

$\beta \in (0, 2]$

Claim 1: $\int (e^{f+\varepsilon u} - 1) \omega_0^n = 0$

$$\text{pf. } \int \eta_R (e^{f+\varepsilon u} - 1) \omega_0^n = n \int \eta_R d\bar{u} \wedge T = -n \int d\bar{u} \wedge d\bar{\eta}_R \wedge T$$

$$|\eta_R| \lesssim \frac{1}{R} \Rightarrow |\text{LHS}| \lesssim \frac{1}{R} \int |\nabla u| \omega_0^n \quad (\omega_0 \approx \omega, \text{ but dep on } \varepsilon)$$

We want to show $\int |\nabla u| \omega_0^n < +\infty$

Recall: $\int \rho^k |u|^{p-2} |\nabla u|^2 + \int \rho^k |u|^p < +\infty$

$$\left. \begin{array}{l} k=3 \Rightarrow \int \rho^{-k} < +\infty \\ p=2 \Rightarrow \int \rho^{-k} |\nabla u|^2 < +\infty \end{array} \right\} \Rightarrow \int |\nabla u| \leq \left(\int \rho^k |\nabla u|^2 \right)^{1/2} \left(\int \rho^{-k} \right)^{1/2} < +\infty \quad \#$$

Claim 2: $\exists C > 0$ indep of ε s.t. $\left(\int \rho^{\alpha(\beta-2-\delta)-\beta} |u-u_{\psi_\delta}|^{2d} \right)^{1/2d} \leq C(\varepsilon)$ for α close to 1

pf Put $v = u - u_{\psi_\delta}$

$$u \times \textcircled{+} \Rightarrow \int u (e^{f+\varepsilon u} - 1) \omega_0^n = n \int u d\bar{u} \wedge T = -n \int d\bar{u} \wedge d\bar{u} \wedge T$$

$$\Rightarrow \int d\bar{u} \wedge d\bar{v} \wedge T = \int u (1 - e^f) \omega_0^n + \int u (1 - e^{\varepsilon u}) e^f \omega_0^n, \quad \text{recall: } u(e^{\varepsilon u} - 1) \gtrsim \varepsilon u^2 > 0$$

$$\Rightarrow \int |\nabla v|^2 \omega_0^n \leq \int |v| |e^f - 1| \omega_0^n \quad \text{since } \int u(1 - e^f) = \int (u - u_{\psi_\delta})(1 - e^f) \omega_0^n$$
$$\int (e^f - 1) \omega_0^n = 0$$

SOB(β), $\beta \in (0, 2]$

$$\hookrightarrow \left(\int \rho^{\alpha(\beta-2-\delta)-\beta} |v|^{2d} \omega_0^n \right)^{1/2d} \leq C(\varepsilon) \int |\nabla v|^2 \omega_0^n \leq C_\delta \int |v| |e^f - 1| \omega_0^n \leq C_\delta \int |v| \rho^{-\mu} \omega_0^n$$

$$\int |v| \rho^{-\mu} \leq \left(\int |v|^{2d} \rho^{m\eta} \right)^{1/m} \left(\int \rho^{-m^*\eta} \rho^{-m^*\mu} \right)^{1/m^*}$$

$$\text{need } m=2d, m\eta = \alpha(\beta-2-\delta)-\beta \Rightarrow m^* = \frac{2d}{2d-1}, \eta = \frac{\alpha(\beta-2-\delta)-\beta}{2d}$$

$$\left(\int \rho^{\alpha(\beta-2-\delta)-\beta} |v|^{2d} \right)^{1/2d} \lesssim \left(\int \rho^{-m^*(\eta+\mu)} \right)^{1/m^*}$$

$$\text{If } m^*(\eta+\mu) \geq \beta + \delta' \Rightarrow \left(\int \rho^{\alpha(\beta-2-\delta)-\beta} |v|^{2d} \right)^{1/2d} \leq C_{\delta, \delta'}$$

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$$\alpha(\beta-2-\delta) - \beta + 2d\mu \geq (2d-1)(\beta+\delta)$$

$$\Leftrightarrow \alpha(2(\beta+\delta) - \beta + 2 + \delta - 2\mu) \leq \delta', \text{ true for } \alpha \text{ close to 1}$$

Lemma 1 \exists constant $C > 0$ indep of ε ($\varepsilon \in \text{supp } f$) s.t.

(1) If $u_{\psi_\delta} \geq 0$, $-C \leq u \leq u_{\psi_\delta} + C$

(2) If $u_{\psi_\delta} \leq 0$, $-C + u_{\psi_\delta} \leq u \leq C$

pf. We show (1) via Moser iteration.

Set $v_+ = (u - u_{\psi_\delta})_+$.

When $v_+ > 0$, $u > u_{\psi_\delta} \geq 0$

$\Rightarrow v_+ (e^{\varepsilon u} - 1) > 0$ on M

$$\hookrightarrow \int |\nabla v_+|^{p_2} \omega_0^n \leq \frac{np^2}{4(p-1)} \int |v_+|^{p-1} |e^f - 1| \omega_0^n$$

SOB(β), $\beta \leq 2$

$$\begin{aligned} & \hookrightarrow \left(\int \rho^{\alpha(\beta-2-\delta)-\beta} |v_+|^{\frac{p}{2}} - (|v_+|^{\frac{p}{2}})_{\psi_\delta} |^{2\alpha} \right)^{\frac{1}{2\alpha}} \leq C_\delta \int |\nabla v_+|^{\frac{p}{2}} \omega_0^n \\ & \leq C_\delta \frac{np^2}{4(p-1)} \int |v_+|^{\frac{p}{2}} |e^f - 1| \omega_0^n \\ & \leq C_\delta \frac{np^2}{4(p-1)} \int |v_+|^{\frac{p}{2}} \rho^{-\mu} \omega_0^n \end{aligned}$$

Note: Set $\|g\|_{p,\alpha,\delta} := \left(\int |g|^p \rho^{\alpha(\beta-2-\delta)-\beta} \right)^{\frac{1}{p}}$

$$\begin{aligned} & \|(v_+)^{\frac{p}{2}}\|_{2\alpha,\alpha,\delta} = \left(\int_M \left\{ \int |v|^{\frac{p}{2}} \cdot \psi_\delta \times \left(\int \psi_\delta \right)^{-1} \right\}^{2\alpha} \cdot \rho^{\alpha(\beta-2-\delta)-\beta} \right)^{\frac{1}{2\alpha}} \\ & = \left(\int \psi_\delta \right)^{-1} \left(\int \rho^{\alpha(\beta-2-\delta)-\beta} \right)^{\frac{1}{2\alpha}} \underbrace{\frac{\int |v|^{\frac{p}{2}} \cdot \psi_\delta}{\left(\int \frac{\psi_\delta^2}{\rho^{\alpha(\beta-2-\delta)-\beta}} \right)^{\frac{1}{2}}} \leq \|v\|_{p,\alpha,\delta}^{\frac{p}{2}}} \\ & \text{product of these 3} \approx \frac{C}{\eta^{\alpha(2-\beta-\delta)}} \text{ if } 1 \leq \alpha < 2 - \frac{\eta}{2-\beta+\delta} \end{aligned}$$

$$\Rightarrow \|v_+\|_{\alpha p, \alpha, \delta} \leq \left(\frac{C_{\alpha, \delta, \eta} p^2}{p-1} \right)^{\frac{1}{p}} \|v_+\|_{p, \alpha, \delta}^{1-\frac{1}{p}} + C_{\alpha, \delta, \eta}^{\frac{1}{p}} \|v_+\|_{p, \alpha, \delta}$$

$$\text{w/ } \alpha(2-\beta+\delta) \leq \mu-\beta$$

Then we can iterate from $p_0 = 2\alpha$ (Claim 2) $\Rightarrow |v_+| \leq C \eta^{\alpha(2-\beta+\delta)}$
 $\Rightarrow u \leq u_{\psi_\delta} + C$

To get " $-C \leq u$ ", we replace v_+ in the above argument by $u_- = \min\{u, 0\}$

$$\Rightarrow u_- \cdot (e^{\xi u} - 1) \geq 0 \quad \text{... (a)}$$

Similarly by SOB(β), $\beta \leq 2$, we have

$$\begin{aligned} & \left(\int \rho^{\alpha(\beta-2-\delta)-\beta} |u_-|^{\frac{p}{2}} - (|u_-|^{\frac{p}{2}})_{\psi_\delta} |^{2\alpha} \right)^{\frac{1}{2\alpha}} \leq C_\delta \int |\nabla |u_-|^{\frac{p}{2}}|^2 \omega_0^n \\ & \leq C_\delta \frac{np^2}{4(p-1)} \int |u_-|^{p-1} |e^f - 1| \omega_0^n \quad \text{... (b)} \end{aligned}$$

With (a)+(b), argue as before

$$\Rightarrow \|u_-\|_{\alpha p, \alpha, \delta} \leq \left(\frac{C_{\alpha, \delta, \eta} p^2}{p-1} \right)^{\frac{1}{p}} \|u_-\|_{p, \alpha, \delta}^{1-\frac{1}{p}} + C_{\alpha, \delta, \eta}^{\frac{1}{p}} \|u_-\|_{p, \alpha, \delta}$$

$$\text{with } \alpha(2-\beta+\delta) \leq \mu-\beta$$

$$\text{Since } u_{f_0} \geq 0 \Rightarrow (u - u_{f_0})_- \leq u_-$$

$$\text{Claim 2: } \left(\int \rho^{\alpha(\beta-2-\delta)-\beta} |u - u_{f_0}|^{2\alpha} \right)^{1/2\alpha} \leq C_\delta \quad \left. \right\} \quad \left(\int \rho^{\alpha(\beta-2-\delta)-\beta} |u|^{2\alpha} \right)^{1/2\alpha} \leq C_\delta$$

$$\|u - u_{f_0}\|_{2\alpha, \alpha, \delta}$$

↪ Moser iteration $\Rightarrow \|u\| \leq C_{\text{Moser}}$

Lemma (Yau's C^2 -estimate) $\exists C_1, C_2 > 0$ independant of $\text{supp}(f)$, ε such that
 $0 = \text{tr}_{\omega_0} \bar{\omega} = n + \Delta_{\omega_0} u \leq C_1 \exp(C_2(u - \inf u))$

Now we consider $f = f_m$ with $\text{supp}(f_m) \nearrow M$, $\int (e^{f_m} - 1) \omega_0^n = 0$

Lemma 2: For each m , $\exists C(m)$ indep of ε st $|u_{m,\varepsilon}| \leq C(m)$

pf. Claim 1: $\int (e^{f_m + 2u_{m,\varepsilon}} - 1) \omega_0^n = 0 \quad \left. \right\} \Rightarrow \int e^{f_m} (e^{u_{m,\varepsilon}} - 1) \omega_0^n = 0$

Assumption: $\int (e^{f_m} - 1) \omega_0^n = 0 \quad \Rightarrow \inf u < 0 < \sup u$

Claim: $\exists x_{\max}, x_{\min} \in \text{supp}(f_m)$ st. $u(x_{\max}) = \sup u$, $u(x_{\min}) = \inf u$

If not either ① $\sup u$ was attained at $x_{\max} \notin \text{supp}(f_m)$

or ② $\exists \{x_k\}_k$, $u(x_k) \rightarrow \sup u$, $x_k \rightarrow \infty$

① at x_{\max} $(\omega_0 + dd^c u)^n(x_{\max}) = \underbrace{e^{\varepsilon u(x_{\max})}}_{\omega_0^n(x_{\max})} > \omega_0^n(x_{\max})$ contradiction

② Similar argument + Yau's max prin
 Same for $\inf u$

If $u_{f_0} \geq 0$, from Lemma 1: $-C \leq u \leq u_{f_0} + C$

Set $V = -\min\{u - \inf u - 1, 0\} \Rightarrow \begin{cases} V(x_{\min}) = 1 \\ 0 \leq V \leq 1 \end{cases}$

$B_r(x_{\min}) =: B_r$, $G(x, y)$: Dirichlet Green's function on B_1

$\eta = \begin{cases} 1 & \text{on } B_{1/2} \\ 0 & \text{outside } B_{3/4} \end{cases}$ a cutoff

Since $-\inf u \leq C$, Yau's C^2 : $n + \Delta u \leq C_1 \exp(C_2(u - \inf u)) \leq C'_1 \exp(C_2 u)$

$$\star \times \eta^2 G(x_{\min}, x) V(x) \quad n - \Delta V \quad \cdots \star$$

\star (note $G(x) > 0$ on $B_{3/4}$)

$$\Rightarrow - \int_{B_1} \Delta V \cdot \eta^2 G \cdot V \leq \int_{B_1} \eta^2 G C'_1 \exp(C_2 u)$$

$$\leq C \int_{B_1} \eta^2 G V \quad \begin{matrix} = u \leq \inf u + 1 \text{ when } V > 0 \\ < 1 \end{matrix}$$

$$\begin{aligned}
 -\int \Delta v \cdot \eta^2 \cdot v \cdot G &= \int \nabla v \cdot \eta^2 v \cdot \nabla G + \int |\nabla v|^2 \eta^2 G + \int v \nabla v \cdot \nabla \eta^2 v \cdot G \\
 &= \frac{1}{2} \int \nabla v^2 \eta^2 \nabla G
 \end{aligned}$$

$$\begin{aligned}
 \text{Hence } \frac{1}{2} \int \nabla(\eta^2 v^2) \cdot \nabla G &= \frac{1}{2} \int \nabla v^2 \cdot \nabla \eta^2 \cdot \nabla G + \frac{1}{2} \int v^2 \nabla \eta^2 \cdot \nabla G \\
 \frac{1}{2} (\eta v)^2 (x_{\min}) &= - \int \Delta v \cdot \nabla \eta^2 \cdot \nabla G - \int |\nabla v|^2 \eta^2 G - \frac{1}{2} \int \nabla v^2 \cdot \nabla \eta^2 G + \frac{1}{2} \int v^2 \nabla \eta^2 \cdot \nabla G \\
 &= \int v^2 \nabla \eta^2 \cdot \nabla G + \frac{1}{2} \int v^2 \Delta \eta^2 G \\
 \frac{1}{2} &\leq C \int_{B_1} v \eta^2 G + C \cdot \int_{B_1} v^2 \\
 &\quad \text{L } G, \nabla G \text{ are under control on } B_{3/4} \setminus B_{1/2}
 \end{aligned}$$

$$\begin{aligned} \stackrel{\text{H\"older}}{\Rightarrow} \quad 1 &\leq C \left(\left(\int_{B_1} |v|^p \right)^{\frac{1}{p}} \left(\int_{B_1} |G|^{p'} \right)^{\frac{1}{p'}} + \int_{B_1} v^2 \right) \\ &\leq v_0 \Big|_{(\text{supp}(v) \cap B_1)} \quad \leq v_0 \Big|_{(\text{supp}(v) \cap B_1)} \quad \text{since } |v| \leq 1 \end{aligned}$$

$$\Rightarrow \text{vol}(\text{supp}(v) \cap B_1) \geq \gamma_c.$$

$$\text{Then } \left(\int \psi_s \omega_0^n \right) u_{\psi_s} = \int_M \psi_s u \omega_0^n = \underbrace{\int_{M \setminus B_1 \cap \text{supp}(v)} \psi_s u}_{(I)} + \underbrace{\int_{B_1 \cap \text{supp}(v)} \psi_s u}_{(II)}$$

$$(I) \leq \sup_{\Omega} u \left(\int \psi_\delta \omega_0^n - \int_{B_1 \cap \text{supp}(u)} \psi_\delta \omega_0^n \right)$$

$$(II) \approx p(x_{min})^{-2-\delta} \times (\inf u+1) \leq C$$

Recall: $u < u_{fs} + C$

$$\text{vol}(B_1 \cap \text{supp}(v)) \geq \frac{1}{C}$$

$$\sup U \lesssim \frac{C}{\int_{B_1 \cap \text{supp}(v)} \psi_s \omega_0^n}, \quad \int_{B_1 \cap \text{supp}(v)} \psi_s \omega_0^n \geq \frac{1}{C} \rho^{-2-\varepsilon}(x_{\min})$$

$$\Rightarrow \sup u \leq C \cdot \rho^{2+\varepsilon} \left(\sup \{ \text{dist}(y, x_0) \mid y \in \text{supp}(f_m) \} \right)$$

Now for each m , we have $|U_{m,\varepsilon}| \leq C(m) \rightarrow$ can do $C^2 + \text{higher order est. (dep on } m)$

$$\Rightarrow u_{m,\varepsilon} \rightarrow u_m \not\rightarrow |u_m| \leq C(m)$$

$$\text{Rmk: } \int_M |\nabla u_m|^2 \omega_0^n < +\infty \quad : \quad \int u_{m,\varepsilon} \cdot dd^c u_{m,\varepsilon} \wedge T = \int u_{m,\varepsilon} (e^{f_m + \varepsilon u_{m,\varepsilon}} - 1) \omega_0^n$$

$$\Rightarrow \int |\nabla u_{m,\varepsilon}|^2 \omega_0^n \leq C \int |u_{m,\varepsilon}| |e^{f_m} - 1| \omega_0^n < +\infty$$

integration by parts formula as before.

Now, we want to show $u_m - (u_m)_{\mathbb{M}_S} \xrightarrow[m \rightarrow \infty]{} u$: sol'n to (MA)

Only need to show $\|u_m - (u_m)_{\mathbb{M}_S}\|_{L^\infty} \leq C_{\text{unif}}$

$u_m - (u_m)_{\mathbb{M}_S}$ is also a sol'n to $(\omega_0 + dd^c u_m)^n = e^{\int u_m} \omega_0^n$, we can assume $(u_m)_{\mathbb{M}_S} = 0$

To do Moser iteration as [Lemma 1](#), we have

$$\int \mathfrak{S} |\nabla u|^{p_2} \omega_0^n = \frac{-n p^2}{4(p-1)} \left\{ \int \mathfrak{S} u |\nabla u|^{p-2} (e^{-1}) \omega_0^n + \int u |\nabla u|^{p-2} d\mathfrak{S} \wedge d^c u \wedge T \right\}$$

To make sure RHS is finite for $\mathfrak{S} = \chi(\frac{\cdot}{R})$ as $R \rightarrow \infty$, we need to control

$$\int |\nabla \chi_R| |\nabla u| \omega_0^n \leq \frac{1}{R} \int_{B_{2R} \setminus B_R} |\nabla u| \rightarrow 0 \text{ as } R \rightarrow \infty$$

Claim: $\lim_{R \rightarrow \infty} \frac{1}{R} \int_{B_{2R} \setminus B_R} |\nabla u| \omega_0^n = 0$

Indeed, $\int_{B_{2R} \setminus B_R} |\nabla u| \omega_0^n \leq \left(\int_{B_{2R} \setminus B_R} |\nabla u|^2 \right)^{1/2} \text{vol}(B_{2R} \setminus B_R)^{1/2}$

$$\text{vol}(B_{2R} \setminus B_R)^{1/2} \sim R^{\frac{\beta}{2}} \quad = \alpha_R \xrightarrow[R \rightarrow \infty]{} 0$$

$$\hookrightarrow \frac{1}{R} \int_{B_{2R} \setminus B_R} |\nabla u| \leq R^{\frac{\beta}{2}-1} \alpha_R \xrightarrow[R \rightarrow \infty]{} 0 \quad \text{for } \beta \leq 2$$