

CS 514 HW 1 (due Sep 25)

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From Foundations

Ex 3.6 (1)  $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ,  $k=3$  (steps). We have  $A = \begin{bmatrix} 1 & 2 \\ -1 & 2 \\ -1 & -2 \end{bmatrix}$

$$\text{let } B = A^T A = \begin{bmatrix} 4 & 0 \\ 0 & 16 \end{bmatrix}$$

$$\text{let } x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$\text{then } x_1 = B x_0 = \begin{bmatrix} 4 & 0 \\ 0 & 16 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 16 \end{bmatrix}$$

$$x_2 = B x_1 = \begin{bmatrix} 4 & 0 \\ 0 & 16 \end{bmatrix} \begin{bmatrix} 4 \\ 16 \end{bmatrix} = \begin{bmatrix} 16 \\ 256 \end{bmatrix}$$

$$x_3 = B x_2 = \begin{bmatrix} 4 & 0 \\ 0 & 16 \end{bmatrix} \begin{bmatrix} 16 \\ 256 \end{bmatrix} = \begin{bmatrix} 64 \\ 16^3 \end{bmatrix} = 64 \begin{bmatrix} 1 \\ 64 \end{bmatrix}$$

$$\text{so we estimate } v_1 \approx \frac{1}{\|x_3\|} x_3 = \frac{1}{\sqrt{1+64^2}} \begin{bmatrix} 1 \\ 64 \end{bmatrix} \\ = \begin{bmatrix} 0.0156 \\ 0.9999 \end{bmatrix}$$

(2) We find the eigenvalues of  $B$ : set  $|B - \lambda I| = 0$

$$\begin{vmatrix} 4-\lambda & 0 \\ 0 & 16-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)(16-\lambda) = 0$$

$$\lambda = 16 \text{ or } \lambda = 4$$

(Hence eigenvalues of  $B$ :  $\lambda_1 = 16$ ,  $\lambda_2 = 4$ )

for  $\lambda_1$ ,  $B \begin{bmatrix} x \\ y \end{bmatrix} = \lambda_1 \begin{bmatrix} x \\ y \end{bmatrix}$

$$\begin{bmatrix} 4 & 0 \\ 0 & 16 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 16 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$4x = 16x \quad \text{--- (1)}$$

$$16y = 16y \quad \text{--- (2)}$$

so  $x=0, y=1$   
 hence  $V_1 = [0 \ 1]^T$  is the eigenvector corresponding to  $\lambda_1$   
 for  $\lambda_2$ ,  $B \begin{bmatrix} x \\ y \end{bmatrix} = \lambda_2 \begin{bmatrix} x \\ y \end{bmatrix}$

$$\begin{bmatrix} 4 & 0 \\ 0 & 16 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 4 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$4x = 4x$$

$$16y = 4y$$

$$\text{so } x=1, y=0$$

hence  $V_2 = [1 \ 0]^T$  is the eigenvector corresponding to  $\lambda_2$

singular values of  $A$ :  $\sigma_1 = \sqrt{\lambda_1} = 4$

$$\sigma_2 = \sqrt{\lambda_2} = 2$$

$$AV_1 = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\text{so } u_1 = \frac{1}{\sigma_1} (AV_1) = \begin{bmatrix} 1/2 & 1/2 & -1/2 & -1/2 \end{bmatrix}^T$$

$$\text{and } u_2 = \frac{1}{\sigma_2} (AV_2) = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ = \begin{bmatrix} 1/2 & -1/2 & 1/2 & -1/2 \end{bmatrix}^T$$

$$\text{Hence, } \sigma_1 = 4, \quad u_1 = \begin{bmatrix} 1/2 & 1/2 & -1/2 & -1/2 \end{bmatrix}^T, \quad v_1 = [0 \ 1]^T \\ \sigma_2 = 2, \quad u_2 = \begin{bmatrix} 1/2 & -1/2 & 1/2 & -1/2 \end{bmatrix}^T, \quad v_2 = [1 \ 0]^T$$

(3)  $V_1$  would represent how much each restaurant conforms to a given "theme" or "concept".

$u_1$  represents how much each person likes the given "theme" of restaurants.

the gap  $\sigma_1 - \sigma_2$  represents how much more significant the first "theme" is compared to the second "theme", in our data matrix.

$$\text{Ex 3.16} \quad A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$B = A^T A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 10 & 14 \\ 14 & 20 \end{bmatrix}$$

$$\text{let } x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$x_1 = \frac{Bx_0}{\|Bx_0\|} = \begin{bmatrix} 0.5767 \\ 0.8170 \end{bmatrix}$$

$$x_2 = \frac{Bx_1}{\|Bx_1\|} = \begin{bmatrix} 0.5761 \\ 0.8174 \end{bmatrix}$$

$$x_3 = \frac{Bx_2}{\|Bx_2\|} = \begin{bmatrix} 0.5760 \\ 0.8174 \end{bmatrix}$$

$$v_1 \text{ of } A = v_1 \text{ of } B = x_3$$

$$Av_1 = \begin{bmatrix} 2.2109 \\ 4.9978 \end{bmatrix} \xrightarrow{\text{Normalize}} \delta_1 = \sqrt{2.2109^2 + 4.9978^2} = 5.465.$$

$$u_1 = \frac{Av_1}{\delta_1} = \begin{bmatrix} 0.4046 \\ 0.9145 \end{bmatrix}$$

$$A' = A - \delta_1 u_1 v_1^T = \begin{bmatrix} -0.2736 & 0.1928 \\ 0.1210 & -0.0853 \end{bmatrix}$$

$$B' = A^T A' = \begin{bmatrix} 0.0895 & -0.0631 \\ -0.0631 & 0.0444 \end{bmatrix}, \text{ let } x'_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$x'_3 = B'(B'(B'x_1)) = \begin{bmatrix} 0.8174 \\ -0.5760 \end{bmatrix} = v_2.$$

$$\Rightarrow \delta_2 = 0.3660, \quad u_2 = \begin{bmatrix} 0.9145 \\ 0.4046 \end{bmatrix}.$$

$$A = \sum_i \delta_i u_i v_i^T = \begin{bmatrix} 0.4046 & -0.9145 \\ 0.9145 & 0.4046 \end{bmatrix} \begin{bmatrix} 5.465 & 0 \\ 0 & 0.366 \end{bmatrix} \begin{bmatrix} 0.5760 & 0.8174 \\ 0.8174 & -0.5760 \end{bmatrix}$$

Ex 4.3

(a)

$$M = \begin{bmatrix} 0.4 & 0.4 & 0 & 0 & 0 \\ 0.6 & 0 & 0.4 & 0 & 0 \\ 0 & 0.6 & 0 & 0.4 & 0 \\ 0 & 0 & 0.6 & 0 & 0.4 \\ 0 & 0 & 0 & 0.6 & 0.6 \end{bmatrix}$$

set  $V_0 = \begin{bmatrix} 0.2 \\ 0.2 \\ 0.2 \\ 0.2 \\ 0.2 \end{bmatrix}$

$$M V_0 = \begin{bmatrix} 0.16 \\ 0.2 \\ 0.2 \\ 0.2 \\ 0.24 \end{bmatrix} \rightarrow M(MV_0) = \begin{bmatrix} 0.144 \\ 0.176 \\ 0.2 \\ 0.216 \\ 0.264 \end{bmatrix} \rightarrow \dots \rightarrow \begin{bmatrix} 0.0758 \\ 0.1137 \\ 0.1706 \\ 0.2559 \\ 0.3839 \end{bmatrix}$$

(b)

$$M = \begin{bmatrix} 0.5 & 0 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 & 1 \end{bmatrix}$$

set  $V_0 = [0.2 \ 0.2 \ 0.2 \ 0.2 \ 0.2]^T$ .

$$V = \begin{bmatrix} 0.2 \\ 0.2 \\ 0.2 \\ 0.2 \\ 0.2 \end{bmatrix} \rightarrow \begin{bmatrix} 0.1 \\ 0.2 \\ 0.2 \\ 0.2 \\ 0.3 \end{bmatrix} \rightarrow \dots \rightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Ex 11.2.1

(a)  $M^T M = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \\ 4 & 16 \end{bmatrix} = \begin{bmatrix} 30 & 100 \\ 100 & 354 \end{bmatrix}$

$$MM^T = \begin{bmatrix} 2 & 6 & 12 & 20 \\ 6 & 20 & 42 & 72 \\ 12 & 42 & 90 & 156 \\ 20 & 72 & 156 & 272 \end{bmatrix}$$

(b)

$$|M^T M - \lambda I| = 0$$

$$\Rightarrow \lambda^2 - 384\lambda + 620 = 0$$

$$\textcircled{1} \Rightarrow \lambda_1 = 382.38$$

$$\begin{bmatrix} 30 & 100 \\ 100 & 354 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 382.38 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \begin{cases} 30x + 100y = 382.38x \\ 100x + 354y = 382.38y \end{cases}$$

$$\text{normalized eigenvector } v_1 = \begin{bmatrix} 0.273 \\ 0.962 \end{bmatrix}$$

\textcircled{2}

$$\Rightarrow \lambda_2 = 1.62$$

$$\begin{bmatrix} 30 & 100 \\ 100 & 354 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 1.62 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\text{normalized eigenvector } v_2 = \begin{bmatrix} -0.962 \\ 0.273 \end{bmatrix}$$

(c)  $\lambda_1 = 382.38, \lambda_2 = 1.62, \lambda_3 = 0, \lambda_4 = 0$

(d)  $MM^T v = \lambda v$

~~$\lambda_1 = 382.38$~~   $\begin{bmatrix} 2 & 6 & 12 & 20 \\ 6 & 20 & 42 & 72 \\ 12 & 42 & 90 & 156 \\ 20 & 72 & 156 & 272 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \lambda_1 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \Rightarrow v_1 = \begin{bmatrix} 0.063 \\ 0.224 \\ 0.484 \\ 0.842 \end{bmatrix} \text{ while } \lambda_1 = 382.38$

$$\lambda_2 = 1.62, v_2 = [0.541 \quad 0.653 \quad 0.336 \quad -0.408]^T$$

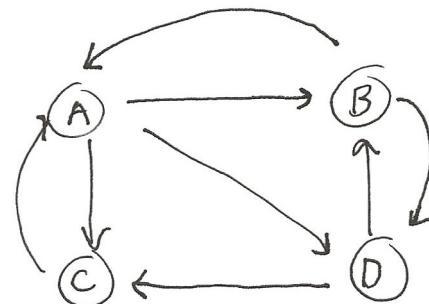
$$\lambda_3 = 0, v_3 = [0.806 \quad -0.263 \quad -0.454 \quad 0.271]^T$$

$$\lambda_4 = 0, v_4 = [0.512 \quad 0.187 \quad -0.761 \quad 0.349]^T$$

5.5.1

we have link matrix  $L =$

$$\begin{array}{c} \text{from} \diagdown \text{to} \\ \begin{array}{l} A \\ B \\ C \\ D \end{array} \end{array} \begin{array}{c} A \quad B \quad C \quad D \\ \left[ \begin{array}{cccc} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array} \right] \end{array}$$



let  $h_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ . then  $a_0 = L^T h_0 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}$

$$h_1 = L a_0 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \\ 4 \end{bmatrix}$$

$$a_1 = L^T h_1 = \begin{bmatrix} 6 \\ 4 \\ 2 \\ 4 \end{bmatrix} \xrightarrow{\text{scale}} a_1 = \begin{bmatrix} 6/10 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$h_2 = L a_1 = \begin{bmatrix} 3 \\ 1.6 \\ 0.6 \\ 2 \end{bmatrix}$$

$$a_2 = L^T h_2 = \begin{bmatrix} 2.2 \\ 5 \\ 5 \\ 4.6 \end{bmatrix} \xrightarrow{\text{scale}} a_2 = \begin{bmatrix} 0.44 \\ 1 \\ 1 \\ 0.92 \end{bmatrix}$$

$$h_3 = L a_2 = \begin{bmatrix} 2.92 \\ 1.36 \\ 0.44 \\ 2 \end{bmatrix}$$

$$a_3 = L^T h_3 = \begin{bmatrix} 1.8 \\ 4.92 \\ 4.92 \\ 4.28 \end{bmatrix} \xrightarrow{\text{scale}} a_3 = \begin{bmatrix} 0.3659 \\ 1 \\ 1 \\ 0.8699 \end{bmatrix}$$

$$h_4 = L a_3 = \begin{bmatrix} 2.87 \\ 1.24 \\ 0.37 \\ 2 \end{bmatrix}$$

$$a_4 = L^T h_4 = \begin{bmatrix} 1.6016 \\ 4.8699 \\ 4.8699 \\ 4.1057 \end{bmatrix} \xrightarrow{\text{scale}} a_4 = \begin{bmatrix} 0.3289 \\ 1 \\ 1 \\ 0.8431 \end{bmatrix}$$

: a few  
: iterations in Python

Hence,  $h = \begin{bmatrix} 2.81 \\ 1.10 \\ 0.29 \\ 2 \end{bmatrix}$ , and  $\alpha = \begin{bmatrix} 0.29 \\ 1 \\ 1 \\ 0.81 \end{bmatrix}$