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Bayesian Econometrics:

Applications in Economics and Finance

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main
+ Normal
Inverse Gamma

Comments on notation

It is common to use upper case letters (ie., X) to denote a random variable (RV) in statistics and a lower case letter to denote a realisation (ie., x) of that random variable. This is a bit of notational overkill and I will only do that in certain occasions when it is important, as, for example, in variable transformations. Also, it is common to distinguish between discrete and continuous RVs when writing down the probability density function (PDF). I will not do so either, and will always use $p(x)$ where X could be either a discrete or continuous RV.

To sum up, I will write $p(x|a,b)$ to denote the PDF of RV X, which can either be discrete or continuous, and which depends on two parameters a and b, rather than the more correct (but also more cumbersome) $p_X(x|a,b)$, where X is a continuous RV and say $\mathbb{P}_X(x|a,b)$ if X is a discrete RV. Also, the notation $X \sim \mathsf{Normal}(\mu, \sigma^2)$ reads as: '*X is distributed as a Normal RV with parameters μ and σ^2*'.

Transformations of random variables

There are a few different ways to transform one (or a set of) random variable to another (ie., based on the distribution functions, the moment generating function, convolution for independent sums of RVs etc. Since we will mainly be using simple inversions and re-scalings of continuous RVs where the transformations will be **one-to-one**, with the inverse function as well as the derivative expected to exist, we can use the following (univariate) transformation relation.

Let X be a RV with PDF $p_X(x)$ (uniform:), and RV $Y = g(X)$ (↳ function). Then, transforming from X to Y

pdf

$$p_Y(y) = p_X(x)\left|\frac{\partial x}{\partial y}\right| \tag{1}$$

random variable goes to a diff. distribution
eg x^2 → chi-sq

where $x = h(y)$ and $h(\cdot)$ is the inverse function of $g(\cdot)$, ie., $g(\cdot)^{-1}$. Note again: $g(\cdot)$ must be **one-to-one**. If it is **many-to-one**, then a sum of terms appears on the right side of (1), with each term corresponding to each possible branch of the inverse.

↳ sum of them

$\frac{\text{Norm RV}}{\sqrt{\text{Gamma RV}}}$ → student dist

As an example, consider the RV X with PDF

gamma function

$$p_X(x|\alpha) = [\Gamma(\alpha)]^{-1}x^{(\alpha-1)}\exp\{-x\},\ \forall\{x,\alpha>0\}\in\mathbb{R}$$

and we are interested in the RV $Y = X\beta$. Then $\partial x/\partial y = 1/\beta$ so that

Gamma with Beta = 1

→ transformation

$$p_Y(y) = p_X\left(x = \frac{y}{\beta}\right)\left|\frac{1}{\beta}\right|$$

$$= [\Gamma(\alpha)]^{-1} \left(\frac{y}{\beta}\right)^{(\alpha-1)} \exp\left\{-\frac{y}{\beta}\right\} \left|\frac{1}{\beta}\right|$$

$$= [\Gamma(\alpha)]^{-1}[\beta^{-\alpha}]y^{(\alpha-1)} \exp\left\{-\frac{y}{\beta}\right\}$$

flexible gamma distri.

which is a $\mathsf{Gamma}(\alpha, \beta)$ density (see below).

Note that, $\Gamma(\alpha) = \int_0^\infty y^{(\alpha-1)} \exp\{-y\}\, dy$ is the Gamma function from mathematics, with the following properties:

$$\Gamma(\alpha + 1) = \alpha\Gamma(\alpha), \forall \alpha > 0 \in \mathbb{R}.$$

When $\alpha = n$ and n is a positive integer (ie., $n \in \mathbb{Z}_{++}$), then we have

$$\Gamma(n) = (n-1)! \text{ (! is the factorial operator) and also } \Gamma(\tfrac{1}{2}) = \sqrt{\pi}, \Gamma(1) = 1.$$

if $\alpha > 1$

Some commonly used Distributions

Uniform

$X \sim \mathsf{Uniform}(a, b)$ if it has the PDF, with $x \in [a, b]$ and $\{a < b\} \in \mathbb{R}$:

$$p(x|a, b) = \frac{1}{b-a} \tag{2}$$

with moments

$$\begin{array}{rcl} E(x) & = & \frac{1}{2}(b+a) \\ \text{Median}(x) & = & \frac{1}{2}(b+a) \\ \text{Mode}(x) & = & \text{any value in } [a,b] \end{array} \qquad \begin{array}{rcl} \text{Var}(x) & = & \frac{1}{12}(b-a)^2 \\ \text{Skewness}(x) & = & 0 \\ \text{Excess Kurtosis}(x) & = & -\frac{6}{5} \end{array}$$

Matlab: $a + (b-a)$`*rand(T,1)` generates a $(T \times 1)$ vector of draws from $\mathsf{Uniform}(a, b)$ in (2).

Normal

$X \sim \mathsf{Normal}(\mu, \sigma^2)$ if it has the PDF, with $x \in \mathbb{R}$ and $\{\sigma^2 > 0, \mu\} \in \mathbb{R}$:

$$p(x|\mu, \sigma^2) = \left(2\pi\sigma^2\right)^{-1/2} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\} \tag{3}$$

with moments

$$\begin{array}{rcl} E(x) & = & \mu \\ \text{Median}(x) & = & \mu \\ \text{Mode}(x) & = & \mu \end{array} \qquad \begin{array}{rcl} \text{Var}(x) & = & \sigma^2 \\ \text{Skewness}(x) & = & 0 \\ \text{Excess Kurtosis}(x) & = & 0 \end{array}$$

Note: With $\mu = 0$ and $\sigma^2 = 1$ we obtain the standard Normal density denoted by $\mathsf{Normal}(0, 1)$.
Matlab: $\mu + \sigma$`*randn(T,1)` generates a $(T \times 1)$ vector of draws from $\mathsf{Normal}(\mu, \sigma^2)$ in (3).

Gamma

$X \sim \mathsf{Gamma}(\alpha, \beta)$ if it has the PDF, with $\{x > 0\} \in \mathbb{R}$ and $\{\alpha > 0, \beta > 0\} \in \mathbb{R}$:[1]

$$p(x|\alpha, \beta) = [\Gamma(\alpha)]^{-1} [\beta^{-\alpha}] x^{(\alpha-1)} \exp\left\{-\frac{x}{\beta}\right\} \tag{4}$$

with moments

$$\begin{aligned} E(x) &= \alpha\beta & \mathrm{Var}(x) &= \alpha\beta^2 \\ \mathrm{Median}(x) &= \text{no closed form} & \mathrm{Skewness}(x) &= \tfrac{2}{\sqrt{\alpha}} \\ \mathrm{Mode}(x) &= (\alpha-1)\beta,\ \alpha \geq 1 & \text{Excess Kurtosis}(x) &= \tfrac{6}{\alpha} \end{aligned}$$

Note: α and β are known as **shape** and **scale** parameters. A Gamma RV can also be defined with $\beta = 1/b$ where b is known as the **inverse scale** or **rate**, so that we get the PDF

$$p(x|\alpha, b) = [\Gamma(\alpha)]^{-1} [b^{\alpha}] x^{(\alpha-1)} \exp\{-xb\}. \tag{5}$$

The moments of (5) are the same as those of (4) with β obviously replaced by $1/b$. We will (mainly) use the definition in (4).
Matlab: `β*randg(α,T,1)` generates a $(T \times 1)$ vector of draws from $\mathsf{Gamma}(\alpha, \beta)$ in (4). That is, $\mathsf{Gamma}(\frac{\nu}{2}, 2) = \mathsf{Chi2}(\nu)$.

Chi-squared

$X \sim \mathsf{Chi2}(\nu)$ if it has the PDF, with $\{x > 0\} \in \mathbb{R}$ and $\nu = 1, 2, 3, \ldots$:

$$p(x|\nu) = \left[\Gamma\left(\tfrac{\nu}{2}\right)\right]^{-1} [2^{-\nu/2}] x^{(\nu/2-1)} \exp\left\{-\frac{x}{2}\right\} \tag{6}$$

with moments

$$\begin{aligned} E(x) &= \nu & \mathrm{Var}(x) &= 2\nu \\ \mathrm{Median}(x) &= \nu\left(1 - \tfrac{2}{9k}\right)^3 & \mathrm{Skewness}(x) &= \tfrac{8}{\sqrt{\nu}} \\ \mathrm{Mode}(x) &= \max\{\nu - 2, 0\} & \text{Excess Kurtosis}(x) &= \tfrac{12}{\nu} \end{aligned}$$

Note: The parameter ν is the **degrees of freedom** parameter. A $\mathsf{Chi2}(\nu)$ RV is a $\mathsf{Gamma}(\alpha, \beta)$ RV with $\alpha = \nu/2$ and $\beta = 2$.
Matlab: `2*randg(ν/2,T,1)` generates a $(T \times 1)$ vector of draws from $\mathsf{Chi2}(\nu)$ in (6).

[1] In Koop's (2003) textbook, a Gamma RV is expressed in terms of mean μ and degrees of freedom ν.

Inverse Gamma

$X \sim \mathsf{InvGam}(\alpha, \beta)$ if it has the PDF, with $\{x > 0\} \in \mathbb{R}$ and $\{\alpha > 0, \beta > 0\} \in \mathbb{R}$:[2]

$$p(x|\alpha,\beta) = [\Gamma(\alpha)]^{-1}[\beta^{\alpha}]x^{-(\alpha+1)} \exp\left\{-\frac{\beta}{x}\right\} \tag{7}$$

with moments

$$\begin{aligned} E(x) &= \tfrac{\beta}{\alpha-1},\ \text{iff } \alpha > 1 & \mathrm{Var}(x) &= \tfrac{\beta^2}{(\alpha-1)^2(\alpha-2)},\ \text{iff } \alpha > 2 \\ \mathrm{Median}(x) &= \text{no closed form} & \mathrm{Skewness}(x) &= \tfrac{4\sqrt{\alpha-2}}{\alpha-3},\ \text{iff } \alpha > 3 \\ \mathrm{Mode}(x) &= \tfrac{\beta}{\alpha+1} & \text{Excess Kurtosis}(x) &= \tfrac{30\alpha-66}{(\alpha-3)(\alpha-4)},\ \text{iff } \alpha > 4 \end{aligned}$$

Note: α and β are again known as **shape** and **scale** parameters. If $Y \sim \mathsf{Gamma}(\alpha, b)$ is from the alternative definition in (5), with $b = \frac{1}{\beta}$, then $X = \frac{1}{Y} \sim \mathsf{InvGam}(\alpha, \beta)$.
Matlab: `β./randg(α,T,1)` generates a $(T \times 1)$ vector of draws from $\mathsf{InvGam}(\alpha, \beta)$ in (7).

Beta

$X \sim \mathsf{Beta}(\alpha, \beta)$ if it has the PDF, with $x \in [0, 1]$ and $\{\alpha > 0, \beta > 0\} \in \mathbb{R}$:

$$p(x|\alpha,\beta) = \mathcal{B}(\alpha,\beta)^{-1}x^{(\alpha-1)}(1-x)^{(\beta-1)} \tag{8}$$

with moments

$$\begin{aligned} E(x) &= \tfrac{\alpha}{\alpha+\beta} & \mathrm{Var}(x) &= \tfrac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} \\ \mathrm{Median}(x) &\approx \begin{array}{l}\text{no closed form,}\\ \tfrac{\alpha-1/3}{\alpha+\beta-2/3} \text{ for } \alpha, \beta, > 1\end{array} & \mathrm{Skewness}(x) &= \tfrac{2(\beta-\alpha)\sqrt{\alpha+\beta+1}}{(\alpha+\beta+2)\sqrt{\alpha\beta}} \\ \mathrm{Mode}(x) &= \tfrac{\alpha-1}{\alpha+\beta-2} \text{ for } \alpha, \beta, > 1 & \text{Excess Kurtosis}(x) &= \tfrac{6(\alpha-\beta)^2(\alpha+\beta+1)-\alpha\beta(\alpha+\beta+2)}{\alpha\beta(\alpha+\beta+2)(\alpha+\beta+3)} \end{aligned}$$

Note: α and β are **shape** parameters in the $\mathsf{Beta}(\alpha, \beta)$ density and $\mathcal{B}(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\alpha)}{\Gamma(\alpha+\beta)}$ is the Beta Function from mathematics.
Matlab: `betarnd(α,β,T,1)` generates a $(T \times 1)$ vector of draws from $\mathsf{Beta}(\alpha, \beta)$ in (8).

[2]Note that when deriving the $\mathsf{InvGam}(\alpha, \beta)$ PDF using the transformations of RV technique as discussed above, we need to start from the alternative definition of the Gamma density in (5) to arrive at $\mathsf{InvGam}(\alpha, \beta)$ in terms of shape and scale.

Student's t

$X \sim \text{Students}(\nu, \mu, \sigma^2)$ if it has the PDF, with $x \in \mathbb{R}$ and $\{\sigma^2 > 0, \mu\} \in \mathbb{R}$ and $\nu = 1, 2, 3, \ldots$:

$$p(x|\nu, \mu, \sigma^2) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \left(\nu\pi\sigma^2\right)^{-1/2} \left[1 + \frac{1}{\nu}\frac{(x-\mu)^2}{\sigma^2}\right]^{-\left(\frac{\nu+1}{2}\right)} \tag{9}$$

with moments

$$\begin{aligned} E(x) &= \mu, \text{ iff } \nu > 1 & \text{Var}(x) &= \sigma^2 \tfrac{\nu}{\nu-2}, \text{ iff } \nu > 2 \\ \text{Median}(x) &= \mu & \text{Skewness}(x) &= 0, \text{ iff } \nu > 3 \\ \text{Mode}(x) &= \mu & \text{Excess Kurtosis}(x) &= \tfrac{6}{\nu-4}, \text{ iff } \nu > 4 \end{aligned}$$

Note: μ, σ^2 and ν are **location**, **scale** and **degrees of freedom** parameters in the Student's t distribution. With $\mu = 0$ and $\sigma^2 = 1$, we obtain the standard Student's t distribution. Also, sometimes the Student's t distribution is expressed as (see for Example in Koop (2003)):

$$p(x|\nu, \mu, \sigma^2) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \left(\pi\sigma^2\right)^{-1/2} \nu^{\frac{\nu}{2}} \left[\nu + \frac{(x-\mu)^2}{\sigma^2}\right]^{-\left(\frac{\nu+1}{2}\right)}. \tag{10}$$

That is, the degrees of freedom parameter $(1/\nu)^{-(\nu+1)/2}$ is taken out of the kernel of the PDF and the normalising constant is re-weighted by $\nu^{(\nu+1)/2}$.
Matlab: $\mu + \sigma$`*trnd(`ν`,T,1)` generates a $(T \times 1)$ vector of draws from $\text{Students}(\nu, \mu, \sigma^2)$ in (9).

Multivariate Normal

$\mathbf{X} \sim \text{MNorm}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ if it has the PDF, with $\underset{(k\times 1)}{\mathbf{x}} = (x_1\ x_2\ \ldots\ x_k) \in \mathbb{R}^k$, $\boldsymbol{\mu} \in \mathbb{R}^k$ and $\boldsymbol{\Sigma}$ a $(k \times k)$ positive definite matrix:

$$p(\mathbf{x}|\boldsymbol{\mu}, \boldsymbol{\Sigma}) = (2\pi)^{-k/2} \det(\boldsymbol{\Sigma})^{-1/2} \exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1} (\mathbf{x}-\boldsymbol{\mu})\right\} \tag{11}$$

with moments

$$\begin{aligned} E(\mathbf{x}) &= \boldsymbol{\mu} & \text{Var}(\mathbf{x}) &= \boldsymbol{\Sigma} \\ \text{Median}(\mathbf{x}) &= \boldsymbol{\mu} & \text{Skewness}(\mathbf{x}) &= 0 \\ \text{Mode}(\mathbf{x}) &= \boldsymbol{\mu} & \text{Excess Kurtosis}(\mathbf{x}) &= 0 \end{aligned}$$

Note: If $\boldsymbol{\Sigma}$ is a diagonal matrix then $\{x_i\}_{i=1}^k$ are uncorrelated and with a multivariate normal distribution this means independent. Also, if $\boldsymbol{\Sigma} = \sigma^2 \mathbf{I}_{(k\times k)}$ then $\det(\boldsymbol{\Sigma})^{-1/2} = \left(\sigma^2\right)^{-k/2}$ because for a general $(n \times n)$ matrix $\mathbf{A}$ we have: $\det(c\mathbf{A}_{(n\times n)}) = c^n \det(\mathbf{A}_{(n\times n)})$.
Matlab: `mvnrnd(`$\boldsymbol{\mu}$`,`$\boldsymbol{\Sigma}$`,T)` generates a $(T \times k)$ vector of draws from $\text{MNorm}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ in (11).

Multivariate Student's t

$\mathbf{X} \sim \text{Mt}(\nu, \boldsymbol{\mu}, \boldsymbol{\Sigma})$ if it has the PDF, with $\underset{(k\times 1)}{\mathbf{x}} = (x_1\ x_2\ \ldots\ x_k) \in \mathbb{R}^k$, $\boldsymbol{\mu} \in \mathbb{R}^k$, $\boldsymbol{\Sigma}$ a $(k \times k)$ positive definite matrix and $\nu = 1, 2, 3, \ldots$:

$$p(\mathbf{x}|\nu, \boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{\Gamma\left(\frac{\nu+k}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} (\nu\pi)^{-k/2} \det(\boldsymbol{\Sigma})^{-1/2} \left[1 + \frac{1}{\nu}(\mathbf{x}-\boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right]^{-\left(\frac{\nu+k}{2}\right)} \tag{12}$$

with moments

$$\begin{aligned} E(\mathbf{x}) &= \boldsymbol{\mu},\ \text{iff } \nu > 1 & \text{Var}(\mathbf{x}) &= \boldsymbol{\Sigma}\tfrac{\nu}{\nu-2},\ \text{iff } \nu > 2 \\ \text{Median}(\mathbf{x}) &= \boldsymbol{\mu} & \text{Skewness}(\mathbf{x}) &= 0,\ \text{iff } \nu > 3 \\ \text{Mode}(\mathbf{x}) &= \boldsymbol{\mu} & \text{Excess Kurtosis}(\mathbf{x}) &= \tfrac{6}{\nu-4},\ \text{iff } \nu > 4 \end{aligned}$$

Note: $\boldsymbol{\mu}$, $\boldsymbol{\Sigma}$ and ν are again **location**, **matrix scale** and **degrees of freedom** parameters. Again, an alternative way to express the Multivariate Student's t distribution is to write it as:

$$p(\mathbf{x}|\nu, \boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{\Gamma\left(\frac{\nu+k}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \pi^{-k/2} \det(\boldsymbol{\Sigma})^{-1/2} \nu^{\frac{\nu}{2}} \left[\nu + (\mathbf{x}-\boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right]^{-\left(\frac{\nu+k}{2}\right)}.$$

Matlab: `(μ+mvtrnd(Σ,ν,T)')'` generates a $(T \times k)$ vector of draws from $\text{Mt}(\nu, \boldsymbol{\mu}, \boldsymbol{\Sigma})$ in (12).

Wishart

$\mathbf{W} \sim \text{Wishart}(\nu, \boldsymbol{\Sigma})$ if it has the PDF, with $\mathbf{w}$ and $\boldsymbol{\Sigma}$ being $(k \times k)$ dimensional positive definite matrices and $\{\nu > k-1\} = 1, 2, 3, \ldots$:

$$p(\mathbf{w}|\nu, \boldsymbol{\Sigma}) = \left[2^{\frac{\nu k}{2}} \boldsymbol{\Gamma}_k\left(\frac{\nu}{2}\right) \det(\boldsymbol{\Sigma})^{\nu/2}\right]^{-1} \det(\mathbf{w})^{\frac{(\nu-k-1)}{2}} \exp\left\{-\frac{1}{2}\text{tr}\left(\boldsymbol{\Sigma}^{-1}\mathbf{w}\right)\right\} \tag{13}$$

with moments

$$\begin{aligned} E(\mathbf{w}) &= \nu\boldsymbol{\Sigma} & \text{Var}(\mathbf{w}_{ij}) &= \nu(\sigma_{ij}^2 + \sigma_{ii}\sigma_{jj}), \forall i, j = 1, \ldots, k \\ \text{Median}(\mathbf{w}) &= \text{no closed form} & \text{Skewness}(\mathbf{w}) &= \text{no closed form} \\ \text{Mode}(\mathbf{w}) &= (\nu-k-1)\boldsymbol{\Sigma} & \text{Excess Kurtosis}(\mathbf{w}) &= \text{no closed form} \end{aligned}$$

Note: $\boldsymbol{\Sigma}$ and ν are again **matrix scale** and **degrees of freedom** parameters. The term $\boldsymbol{\Gamma}_k\left(\frac{\nu}{2}\right) = \pi^{k(k-1)/4}\prod_{j=1}^{k}\Gamma\left[\nu/2 + (1-j)/2\right]$ is the multivariate Gamma function, $\text{tr}(\cdot)$ is the trace operator (sum of diagonal elements) and σ_{ij} denotes the i^{th}, j^{th} element in the covariance matrix $\boldsymbol{\Sigma}$.

Matlab: `wishrnd(Σ,ν)` generates a $(k \times k)$ vector of draws from $\text{Wishart}(\nu, \boldsymbol{\Sigma})$ in (13).

Some useful integrals

$$\int_a^b c\,dx = c(b-a)$$

$$\underbrace{\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\} dx}_{\text{Normal Integral}} = \left(2\pi\sigma^2\right)^{1/2}$$

$$\underbrace{\int_0^{\infty} x^{(\alpha-1)} \exp\left\{-\frac{x}{\beta}\right\} dx}_{\text{Gamma Integral}} = \Gamma(\alpha)\beta^{\alpha}$$

$$\underbrace{\int_0^{\infty} x^{-(\alpha+1)} \exp\left\{-\frac{\beta}{x}\right\} dx}_{\text{Inverse Gamma Integral}} = \Gamma(\alpha)\beta^{-\alpha}$$

$$\underbrace{\int_0^{\infty} x^{(\nu/2-1)} \exp\left\{-\frac{x}{2}\right\} dx}_{\text{Chi-Square Integral}} = \Gamma(\nu/2)2^{\nu/2}$$

$$\underbrace{\int_0^{1} x^{(\alpha-1)}(1-x)^{(\beta-1)} dx}_{\text{Beta Integral}} = \mathcal{B}(\alpha,\beta)$$

$$\underbrace{\int_{-\infty}^{\infty} \left[1+\frac{1}{\nu}\frac{(x-\mu)^2}{\sigma^2}\right]^{-\left(\frac{\nu+1}{2}\right)} dx}_{\text{Student's t Integral}} = \frac{\Gamma\left(\frac{\nu}{2}\right)}{\Gamma\left(\frac{\nu+1}{2}\right)}\left(\nu\pi\sigma^2\right)^{1/2}$$

$$\underbrace{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \left[1+\frac{1}{\nu}(\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right]^{-\left(\frac{\nu+k}{2}\right)} dx_1\ldots dx_k}_{\text{Multivariate t Integral}} = \frac{\Gamma\left(\frac{\nu}{2}\right)}{\Gamma\left(\frac{\nu+k}{2}\right)}(\nu\pi)^{k/2}\det(\boldsymbol{\Sigma})^{1/2}$$

$$\underbrace{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right\} dx_1\ldots dx_k}_{\text{Multivariate Normal Integral}} = (2\pi)^{k/2}\det(\boldsymbol{\Sigma})^{1/2}$$

$$\underbrace{\int_0^{\infty}\cdots\int_0^{\infty} \det(\mathbf{w})^{\frac{(\nu-p-1)}{2}} \exp\left\{-\frac{1}{2}\mathrm{tr}\left(\boldsymbol{\Sigma}^{-1}\mathbf{w}\right)\right\} dw_1\ldots dw_k}_{\text{Wishart Integral}} = \left[2^{\frac{\nu k}{2}}\boldsymbol{\Gamma}_k\left(\frac{\nu}{2}\right)\det(\boldsymbol{\Sigma})^{\nu/2}\right]$$

Quick reference for densities

$$\text{Uniform}(a,b): \quad p(x|a,b) = \frac{1}{b-a}$$

$$\text{Normal}(\mu,\sigma^2): \quad p(x|\mu,\sigma^2) = \left(2\pi\sigma^2\right)^{-1/2} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\}$$

$$\text{Chi2}(\nu): \quad p(x|\nu) = \left[\Gamma\left(\tfrac{\nu}{2}\right)\right]^{-1} [b^{-\nu/2}] x^{(\nu/2-1)} \exp\left\{-\frac{x}{2}\right\}$$

$$\text{Gamma}(\alpha,\beta): \quad p(x|\alpha,\beta) = [\Gamma(\alpha)]^{-1}[\beta^{-\alpha}]x^{(\alpha-1)} \exp\left\{-\frac{x}{\beta}\right\}$$

$$\text{Alt. Gamma}(\alpha,b): \quad p(x|\alpha,b) = [\Gamma(\alpha)]^{-1}\,[b^{\alpha}]x^{(\alpha-1)} \exp\{-xb\}$$

$$\text{InvGam}(\alpha,\beta): \quad p(x|\alpha,\beta) = [\Gamma(\alpha)]^{-1}[\beta^{\alpha}]x^{-(\alpha+1)} \exp\left\{-\frac{\beta}{x}\right\}$$

$$\text{Beta}(\alpha,\beta): \quad p(x|\alpha,\beta) = \mathcal{B}(\alpha,\beta)^{-1}x^{(\alpha-1)}(1-x)^{(\beta-1)}$$

$$\text{Students}(\nu,\mu,\sigma^2): \quad p(x|\nu,\mu,\sigma^2) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \left(\nu\pi\sigma^2\right)^{-1/2} \left[1 + \frac{1}{\nu}\frac{(x-\mu)^2}{\sigma^2}\right]^{-\left(\frac{\nu+1}{2}\right)}$$

$$\text{Alt. Students}(\nu,\mu,\sigma^2): \quad p(x|\nu,\mu,\sigma^2) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \left(\pi\sigma^2\right)^{-1/2} \nu^{\frac{\nu}{2}} \left[\nu + \frac{(x-\mu)^2}{\sigma^2}\right]^{-\left(\frac{\nu+1}{2}\right)}$$

$$\text{MNorm}(\boldsymbol{\mu},\boldsymbol{\Sigma}): \quad p(\mathbf{x}|\boldsymbol{\mu},\boldsymbol{\Sigma}) = (2\pi)^{-k/2}\det(\boldsymbol{\Sigma})^{-1/2} \exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})'\,\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right\}$$

$$\text{Mt}(\nu,\boldsymbol{\mu},\boldsymbol{\Sigma}): \quad p(\mathbf{x}|\nu,\boldsymbol{\mu},\boldsymbol{\Sigma}) = \frac{\Gamma\left(\frac{\nu+k}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} (\nu\pi)^{-k/2}\det(\boldsymbol{\Sigma})^{-1/2} \left[1 + \frac{1}{\nu}(\mathbf{x}-\boldsymbol{\mu})'\,\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right]^{-\left(\frac{\nu+k}{2}\right)}$$

$$\text{Alt. Mt}(\nu,\boldsymbol{\mu},\boldsymbol{\Sigma}): \quad p(\mathbf{x}|\nu,\boldsymbol{\mu},\boldsymbol{\Sigma}) = \frac{\Gamma\left(\frac{\nu+k}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \pi^{-k/2}\det(\boldsymbol{\Sigma})^{-1/2}\nu^{\frac{\nu}{2}} \left[\nu + (\mathbf{x}-\boldsymbol{\mu})'\,\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right]^{-\left(\frac{\nu+k}{2}\right)}$$

$$\text{Wishart}(\nu,\boldsymbol{\Sigma}): \quad p(\mathbf{w}|\nu,\boldsymbol{\Sigma}) = \left[2^{\frac{\nu k}{2}}\boldsymbol{\Gamma}_k\left(\frac{\nu}{2}\right)\det(\boldsymbol{\Sigma})^{\nu/2}\right]^{-1}\det(\mathbf{w})^{\frac{(\nu-k-1)}{2}} \exp\left\{-\frac{1}{2}\text{tr}\left(\boldsymbol{\Sigma}^{-1}\mathbf{w}\right)\right\}$$