Analysis for the complex Monge-Ampère equation

Let (M,ω_0) be a complete non-compact Kähler manifold satisfying SOB(β), $\beta>0$
& with $C^{3,\alpha}$ quasi-atlas

Tian-Yau-Hein package

Let $f\in C^{2,\alpha}(M)$ s.t. $|f|\le C r^{-\mu}$ on $\{r>1\}$ for some $\mu>2$

If $\begin{cases}\beta>2 \\ \beta\le 2 \ \& \ \int (e^f-1)\omega_0^n=0\end{cases}$, then $\exists\ \alpha\in(0,\bar\alpha]$, $u\in C^{4,\bar\alpha}(M)$ s.t. $(\omega_0+dd^c u)^n=e^f\omega_0^n$

Moreover for $\beta\le 2$, $\int |\nabla u|^2\omega_0^n<+\infty$

Independent of the value of β, if $\underset{k\ge 3}{f\in C^{k,\alpha}_{loc}} \Rightarrow u\in C^{k+2,\alpha}_{loc}$

Defn: Let (M,ω_0) be a complete Kähler mfd.
A $C^{k,\alpha}$ quasi-atlas for (M,ω_0) is a collection $\{\Phi_x,\ x\in A\}$, $A\subset M$ of loc. holo. diffeo
$\Phi_x: B\to M$, $\Phi_x(0)=x$ & $\exists\ C\ge 1$ with $\begin{cases}\mathrm{inj}(\Phi_x^* g_0)\ge 1/C \\ \frac{1}{C}g_{euc}\le \Phi_x^* g_0\le C g_{euc} \quad \forall x\in A \\ \|\Phi_x^* g_0\|_{C^{k,\alpha}(B,g_{euc})}\le C\end{cases}$
$B(0,1)\subset\mathbb{C}^n$ (with $B = B(0,1)$)

& $\forall y\in M$, $\exists\ x\in A$ s.t. $y\in\Phi_x(B)$, $\mathrm{dist}_{g_0}(y,\partial\Phi_x(B))\ge 1/C$

↳ Given $C^{k,\alpha}$ quasi-atlas, define $\|u\|_{C^{l,r}(M)}:=\sup_{x\in A}\{\|u\circ\Phi_x\|_{C^{l,r}(B)}\}$

Schauder: for a 2nd order (unif) elliptic operator L with coeff $\in C^{l,r}$
$$\|u\|_{C^{l+2,r}(M)}\le C\left(\|Lu\|_{C^{l,r}(M)}+\|u\|_{L^\infty}\right)\quad\forall\ u\in C^\infty_{loc}(M)$$
(if $l\le k-1$, $r\in(0,\alpha]$)
↳ s.t. $C^{l,r}$-norm on B is compariable w/ $C^{l,r}$-norm on B induced by $\Phi_x^* g_0$

Lemma (Tian-Yau) If $|Rm|\le C$, then $\exists\ C^{1,\alpha}$ quasi-atlas $\forall\ \alpha$
If moreover $\sum_{i=1}^k|\nabla^i Scal|\le C$, $\exists\ C^{k+1,\alpha}$ quasi-atlas.
we discuss its pf after pf of TYH

§ Existence I : ε-perturbed equation $(\omega_0+dd^c u_\varepsilon)^n=e^{f+\varepsilon u}\omega_0^n$ ... (MA_ε)

Continuity method by Cheng-Yau

(M,ω_0) complete Kähler manifold with $C^{2,\alpha}$ quasi-atlas, $f\in C^{2,\alpha}$

Goal: Show $\exists\ u_\varepsilon\in C^{4,\bar\alpha}$, for some $\bar\alpha\in(0,\alpha]$ to (MA_ε)
↳ depend on M,ω_0,α,f, but not on ε.

Consider $(\omega_0+dd^c u_{\varepsilon,t})^n=e^{tf+\varepsilon u_{\varepsilon,t}}\omega_0^n$ ... $(MA_{\varepsilon,t})$

For $r\in(0,\alpha]$, $J_r=\{t\in[0,1]\mid (MA_{\varepsilon,t})$ admits a sol'n $\in C^{4,r}\}$

$0\in J_r$, want to prove openness & closedness

Lemma (Yau's maximum principle)
Let (M,g) be a complete mfd with sectional curvature bounded below, $x_0\in M$.
If $u\in C^2_{loc}$ with $|u|+|\nabla u|+|\nabla^2 u|\le C$ does not attain its supremum, then $\exists\ \{x_k\}\subset M$,
with $\mathrm{dist}(x_0,x_k)\to\infty$ & s.t. $u(x_k)\to\sup u$, $|\nabla u|(x_k)\to 0$, $\limsup_k \max\mathrm{spec}(\nabla^2 u)(x_k)\le 0$

Claim 1 : $\forall\, r\in(0,\alpha]$, $t\in J_r$, $\exists$ bdd linear operator $G: C^{2,r}\to C^{4,r}$ with $(\Delta_\omega-\varepsilon)\circ G = id$
where $\omega=\omega_0+dd^c u_{\varepsilon,t}$

$\hookrightarrow$ this gives the openness (linearized operator is an isomorphism $\rightsquigarrow$ implicit fcn thm is good)

pf. Since $\varepsilon>0$, $\forall$ bdd domain w/ sufficiently smooth bdry $\Omega\subset M$
$\exists\, u=u_\Omega\in C^{4,r}(\Omega)$ solves $\begin{cases}(\Delta-\varepsilon)u=f|_\Omega\\ u=0 \text{ on } \partial\Omega\end{cases}$

Schauder $\rightsquigarrow$ there are local estimates (w.r.t. charts of quasi-atlas)
for $\|u\|_{C^{4,r}}$ in terms of $\|f\|_{C^{2,r}}$ & $\|u\|_{L^\infty}$

Max prin : at max $x_{max}\in\Omega$, $f(x_{max}) = (\Delta-\varepsilon)u|_{x_{max}} \le -\varepsilon\, u(x_{max}) \Rightarrow u(x_{max}) \le -\frac{1}{\varepsilon}f(x_{max})$
similarly, $u(x_{min}) \ge -\frac{1}{\varepsilon}f(x_{min})$

$\Rightarrow \|u\|_{L^\infty} \le \frac{1}{\varepsilon}\|f\|_{L^\infty}$

Arzela-Ascoli : $\{u_{\Omega_k}\}\to u$ s.t. $(\Delta_\omega-\varepsilon)u=f$ on M
uniqueness : for $(\Delta-\varepsilon)u=0$ on M
if $x_{max}\in M \Rightarrow u(x_{max})\le 0$
$x_{min}\in M \Rightarrow u(x_{min})\ge 0$
| if sup not in M, use Yau's max prin
inf

#

Claim 2 : $\exists\,\bar\alpha\in(0,\alpha]$ s.t. if $u=u_{\varepsilon,t}\in C^4$ solves $(MA_{\varepsilon,t})$ for $t\in[0,1]$, $\varepsilon\in(0,1]$,
then $\|u\|_{C^{2,\bar\alpha}(M)} \le C(M,\omega_0,\|f\|_{C^{2,\alpha}(M)},\varepsilon)$

$\hookrightarrow$ this gives the closedness.

pf : By max prin (or Yau's max prin), $\|u\|_{L^\infty}\le\frac{1}{\varepsilon}\|f\|_{L^\infty}$

C^2 : Yau $0\le tr_{\omega_0}\omega = n+\Delta_{\omega_0}u \le C\cdot\exp(C(u-\inf u))$, $C=C(M,\omega_0,\|f\|_{C^2(M)})$ if $\varepsilon\le 1$
$\hookrightarrow \frac{1}{C}\omega_0\le\omega\le C\omega_0$
Bounded Δ + Linear theory $\Rightarrow \|u\|_{C^{1,r}(M)}\le C(r)$ $\forall\, r\in(0,1)$
Schauder $\rightsquigarrow u\in C^{2,\alpha}$ uniformly #

§ Existence 2 : the limit $\varepsilon\to 0$

Rmk : only need to make L^∞-estimate unif in ε
Step 1 : high integrability of the sol'ns (L^p might still blow up as $\varepsilon\to 0$)
Step 2 : Moser iteration

Recall : $\omega^n-\omega_0^n = dd^c u\wedge \overbrace{\sum_{k=0}^{n-1}(\omega^k\wedge\omega_0^{n-1-k})}^{=T}$ $\cdots$ ⊛

For $u=u_\varepsilon\in C^2(M)$ solving (MA_ε), $\zeta\in C_c^\infty(M)$

Step 1: ⊛ $\times\, \zeta u|u|^{p-2}$, $p>1$, $\zeta u|u|^{p-2}(e^{f+\varepsilon u}-1)\omega_0^n = \zeta u|u|^{p-2}dd^c u\wedge T$

$$\int \zeta u|u|^{p-2}dd^c u\wedge T = -\int u|u|^{p-2}d\zeta\wedge d^c u\wedge T - (p-1)\int \zeta|u|^{p-2}du\wedge d^c u\wedge T$$

$$\begin{aligned}\int \zeta\,|\nabla|u|^{\frac{p}{2}}|^2_{\omega_0}\,\omega_0^n &= n\frac{p^2}{4}\int \zeta|u|^{p-2}du\wedge d^c u\wedge\omega_0^{n-1}\\ &\le n\frac{p^2}{4}\int\zeta|u|^{p-2}du\wedge d^c u\wedge T\\ &= \frac{-np^2}{4(p-1)}\Big\{\int \underbrace{\zeta u|u|^{p-2}dd^c u\wedge T}_{=(e^{f+\varepsilon u}-1)\,\omega_0^n} + \int u|u|^{p-2}d\zeta\wedge d^c u\wedge T\Big\}\end{aligned}$$

$$\Rightarrow \int \zeta \left|\nabla |u|^{p/2}\right|^2_{\omega_0} \omega_0^n + \frac{np^2}{4(p-1)} \int \zeta \underline{u} |u|^{p-2} \underline{(e^{\varepsilon u}-1)}\, e^f \omega_0^n$$
$$\leq -\frac{np^2}{4(p-1)} \left\{ \int \zeta u |u|^{p-2}(e^f-1)\, \omega_0^n + \int u |u|^{p-2} d\zeta \wedge d^c u \wedge T \right\}$$

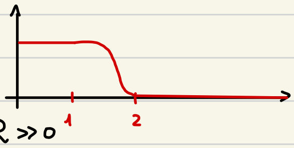
Note: $\forall C < \infty,\ \exists \delta = \delta(C) > 0$ s.t. if $|x| \leq C$, $x(e^x-1) \geq \delta x^2$

$\varepsilon \|u\|_{L^\infty} \leq \|f\|_{L^\infty} \Rightarrow u(e^{\varepsilon u}-1) \geq \delta \varepsilon u^2$

(1) High integrability

Schoen-Yau : if $\mathrm{Ric}(\omega_0) \geq -C \Rightarrow \exists$ smooth $\rho : M \to \mathbb{R}$ s.t. $\rho \sim 1 + \mathrm{dist}(x_0, \cdot)$ for any fixed x_0 & $|\nabla \rho| + \rho |\Delta \rho| \leq C$

Fix a cutoff $\chi : \mathbb{R}_+ \to \mathbb{R}_+$



$\zeta_k = \chi(\rho/R) \cdot \rho^k$, $k \in \mathbb{R}$, $R \gg 0$

$$\hookrightarrow \int \zeta_k \left|\nabla |u|^{p/2}\right|^2_{\omega_0} \omega_0^n + \frac{np^2}{4(p-1)} \underbrace{\int \zeta_k u |u|^{p-2} (e^{\varepsilon u}-1) e^f \omega_0^n}_{\geq\ \delta\varepsilon \int \zeta_k |u|^p e^f \omega_0^n} \quad \cdots (\star\star)$$
$$\leq -\frac{np^2}{4(p-1)} \left\{ \int \zeta_k u|u|^{p-2}(e^f-1)\, \omega_0^n + \underbrace{\int u |u|^{p-2} d\zeta_k \wedge d^c u \wedge T}_{= \frac{1}{p} \int |u|^p\, dd^c \zeta_k \wedge T} \right\}$$

$$dd^c\zeta_k = d\left(\chi'(\rho/R) \cdot \rho^k \frac{d^c\rho}{R} + k \chi(\rho/R) \cdot \rho^{k-1} d^c \rho \right)$$
$$= \chi''(\rho/R) \cdot \rho^k \cdot \frac{d\rho \wedge d^c\rho}{R} + 2k \chi'(\rho/R) \cdot \rho^{k-1} \frac{d\rho \wedge d^c \rho}{R} + \chi'(\rho/R) \cdot \rho^k \frac{dd^c\rho}{R}$$
$$+ k(k-1) \chi(\rho/R) \rho^{k-2} d\rho \wedge d^c \rho + k \chi(\rho/R) \rho^{k-1} dd^c \rho$$

(depends on ε)

$T \approx \omega_0^n$

Assume $\|u\|_{C^{2,\alpha}} < +\infty$, $\|f\|_{L^\infty} < +\infty$ as $R \to +\infty$

$$\int \rho^k \left|\nabla |u|^{p/2}\right|^2_{\omega_0} \omega_0^n + \frac{np^2}{4(p-1)} \delta \cdot \varepsilon \int \rho^k |u|^p \omega_0^n \leq \frac{np^2}{4(p-1)} \left\{ \int \rho^k |u|^{p-1} |e^f - 1|\, \omega_0^n + \frac{C(k)}{p} \int \rho^{k-2} |u|^p \omega_0^n \right\}$$

(M, ω_0) has poly vol growth $|B_{\omega_0}(x_0, s)| \leq C s^\beta$

Temporary assumption : $f \in C_c^\infty(M)$

For $k_0 \in \mathbb{Z}$ sufficiently negative s.t. $\int \rho^{k_0-2} |u|^p < +\infty$

$$\int \rho^k |u|^{p-2} |\nabla u|^2 + \varepsilon \int \rho^k |u|^p \lesssim \int \rho^k |u|^p |e^f - 1| + \int \rho^{k-2} |u|^p$$

Starting from k_0, can iterate

$$\Rightarrow \int \rho^k |u|^{p-2} |\nabla u|^2 + \int \rho^k |u|^p < +\infty \quad \forall k \in \mathbb{N},\ p > 1$$

(2) Uniform L^∞-estimate

Take $\zeta = \zeta_0 = \chi(\rho/R)$ in (A) with $k=0$, $R\to\infty$ & drop the ε term on the LHS :

$$\int |\nabla |u|^{p/2}|^2_{\omega_0}\, \omega_0^n \leq -\frac{np^2}{4(p-1)} \int u|u|^{p-2}(e^f-1)\,\omega_0^n$$

$\beta > 2$: SOB(β), $\beta>2$ $\rightsquigarrow$ $\displaystyle \underbrace{\left(\int \rho^{\alpha(\beta-2)-\beta} |\,|u|^{p/2}|^{2\alpha}\omega_0^n\right)^{1/\alpha}}_{\|u\|^p_{\alpha p,\alpha}} \leq C\int |\nabla |u|^{p/2}|^2_{\omega_0}\,\omega_0^n \leq \frac{Cnp^2}{(p-1)}\int |u|^{p-1}|e^f-1|\,\omega_0^n$ ... (I)

$\|v\|_{p,\alpha} := \left(\int \rho^{\alpha(\beta-2)-\beta}|v|^p\right)^{1/p}$

note that $|e^f-1| \leq C\cdot|f|$
$\uparrow$ depend on $\sup|f|$

step I (I) $\Rightarrow$ $\displaystyle \|u\|^p_{\alpha p,\alpha} \leq \frac{Cnp^2}{(p-1)}\int |u|^{p-1}\rho^{-\mu}\,\omega_0^n \leq \frac{Cnp^2}{(p-1)} \underbrace{\left(\int |u|^{\overset{m(p-1)}{\overset{\|}{\alpha p}}}\rho^{m\zeta}\right)^{\frac{1}{m}}}_{\|u\|^{p-1}_{\alpha p,\alpha}} \underbrace{\left(\int \rho^{-m^*\zeta}\rho^{-m^*\mu}\right)^{\frac{1}{m^*}}}_{=A}$

need $m(p-1)=\alpha p \Rightarrow m = \frac{\alpha p}{p-1}$, $m^* = \frac{\alpha p}{\alpha p - p + 1}$
$m\zeta = \alpha(\beta-2)-\beta \Rightarrow \zeta = \frac{p-1}{\alpha p}(\alpha(\beta-2)-\beta)$

If $r>\beta$, $\int\rho^{-r} < C$. So we want $m^*(\zeta+\mu) \geq \beta+\varepsilon$
$\Leftrightarrow p\alpha(\zeta+\mu) \geq (\beta+\varepsilon)(p(\alpha-1)+1)$
$\Leftrightarrow p\{\alpha(\beta-2)-\beta+\alpha\mu\} - \alpha(\beta-2) + \cancel{\beta} \geq p(\beta+\varepsilon)(\alpha-1) + \cancel{\beta}+\varepsilon$
$\Leftrightarrow \underbrace{p\{\alpha\mu + \alpha(\beta-2) - \beta - (\beta+\varepsilon)(\alpha-1)\}}_{p\{\alpha(\mu-2)-\varepsilon(\alpha-1)\}} \geq \alpha(\beta-2)+\varepsilon$
$\Leftrightarrow p \geq \frac{\alpha(\beta-2)+\varepsilon}{\alpha(\mu-2)-\varepsilon(\alpha-1)}$ if ε sufficiently small s.t. $(\mu-2)-\varepsilon(\alpha-1)>0$

$\hookrightarrow$ if $p > \frac{(\beta-2)+\varepsilon}{(\mu-2)-\varepsilon(\alpha-1)}$, $A \leq \left(\int\rho^{-(\beta+\varepsilon)}\right)^{1/m^*}$

$$\int\rho^{-(\beta+\varepsilon)} \sim \sum_{i=0}^\infty \int_{A(\kappa^i r_0,\kappa^{i+1}r_0)} \rho^{-(\beta+\varepsilon)} \lesssim \sum_{i=0}^\infty (\kappa^{i+1}r_0)^\beta\cdot(\kappa^i r_0)^{-(\beta+\varepsilon)} = \kappa^\beta r_0^{-\varepsilon}\sum_{i=0}^\infty \kappa^{-i\varepsilon}$$
$$\sim r_0^{-\varepsilon} = \exp(-\varepsilon\log r_0) \sim \frac{1}{\varepsilon}$$

$\frac{1}{m^*} = \frac{p(\alpha-1)+1}{\alpha p}$ $\searrow$ in p $\Rightarrow$ $\left(\int\rho^{-(\beta+\varepsilon)}\right)^{1/m^*} \lesssim \frac{1}{\varepsilon}$ if $p>1$, $p > \frac{\beta-2+\varepsilon}{\mu-2-\varepsilon(\alpha-1)}$

All in all $\|u\|_{\alpha p,\alpha} \leq \frac{Cnp^2}{\varepsilon(p-1)}$ ... (1)

step II (I) $\Rightarrow$ $\displaystyle \|u\|^p_{\alpha p,\alpha} \leq \frac{Cp^2}{(p-1)}\int |u|^{p-1}\rho^{-\mu} \leq \frac{Cp^2}{p-1}\underbrace{\left(\int |u|^{\overset{m(p-1)}{\overset{\|}{p}}}\rho^{m\eta}\right)^{1/m}}_{\|u\|^{p-1}_{p,\alpha}}\left(\int\rho^{-m^*\eta}\rho^{-m^*\mu}\right)^{1/m^*}$

need $m(p-1)=p \Rightarrow m = \frac{p}{p-1}$, $m^*=p$
$m\eta = \alpha(\beta-2)-\beta \Rightarrow \eta = \frac{p-1}{p}(\alpha(\beta-2)-\beta)$

Want $m^*(\eta+\mu)\geq\beta+\varepsilon \Leftrightarrow (p-1)(\alpha(\beta-2)-\beta) + p\mu \geq \beta+\varepsilon$
$\Leftrightarrow p\{\mu+\alpha(\beta-2)-\beta\} - \alpha(\beta-2) + \cancel{\beta} \geq \cancel{\beta}+\varepsilon$
$\Leftrightarrow p \geq \frac{\alpha(\beta-2)+\varepsilon}{\mu+\alpha(\beta-2)-\beta}$ if $\mu+\alpha(\beta-2)-\beta>0$ true if α close to 1

Then $\|u\|_{\alpha p,\alpha} \leq \left(\frac{Cp^2}{\varepsilon(p-1)}\right)^{1/p}\|u\|^{1-1/p}_{p,\alpha}$ if $p > \frac{\alpha(\beta-2)+\varepsilon}{\mu+\alpha(\beta-2)-\beta}$ & $\underset{[1,\frac{n}{n-1}]}{\overset{}{\alpha}}$ s.t. $\mu+\alpha(\beta-2)-\beta>0$

Step I + II + Iteration $\Rightarrow$ $\|u\|_{L^\infty} \leq C$: indep of ε

Decay estimate for $\beta > 2$: if $2<\mu<\beta$, $\forall \delta>0$, $|u| < C(\delta)\, \rho^{2-\mu+\delta}$

$$\xi = \chi(\rho/R)\cdot \rho^{l}$$

$$\star \times \xi u |\xi u|^{p-2}\xi \Rightarrow \int \xi u|\xi u|^{p-2}\xi\,(e^{f+\varepsilon u}-1)\,\omega_0^n = \int \xi u|\xi u|^{p-2}\xi\, dd^c u\wedge T$$

$$\int |\nabla |\xi u|^{p/2}|^2_{\omega_0}\omega_0^n = \frac{np^2}{4}\int |\xi u|^{p-2} d(\xi u)\wedge d^c(\xi u)\wedge \omega_0^{n}$$

$$\leq \frac{np^2}{4}\int |\xi u|^{p-2} d(\xi u)\wedge d^c(\xi u)\wedge T = \frac{-np^2}{4(p-1)}\int \xi u|\xi u|^{p-2}\, dd^c(\xi u)\wedge T$$

$$= \frac{-np^2}{4(p-1)}\int \xi u\,|\xi u|^{p-2}\left(\xi\, dd^c u\wedge T + u\, dd^c\xi\wedge T + 2\, du\wedge d^c\xi\wedge T\right)$$

$$= \frac{-np^2}{4(p-1)}\left\{\int \xi u|\xi u|^{p-2}\xi\,(e^{f+\varepsilon u}-1)\,\omega_0^n + \int \xi u|\xi u|^{p-2}\, u\, dd^c\xi\wedge T + \frac{2}{p}\int |u|^p\, dd^c\xi^p\wedge T\right\}$$

$$\Rightarrow \int |\nabla |\xi u|^{p/2}|^2_{\omega_0}\omega_0^n + \frac{np^2}{4(p-1)}\int \xi^2|\xi u|^{p-2} u(e^{\varepsilon u}-1)\, e^{f}\omega_0^n$$

$$\leq \frac{-np^2}{4(p-1)}\left\{\int \xi u|\xi u|^{p-2}\xi\,(e^{f}-1)\,\omega_0^n + \int \xi u|\xi u|^{p-2} u\, dd^c\xi\wedge T + \frac{2}{p^2}\int |u|^p\, dd^c\xi^p\wedge T\right\}$$

Note: ① $\varepsilon\|u\|_{L^\infty} \leq \|f\|_{L^\infty} \Rightarrow u(e^{\varepsilon u}-1) \geq \delta\cdot\varepsilon u^2 \geq 0$

② $\|u\|_{L^\infty} \leq C$ indep of ε $\Rightarrow$ Yau's C^2 : $\omega \approx \omega_0$; thus, $T \approx \omega_0^{n-1}$

③ $$dd^c\xi = d\left(\chi'(\rho/R)\rho^{l}\, \frac{d^c\rho}{R} + \chi(\rho/R)\, l\rho^{l-1}d^c\rho\right)$$

$$= \chi''(\rho/R)\rho^{l}\frac{d\rho\wedge d^c\rho}{R^2} + \chi'(\rho/R)\, l\rho^{l-1}\frac{d\rho\wedge d^c\rho}{R} + \chi'(\rho/R)\rho^{l}\frac{dd^c\rho}{R}$$
$$+ \chi'(\rho/R)\, l\rho^{l-1}\frac{d\rho\wedge d^c\rho}{R} + \chi(\rho/R)\, l(l-1)\rho^{l-2}d\rho\wedge d^c\rho + \chi(\rho/R)\, l\rho^{l-1}dd^c\rho$$

$$|dd^c\xi|_{\omega_0} \leq C\cdot\rho^{l-2} + C|l|(|l|+1)\rho^{l-2}$$

$$dd^c\xi^p = dd^c\left(\chi(\rho/R)^p\rho^{pl}\right) = d\left(p\,\chi(\rho/R)^{p-1}\chi'(\rho/R)\rho^{pl}\, d^c\rho/R + pl\,\chi(\rho/R)^p\rho^{pl-1}d^c\rho\right)$$

$$= p(p-1)\chi(\rho/R)^{p-2}\chi'(\rho/R)^2\rho^{pl}\frac{d\rho\wedge d^c\rho}{R^2} + p\,\chi(\rho/R)^{p-1}\chi''(\rho/R)\rho^{pl}\frac{d\rho\wedge d^c\rho}{R^2}$$
$$+ p\,\chi(\rho/R)^{p-1}\chi'(\rho/R)\, pl\,\rho^{pl-1}\frac{d\rho\wedge d^c\rho}{R} + p\,\chi(\rho/R)^{p-1}\chi'(\rho/R)\rho^{pl}\frac{dd^c\rho}{R}$$
$+ \cdots$ similar computation

$$|dd^c\xi^p|_{\omega_0} \leq C\, p|l|\,(p|l|+1)\,\rho^{pl-2}$$

$$\Rightarrow \int |\nabla|\xi u|^{p/2}|^2_{\omega_0}\omega_0^n \leq \frac{np^2}{4(p-1)}\left\{\int |\xi u|^{p-1}\underbrace{|e^{f}-1|}_{\lesssim C\rho^{-\mu}}\,\omega_0^n + C|l|(|l|+1)\int \rho^{-2}|\rho^{l} u|^p\,\omega_0^n\right\}$$

vs SoB(β), $\beta>2$

$$\left(\int \rho^{\alpha(\beta-2)-\beta}|\xi u|^{p\alpha}\,\omega_0^n\right)^{1/\alpha}$$

$$\eta := \rho^{l} \Rightarrow \left(\int \rho^{\alpha(\beta-2)-\beta}|\eta u|^{p\alpha}\right)^{1/\alpha} \lesssim \frac{p^2}{p-1}\left(\int |\eta u|^{p-1}\rho^{-\mu} + |l|(|l|+1)\int \rho^{-2}|\eta u|^p\right)$$

Now $\eta_k = \rho^{l_k}$, l_k : TBD, $p_k = \alpha^k p_0$, $k\in\mathbb{N}$

Then $\underbrace{\left(\int \rho^{\alpha(\beta-2)-\beta+l_k\alpha^{k+1}p_0}\,|u|^{\alpha^{k+1}p_0}\right)^{1/\alpha}}_{=\,C} \lesssim \frac{p_k^2}{p_k-1}\left(\overbrace{\int \rho^{l_k\alpha^k p_0-l_k-\mu}\,|u|^{\alpha^k p_0-1}}^{A} + |l_k|(|l_k|+1)\underbrace{\int \rho^{l_k\alpha^k p_0-2}\,|u|^{\alpha^k p_0}}_{B}\right)$

For $l_k \leq \frac{\beta-2}{p_0}\left(1-1/\alpha^k\right) \Rightarrow l_k\alpha^k p_0 - 2 \leq \alpha^k(\beta-2)-\beta$

$\Rightarrow B \leq \int \rho^{\alpha^k(\beta-2)-\beta}\,|u|^{\alpha^k p_0}$

$l_k \geq \frac{\beta-2}{p_0}\left(1-1/\alpha^k\right) \Rightarrow l_k\alpha^{k+1}p_0 + \alpha(\beta-2)-\beta \geq \alpha^{k+1}(\beta-2)-\beta$

$\Rightarrow C \geq \int \rho^{\alpha^{k+1}(\beta-2)-\beta}\,|u|^{\alpha^{k+1}p_0}$

Take $l_k = \frac{\beta-2}{p_0}\left(1-1/\alpha^k\right)$

$A = \int |\rho^{l_k}u|^{p_k-1}\cdot\rho^{-\mu} \leq \left(\int |\rho^{l_k}u|^{p_k}\rho^{-2}\right)^{\frac{p_k-1}{p_k}}\left(\int \rho^{2(p_k-1)}\rho^{l_kp_k}\rho^{-p_k\mu}\right)^{1/p_k}$

$\leq \max\{1, B\}\left(\int \rho^{p_k(2+l_k-\mu)-2}\right)^{1/p_k}$

If $p_0(2-\mu)+\beta-2 = -\varepsilon < 0 \Rightarrow p_k(2+l_k-\mu)-2 = \alpha^k p_0\left(2+\left(\frac{\beta-2}{p_0}\right)\left(1-1/\alpha^k\right)-\mu\right)-2$

$= -\alpha^k\varepsilon-\beta$

$\Rightarrow \int \rho^{p_k(2+l_k-\mu)-2} = \int \rho^{-\beta-\alpha^k\varepsilon} \leq C \quad \forall\, k\geq 1$

$\Rightarrow \int \rho^{-\beta}\left|\rho^{\frac{\beta-2}{p_0}}u\right|^{\alpha^{k+1}p_0} \lesssim \left(\frac{p_k^2}{p_k-1}\left(\frac{\beta-2}{p_0}\right)\left(\frac{\beta-2+p_0}{p_0}\right)\right)^{1/p_k} \max\left\{1, \int \rho^{-\beta}\left|\rho^{\frac{\beta-2}{p_0}}u\right|^{\alpha^k p_0}\right\}$

Recall that $\|u\|_{\alpha p,\alpha} \leq \frac{Cnp^2}{\varepsilon(p-1)}$ if $p>1$, $p>\frac{\beta-2+\varepsilon}{\mu-2-\varepsilon(\alpha-1)}$, $\mu-2-\varepsilon(\alpha-1)>0$

$\|$

$\left(\int \rho^{\alpha(\beta-2)-\beta}\,|u|^{\alpha p}\right)^{1/\alpha p} = \left(\int \rho^{-\beta}\left|\rho^{\frac{\beta-2}{p}}u\right|^{\alpha p}\right)^{1/\alpha p}$

take $p_0>1$ s.t. $p_0 > \frac{\beta-2+\varepsilon}{\mu-2-\varepsilon(\alpha-1)}$ & $p_0(2-\mu)+\beta-2 = -\delta < 0$ $\quad \varepsilon$ s.t.

↳ Moser iteration $\Rightarrow \left|\rho^{\frac{\beta-2}{p_0}}u\right| \leq C$ & if $2<\mu<\beta$, we can always find such a p_0

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