\,dr\,d\theta =$$

$$= 4\int_0^{\pi/2}\int_0^2 (4-r^2)\,r\,dr\,d\theta = 4\int_0^{\pi/2} d\theta \int_0^2 (4r-r^3)\,dr$$

$$= 4\cdot\frac{\pi}{2}\left(\frac{4r^2}{2}-\frac{r^4}{4}\right)\Big|_0^2 = 2\pi\cdot\left(2r^2-\frac{r^4}{4}\right)\Big|_0^2 = 2\pi\cdot\left(8-\frac{16}{4}\right) = 8\pi$$

$8-4=4$

⑤ $$\iint_S \text{rot}\,F\cdot dS = \iint_S \text{rot}\,F\cdot(r_x\times r_y)\,dA$$

onde $\text{rot}\,F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2z & y^2 & xy \end{vmatrix} = (x-0)\vec{i}-(y-x^2)\vec{j}+(0)\vec{k} = \langle x, x^2-y, 0\rangle$

onde $r(x,y) = \begin{cases} x=x \\ y=y \\ z=1-x^2-y^2 \end{cases} \Rightarrow r_x=\langle 1,0,-2x\rangle$, $r_y=\langle 0,1,-2y\rangle$

$$r_x\times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix} = 2x\vec{i}-(-2y)\vec{j}+\vec{k} = \langle 2x, 2y, 1\rangle$$

Daí temos $(\text{rot}\,F)\cdot(r_x\times r_y) = 2x^2+2x^2y-2y^2$

com $S \Rightarrow x^2+y^2=1$

$$\iint_S \text{rot}\,F\cdot dS = 2\int_0^{2\pi}\int_0^1 (r^2\cos^2\theta + r^3\cos^2\theta\,\text{sen}\,\theta - r^2\text{sen}^2\theta)\,r\,dr\,d\theta$$

$$=2\int_0^{2\pi}\int_0^1 (r^2\cos 2\theta + r^3\cos^2\theta\,\text{sen}\,\theta)\,r\,dr\,d\theta$$

$$=2\left[\int_0^{2\pi}\int_0^1 r^3\cos 2\theta\,dr\,d\theta + \int_0^{2\pi}\int_0^1 r^4\cos^2\theta\,\text{sen}\,\theta\,dr\,d\theta\right]$$

$$= 2\left[\int_0^{2\pi} \cos 2\theta \, d\theta \int_0^1 r^3 dr + \int_0^{2\pi} \underbrace{\cos^2\theta}_{u^2} \underbrace{\operatorname{sen}\theta \, d\theta}_{du} \int_0^1 r^4 dr\right]$$

$$= 2\left(\frac{\operatorname{sen} 2\theta}{2}\Big|_0^{2\pi} \cdot \frac{r^4}{4}\Big|_0^1 + \left(\frac{-u^3}{3}\right)\Big|_1^1 \cdot \frac{r^5}{5}\Big|_0^1\right)$$

$$= 2(0+0) = 0$$

Outro jeito de fazer é usando Teor. Stokes:

$$\iint_S \operatorname{rot} F \cdot dS = \int_C F \cdot dr = \int_C F \cdot r'(t) dt =$$

Assim $S \Rightarrow x^2+y^2=1$ no plano $z=0$

$$\Rightarrow r(\theta) = \begin{cases} x = \cos\theta \\ y = \operatorname{sen}\theta \\ z = 0 \end{cases} \Rightarrow r'(t) = \langle -\operatorname{sen}\theta, \cos\theta, 0\rangle$$

$0 \leq \theta \leq 2\pi$

$$F(r(\theta)) = (\cos^2\theta \cdot 0)\vec{i} + (\operatorname{sen}^2\theta)\vec{j} + (\cos\theta \operatorname{sen}\theta)\vec{k}$$

$$\text{Daí } F \cdot r'(\theta) = \int_0^{2\pi} \operatorname{sen}^2\theta \cos\theta \, d\theta = \int_0^0 u^2 du = 0$$

⑥ Fluxo:

$$\iint_S F \cdot dS = \iint_S F \cdot (r_u \times r_v) dA$$

$$\text{Elipse } r(\theta, z) = \begin{cases} x = 2\cos\theta \\ y = \operatorname{sen}\theta \\ z = z \end{cases} \Rightarrow r_\theta = \langle -2\operatorname{sen}\theta, \cos\theta, 0\rangle$$

$$r_z = \langle 0, 0, 1\rangle$$

$0 \leq \theta \leq 2\pi$

$0 \leq z \leq 2$

$$r_\theta \times r_z = \begin{vmatrix} i & j & k \\ -2\operatorname{sen}\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \cos\theta \,\vec{i} - (-2\operatorname{sen}\theta)\vec{j} + 0\vec{k}$$

$$\langle \cos\theta, 2\operatorname{sen}\theta, 0\rangle$$

$$F(r(\theta, z)) = 2z\cos\theta\,\vec{i} + \operatorname{sen}\theta\, z\,\vec{j} + 2018z\,\vec{k}$$

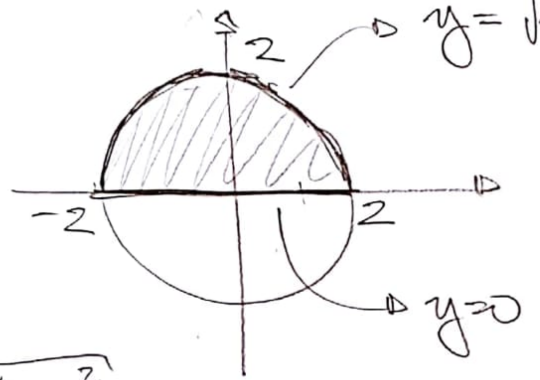
$$F \cdot (r_\theta \times r_z) = 2z\cos^2\theta + 2\operatorname{sen}^2\theta\, z = 2z$$

$$\iint_S 2z\,dA = \int_0^{2\pi}\int_0^{2} 2z\,dz\,d\theta = 2\int_0^{2} z\,dz \int_0^{2\pi} d\theta =$$

$$= \cancel{2}\,\frac{z^2}{\cancel{2}}\Big|_0^2 \cdot 2\pi = 4\cdot 2\pi = 8\pi$$

(7) Região de Integração:

$y = \sqrt{4-x^2} \Rightarrow x^2+y^2=4$



$$\int_{-2}^{2}\int_0^{\sqrt{4-x^2}} y\,dy\,dx = \int_0^{\pi}\int_0^{2} r\,\text{sen}\,\theta\; r\,dr\,d\theta$$

$$= \int_0^{\pi} \text{sen}\,\theta\,d\theta \int_0^{2} r^2\,dr = -\cos\theta\Big|_0^{\pi} \cdot \frac{r^3}{3}\Big|_0^2$$

$$= \cos\theta\Big|_{\pi}^{0} \cdot \frac{r^3}{3}\Big|_0^2 = [1-(-1)]\left(\frac{8}{3}-0\right) = 2\cdot\frac{8}{3} = \frac{16}{3}$$