Homework 02

CS 514 Fall 2018
Algorithms for Data Science

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Foundations of Data Science

7.11

(1) Example where x minimizing $\sum_{i=1}^{n} |a_i - x|$ is not unique[figure 1 (a)]:
x' can be any number within [-1, 1] and they all minimize $\sum_{i=1}^{n} |a_i - x|$, where $a_1 = -1, a_2 = 1$
(2) Example where centroid is different from x that minimize$\sum_{i=1}^{n} |a_i - x|$ is not unique[figure 1 (b)]:
Data points are $\{-2, -1, 1, 2, 100\}$
Centroid is $\mu = \frac{-2-1+1+2+100}{5} = 20$
$argmin_x \sum_{i=1}^{n} |a_i - x| = 1$ (which is the median of five data points)
The point 1 and 20 are quite far apart.

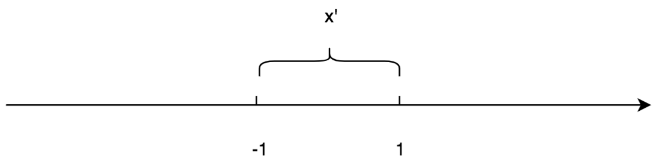

Figure 1 (a)

Figure 1 (b)

Figure 1: Examples

7.12

Want to show that $\frac{1}{n^2} \sum_{i,j} a_i a_j^T = \frac{1}{n} \sum_{i=1}^{n} a_i c^T$, $i, j = 1, 2, ..., n$
Known $c = \frac{1}{n} \sum_{i=1}^{n} a_i = \frac{1}{n} \sum_{j=1}^{n} a_j$
Proof:
$RHS = \frac{1}{n} \sum_{i=1}^{n} a_i c^T = \frac{1}{n} \sum_{i=1}^{n} a_i (\frac{1}{n} \sum_{j=1}^{n} a_j^T) = \frac{1}{n^2} \sum_{i=1}^{n} a_i \sum_{j=1}^{n} a_j^T = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{J=1}^{n} a_i a_j^T$
$= \frac{1}{n^2} \sum_{i,j} a_i a_j^T = LHS$
Hence, the average cluster similarity is the same by computing average of all pairs, or average similarity of each point with the centroid of the cluster.

7.17

The distance of the old and the new centroid is the difference of their weighted average value. $|\mu(S \cup T) - \mu(S)| = |\frac{\mu(S)|S|+\mu(T)|T|}{|S|+|T|} - \mu(S)| = |\frac{\mu(S)|S|+\mu(T)|T|-\mu(S)(|S|+|T|)}{|S|+|T|}| = \frac{|T|}{|T|+|S|}|\mu(S) - \mu(T)|$. Thus, the centroid $\mu(S \cup T)$ of $S \cup T$ is at distance at most $\frac{|T|}{|T|+|S|}|\mu(S) - \mu(T)|$ from $\mu(S)$.

Mining of Massive Datasets

3.1.1

For {1,2,3,4} and {2,3,5,7}, $J = \frac{2}{6} = \frac{1}{3}$. For {1,2,3,4} and {2,4,6}, $J = \frac{2}{5}$. For {2,3,5,7} and {2,4,6}, $J = \frac{1}{6}$.

3.1.3

$$SIM(S,T) = \frac{|S \cap T|}{|S \cup T|} = \frac{k}{2m-k}$$

Suppose $|S \cap T| = k$, where $0 \leq k \leq m$. Then S has $\binom{n}{m}$ choices and T has $\binom{m}{k}\binom{n-m}{m-k}$ choices.
Therefore $P(SIM(S,T) = \frac{k}{2m-k}) = \frac{\binom{m}{k}\binom{n-m}{m-k}}{\binom{n}{m}}$
and $E(SIM(S,T)) = \sum_{k=0}^{m} \frac{\binom{m}{k}\binom{n-m}{m-k}}{\binom{n}{m}} \frac{k}{2m-k}$

3.3.2

The values of the two hash functions applied to the row numbers are given in the last two columns below

Rows	S_1	S_2	S_3	S_4	$2x + 4 \ mod \ 5$	$3x - 1 \ mod \ 5$
0	1	0	0	1	4	4
1	0	0	1	0	1	2
2	0	1	0	1	3	0
3	1	0	1	1	0	3
4	0	0	1	0	2	1

The added signature matrix is

	S_1	S_2	S_3	S_4
h_1	1	3	0	1
h_2	0	2	0	0
h_3	0	3	0	0
h_4	3	0	1	0

The calculating process is as follows

$$\begin{bmatrix} \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty \end{bmatrix} \xrightarrow{\text{row}(0)} \begin{bmatrix} 4 & \infty & \infty & 4 \\ 4 & \infty & \infty & 4 \end{bmatrix} \xrightarrow{\text{row}(1)} \begin{bmatrix} 4 & \infty & 1 & 4 \\ 4 & \infty & 2 & 4 \end{bmatrix} \xrightarrow{\text{row}(2)} \begin{bmatrix} 4 & 3 & 1 & 3 \\ 4 & 0 & 2 & 0 \end{bmatrix} \xrightarrow{\text{row}(3)}$$
$$\begin{bmatrix} 0 & 3 & 0 & 0 \\ 3 & 0 & 2 & 0 \end{bmatrix} \xrightarrow{\text{row}(4)} \begin{bmatrix} 0 & 3 & 0 & 0 \\ 3 & 0 & 1 & 0 \end{bmatrix}$$

3.3.3

(a) The matrix with hash functions values is

Element	S_1	S_2	S_3	S_4	$2x+1\ mod\ 6$	$3x+2\ mod\ 6$	$5x+2\ mod\ 6$
0	0	1	0	1	1	2	2
1	0	1	0	0	3	5	1
2	1	0	0	1	5	2	0
3	0	0	1	0	1	5	5
4	0	0	1	1	3	2	4
5	1	0	0	0	5	5	3

The signature matrix is

	S_1	S_2	S_3	S_4
h_1	5	1	1	1
h_2	2	2	2	2
h_3	0	1	4	0

The calculating process is as follows

$$\begin{bmatrix} \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty \end{bmatrix} \xrightarrow{\text{row}(0)} \begin{bmatrix} \infty & 1 & \infty & 1 \\ \infty & 2 & \infty & 2 \\ \infty & 2 & \infty & 2 \end{bmatrix} \xrightarrow{\text{row}(1)} \begin{bmatrix} \infty & 1 & \infty & 1 \\ \infty & 2 & \infty & 2 \\ \infty & 1 & \infty & 2 \end{bmatrix} \xrightarrow{\text{row}(2)} \begin{bmatrix} 5 & 1 & \infty & 1 \\ 2 & 2 & \infty & 2 \\ 0 & 1 & \infty & 0 \end{bmatrix} \xrightarrow{\text{row}(3)}$$
$$\begin{bmatrix} 5 & 1 & 1 & 1 \\ 2 & 2 & 5 & 2 \\ 0 & 1 & 5 & 0 \end{bmatrix} \xrightarrow{\text{row}(4)} \begin{bmatrix} 5 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 0 & 1 & 4 & 0 \end{bmatrix} \xrightarrow{\text{row}(5)} \begin{bmatrix} 5 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 0 & 1 & 4 & 0 \end{bmatrix}$$

(b) Only $\{h_3(x) = 5x + 2\ mod\ 6\}$ is a true permutation, since there is no collision, i.e. no two rows get the same hash value.
(c) The matrix of true Jaccard similarities and estimated Jaccard similarities matrix is

Pairs	(S_1, S_2)	(S_1, S_3)	(S_1, S_4)	(S_2, S_3)	(S_2, S_4)	(S_3, S_4)
True Jaccard Similarity	0	0	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{4}$
Estimated Jaccard Similarity	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$

10.4.1

(a) The adjacency matrix:

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

(b) The degree matrix:

$$D = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

(c) The Laplacian matrix

$$L = D - A = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & -1 & 3 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 3 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 3 & -1 & -1 \\ 0 & -1 & 0 & 0 & 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 2 \end{bmatrix}$$

10.4.2

The code file: q10_4_2.py
For the Laplacian matrix above, implement eigendecomposition, obtain the second-smallest eigenvalue is 0.69722436226800433, the corresponding vector is [-0.15728598, -0.16666667, 0.29389153, -0.33333333, 0.28305594, -0.40824829, 0.50834187, 0.00210742, -0.48643259].
The second eigenvector has four positive and five negative components, which suggests that one group should be {C, E, G, H}, the nodes with positive components; and the other group should be {A, B, D, F, I}, the nodes with positive components.
The partition is however unbalanced, even by changing the threshold. Check the eigendecomposition again, the third-smallest eigenvalue is 0.69722436226800577, which is very close to the second-smallest eigenvalue, the corresponding vector is [-0.36219431, 0.33333333, -0.38287473, -0.33333333 0.10413675, -0.40824829, -0.3923125, 0.23920786, -0.07430338]
Thus consider the second and the third vectors together, the new partition is {A, D, F, I} (negative values in both vectors), {B} (negative in the second, positive in the third vector), {C, G} (positive in the second, negative in the third vector), {E, H} (positive in both vectors).
The new partition is either balanced for the graph, further eigenvalues and vectors are ought to be considered.

Coding Assignment

7.4

[See Python file q7_4.py]

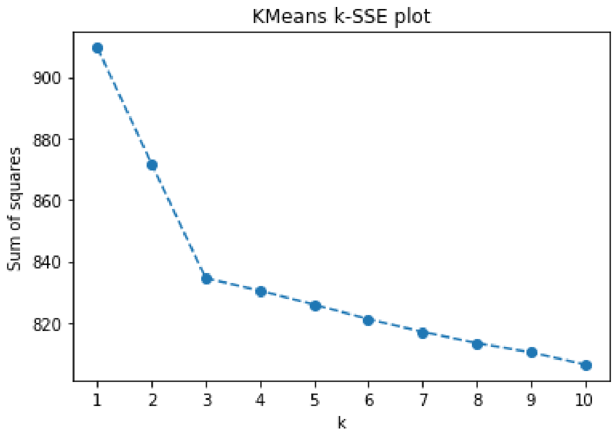

Figure 2: The k against sum of sqaures, k-means clustering

(1) Use the *numpy.random.seed*(86) of module *numpy* as permutation peudo-random number, to randomly distribute the matrix A. Apply the k-means algorithm to A with $k = 3$. The clusters and their indices are

First cluster indices: [61 55 59 84 72 53 69 91 87 74 51 81 50 98 85 89 94 90 80 77 54 86 83 79 71 97 62 82 95 63 52 75 70 68 58 88 73 56 60 92 57 93 65 66 67 64 78 76 99 96]
Second cluster indices: [32 30 1 8 39 9 19 34 35 40 37 25 20 21 47 0 11 45 3 15 7 33 6 36 42 10 18 23 46 38 14 31 5 26 12 49 43 2 22 48 24 16 4 29 44 41 13 17 27 28]
Third cluster indices: [113 130 132 128 138 136 129 143 146 117 118 100 133 102 112 139 125 123 110 142 111 134 103 106 135 121 107 149 108 114 147 126 104 105 131 141 144 140 115 145 101 127 109 116 124 120 119 122 137 148]

Since the first cluster contains data points with indices 50 to 99, the second cluster contains data points with indices 0 to 49 and the third cluster contains data points with indices 100 to 149, which are the same as the natural clusters of A, it finds the correct clusters.

(2) Apply the k-means algorithm to A for $1 \leq k \leq 10$. The plot of the value of the sum of squares to the cluster centers versus k is in the Figure 2. Since the sum of square decreases abruptly at $k = 3$, which is the elbow point, three is the correct value for k.

Attachments

q7_4.py, q10_4_2.py