Analysis for the complex Monge-Ampère equation

Let (M,ω_0) be a complete non-compact Kähler manifold satisfying SOB(β), $\beta>0$
& with $C^{3,\alpha}$ quasi-atlas

Tian-Yau-Hein package

Let $f\in C^{2,\alpha}(M)$ s.t. $|f|\le C r^{-\mu}$ on $\{r>1\}$ for some $\mu>2$

If $\begin{cases}\beta>2\\ \beta\le 2 \ \& \ \int (e^f-1)\omega_0^n=0\end{cases}$, then $\exists\ \alpha\in(0,\bar\alpha]$, $u\in C^{4,\bar\alpha}(M)$ s.t. $(\omega_0+dd^c u)^n=e^f\omega_0^n$

Moreover for $\beta\le 2$, $\int |\nabla u|^2\omega_0^n<+\infty$

Independent of the value of β, if $f\in C^{k,\alpha}_{loc}$ $\Rightarrow$ $u\in C^{k+2,\alpha}_{loc}$
$k\ge 3$

Defn: Let (M,ω_0) be a complete Kähler mfd.

A $C^{k,\alpha}$ quasi-atlas for (M,ω_0) is a collection $\{\Phi_x,\ x\in A\}$, $A\subset M$ of loc. holo. diffeo
$\Phi_x: B\to M$, $\Phi_x(0)=x$ & $\exists\, C\ge 1$ with $\begin{cases}\mathrm{inj}(\Phi_x^* g_0)\ge 1/C\\ \frac{1}{C}g_{euc}\le \Phi_x^* g_0\le C g_{euc} & \forall x\in A\\ \|\Phi_x^* g_0\|_{C^{k,\alpha}(B,g_{euc})}\le C\end{cases}$

$B(0,1)\subset\mathbb{C}^n$ ($= B$)

& $\forall y\in M$, $\exists\, x\in A$ s.t. $y\in\Phi_x(B)$, $\mathrm{dist}_{g_0}(y,\partial\Phi_x(B))\ge 1/C$

↳ Given $C^{k,\alpha}$ quasi-atlas, define $\|u\|_{C^{l,r}(M)}:=\sup_{x\in A}\{\|u\circ\Phi_x\|_{C^{l,r}(B)}\}$

Schauder: for a 2nd order (unif) elliptic operator L with coeff $\in C^{l,r}$

$$\|u\|_{C^{l+2,r}(M)}\le C\left(\|Lu\|_{C^{l,r}(M)}+\|u\|_{L^\infty}\right)\quad\forall\ u\in C^\infty_{loc}(M)$$

(if $l\le k-1$, $r\in(0,\alpha]$)
↳ s.t. $C^{l,r}$-norm on B is comparable w/ $C^{l,r}$-norm on B induced by $\Phi_x^* g_0$

Lemma (Tian-Yau) If $|Rm|\le C$, then $\exists\ C^{1,\alpha}$ quasi-atlas $\forall\ \alpha$
If moreover $\sum_{i=1}^k |\nabla^i Scal|\le C$, $\exists\ C^{k+1,\alpha}$ quasi-atlas.

we discuss its pf after pf of TYH

§ Existence I : ε-perturbed equation $(\omega_0+dd^c u_\varepsilon)^n=e^{f+\varepsilon u}\omega_0^n$ ... (MA_ε)

Continuity method by Cheng-Yau

(M,ω_0) complete Kähler manifold with $C^{2,\alpha}$ quasi-atlas, $f\in C^{2,\alpha}$

Goal : Show $\exists\ u_\varepsilon\in C^{4,\bar\alpha}$, for some $\bar\alpha\in(0,\alpha]$ to (MA_ε)
↳ depend on M, ω_0, α, f, but not on ε.

Consider $(\omega_0+dd^c u_{\varepsilon,t})^n=e^{tf+\varepsilon u_{\varepsilon,t}}\omega_0^n$ ... $(MA_{\varepsilon,t})$

For $r\in(0,\alpha]$, $J_r=\{t\in[0,1] \mid (MA_{\varepsilon,t})$ admits a sol'n $\in C^{4,r}\}$

$0\in J_r$, want to prove openness & closedness

Lemma (Yau's maximum principle)

Let (M,g) be a complete mfd with sectional curvature bounded below, $x_0\in M$.
If $u\in C^2_{loc}$ with $|u|+|\nabla u|+|\nabla^2 u|\le C$ does not attain its supremum, then $\exists\ \{x_k\}\subset M$,
with $\mathrm{dist}(x_0,x_k)\to\infty$ & s.t. $u(x_k)\to\sup u$, $|\nabla u|(x_k)\to 0$, $\limsup_k \max\mathrm{spec}(\nabla^2 u)(x_k)\le 0$

Claim 1 : $\forall\, r \in (0,\alpha]$, $t \in J_r$, $\exists$ bdd linear operator $G : C^{2,r} \to C^{4,r}$ with $(\Delta_\omega - \varepsilon)\circ G = id$ where $\omega = \omega_0 + dd^c u_{\varepsilon,t}$

↳ this gives the openness (linearized operator is an isomorphism ⇝ implicit fcn thm is good)

pf · Since $\varepsilon > 0$, $\forall$ bdd domain w/ sufficiently smooth bdry $\Omega \subset M$

$\exists\, u = u_\Omega \in C^{4,r}(\Omega)$ solves $\begin{cases} (\Delta - \varepsilon)u = f|_\Omega \\ u = 0 \text{ on } \partial\Omega \end{cases}$

Schauder ⇝ there are local estimates (w.r.t. charts of quasi-atlas) for $\|u\|_{C^{4,r}}$ in terms of $\|f\|_{C^{2,r}}$ & $\|u\|_{L^\infty}$

Max prin : at max $x_{max} \in \Omega$, $f(x_{max}) = (\Delta - \varepsilon)u|_{x_{max}} \leq -\varepsilon\, u(x_{max}) \Rightarrow u(x_{max}) \leq -\frac{1}{\varepsilon} f(x_{max})$

similarly, $u(x_{min}) \geq -\frac{1}{\varepsilon} f(x_{min})$

$\Rightarrow \|u\|_{L^\infty} \leq \frac{1}{\varepsilon}\|f\|_{L^\infty}$

Arzela-Ascoli : $\{u_{\Omega_k}\} \to u$ s.t. $(\Delta_\omega - \varepsilon)u = f$ on M

uniqueness : for $(\Delta - \varepsilon)u = 0$ on M

if $x_{max} \in M \Rightarrow u(x_{max}) \leq 0$ | if sup not in M, use Yau's max prin

$x_{min} \in M \Rightarrow u(x_{min}) \geq 0$ | inf

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Claim 2 : $\exists\, \bar\alpha \in (0,\alpha]$ s.t. if $u = u_{\varepsilon,t} \in C^4$ solves $(MA_{\varepsilon,t})$ for $t \in [0,1]$, $\varepsilon \in (0,1]$, then $\|u\|_{C^{2,\bar\alpha}(M)} \leq C(M, \omega_0, \|f\|_{C^{2,\alpha}(M)}, \varepsilon)$

↳ this gives the closedness.

pf : By max prin (or Yau's max prin), $\|u\|_{L^\infty} \leq \frac{1}{\varepsilon}\|f\|_{L^\infty}$

C^2 : Yau $0 \leq tr_{\omega_0}\omega = n + \Delta_{\omega_0} u \leq C\cdot\exp(C(u - \inf u))$, $C = C(M, \omega_0, \|f\|_{C^2(M)})$ if $\varepsilon \leq 1$

↳ $\frac{1}{C}\omega_0 \leq \omega \leq C\omega_0$

Bounded Δ + Linear theory $\Rightarrow \|u\|_{C^{1,r}(M)} \leq C(r)$ $\forall\, r \in (0,1)$

Schauder ⇝ $u \in C^{2,\alpha}$ uniformly #

§ Existence 2 : the limit $\varepsilon \to 0$

Rmk : only need to make L^∞-estimate unif in ε

Step 1 : high integrability of the sol'ns (L^p might still blow up as $\varepsilon \to 0$)

Step 2 : Moser iteration

Recall : $\omega^n - \omega_0^n = dd^c u \wedge \overbrace{\sum_{k=0}^{n-1} (\omega^k \wedge \omega_0^{n-1-k})}^{=T}$ ⋯ ⊛

For $u = u_\varepsilon \in C^2(M)$ solving (MA_ε), $\zeta \in C_c^\infty(M)$

Step 1 : ⊛ $\times\ \zeta u|u|^{p-2}$, $p > 1$, $\zeta u|u|^{p-2}(e^{f+\varepsilon u} - 1)\,\omega_0^n = \zeta u|u|^{p-2} dd^c u \wedge T$

$$\int \zeta u|u|^{p-2} dd^c u \wedge T = -\int u|u|^{p-2} d\zeta \wedge d^c u \wedge T - (p-1)\int \zeta |u|^{p-2} du \wedge d^c u \wedge T$$

$$\begin{aligned}\int \zeta |\nabla |u|^{\frac{p}{2}}|^2_{\omega_0}\, \omega_0^n &= n\frac{p^2}{4}\int \zeta |u|^{p-2} du \wedge d^c u \wedge \omega_0^{n-1} \\ &\leq n\frac{p^2}{4}\int \zeta |u|^{p-2} du \wedge d^c u \wedge T \\ &= \frac{-np^2}{4(p-1)}\Big\{\int \zeta u|u|^{p-2} \underbrace{dd^c u \wedge T}_{=(e^{f+\varepsilon u}-1)\,\omega_0^n} + \int u|u|^{p-2} d\zeta \wedge d^c u \wedge T\Big\}\end{aligned}$$

$$\Rightarrow \int \zeta \left|\nabla |u|^{p/2}\right|^2_{\omega_0} \omega_0^n + \frac{np^2}{4(p-1)} \int \zeta\, \underline{u} |u|^{p-2} \underline{(e^{\varepsilon u}-1)}\, e^f \omega_0^n$$

$$\leq -\frac{np^2}{4(p-1)} \left\{ \int \zeta\, u|u|^{p-2}(e^f-1)\,\omega_0^n + \int u|u|^{p-2} d\zeta \wedge d^c u \wedge T \right\}$$

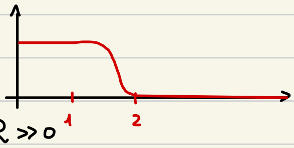
Note: $\forall C<\infty$, $\exists \delta=\delta(C)>0$ s.t. if $|x|\leq C$, $x(e^x-1)\geq \delta x^2$

$\varepsilon \|u\|_{L^\infty} \leq \|f\|_{L^\infty} \Rightarrow u(e^{\varepsilon u}-1) \geq \delta \varepsilon u^2$

(1) High integrability

Schoen-Yau : if $\mathrm{Ric}(\omega_0) \geq -C \Rightarrow \exists$ smooth $\rho : M \to \mathbb{R}$ s.t. $\rho \sim 1+\mathrm{dist}(x_0,\cdot)$ for any fixed x_0
& $|\nabla\rho| + \rho|\Delta\rho| \leq C$

Fix a cutoff $\chi : \mathbb{R}_+ \to \mathbb{R}_+$



$\zeta_k = \chi(\rho/R)\cdot \rho^k$, $k\in\mathbb{R}$, $R \gg 0$

$$\hookrightarrow \int \zeta_k \left|\nabla|u|^{p/2}\right|^2_{\omega_0}\omega_0^n + \frac{np^2}{4(p-1)} \underbrace{\int \zeta_k\, u|u|^{p-2}(e^{\varepsilon u}-1)e^f\omega_0^n}_{\geq\ \delta\varepsilon \int \zeta_k |u|^p e^f \omega_0^n} \quad \cdots (\star\star)$$

$$\leq -\frac{np^2}{4(p-1)}\left\{ \int \zeta_k\, u|u|^{p-2}(e^f-1)\,\omega_0^n + \underbrace{\int u|u|^{p-2} d\zeta_k \wedge d^c u \wedge T}_{= \frac{1}{p}\int |u|^p dd^c\zeta_k \wedge T} \right\}$$

$$dd^c\zeta_k = d\left(\chi'(\rho/R)\cdot\rho^k \frac{d^c\rho}{R} + k\chi(\rho/R)\cdot\rho^{k-1}d^c\rho\right)$$

$$= \chi''(\rho/R)\cdot\rho^k\cdot\frac{d\rho\wedge d^c\rho}{R} + 2k\chi'(\rho/R)\cdot\rho^{k-1}\frac{d\rho\wedge d^c\rho}{R} + \chi'(\rho/R)\cdot\rho^k\frac{dd^c\rho}{R}$$
$$+ k(k-1)\chi(\rho/R)\rho^{k-2}d\rho\wedge d^c\rho + k\chi(\rho/R)\rho^{k-1}dd^c\rho$$

(depends on ε)
$T \approx \omega_0^n$

Assume $\|u\|_{C^{2,\alpha}} < +\infty$, $\|f\|_{L^\infty} < +\infty$ as $R\to+\infty$

$$\int \rho^k \left|\nabla|u|^{p/2}\right|^2_{\omega_0}\omega_0^n + \frac{np^2}{4(p-1)}\,\delta\cdot\varepsilon \int \rho^k |u|^p\omega_0^n \leq \frac{np^2}{4(p-1)}\left\{\int\rho^k|u|^{p-1}|e^f-1|\,\omega_0^n + \frac{C(k)}{p}\int\rho^{k-2}|u|^p\omega_0^n\right\}$$

(M,ω_0) has poly vol growth $|B_{\omega_0}(x_0,s)| \leq C s^\beta$

Temporary assumption : $f\in C_c^\infty(M)$

For $k_0\in\mathbb{Z}$ sufficiently negative s.t. $\int \rho^{k_0-2}|u|^p < +\infty$

$$\int \rho^k |u|^{p-2}|\nabla u|^2 + \varepsilon\int\rho^k|u|^p \lesssim \int\rho^k|u|^p|e^f-1| + \int\rho^{k-2}|u|^p$$

Starting from k_0, can iterate

$$\Rightarrow \int\rho^k|u|^{p-2}|\nabla u|^2 + \int\rho^k|u|^p < +\infty \quad \forall k\in\mathbb{N},\ p>1$$

(2) Uniform L^∞-estimate

Take $\zeta = \zeta_0 = \chi(\rho/R)$ $\rightsquigarrow$ (A) with $k=0$, $R\to\infty$ & drop the ε term on the LHS:

$$\int |\nabla |u|^{\frac{p}{2}}|^2_{\omega_0}\, \omega_0^n \leq -\frac{np^2}{4(p-1)}\int u|u|^{p-2}(e^f-1)\,\omega_0^n$$

$\boxed{\beta>2}$: $SOB(\beta)$, $\beta>2$ $\rightsquigarrow$ $\underbrace{\left(\int \rho^{\alpha(\beta-2)-\beta}\,||u|^{\frac{p}{2}}|^{2\alpha}\omega_0^n\right)^{1/\alpha}}_{=\|u\|^p_{\alpha p,\alpha}} \leq C\int |\nabla |u|^{\frac{p}{2}}|^2_{\omega_0}\,\omega_0^n \leq \frac{Cnp^2}{(p-1)}\int |u|^{p-1}|e^f-1|\,\omega_0^n$ ... (I)

$\|v\|_{p,\alpha} := \left(\int \rho^{\alpha(\beta-2)-\beta}|v|^p\right)^{1/p}$

note that $|e^f-1| \leq C\cdot|f|$
↑ depend on $\sup|f|$

step I (I) $\Rightarrow$ $\|u\|^p_{\alpha p,\alpha} \leq \frac{Cnp^2}{(p-1)}\int |u|^{p-1}\rho^{-\mu}\,\omega_0^n \leq \frac{Cnp^2}{(p-1)}\underbrace{\left(\int |u|^{\overset{m(p-1)}{\overset{\parallel}{\alpha p}}}\rho^{m\zeta}\right)^{\frac{1}{m}}}_{=\|u\|^{p-1}_{\alpha p,\alpha}}\underbrace{\left(\int \rho^{-m^*\zeta}\rho^{-m^*\mu}\right)^{\frac{1}{m^*}}}_{=A}$

need $m(p-1)=\alpha p$ $\Rightarrow$ $m=\frac{\alpha p}{p-1}$, $m^*=\frac{\alpha p}{\alpha p-p+1}$
$m\zeta=\alpha(\beta-2)-\beta$ $\Rightarrow$ $\zeta=\frac{p-1}{\alpha p}(\alpha(\beta-2)-\beta)$

If $r>\beta$, $\int\rho^{-r}<C$. So we want $m^*(\overset{s}{\overset{/\!/}{\zeta+\mu}})\geq\beta+\varepsilon$

$\Leftrightarrow$ $p\alpha(\zeta+\mu)\geq(\beta+\varepsilon)(p(\alpha-1)+1)$

$\Leftrightarrow$ $p\{\alpha(\beta-2)-\beta+\alpha\mu\}-\alpha(\beta-2)+\cancel{\beta} \geq p(\beta+\varepsilon)(\alpha-1)+\cancel{\beta}+\varepsilon$

$\Leftrightarrow$ $p\underbrace{\{\alpha\mu+\alpha(\beta-2)-\beta-(\beta+\varepsilon)(\alpha-1)\}}_{=\{\alpha(\mu-2)-\varepsilon(\alpha-1)\}}\geq\alpha(\beta-2)+\varepsilon$

$\Leftrightarrow$ $p\geq\frac{\alpha(\beta-2)+\varepsilon}{\alpha(\mu-2)-\varepsilon(\alpha-1)}$ if ε sufficiently small s.t. $(\mu-2)-\varepsilon(\alpha-1)>0$

↳ if $p>\frac{(\beta-2)+\varepsilon}{(\mu-2)-\varepsilon(\alpha-1)}$, $A\leq\left(\int\rho^{-(\beta+\varepsilon)}\right)^{1/m^*}$

$$\int\rho^{-(\beta+\varepsilon)}\sim\Sigma_{i=0}^\infty\int_{A(\kappa^i r_0,\kappa^{i+1}r_0)}\rho^{-(\beta+\varepsilon)}\lesssim\Sigma_{i=0}^\infty(\kappa^{i+1}r_0)^\beta\cdot(\kappa^i r_0)^{-(\beta+\varepsilon)}=\kappa^\beta r_0^{-\varepsilon}\sum_{i=0}^\infty\kappa^{-i\varepsilon}$$
$$\sim r_0^{-\varepsilon}=\exp(-\varepsilon\log r_0)\sim\frac{1}{\varepsilon}$$

$\frac{1}{m^*}=\frac{p(\alpha-1)+1}{\alpha p}$ ↘ in p $\Rightarrow$ $\left(\int\rho^{-(\beta+\varepsilon)}\right)^{1/m^*}\lesssim\frac{1}{\varepsilon}$

if $p>1$, $p>\frac{\beta-2+\varepsilon}{\mu-2-\varepsilon(\alpha-1)}$

All in all $\|u\|_{\alpha p,\alpha}\leq\frac{Cnp^2}{\varepsilon(p-1)}$ ... (1)

step II (I) $\Rightarrow$ $\|u\|^p_{\alpha p,\alpha}\leq\frac{Cp^2}{(p-1)}\int|u|^{p-1}\rho^{-\mu}\leq\frac{Cp^2}{p-1}\underbrace{\left(\int|u|^{\overset{m(p-1)}{\overset{\parallel}{p}}}\rho^{m\eta}\right)^{1/m}}_{=\|u\|^{p-1}_{p,\alpha}}\left(\int\rho^{-m^*\eta}\rho^{-m^*\mu}\right)^{1/m^*}$

need $m(p-1)=p$ $\Rightarrow$ $m=\frac{p}{p-1}$, $m^*=p$
$m\eta=\alpha(\beta-2)-\beta$ $\Rightarrow$ $\eta=\frac{p-1}{p}(\alpha(\beta-2)-\beta)$

Want $m^*(\eta+\mu)\geq\beta+\varepsilon$ $\Leftrightarrow$ $(p-1)(\alpha(\beta-2)-\beta)+p\mu\geq\beta+\varepsilon$

$\Leftrightarrow$ $p\{\mu+\alpha(\beta-2)-\beta\}-\alpha(\beta-2)+\cancel{\beta}\geq\cancel{\beta}+\varepsilon$

$\Leftrightarrow$ $p\geq\frac{\alpha(\beta-2)+\varepsilon}{\mu+\alpha(\beta-2)-\beta}$ if $\mu+\alpha(\beta-2)-\beta>0$ true if α close to 1

Then $\|u\|_{\alpha p,\alpha}\leq\left(\frac{Cp^2}{\varepsilon(p-1)}\right)^{1/p}\|u\|^{1-1/p}_{p,\alpha}$ if $p>\frac{\alpha(\beta-2)+\varepsilon}{\mu+\alpha(\beta-2)-\beta}$ & $\underset{[1,\frac{n}{n-1}]}{\overset{\uparrow}{\alpha}}$ s.t. $\mu+\alpha(\beta-2)-\beta>0$

Step I + II + Iteration $\Rightarrow$ $\|u\|_{L^\infty}\leq C$: indep of ε

Decay estimate for $\beta>2$: Suppose that $|d\rho| + \rho|dd^c\rho| \leq C$.

If $\mu \in (0,\beta) \Rightarrow \forall \delta>0$, $|u| \lesssim C(\delta)\,\rho^{2-\mu+\delta}$

$$\zeta = \chi(\rho/R)\cdot\rho^{l}$$

$$\bigstar \times \zeta u|\zeta u|^{p-2}\zeta \Rightarrow \int \zeta u|\zeta u|^{p-2}\zeta\,(e^{f+\varepsilon u}-1)\,\omega_0^n = \int \zeta u|\zeta u|^{p-2}\zeta\, dd^c u\wedge T$$

$$\begin{aligned}\int |\nabla|\zeta u|^{p/2}|^2_{\omega_0}\omega_0^n &= n\int d|\zeta u|^{p/2}\wedge d^c|\zeta u|^{p/2}\wedge\omega_0^{n-1}\\ &= n\int |u|^p\cdot\left(\tfrac{p}{2}\zeta|\zeta|^{\frac{p}{2}-2}\right)^2 d\zeta\wedge d^c\zeta\wedge\omega_0^{n-1} + \underline{n\int \zeta^p\left(\tfrac{p}{2}u|u|^{\frac{p}{2}-2}\right)^2 du\wedge d^cu\wedge\omega_0^{n-1}}\\ &\quad + 2n\int \tfrac{p^2}{4}\zeta u|\zeta u|^{p-2} d\zeta\wedge d^cu\wedge\omega_0^{n-1}\end{aligned}$$

$$\begin{aligned}&\leq \frac{p^2}{4}\int \zeta^p|u|^{p-2}du\wedge d^cu\wedge T\\ &= \frac{p^2}{4(p-1)}\int \zeta^p\, d|u|^{p-1}\wedge d^cu\wedge T\\ &= -\frac{p^2}{4(p-1)}\left\{\begin{aligned}&\int |\zeta u|^{p-1}\zeta\, dd^cu\wedge T\\ &+p\int |\zeta u|^{p-1}d\zeta\wedge d^cu\wedge T\end{aligned}\right\}\end{aligned}$$

$$\begin{aligned}\leq &-\frac{np^2}{4(p-1)}\int |\zeta u|^{p-1}\zeta\,(e^{f+\varepsilon u}-1)\,\omega_0^n + \frac{np^2}{4}\int |\zeta u|^{p-2}|u|^2 d\zeta\wedge d^c\zeta\wedge\omega_0^{n-1}\\ &+ \frac{np^2}{2}\underbrace{\int \zeta u|\zeta u|^{p-2}d\zeta\wedge d^cu\wedge\omega_0^{n-1}}_{\hookrightarrow\ \text{similar}} - \underline{\frac{np^3}{4(p-1)}\int |\zeta u|^{p-1}d\zeta\wedge d^cu\wedge T}\end{aligned}$$

$$= -\frac{np^2}{4(p-1)}\left\{\begin{aligned}&\int (p-1)|\zeta u|^{p-2}|u|^2 d\zeta\wedge d^c\zeta\wedge T\\ &+\int |\zeta u|^{p-1}u\, dd^c\zeta\wedge T\end{aligned}\right\}$$

Note: ① $\varepsilon\|u\|_{L^\infty} \leq \|f\|_{L^\infty} \Rightarrow u(e^{\varepsilon u}-1) \geq \delta\cdot\varepsilon u^2 \geq 0$

② $\|u\|_{L^\infty} \leq C$ indep of ε $\Rightarrow$ Yau's C^2: $\omega \approx \omega_0$; thus, $T \approx \omega_0^{n-1}$

③ $d\zeta = \chi'(\rho/R)\,\rho^{l}\,\frac{d\rho}{R} + \chi(\rho/R)\,l\rho^{l-1}d\rho \Rightarrow d\zeta\wedge d^c\zeta \lesssim (|l|+1)^2\rho^{2l-2}\omega_0$

$dd^c\zeta = \chi''(\rho/R)\,\rho^{l}\,\frac{d\rho\wedge d^c\rho}{R^2} + \cdots \Rightarrow |dd^c\zeta|_{\omega_0} \lesssim (|l|+1)^2\rho^{2l-2}$

$$\hookrightarrow \int |\nabla|\zeta u|^{p/2}|^2_{\omega_0}\omega_0^n + \frac{np^2}{4(p-1)}\int \zeta^2|\zeta u|^{p-2}u(e^{\varepsilon u}-1)\,e^f\omega_0^n$$

$$\leq \frac{-np^2}{4(p-1)}\int \zeta u|\zeta u|^{p-2}\zeta\,(e^f-1)\,\omega_0^n + Cnp^2(|l|+1)^2\int |\rho^l u|^p\rho^{-2}\omega_0^n$$

$$\hookrightarrow \int |\nabla|\zeta u|^{p/2}|^2_{\omega_0}\omega_0^n \leq \frac{np^2}{4(p-1)}\left\{\int |\zeta u|^{p-1}\underbrace{|e^f-1|}_{\lesssim C\rho^{-\mu}}\omega_0^n + Cp(|l|+1)^2\int \rho^{-2}|\rho^l u|^p\omega_0^n\right\}$$

VS $\leftarrow$ SOB(β), $\beta>2$

$$\left(\int \rho^{\alpha(\beta-2)-\beta}|\zeta u|^{p\alpha}\omega_0^n\right)^{1/\alpha}$$

$$\eta := \rho^l \Rightarrow \left(\int \rho^{\alpha(\beta-2)-\beta}|\eta u|^{p\alpha}\right)^{1/\alpha} \lesssim \frac{p^3}{p-1}\left(\int |\eta u|^{p-1}\rho^{-\mu} + (|l|+1)^2\int \rho^{-2}|\eta u|^p\right)$$

Now $\eta_k = \rho^{l_k}$, l_k: TBD, $p_k = \alpha^k p_0$, $k\in\mathbb{N}$

$$\text{Then}\ \Big(\underbrace{\int \rho^{\alpha(\beta-2)-\beta+l_k\alpha^{k+1}p_0}|u|^{\alpha^{k+1}p_0}}_{=C}\Big)^{1/\alpha} \lesssim \frac{p_k^3}{p_k-1}\left(\overbrace{\int \rho^{l_k\alpha^k p_0-l_k-\mu}|u|^{\alpha^k p_0-1}}^{A} + (|l_k|+1)^2\underbrace{\int \rho^{l_k\alpha^k p_0-2}|u|^{\alpha^k p_0}}_{B}\right)$$

For $l_k \leq \frac{\beta-2}{p_0}\left(1-\frac{1}{\alpha^k}\right) \Rightarrow l_k \alpha^k p_0 - 2 \leq \alpha^k(\beta-2)-\beta$

$\Rightarrow B \leq \int \rho^{\alpha^k(\beta-2)-\beta} |u|^{\alpha^k p_0}$

$l_k \geq \frac{\beta-2}{p_0}\left(1-\frac{1}{\alpha^k}\right) \Rightarrow l_k \alpha^{k+1} p_0 + \alpha(\beta-2) - \beta \geq \alpha^{k+1}(\beta-2)-\beta$

$\Rightarrow C \geq \int \rho^{\alpha^{k+1}(\beta-2)-\beta} |u|^{\alpha^{k+1} p_0}$

Take $l_k = \frac{\beta-2}{p_0}\left(1-\frac{1}{\alpha^k}\right)$

$$A = \int |\rho^{l_k} u|^{p_k-1} \cdot \rho^{-\mu} \leq \left(\int |\rho^{l_k} u|^{p_k} \rho^{-2}\right)^{\frac{p_k-1}{p_k}} \left(\int \rho^{2(p_k-1)} \rho^{l_k p_k} \rho^{-p_k \mu}\right)^{1/p_k}$$

$$\leq \max\{1, B\} \left(\int \rho^{p_k(2+l_k-\mu)-2}\right)^{1/p_k}$$

If $p_0(2-\mu)+\beta-2 = -\varepsilon < 0 \Rightarrow p_k(2+l_k-\mu)-2 = \alpha^k p_0\left(2+\left(\frac{\beta-2}{p_0}\right)\left(1-\frac{1}{\alpha^k}\right)-\mu\right)-2$

$= -\alpha^k \varepsilon - \beta$

$\Rightarrow \int \rho^{p_k(2+l_k-\mu)-2} = \int \rho^{-\beta-\alpha^k \varepsilon} \leq C \quad \forall k \geq 1$

$$\Rightarrow \int \rho^{-\beta} |\rho^{\frac{\beta-2}{p_0}} u|^{\alpha^{k+1} p_0} \lesssim \left(\frac{p_k^3}{p_k-1}\left(\frac{\beta-2+p_0}{p_0}\right)^2\right)^{1/p_k} \max\left\{1, \int \rho^{-\beta} |\rho^{\frac{\beta-2}{p_0}} u|^{\alpha^k p_0}\right\}$$

Recall that $\|u\|_{\alpha p, \alpha} \leq \frac{Cnp^2}{\varepsilon(p-1)}$ if $p>1$, $p > \frac{\beta-2+\varepsilon}{\mu-2-\varepsilon(\alpha-1)}$, $\mu-2-\varepsilon(\alpha-1)>0$

$\|$

$$\left(\int \rho^{\alpha(\beta-2)-\beta} |u|^{\alpha p}\right)^{1/\alpha p} = \left(\int \rho^{-\beta} |\rho^{\frac{\beta-2}{p}} u|^{\alpha p}\right)^{1/\alpha p}$$

take $p_0 > 1$ s.t. $p_0 > \frac{\beta-2+\varepsilon}{\mu-2-\varepsilon(\alpha-1)}$ & $p_0(2-\mu)+\beta-2 = -\delta < 0$

↳ Moser iteration $\Rightarrow |\rho^{\frac{\beta-2}{p_0}} u| \leq C$ & if $2<\mu<\beta$, we can always find such a p_0 s.t. $\ast$

#.

Decay estimate 2: Same assumption on ρ, if $\mu \geq \beta \not\Rightarrow \forall \delta > 0$, $|u| \leq C(\delta) \rho^{2-\mu+\frac{2+\delta}{p_0}}$

where $p_0 > 1$, $p_0(2-\mu)+\beta-2 = -2-\delta$

not good..

We only need to check $p_k(2-l_k-\mu)-2 \leq -\beta-\delta$ for some $\delta > 0$

$\|$

$\alpha^k p_0\left(2-\mu+\frac{\beta-2}{p_0}\left(1-\frac{1}{\alpha^k}\right)\right) = \alpha^k(p_0(2-\mu)+\beta-2)-\beta+2$

$\Leftrightarrow \alpha^k(p_0(2-\mu)+\beta-2) \leq -2-\delta \quad \forall k$

for $\varepsilon \ll 1$, this is fine

$\Leftrightarrow p_0(2-\mu)+\beta-2 \leq -2-\delta \Leftrightarrow p_0(2-\mu)+\beta \leq -\delta$

Take $p_0 > 1$, $p_0 > \frac{\beta-2+\varepsilon}{\mu-2-\varepsilon(\alpha-1)}$, $p_0(2-\mu)+\beta-2 = -2-\delta$

↳ $|\rho^{\frac{\beta-2}{p_0}} u| \leq C \Rightarrow |u| \leq C\rho^{-\frac{\beta-2}{p_0}} \lesssim \rho^{2-\mu+\frac{2+\delta}{p_0}}$