

$\beta \in (0, 2]$

Claim 1: $\int (e^{t+\varepsilon u} - 1) \omega_0^n = 0$

pf. $\gamma_R = \gamma(p_R) \times \star \Rightarrow \int \eta_R (e^{t+\varepsilon u} - 1) \omega_0^n = n \int \eta_R dd^c u \wedge T = -n \int du \wedge d^c \eta_R \wedge T$

$|d\eta_R| \lesssim 1/R \rightsquigarrow |\text{LHS}| \lesssim \frac{1}{R} \int |\nabla u| \omega_0^n$ ($\omega_0 \approx \omega$, but dep on ε)

We want to show $\int |\nabla u| \omega_0^n < +\infty$

Recall: $\int \rho^k |u|^{p-2} |\nabla u|^2 + \int \rho^k |u|^p < +\infty$

$$\left. \begin{array}{l} k=3 \Rightarrow \int \rho^{-k} < +\infty \\ p=2 \Rightarrow \int \rho^{-k} |\nabla u|^2 < +\infty \end{array} \right\} \Rightarrow \int |\nabla u| \leq \left(\int \rho^k |\nabla u|^2 \right)^{1/2} \left(\int \rho^{-k} \right)^{1/2} < +\infty \quad \#$$

Claim 2: $\exists C > 0$ indep of ε s.t. $\left(\int \rho^{\alpha(\beta-2-\delta)-\beta} |u-u_{\beta}|^{2\alpha} \right)^{1/2\alpha} \leq C(\delta)$ for α close to 1

pf Put $v = u - u_{\beta}$

$u \times \star \Rightarrow \int u (e^{t+\varepsilon u} - 1) \omega_0^n = n \int u dd^c u \wedge T = -n \int du \wedge d^c u \wedge T$

$\Rightarrow \int dv \wedge d^c v \wedge T = \int u(1-e^t) \omega_0^n + \int u(1-e^{\varepsilon u}) e^t \omega_0^n$, recall: $u(e^{\varepsilon u} - 1) \gtrsim \varepsilon u^2 \geq 0$

$\Rightarrow \int |\nabla v|^2 \omega_0^n \leq \int |v| |e^t - 1| \omega_0^n$ since $\int u(1-e^t) = \int (u-u_{\beta})(1-e^t) \omega_0^n$
 \uparrow
 $\int (e^t - 1) \omega_0^n = 0$

SOB(β), $\beta \in (0, 2]$

$$\hookrightarrow \left(\int \rho^{\alpha(\beta-2-\delta)-\beta} |v|^{2\alpha} \omega_0^n \right)^{1/2\alpha} \leq C(\delta) \int |\nabla v|^2 \omega_0^n \leq C_\delta \int |v| |e^t - 1| \omega_0^n \leq C_\delta \int |v| \rho^{-\mu} \omega_0^n$$

$$\int |v| \rho^{-\mu} \leq \left(\int |v|^{2\alpha} \rho^{m\eta} \right)^{1/m} \left(\int \rho^{-m^*\eta} \rho^{-m^*\mu} \right)^{1/m^*}$$

need $m=2\alpha$, $m\eta = \alpha(\beta-2-\delta)-\beta$
 $\Rightarrow m^* = \frac{2\alpha}{2\alpha-1}$, $\eta = \frac{\alpha(\beta-2-\delta)-\beta}{2\alpha}$

$$\left(\int \rho^{\alpha(\beta-2-\delta)-\beta} |v|^{2\alpha} \right)^{1/2\alpha} \lesssim \left(\int \rho^{-m^*(\eta+\mu)} \right)^{1/m^*}$$

If $m^*(\eta+\mu) \geq \beta+\delta' \Rightarrow \left(\int \rho^{\alpha(\beta-2-\delta)-\beta} |v|^{2\alpha} \right)^{1/2\alpha} \leq C_{\delta, \delta'}$

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$$\alpha(\beta-2-\delta)-\beta+2\alpha\mu \geq (2\alpha-1)(\beta+\delta')$$

$$\Leftrightarrow \alpha(2(\beta+\delta')-\beta+2+\delta-2\mu) \leq \delta', \text{ true for } \alpha \text{ close to } 1$$

Lemma 1 \exists constant $C > 0$ indep of ε (a supp f) s.t.

(1) If $u_{\beta} \geq 0$, $-C \leq u \leq u_{\beta} + C$

(2) If $u_{\beta} \leq 0$, $-C + u_{\beta} \leq u \leq C$

pf. We show (1) via Moser iteration.

Set $v_t = (u - u_{\beta})_+$.

When $v_t > 0$, $u > u_{\beta} \geq 0$

$\Rightarrow v_t (e^{\varepsilon u} - 1) \geq 0$ on M

$$\hookrightarrow \int |\nabla v_t|^{p_t} \omega_0^n \leq \frac{np^2}{4(p-1)} \int |v_t|^{p_t} |e^t - 1| \omega_0^n$$

SOB(β), $\beta \leq 2$

$$\begin{aligned} \hookrightarrow \left(\int \rho^{\alpha(\beta-2-\delta)-\beta} |v_+^{\frac{p_2}{2}} - (v_+^{\frac{p_2}{2}})_{\psi_s}|^{2\alpha} \right)^{\frac{1}{2\alpha}} &\leq C_s \int |\nabla v_+^{\frac{p_2}{2}}|_{\omega_0}^2 \omega_0^n \\ &\leq C_s \frac{np^2}{4(p-1)} \int |v_+|^{\frac{p_2}{2}} |e^f - 1| \omega_0^n \\ &\leq C_s \frac{np^2}{4(p-1)} \int |v_+|^{\frac{p_2}{2}} \rho^{-\mu} \omega_0^n \end{aligned}$$

Note: Set $\|g\|_{p,\alpha,\delta} := \left(\int |g|^p \rho^{\alpha(\beta-2-\delta)-\beta} \right)^{\frac{1}{p}}$

$$\begin{aligned} \hookrightarrow \|(v_+^{\frac{p_2}{2}})_{\psi_s}\|_{2\alpha,\alpha,\delta} &= \left(\int_M \left\{ \int |v_+|^{\frac{p_2}{2}} \cdot \psi_s \times \left(\int \psi_s \right)^{-1} \right\}^{2\alpha} \cdot \rho^{\alpha(\beta-2-\delta)-\beta} \right)^{\frac{1}{2\alpha}} \\ &= \left(\int \psi_s \right)^{-1} \left(\int \rho^{\alpha(\beta-2-\delta)-\beta} \right)^{\frac{1}{2\alpha}} \int |v_+|^{\frac{p_2}{2}} \cdot \psi_s \\ &\quad = \left(\int \frac{\psi_s^2}{\rho^{\alpha(\beta-2-\delta)-\beta}} \right)^{\frac{1}{2}} \|v_+\|_{p,\alpha,\delta}^{\frac{p_2}{2}} \end{aligned}$$

product of these 3

$\approx \frac{C}{\eta^{\alpha(2-\beta-\delta)}} \text{ if } 1 \leq \alpha < 2 - \frac{2}{2-\beta+\delta}$

$$\Rightarrow \|v_+\|_{\alpha p,\alpha,\delta} \leq \left(\frac{C_{\alpha,\delta,\eta} p^2}{p-1} \right)^{\frac{1}{p}} \|v_+\|_{p,\alpha,\delta}^{1-\frac{1}{p}} + C_{\alpha,\delta,\eta}^{\frac{1}{p}} \|v_+\|_{p,\alpha,\delta}$$

$$\text{w/ } \alpha(2-\beta+\delta) \leq \mu - \beta$$

Then we can iterate from $p_0 = 2\alpha$ (Claim 2) $\Rightarrow |v_+| \leq C_{\mu,\eta}$

$$\Rightarrow u \leq u_{\psi_s} + C$$

To get " $-C \leq u$ ", we replace v_+ in the above argument by $u_- = \min\{u, 0\}$

$$\Rightarrow u_- \cdot (e^{\varepsilon u} - 1) \geq 0 \quad \dots (a)$$

Similarly by SOB(β), $\beta \leq 2$, we have

$$\begin{aligned} \left(\int \rho^{\alpha(\beta-2-\delta)-\beta} |u_-|^{\frac{p_2}{2}} - (u_-^{\frac{p_2}{2}})_{\psi_s}|^{2\alpha} \right)^{\frac{1}{2\alpha}} &\leq C_s \int |\nabla |u_-|^{\frac{p_2}{2}}|_{\omega_0}^2 \omega_0^n \\ &\leq C_s \frac{np^2}{4(p-1)} \int |u_-|^{p-1} |e^f - 1| \omega_0^n \quad \dots (b) \end{aligned}$$

With (a) + (b), argue as before

$$\Rightarrow \|u_-\|_{\alpha p,\alpha,\delta} \leq \left(\frac{C_{\alpha,\delta,\eta} p^2}{p-1} \right)^{\frac{1}{p}} \|u_-\|_{p,\alpha,\delta}^{1-\frac{1}{p}} + C_{\alpha,\delta,\eta}^{\frac{1}{p}} \|u_-\|_{p,\alpha,\delta}$$

$$\text{with } \alpha(2-\beta+\delta) \leq \mu - \beta$$

Since $u_{\gamma_s} \geq 0 \Rightarrow (u - u_{\gamma_s})_- \leq u_-$

Claim 2: $\left(\int \rho^{\alpha(\beta-2-\delta)-\beta} |u - u_{\gamma_s}|^{2\alpha} \right)^{1/2\alpha} \leq C_s$ } $\left(\int \rho^{\alpha(\beta-2-\delta)-\beta} |u|^{2\alpha} \right)^{1/2\alpha} \leq C_s$

$\parallel u_- \parallel_{2\alpha, \alpha, \delta}$

\hookrightarrow Moser iteration $\Rightarrow \|u\| \leq C_{\text{unif}} \#$

Lemma (Yau's C^2 -estimate) $\exists C_1, C_2 > 0$ independent of $\text{supp}(f)$, ε such that
 $0 = \text{tr}_{\omega_0} \omega = n + \Delta_{\omega_0} u \leq C_1 \exp(C_2(u - \inf u))$

Now we consider $f = f_m$ with $\text{supp}(f_m) \not\supset M$, $\int (e^{f_m} - 1) \omega_0^n = 0$

Lemma 2: For each m , $\exists C(m)$ indep of ε s.t. $|u_{m,\varepsilon}| \leq C(m)$

pf. Claim 1: $\int (e^{f_m + 2u_{m,\varepsilon}} - 1) \omega_0^n = 0$ } $\Rightarrow \int e^{f_m} (e^{u_{m,\varepsilon}} - 1) \omega_0^n = 0$

Assumption: $\int (e^{f_m} - 1) \omega_0^n = 0$ } $\Rightarrow \inf u < 0 < \sup u$

Claim: $\exists x_{\max}, x_{\min} \in \text{supp}(f_m)$ s.t. $u(x_{\max}) = \sup u$, $u(x_{\min}) = \inf u$

If not either Φ $\sup u$ was attained at $x_{\max} \notin \text{supp}(f_m)$

or Θ $\exists \{x_k\}_k$, $u(x_k) \rightarrow \sup u$, $x_k \rightarrow \infty$

Φ at x_{\max} $(\omega_0 + dd^c u)^n(x_{\max}) = e^{\varepsilon u(x_{\max})} \omega_0^n(x_{\max}) > \omega_0^n(x_{\max})$ contradiction

Θ Similar argument + Yau's max prin
 Same for \inf . #

If $u_{\gamma_s} \geq 0$, from Lemma 1: $-C \leq u \leq u_{\gamma_s} + C$

Set $v = -\min\{u - \inf u - 1, 0\} \Rightarrow \begin{cases} v(x_{\min}) = 1 \\ 0 \leq v \leq 1 \end{cases}$

$B_r(x_{\min}) =: B_r$, $G(x, y)$: Dirichlet Green's function on B_1

$\eta = \begin{cases} 1 & \text{on } B_{1/2} \\ 0 & \text{outside } B_{3/4} \end{cases}$ a cutoff

Since $-\inf u \leq C$, Yau's C^2 : $n + \Delta u \leq C_1 \exp(C_2(u - \inf u)) \leq C' \exp(C_2 u)$

$\star \times \eta^2 \overset{= G(x)}{G(x_{\min}, x)} v(x)$ (note $G(x) > 0$ on $B_{3/4}$)

$\Rightarrow -\int_{B_1} \Delta v \cdot \eta^2 G \cdot v \leq \int_{B_1} \eta^2 G C' \exp(C_2 u)$

$\leq C \int_{B_1} \eta^2 G v$

$u \leq \inf u + 1$ when $v > 0$
 ≤ 1

$$-\int \Delta v \cdot \eta^2 \cdot v \cdot G = \int \nabla v \cdot \eta^2 v \nabla G + \int |\nabla v|^2 \eta^2 G + \int \nabla v \nabla \eta^2 v \cdot G$$

$$= \frac{1}{2} \int \nabla v^2 \eta^2 \nabla G$$

$$\text{Hence } \frac{1}{2} \int \nabla(\eta^2 v^2) \nabla G = \frac{1}{2} \int \nabla v^2 \eta^2 \nabla G + \frac{1}{2} \int v^2 \nabla \eta^2 \nabla G$$

$$\frac{1}{2} (\eta v)^2(x_{\min}) \stackrel{\parallel}{=} - \int \Delta v \cdot \eta^2 v G - \int |\nabla v|^2 \eta^2 G - \frac{1}{2} \int \nabla v^2 \nabla \eta^2 G + \frac{1}{2} \int v^2 \nabla \eta^2 \nabla G$$

$$\stackrel{1/2}{=} \int v^2 \nabla \eta^2 \nabla G + \frac{1}{2} \int v^2 \Delta \eta^2 G$$

$$\leq C \int_{B_1} v \eta^2 G + C \cdot \int_{B_1} v^2$$

$\uparrow G, \nabla G$ are under control on $B_{3/4} \setminus B_{1/2}$

$\text{supp}(\nabla \eta), \text{supp}(\Delta \eta)$

$$\stackrel{\text{Holder}}{\Rightarrow} 1 \leq C \left(\left(\int_{B_1} |v|^p \right)^{1/p} \left(\int_{B_1} |G|^{p'} \right)^{1/p'} + \int_{B_1} v^2 \right)$$

$$\leq \text{vol}(\text{supp}(v) \cap B_1)$$

$$\leq \text{vol}(\text{supp}(v) \cap B_1) \text{ since } |v| \leq 1$$

$$\Rightarrow \text{vol}(\text{supp}(v) \cap B_1) \geq 1/e.$$

Then

$$\left(\int \psi_s \omega_0^n \right) u_{\psi_s} = \int_M \psi_s u \omega_0^n = \underbrace{\int_{M \setminus B_1 \cap \text{supp}(v)} \psi_s u}_{(I)} + \underbrace{\int_{B_1 \cap \text{supp}(v)} \psi_s u}_{(II)}$$

$$(I) \leq \sup u \left(\int \psi_s \omega_0^n - \int_{B_1 \cap \text{supp}(v)} \psi_s \omega_0^n \right)$$

$$(II) \lesssim \rho(x_{\min})^{-2-\delta} \times (\inf u + 1) \leq C$$

Recall: $u < u_{\psi_s} + C$

$$\text{vol}(B_1 \cap \text{supp}(v)) \geq 1/C$$

$$\sup u \lesssim \frac{C}{\int_{B_1 \cap \text{supp}(v)} \psi_s \omega_0^n}, \quad \int_{B_1 \cap \text{supp}(v)} \psi_s \omega_0^n \geq \frac{1}{C} \rho(x_{\min})^{-2-\varepsilon}$$

$$\Rightarrow \sup u \leq C \cdot \rho^{2+\varepsilon} \left(\sup \{ \text{dist}(y, x_0) \mid y \in \text{supp}(f_m) \} \right)$$

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Now for each m , we have $|u_{m,\varepsilon}| \leq C(m) \rightarrow$ can do C^2 + higher order est. (dep on m)

$$\Rightarrow u_{m,\varepsilon} \rightarrow u_m \text{ \& } |u_m| \leq C(m)$$

$$\text{Prmk: } \int_M |\nabla u_m|^2 \omega_0^n < +\infty : \int u_{m,\varepsilon} \cdot dd^c u_{m,\varepsilon} \wedge T = \int u_{m,\varepsilon} (e^{f_m + \varepsilon u_{m,\varepsilon}} - 1) \omega_0^n$$

$$\Rightarrow \int |\nabla u_{m,\varepsilon}|^2 \omega_0^n \leq C \int |u_{m,\varepsilon}| |e^{f_m} - 1| \omega_0^n < +\infty$$

\uparrow integration by parts formula as before.

Now, we want to show $u_m - (u_m)_{\mathcal{U}_S} \xrightarrow{m \rightarrow \infty} u$: sol'n to (MA)

Only need to show $\|u_m - (u_m)_{\mathcal{U}_S}\|_{L^\infty} \leq C_{\text{unif}}$

$u_m - (u_m)_{\mathcal{U}_S}$ is also a sol'n to $(\omega_0 + dd^c u_m)^n = e^{f_m} \omega_0^n$, we can assume $(u_m)_{\mathcal{U}_S} = 0$

To do Moser iteration as **Lemma 1**, we have

$$\int \mathcal{S} |\nabla u|^{p/2} \omega_0^n = \frac{-n p^2}{4(p-1)} \left\{ \int \mathcal{S} |u|^{p-2} (e^f - 1) \omega_0^n + \int u |u|^{p-2} d\mathcal{S} \wedge du \wedge T \right\}$$

To make sure RHS is finite for $\mathcal{S} = \chi(p/R)$ as $R \rightarrow \infty$,

we need to control $\int |\nabla \mathcal{S}_R| |\nabla u| \omega_0^n \leq \frac{1}{R} \int_{B_{2R} \setminus B_R} |\nabla u| \rightarrow 0$ as $R \rightarrow \infty$

Claim: $\lim_{R \rightarrow \infty} \frac{1}{R} \int_{B_{2R} \setminus B_R} |\nabla u| \omega_0^n = 0$

↑ indeed, $\int_{B_{2R} \setminus B_R} |\nabla u| \omega_0^n \leq \left(\int_{B_{2R} \setminus B_R} |\nabla u|^2 \right)^{1/2} \text{vol}(B_{2R} \setminus B_R)^{1/2}$

$$\text{vol}(B_{2R} \setminus B_R)^{1/2} \sim R^{\beta/2} = a_R \xrightarrow{R \rightarrow \infty} 0$$

$$\hookrightarrow \frac{1}{R} \int_{B_{2R} \setminus B_R} |\nabla u| \leq R^{\beta/2 - 1} a_R \xrightarrow{R \rightarrow \infty} 0 \quad \text{for } \beta \leq 2 \quad \#$$