CS 514 HW 1 (due Sep 25)

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From Foundations

Ex 3.6 (1) $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $k=3$ (steps). We have $A = \begin{bmatrix} 1 & 2 \\ -1 & 2 \\ 1 & -2 \\ -1 & -2 \end{bmatrix}$

let $B = A^T A = \begin{bmatrix} 4 & 0 \\ 0 & 16 \end{bmatrix}$

let $x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$,

then $x_1 = B x_0 = \begin{bmatrix} 4 & 0 \\ 0 & 16 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 16 \end{bmatrix}$

$x_2 = B x_1 = \begin{bmatrix} 4 & 0 \\ 0 & 16 \end{bmatrix} \begin{bmatrix} 4 \\ 16 \end{bmatrix} = \begin{bmatrix} 16 \\ 256 \end{bmatrix}$

$x_3 = B x_2 = \begin{bmatrix} 4 & 0 \\ 0 & 16 \end{bmatrix} \begin{bmatrix} 16 \\ 256 \end{bmatrix} = \begin{bmatrix} 64 \\ 16^3 \end{bmatrix} = 64 \begin{bmatrix} 1 \\ 64 \end{bmatrix}$

so we estimate $v_1 \approx \frac{1}{\|x_3\|} x_3 = \frac{1}{\sqrt{1+64^2}} \begin{bmatrix} 1 \\ 64 \end{bmatrix}$

$= \begin{bmatrix} 0.0156 \\ 0.9999 \end{bmatrix}$

(2) We find the eigenvals of B: set $|B - \lambda I| = 0$

$$\begin{vmatrix} 4-\lambda & 0 \\ 0 & 16-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)(16-\lambda) = 0$$

$$\lambda = 16 \text{ or } \lambda = 4$$

Hence eigenvals of B: $\lambda_1 = 16$, $\lambda_2 = 4$

for λ_1, $B \begin{bmatrix} x \\ y \end{bmatrix} = \lambda_1 \begin{bmatrix} x \\ y \end{bmatrix}$

$$\begin{bmatrix} 4 & 0 \\ 0 & 16 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 16 \begin{bmatrix} x \\ y \end{bmatrix}$$

$4x = 16x$ —— ①

$16y = 16y$ —— ②

So $x = 0, \; y = 1$

hence $V_1 = [0 \quad 1]^T$ is the eigenvector corresponding to λ_1

for λ_2, $B \begin{bmatrix} x \\ y \end{bmatrix} = \lambda_2 \begin{bmatrix} x \\ y \end{bmatrix}$

$$\begin{bmatrix} 4 & 0 \\ 0 & 16 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 4 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$4x = 4x$$
$$16y = 4y$$

so $x = 1, \; y = 0$

hence $V_2 = [1 \quad 0]^T$ is the eigenvector corresponding to λ_2

singular values of A: $\sigma_1 = \sqrt{\lambda_1} = 4$

$\sigma_2 = \sqrt{\lambda_2} = 2$

$$AV_1 = \begin{bmatrix} 1 & 2 \\ -1 & 2 \\ 1 & -2 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -2 \\ -2 \end{bmatrix}$$

so $u_1 = \frac{1}{\sigma_1} (AV_1) = [\frac{1}{2} \quad \frac{1}{2} \quad -\frac{1}{2} \quad -\frac{1}{2}]^T$

and $u_2 = \frac{1}{\sigma_2} (AV_2) = \frac{1}{2} \begin{bmatrix} 1 & 2 \\ -1 & 2 \\ 1 & -2 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$= [\frac{1}{2} \quad -\frac{1}{2} \quad \frac{1}{2} \quad -\frac{1}{2}]^T$

Hence, $\sigma_1 = 4, \; u_1 = [\frac{1}{2} \quad \frac{1}{2} \quad -\frac{1}{2} \quad -\frac{1}{2}]^T, \; v_1 = [0 \quad 1]^T$

$\sigma_2 = 2, \; u_2 = [\frac{1}{2} \quad -\frac{1}{2} \quad \frac{1}{2} \quad -\frac{1}{2}]^T, \; V_2 = [1 \quad 0]^T$

(3) V_1 would represent how much each ~~restaurants~~ restaurant conforms to a given "theme" or "concept".

u_1 represents how much each person likes the given "theme" of restaurants.

the gap $\sigma_1 - \sigma_2$ represents how much more significant the first "theme" is compared to the second "theme", in our data matrix.

Ex 3.16 $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$B = A^T A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 10 & 14 \\ 14 & 20 \end{bmatrix}$

let $x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$x_1 = \frac{Bx_0}{\|Bx_0\|} = \begin{bmatrix} 0.5767 \\ 0.8170 \end{bmatrix}$.

$x_2 = \frac{Bx_1}{\|Bx_1\|} = \begin{bmatrix} 0.5761 \\ 0.8174 \end{bmatrix}$

$x_3 = \frac{Bx_2}{\|Bx_2\|} = \begin{bmatrix} 0.5760 \\ 0.8174 \end{bmatrix}$.

v_1 of $A = v_1$ of $B = x_3$

$Av_1 = \begin{bmatrix} 2.2109 \\ 4.9978 \end{bmatrix} \xrightarrow{\text{Normalize}} \sigma_1 = \sqrt{2.2109^2 + 4.9978^2} = 5.465$.

$u_1 = \frac{Av_1}{\sigma_1} = \begin{bmatrix} 0.4046 \\ 0.9145 \end{bmatrix}$.

$A' = A - \sigma_1 u_1 v_1^T = \begin{bmatrix} -0.2736 & 0.1928 \\ 0.1210 & -0.0853 \end{bmatrix}$

$B' = A'^T A' = \begin{bmatrix} 0.0895 & -0.0631 \\ -0.0631 & 0.0444 \end{bmatrix}$, let $x_0' = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$x_3' = B'(B'(B'x_1)) = \begin{bmatrix} 0.8174 \\ -0.5760 \end{bmatrix} = v_2$.

$\Rightarrow \sigma_2 = 0.3660, \quad u_2 = \begin{bmatrix} -0.9145 \\ 0.4046 \end{bmatrix}$.

$A = \sum_i \sigma_i u_i v_i^T = \begin{bmatrix} 0.4046 & -0.9145 \\ 0.9145 & 0.4046 \end{bmatrix}\begin{bmatrix} 5.465 & 0 \\ 0 & 0.366 \end{bmatrix}\begin{bmatrix} 0.5760 & 0.8174 \\ 0.8174 & -0.5760 \end{bmatrix}$

Ex 4.3

(a)

$$M = \begin{bmatrix} 0.4 & 0.4 & 0 & 0 & 0 \\ 0.6 & 0 & 0.4 & 0 & 0 \\ 0 & 0.6 & 0 & 0.4 & 0 \\ 0 & 0 & 0.6 & 0 & 0.4 \\ 0 & 0 & 0 & 0.6 & 0.6 \end{bmatrix}$$

set $v_0 = \begin{bmatrix} 0.2 \\ 0.2 \\ 0.2 \\ 0.2 \\ 0.2 \end{bmatrix}$

$$Mv_0 = \begin{bmatrix} 0.16 \\ 0.2 \\ 0.2 \\ 0.2 \\ 0.24 \end{bmatrix} \Rightarrow M(Mv_0) = \begin{bmatrix} 0.144 \\ 0.176 \\ 0.2 \\ 0.216 \\ 0.264 \end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix} 0.0758 \\ 0.1137 \\ 0.1706 \\ 0.2559 \\ 0.3839 \end{bmatrix}$$

(b)

$$M = \begin{bmatrix} 0.5 & 0 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 & 1 \end{bmatrix}$$

set $v_0 = \begin{bmatrix} 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \end{bmatrix}^T$.

$$v = \begin{bmatrix} 0.2 \\ 0.2 \\ 0.2 \\ 0.2 \\ 0.2 \end{bmatrix} \rightarrow \begin{bmatrix} 0.1 \\ 0.2 \\ 0.2 \\ 0.2 \\ 0.3 \end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Ex 11.2.1

(a) $$M^TM = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \\ 4 & 16 \end{bmatrix} = \begin{bmatrix} 30 & 100 \\ 100 & 354 \end{bmatrix}$$

$$MM^T = \begin{bmatrix} 2 & 6 & 12 & 20 \\ 6 & 20 & 42 & 72 \\ 12 & 42 & 90 & 156 \\ 20 & 72 & 156 & 272 \end{bmatrix}$$

(b)

$$|M^TM - \lambda I| = 0$$

$$\Rightarrow \lambda^2 - 384\lambda + 620 = 0$$

① $\Rightarrow \lambda_1 = 382.38$

$$\begin{bmatrix} 30 & 100 \\ 100 & 354 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 382.38 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \begin{cases} 30x + 100y = 382.38x \\ 100x + 354y = 382.38y \end{cases}$$

normalized eigenvector $V_1 = \begin{bmatrix} 0.273 \\ 0.962 \end{bmatrix}$

②

$\Rightarrow \lambda_2 = 1.62$

$$\begin{bmatrix} 30 & 100 \\ 100 & 354 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 1.62 \begin{bmatrix} x \\ y \end{bmatrix}$$

normalized eigenvector $V_2 = \begin{bmatrix} -0.962 \\ 0.273 \end{bmatrix}$

(c) $\lambda_1 = 382.38,\ \lambda_2 = 1.62,\ \lambda_3 = 0,\ \lambda_4 = 0$

(d) $MM^T v = \lambda v$

~~$\lambda_1 = 38$~~ $$\begin{bmatrix} 2 & 6 & 12 & 20 \\ 6 & 20 & 42 & 72 \\ 12 & 42 & 90 & 156 \\ 20 & 72 & 156 & 272 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \lambda_1 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \Rightarrow V_1 = \begin{bmatrix} 0.063 \\ 0.224 \\ 0.484 \\ 0.842 \end{bmatrix}$$ while $\lambda_1 = 382.38$

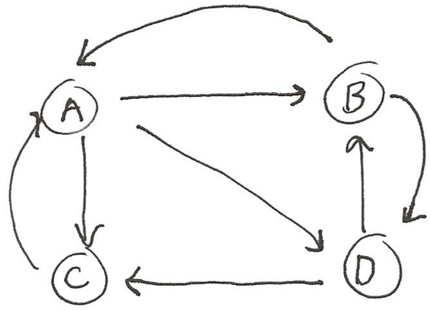
$\lambda_2 = 1.62,\quad V_2 = [0.541 \quad 0.653 \quad 0.336 \quad -0.408]^T$

$\lambda_3 = 0,\quad V_3 = [0.806 \quad -0.263 \quad -0.454 \quad 0.271]^T$

$\lambda_4 = 0,\quad V_4 = [0.512 \quad 0.187 \quad -0.761 \quad 0.349]^T$

5.5.1

we have link matrix
$$L = \begin{array}{c|cccc} \text{from} \backslash \text{to} & A & B & C & D \\ \hline A & & & & \\ B & & & & \\ C & & & & \\ D & & & & \end{array} \quad \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$



let $h_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. then $a_0 = L^T h_0 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}$

$$h_1 = L a_0 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \\ 4 \end{bmatrix}$$

$$a_1 = L^T h_1 = \begin{bmatrix} 6 \\ 10 \\ 10 \\ 10 \end{bmatrix} \xrightarrow{\text{scale}} a_1 = \begin{bmatrix} 6/10 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$h_2 = L a_1 = \begin{bmatrix} 3 \\ 1.6 \\ 0.6 \\ 2 \end{bmatrix}$$

$$a_2 = L^T h_2 = \begin{bmatrix} 2.2 \\ 5 \\ 5 \\ 4.6 \end{bmatrix} \xrightarrow{\text{scale}} a_2 = \begin{bmatrix} 0.44 \\ 1 \\ 1 \\ 0.92 \end{bmatrix}$$

$$h_3 = L a_2 = \begin{bmatrix} 2.92 \\ 1.36 \\ 0.44 \\ 2 \end{bmatrix}$$

$$a_3 = L^T h_3 = \begin{bmatrix} 1.8 \\ 4.92 \\ 4.92 \\ 4.28 \end{bmatrix} \xrightarrow{\text{scale}} a_3 = \begin{bmatrix} 0.3659 \\ 1 \\ 1 \\ 0.8699 \end{bmatrix}$$

$$h_4 = L a_3 = \begin{bmatrix} 2.87 \\ 1.24 \\ 0.37 \\ 2 \end{bmatrix}$$

$$a_4 = L^T h_4 = \begin{bmatrix} 1.6016 \\ 4.8699 \\ 4.8699 \\ 4.1057 \end{bmatrix} \xrightarrow{\text{scale}} a_4 = \begin{bmatrix} 0.3289 \\ 1 \\ 1 \\ 0.8431 \end{bmatrix}$$

⋮ a few iterations in Python

Hence, $h = \begin{bmatrix} 2.81 \\ 1.10 \\ 0.29 \\ 2 \end{bmatrix}$, and $a = \begin{bmatrix} 0.29 \\ 1 \\ 1 \\ 0.81 \end{bmatrix}$

Coding Question Results

3.17

1. The first left singular vector of a matrix A:

$$v1 = \begin{bmatrix} 0.31975064 \\ 0.36962506 \\ 0.39811311 \\ 0.40391891 \\ 0.38728042 \\ 0.34995868 \\ 0.29512622 \\ 0.22716235 \\ 0.15136861 \\ 0.07362361 \end{bmatrix}$$

   The first right singular vector of a matrix A: $v1.T$

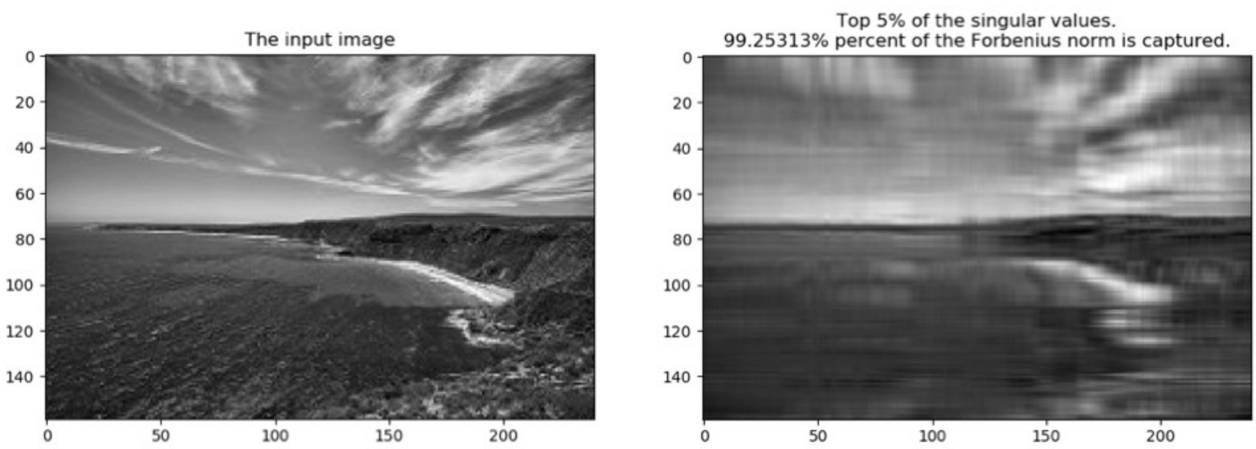
2. The first four left singular vectors of matrix A:

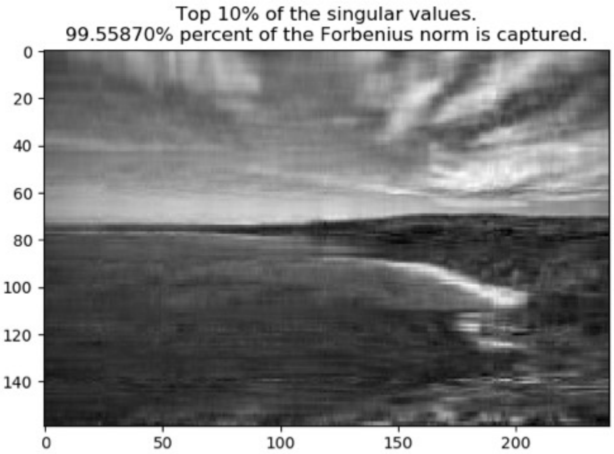
$$V_4 = \begin{bmatrix} -0.3197506 & -0.45784552 & -0.42415456 & -0.39363343 \\ -0.36962502 & -0.3936509 & -0.24288284 & -0.02849202 \\ -0.39811309 & -0.25497036 & 0.07043602 & 0.36152052 \\ -0.4039189 & -0.06980555 & 0.33936334 & 0.38322642 \\ -0.38728043 & 0.12450888 & 0.41233765 & 0.01440651 \\ -0.3499587 & 0.2887672 & 0.24775341 & -0.37382104 \\ -0.29512626 & 0.38972804 & -0.06268814 & -0.39083236 \\ -0.22716239 & 0.40675711 & -0.34546459 & -0.0199787 \\ -0.15136864 & 0.33594883 & -0.44265159 & 0.36463826 \\ -0.07362363 & 0.19090282 & -0.30058613 & 0.37499533 \end{bmatrix}$$

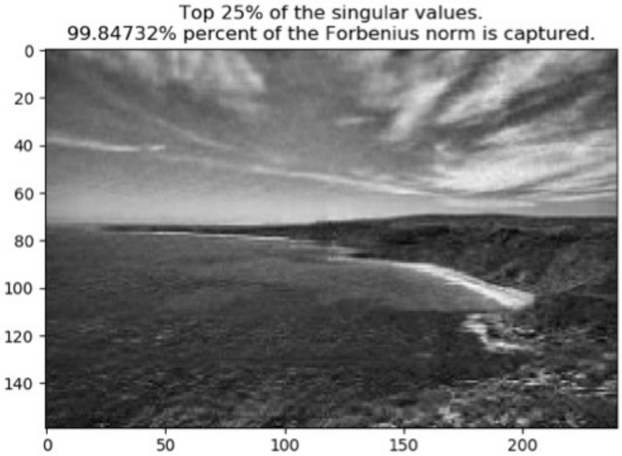
   The first four right singular vectors of matrix A: $V_4.T$

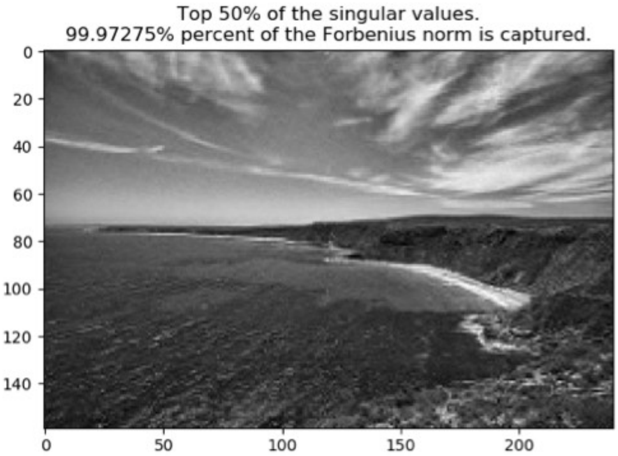
3.27

Print the reconstructed photo.









3.32

The distance matrix is a 3×10 matrix. After performing SVD on the distance matrix, we find that the 3 singular values are $0.330032787, 0.0217985474, 3.86625081 \times 10^{-5}$.
Note that the 3rd singular value is significantly smaller than the first 2 singular values.
Hence we can approximate the distance matrix by a rank 2 matrix, which is 2×10.
Hence the data lies on a lower dimensional "sheath".