Most Slides from

- Thomas Opsahl, UNIK
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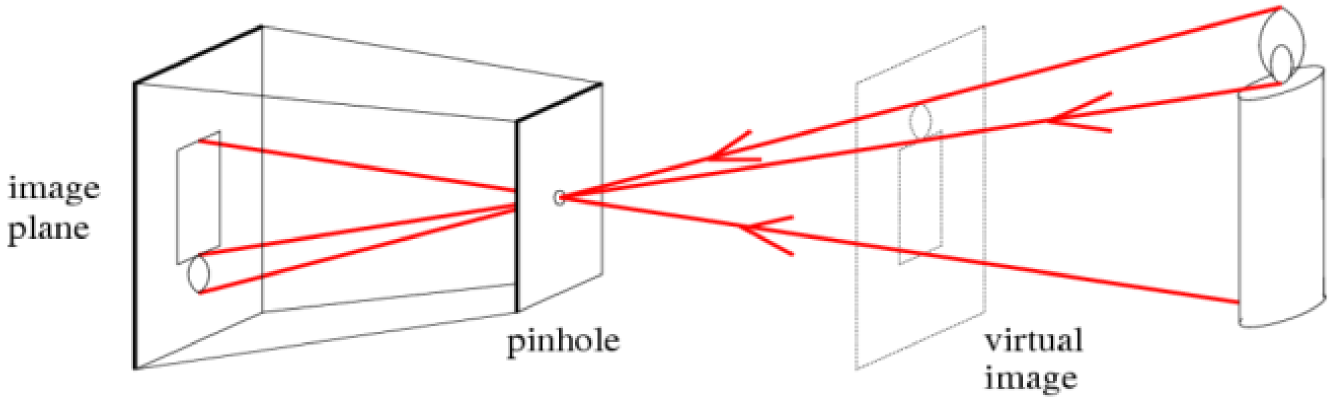
Basic projective geometry

Motivation



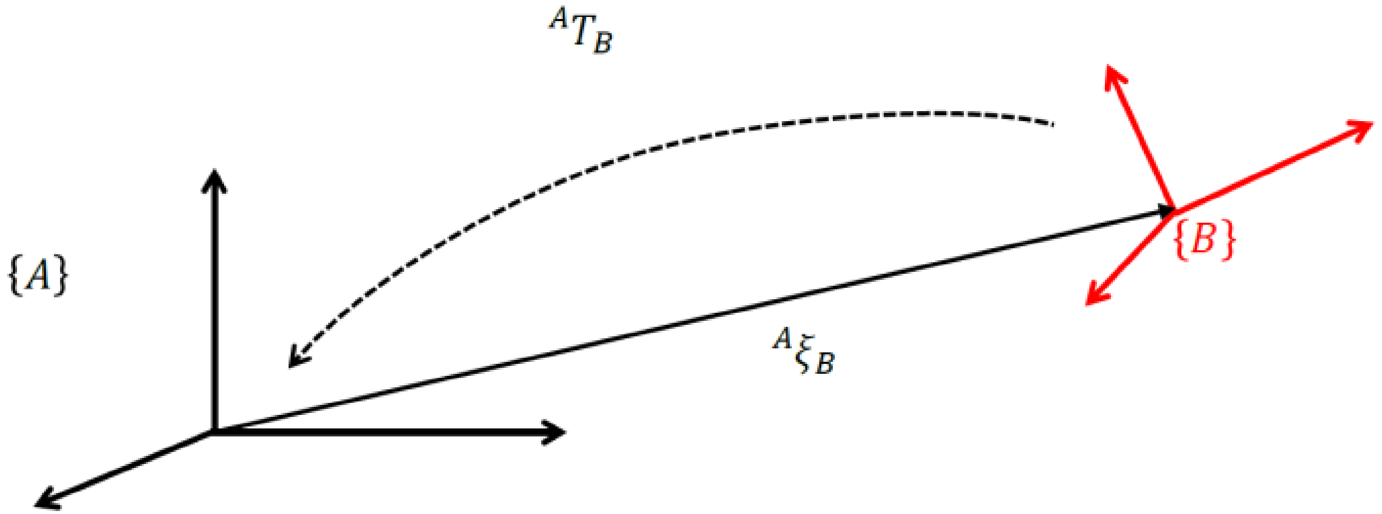
- For the pinhole camera, the correspondence between observed 3D points in the world and 2D points in the captured image is given by straight lines through a common point (pinhole)

- This correspondence can be described by a mathematical model known as "*the perspective camera model*" or "*the pinhole camera model*"

- This model can be used to describe the imaging geometry of many modern cameras, hence it plays a central part in computer vision

Basic projective geometry

Introduction



- We have seen that the pose of a coordinate frame $\{B\}$ relative to a coordinate frame $\{A\}$, denoted ${}^{A}\xi_{B}$, can be represented as a homogeneous transformation ${}^{A}T_{B}$ in 2D and 3D

$$ {}^{A}\xi_{B} \quad \mapsto \quad {}^{A}T_{B} = \begin{bmatrix} {}^{A}R_{B} & {}^{A}\boldsymbol{t}_{B} \\ \boldsymbol{0} & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & {}^{A}t_{Bx} \\ r_{21} & r_{22} & r_{23} & {}^{A}t_{By} \\ r_{31} & r_{32} & r_{33} & {}^{A}t_{Bz} \\ 0 & 0 & 0 & 1 \end{bmatrix} \in SE(3) $$

Basic projective geometry

Introduction

- And we have seen how they can transform points from one reference frame to another if we represent points in homogeneous coordinates

$$\boldsymbol{p} = \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \tilde{\boldsymbol{p}} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \qquad \boldsymbol{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \tilde{\boldsymbol{p}} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

- The main reason for representing pose as homogeneous transformations, was the nice algebraic properties that came with the representation

Basic projective geometry

Introduction

- Euclidean geometry
  - ${}^A\xi_B \longmapsto ({}^A R_B, {}^A\boldsymbol{t}_B)$
  - Complicated algebra

$$
\begin{aligned}
{}^A\boldsymbol{p} = {}^A\xi_B \cdot {}^B\boldsymbol{p} \quad &\mapsto \quad {}^A\boldsymbol{p} = {}^A R_B\, {}^B\boldsymbol{p} + {}^A\boldsymbol{t}_B \\
{}^A\xi_C = {}^A\xi_B \oplus {}^B\xi_C \quad &\mapsto \quad \left({}^A R_C, {}^A\boldsymbol{t}_C\right) = \left({}^A R_B\, {}^B R_C, {}^A R_B\, {}^B\boldsymbol{t}_C + {}^A\boldsymbol{t}_B\right) \\
\ominus {}^A\xi_B \quad &\mapsto \quad \left({}^A R_C{}^T, -{}^A R_C{}^T\, {}^A\boldsymbol{t}_C\right)
\end{aligned}
$$

- Projective geometry
  - ${}^A\xi_B \longmapsto {}^A T_B = \begin{bmatrix} {}^A R_B & {}^A\boldsymbol{t}_B \\ \boldsymbol{0} & 1 \end{bmatrix}$
  - Simple algebra

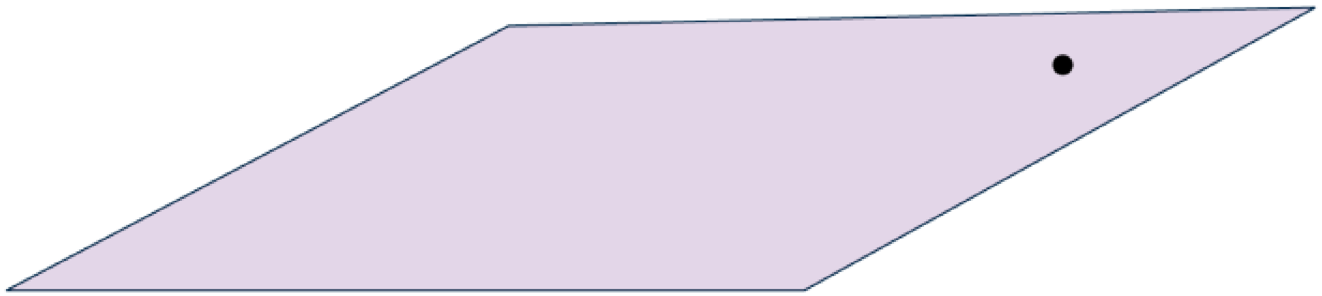
$$
\begin{aligned}
{}^A\boldsymbol{p} = {}^A\xi_B \cdot {}^B\boldsymbol{p} \quad &\mapsto \quad {}^A\tilde{\boldsymbol{p}} = {}^A T_B\, {}^B\tilde{\boldsymbol{p}} \\
{}^A\xi_C = {}^A\xi_B \oplus {}^B\xi_C \quad &\mapsto \quad {}^A T_C = {}^A T_B\, {}^B T_C \\
\ominus {}^A\xi_B \quad &\mapsto \quad {}^A T_B{}^{-1}
\end{aligned}
$$

- In the following we will take a closer look at some basic elements of projective geometry that we will encounter when we study the geometrical aspects of imaging
  - Homogeneous coordinates, homogeneous transformations

Basic projective geometry - points

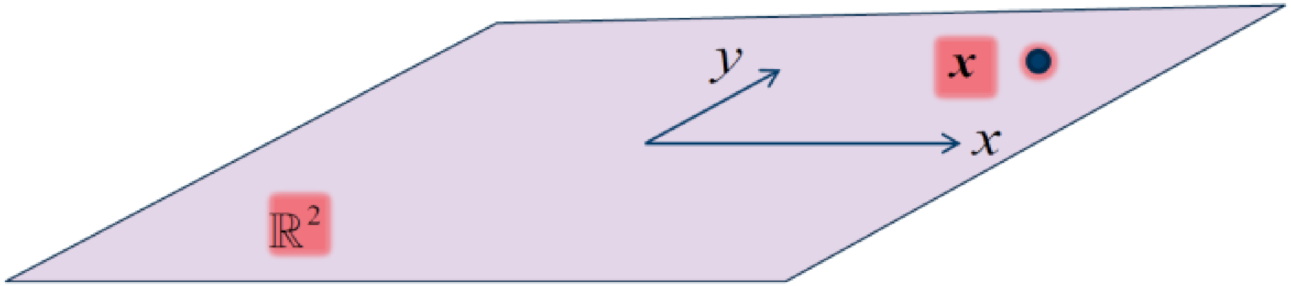
The projective plane
Points

How to describe points in the plane?



Basic projective geometry - points

The projective plane
Points



How to describe points in the plane?

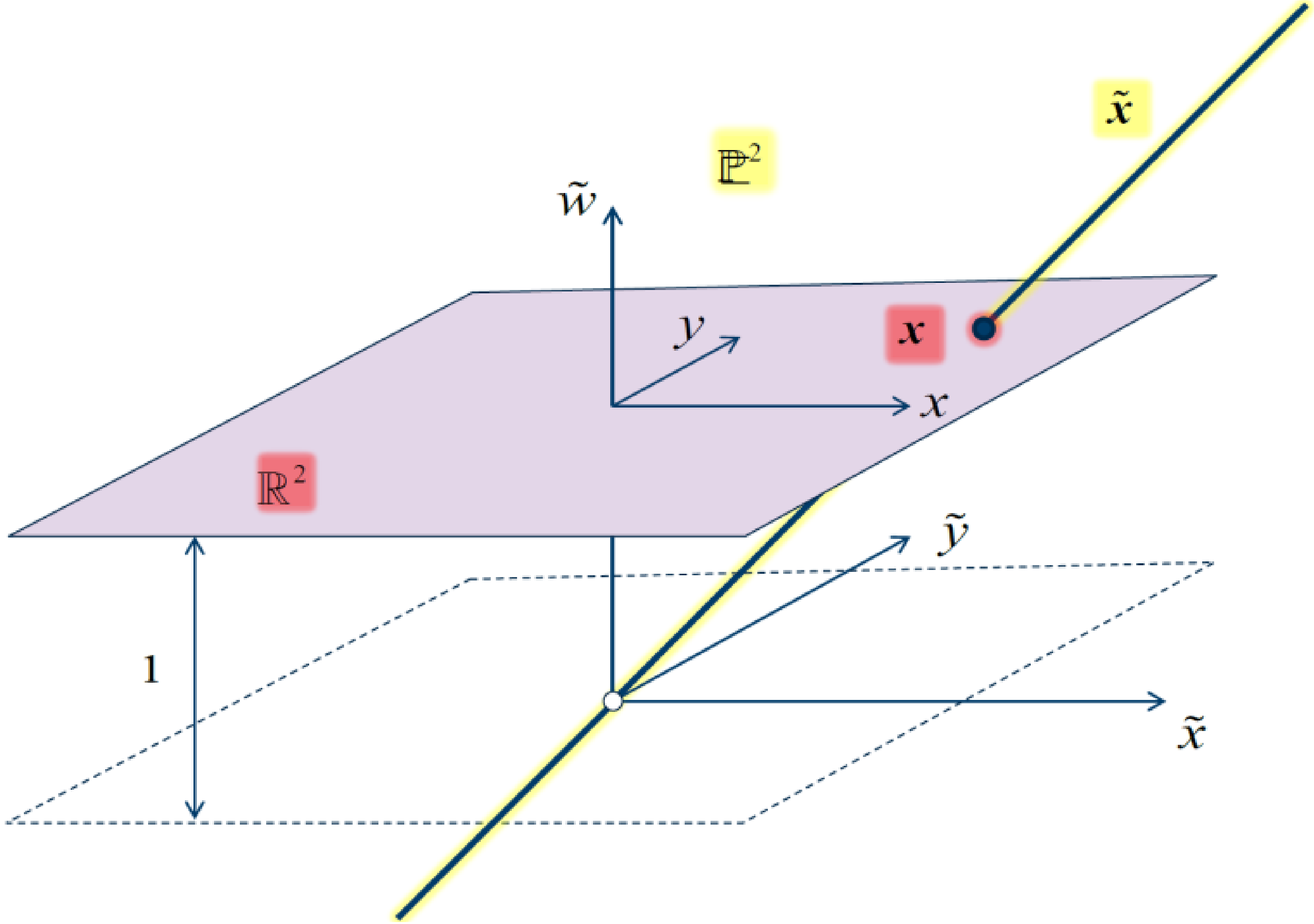
Euclidean plane $\mathbb{R}^2$

- Choose a 2D coordinate frame
- Each point corresponds to a unique pair of Cartesian coordinates

$$\boldsymbol{x} = (x, y) \in \mathbb{R}^2 \longmapsto \boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Basic projective geometry - points

The projective plane
Points



How to describe points in the plane?

Euclidean plane $\mathbb{R}^2$

- Choose a 2D coordinate frame
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$$\boldsymbol{x} = (x, y) \in \mathbb{R}^2 \longmapsto \boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Projective plane $\mathbb{P}^2$

- Expand coordinate frame to 3D
- Each point corresponds to a triple of homogeneous coordinates

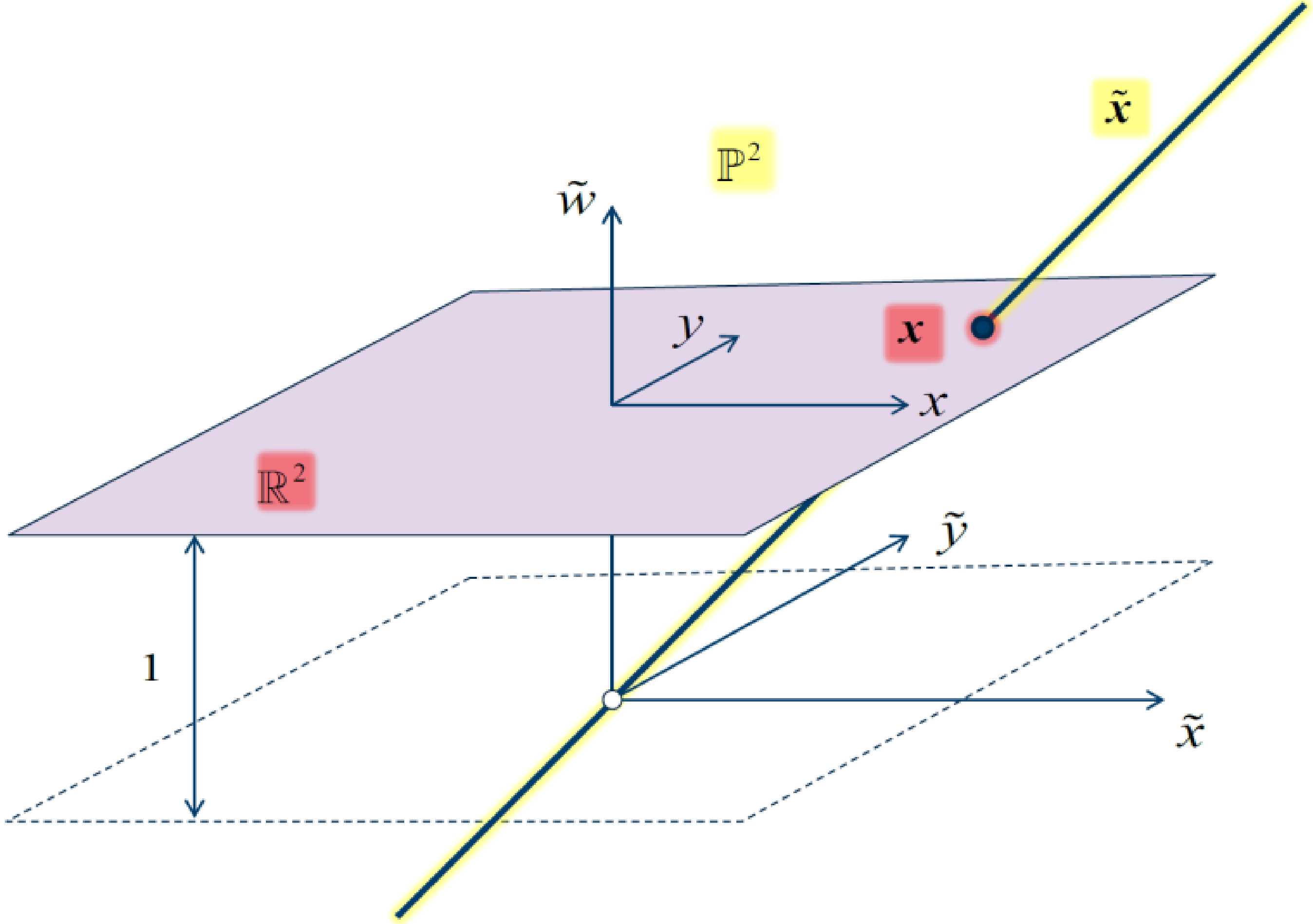
$$\tilde{\boldsymbol{x}} = (\tilde{x}, \tilde{y}, \tilde{w}) \in \mathbb{R}^2 \longmapsto \tilde{\boldsymbol{x}} = \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{w} \end{bmatrix}$$

s.t.

$$(\tilde{x}, \tilde{y}, \tilde{w}) = \lambda(\tilde{x}, \tilde{y}, \tilde{w}) \;\; \forall \lambda \in \mathbb{R}\backslash\{0\}$$

Basic projective geometry - points

The projective plane
Points



Observations

1. Any point $\boldsymbol{x} = (x, y)$ in the Euclidean plane has a corresponding homogeneous point $\tilde{\boldsymbol{x}} = (x, y, 1)$ in the projective plane

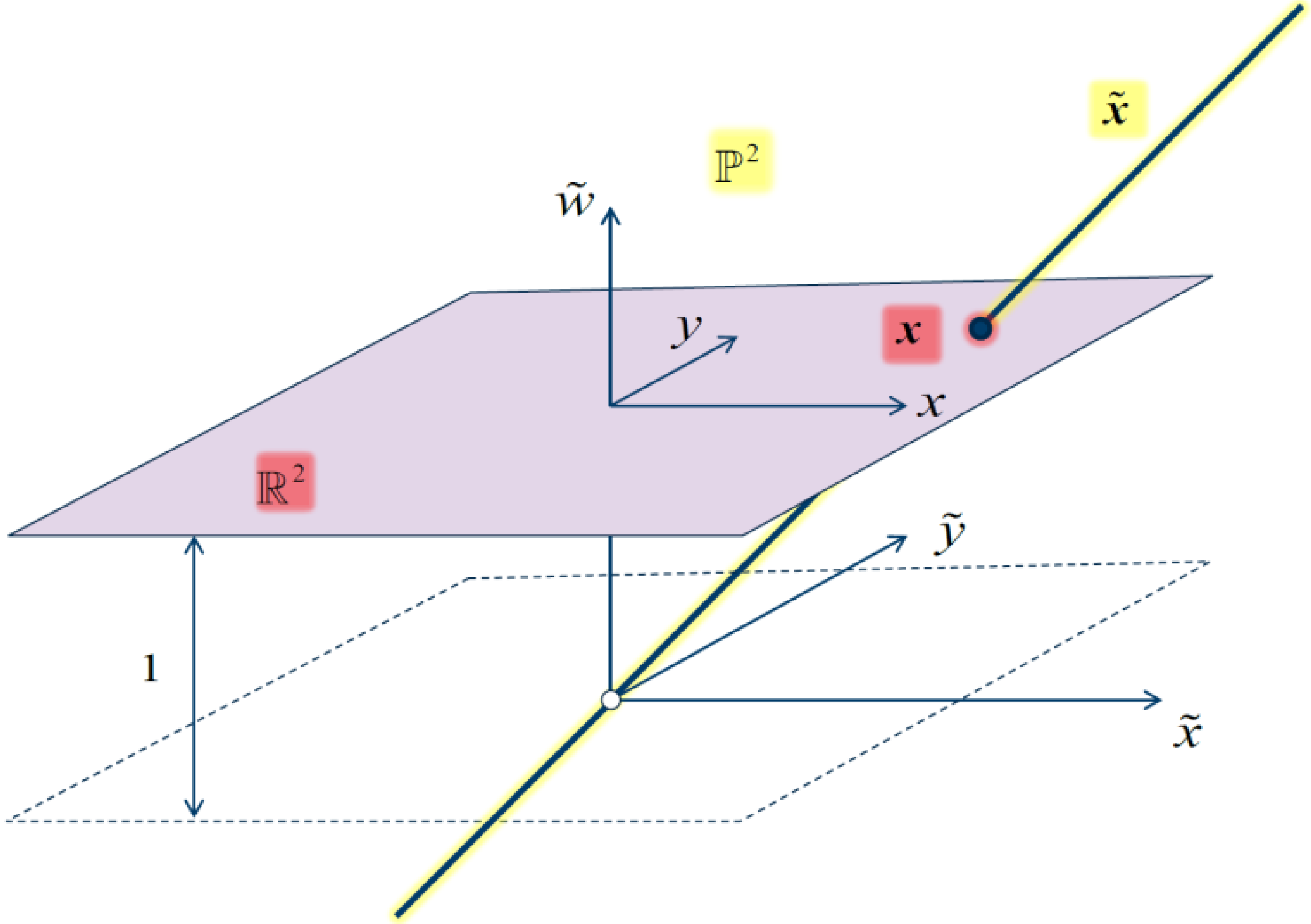
2. Homogeneous points of the form $(\tilde{x}, \tilde{y}, 0)$ does not have counterparts in the Euclidean plane

   They correspond to points at infinity and are called *ideal points*

Basic projective geometry - points

The projective plane

Points



Observations

3. When we work with geometrical problems in the plane, we can swap between the Euclidean representation and the projective representation

$$\mathbb{R}^2 \ni \boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \mapsto \quad \tilde{\boldsymbol{x}} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \in \mathbb{P}^2$$

$$\mathbb{P}^2 \ni \tilde{\boldsymbol{x}} = \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{w} \end{bmatrix} \quad \mapsto \quad \boldsymbol{x} = \begin{bmatrix} \tilde{x}/\tilde{w} \\ \tilde{y}/\tilde{w} \end{bmatrix} \mathbb{R}^2$$

Basic projective geometry - points

Example

1. These homogeneous vectors are different numerical representations of the same point in the plane

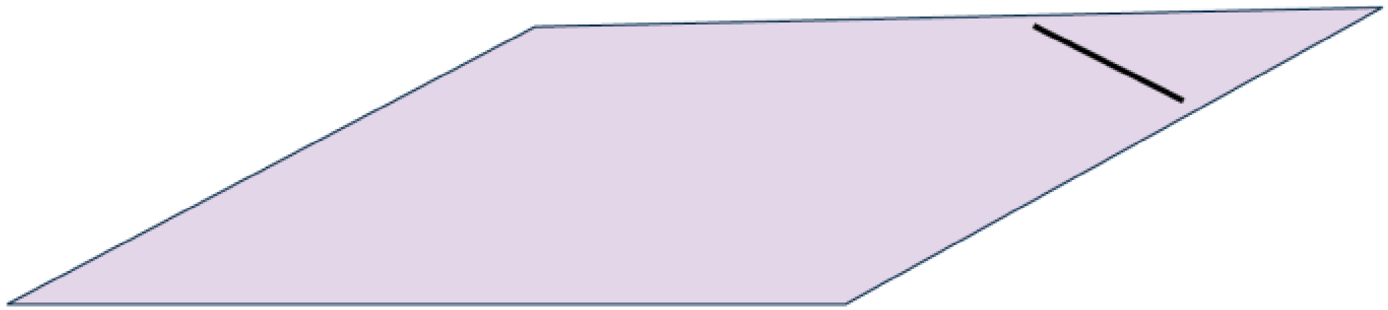
$$\tilde{\boldsymbol{x}} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} -30 \\ -20 \\ -10 \end{bmatrix} \in \mathbb{P}^2$$

2. The homogeneous point $(1,2,3) \in \mathbb{P}^2$ represents the same point as $\left(\frac{1}{3}, \frac{2}{3}\right) \in \mathbb{R}^2$

Basic projective geometry - lines

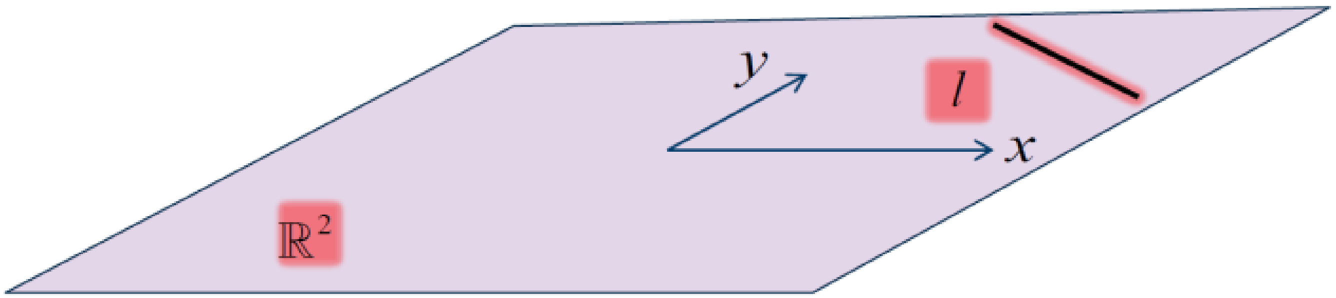
The projective plane
Lines

How to describe lines in the plane?



Basic projective geometry - lines

The projective plane
Lines



How to describe lines in the plane?

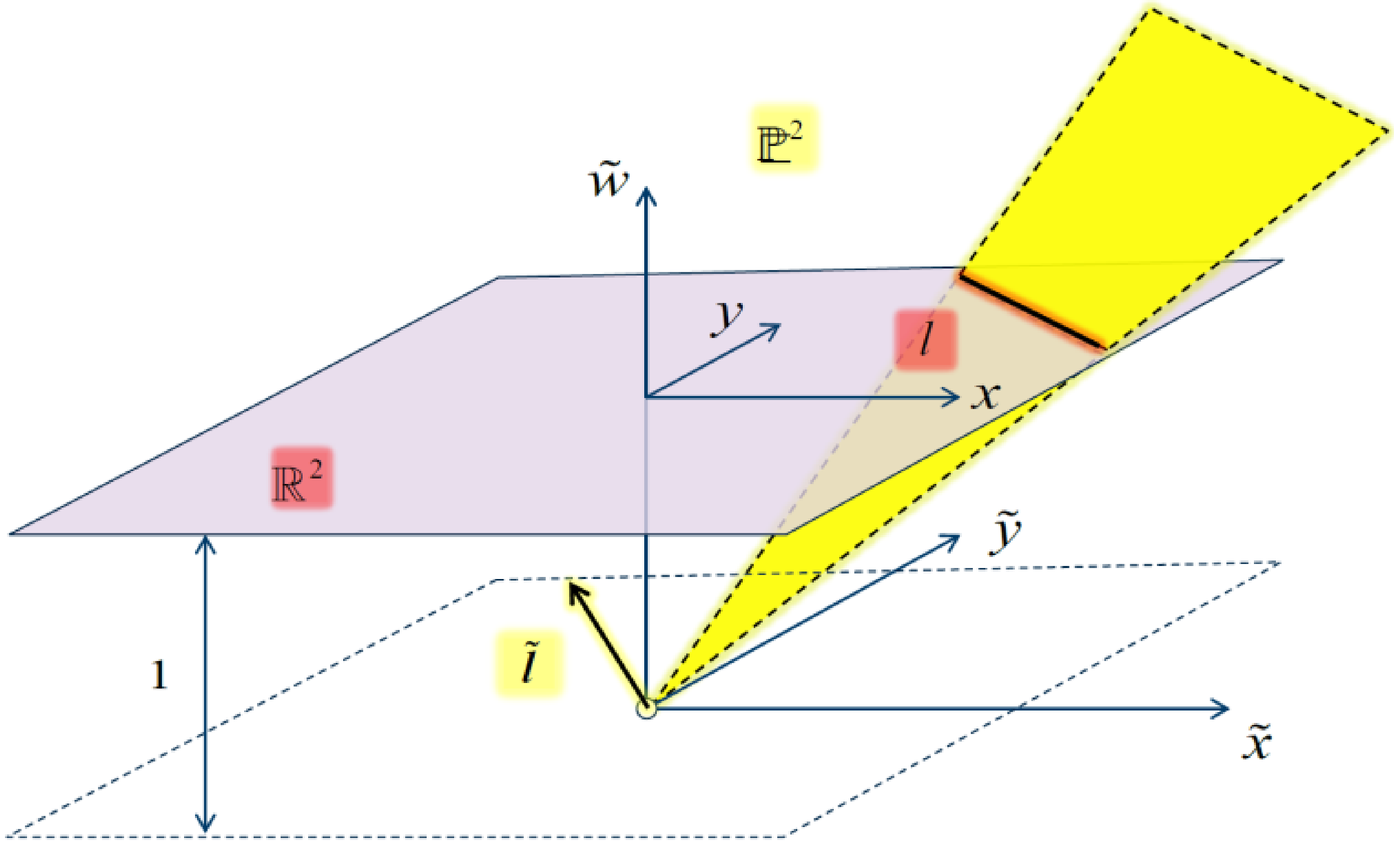
Euclidean plane $\mathbb{R}^2$

- 3 parameters $a, b, c \in \mathbb{R}$

$$l = \{(x, y) \mid ax + by + c = 0\}$$

Basic projective geometry - lines

The projective plane
Lines



How to describe lines in the plane?

Euclidean plane $\mathbb{R}^2$

- Triple $(a, b, c) \in \mathbb{R}^3 \backslash \{\mathbf{0}\}$

$$l = \{(x, y) \mid ax + by + c = 0\}$$

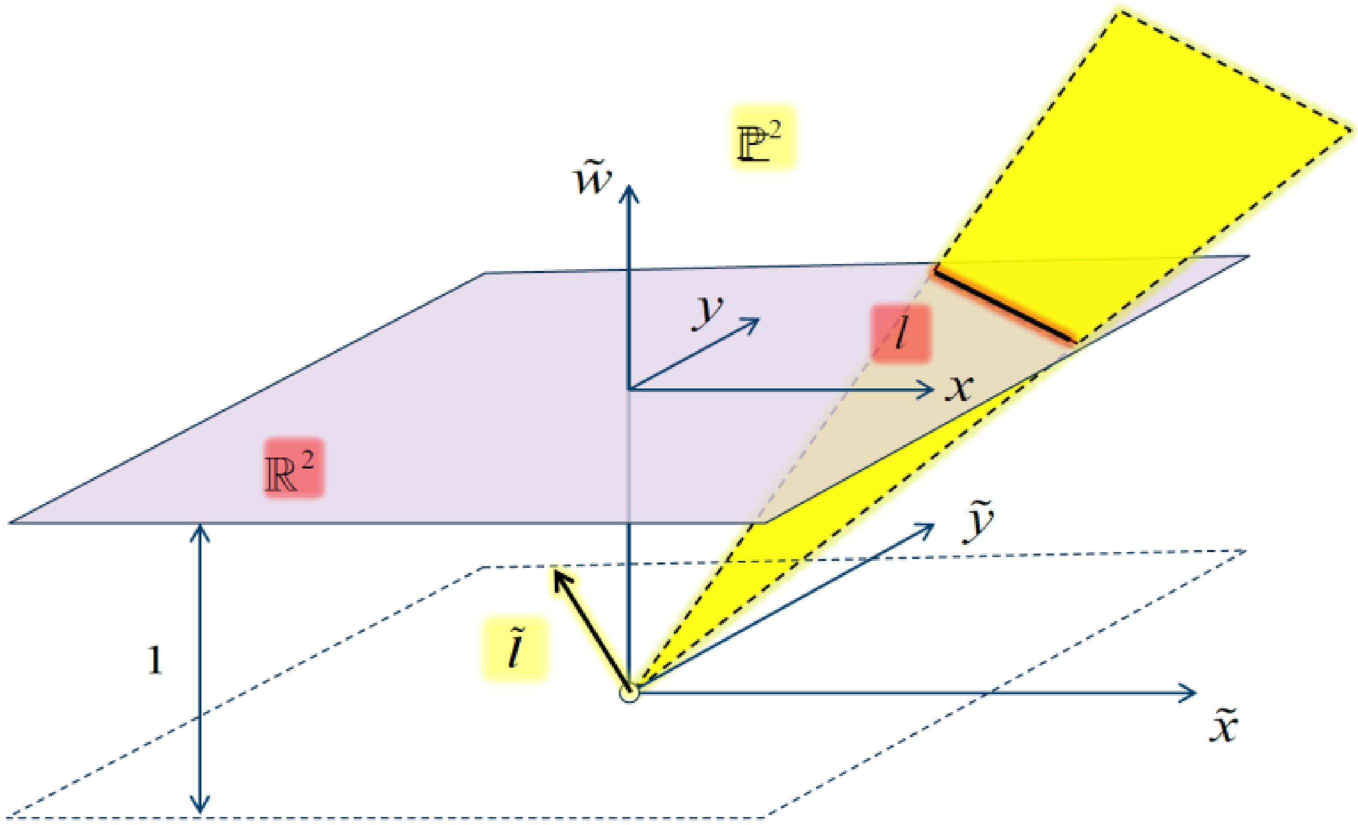
Projective plane $\mathbb{P}^2$

- Homogeneous vector $\tilde{\boldsymbol{l}} = [a, b, c]^T$

$$l = \{\tilde{\boldsymbol{x}} \in \mathbb{P}^2 \mid \tilde{\boldsymbol{l}}^T \tilde{\boldsymbol{x}} = 0\}$$

Basic projective geometry - lines

The projective plane
Lines



Observations

1. Points and lines in the projective plane have the same representation, we say that points and lines are dual objects in $\mathbb{P}^2$

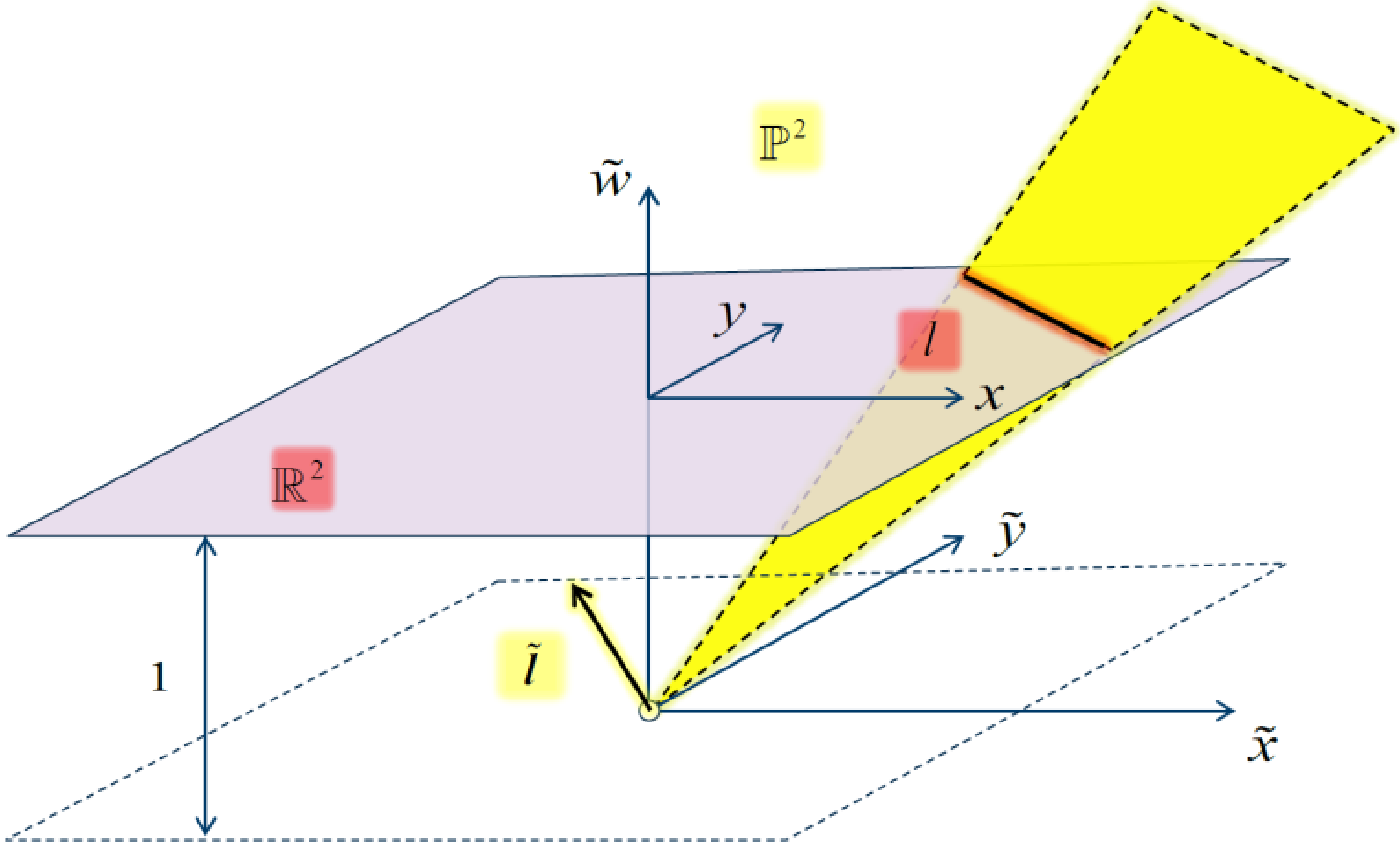
2. All lines in the Euclidean plane have a corresponding line in the projective plane

3. The line $\tilde{l} = [0,0,1]^T$ in the projective plane does not have an Euclidean counterpart

   This line consists entirely of ideal points, and is know as *the line at infinity*

Basic projective geometry - lines

The projective plane
Lines



Properties of lines in the projective plane

1. In the projective plane, all lines intersect, parallel lines intersect at infinity

   Two lines $\tilde{\boldsymbol{l}}_1$ and $\tilde{\boldsymbol{l}}_2$ intersect in the point
   $$\tilde{\boldsymbol{x}} = \tilde{\boldsymbol{l}}_1 \times \tilde{\boldsymbol{l}}_2$$

2. The line passing through points $\tilde{\boldsymbol{x}}_1$ and $\tilde{\boldsymbol{x}}_2$ is given by
   $$\tilde{\boldsymbol{l}} = \tilde{\boldsymbol{x}}_1 \times \tilde{\boldsymbol{x}}_2$$

Duality

$$\mathbf{x} \longleftrightarrow \mathbf{l}$$

$$\mathbf{x}^{\mathsf{T}}\mathbf{l} = 0 \longleftrightarrow \mathbf{l}^{\mathsf{T}}\mathbf{x} = 0$$

$$\mathbf{x} = \mathbf{l} \times \mathbf{l}' \longleftrightarrow \mathbf{l} = \mathbf{x} \times \mathbf{x}'$$

Duality principle:

To any theorem of 2-dimensional projective geometry there corresponds a dual theorem, which may be derived by interchanging the role of points and lines in the original theorem

Basic projective geometry - lines

Example

Determine the line passing through the two points $(2, 4)$ and $(5, 13)$

Homogeneous representation of points

$$\tilde{\boldsymbol{x}}_1 = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} \in \mathbb{P}^2 \qquad \tilde{\boldsymbol{x}}_2 = \begin{bmatrix} 5 \\ 13 \\ 1 \end{bmatrix} \in \mathbb{P}^2$$

Homogeneous representation of line

$$\tilde{\boldsymbol{l}} = \tilde{\boldsymbol{x}}_1 \times \tilde{\boldsymbol{x}}_2 = [\tilde{\boldsymbol{x}}_1]_\times \tilde{\boldsymbol{x}}_2 = \begin{bmatrix} 0 & -1 & 4 \\ 1 & 0 & -2 \\ -4 & 2 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 13 \\ 1 \end{bmatrix} = \begin{bmatrix} -9 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$$

Equation of the line

$$-3x + y + 2 = 0 \quad \Leftrightarrow \quad y = 3x - 2$$

Matrix representation of the cross product

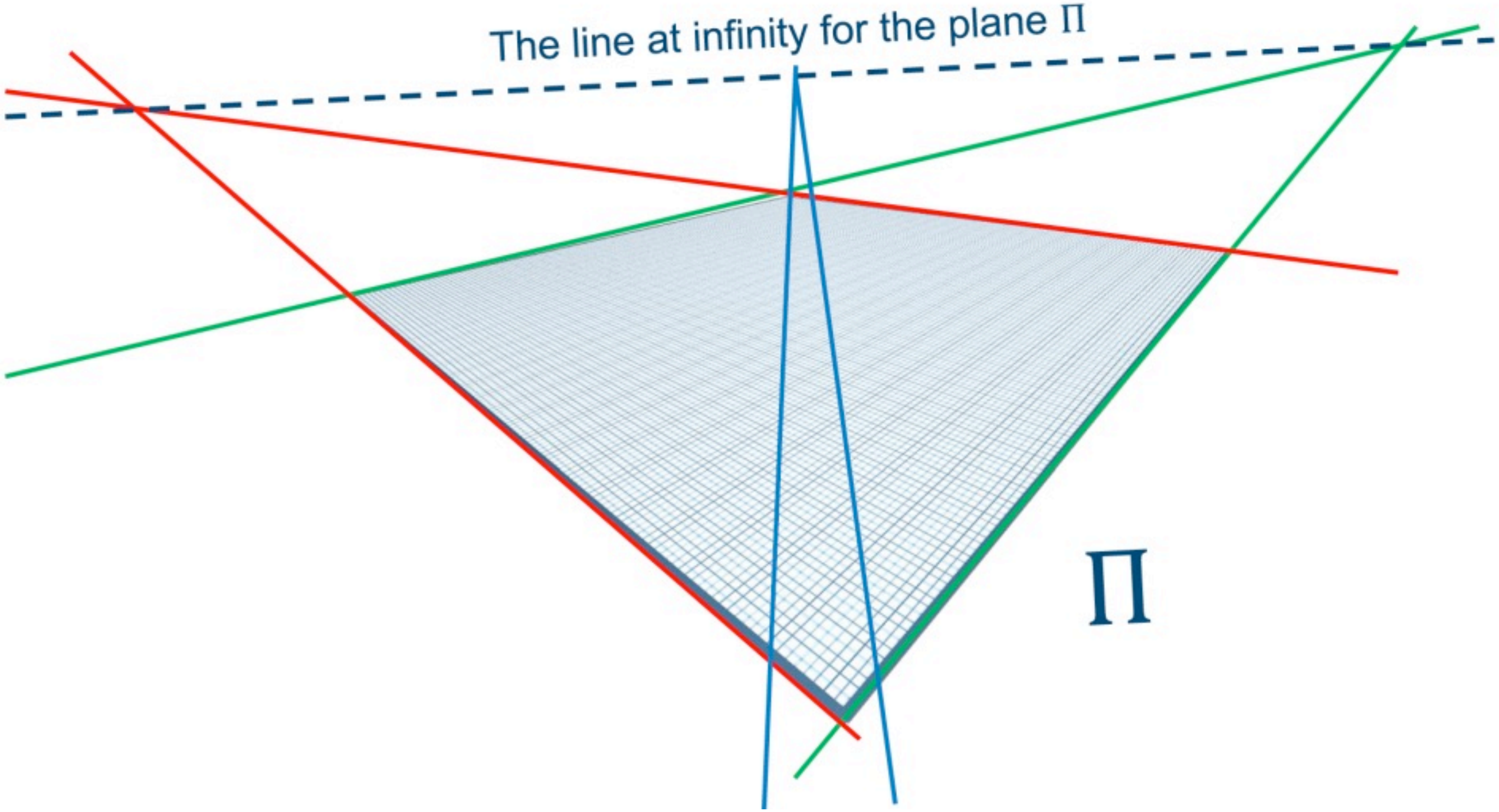
$$\boldsymbol{u} \times \boldsymbol{v} \longmapsto [\boldsymbol{u}]_\times \boldsymbol{v}$$

where

$$[\boldsymbol{u}]_\times \overset{def}{=} \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}$$

Basic projective geometry - lines

Example



A point at infinity

Basic projective geometry – planar homography

The projective plane
Transformations

- Some important transformations – like the action of a pose ξ on points in the plane – happen to be linear in the projective plane and non-linear in the Euclidean plane

- The most general invertible transformations of the projective plane are known as homographies
  - or projective transformations / linear projective transformations / projectivities / collineations

Definition

A homography of $\mathbb{P}^2$ is a linear transformation on homogeneous 3-vectors represented by a homogeneous, non-singular 3×3 matrix H

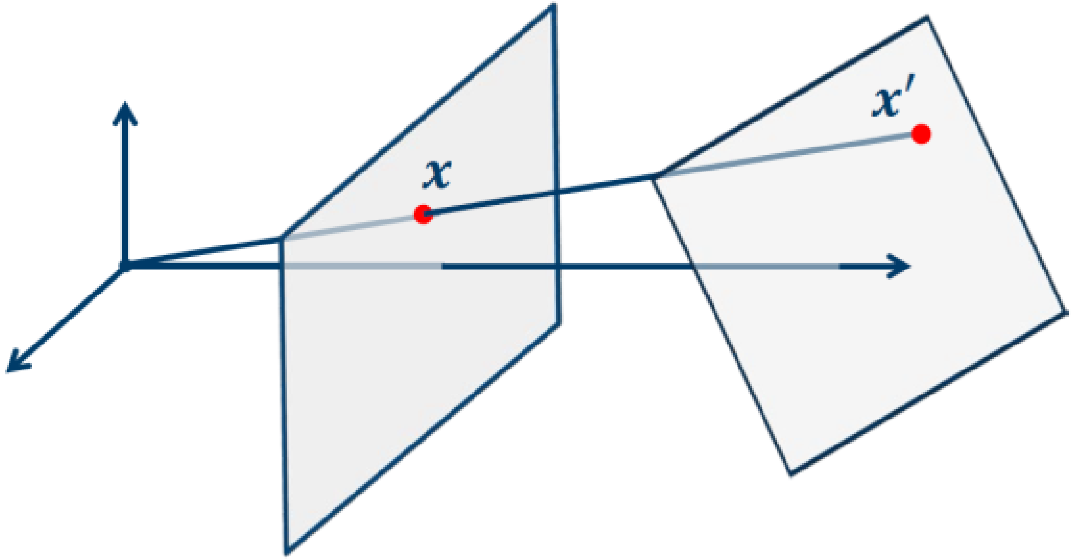
$$\begin{bmatrix} \tilde{x}' \\ \tilde{y}' \\ \tilde{w}' \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{w} \end{bmatrix}$$

So H is unique up to scale, i.e. $H = \lambda H \;\forall\; \lambda \in \mathbb{R}\backslash\{0\}$

Basic projective geometry – planar homography

- One characteristic of homographies is that they preserve lines, in fact any invertible transformation of $\mathbb{P}^2$ that preserves lines is a homography

- Examples
  - Central projection from one plane to another is a homography
    Hence if we take an image with a perspective camera of a flat surface from an angle, we can remove the perspective distortion with a homography



Perspective distortion

Without distortion

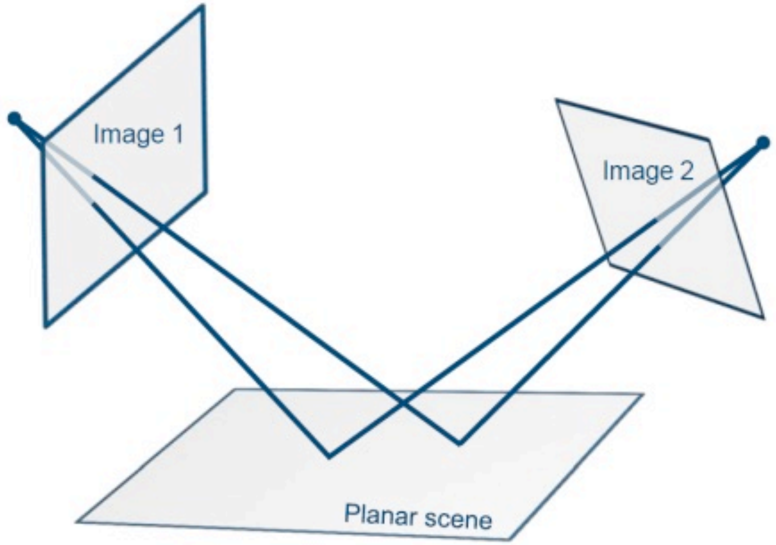
Images from http://www.robots.ox.ac.uk/~vgg/hzbook.html

Basic projective geometry – planar homography

The projective plane
Transformations

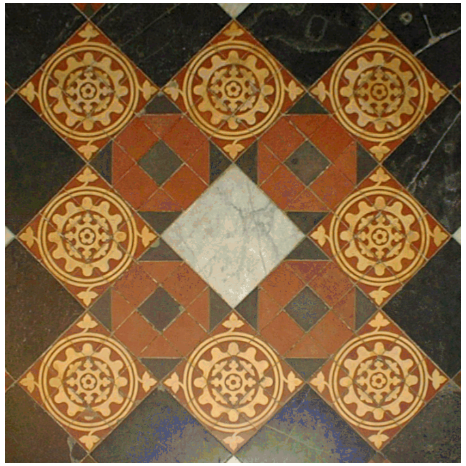
- One characteristic of homographies is that they preserve lines, in fact any invertible transformation of $\mathbb{P}^2$ that preserves lines is a homography

- Examples
  - Central projection from one plane to another is a homography
  - Two images, captured by perspective cameras, of the same planar scene is related by a homography



Basic projective geometry – planar homography

- **Multiple views of a plane**



Basic projective geometry – planar homography

The projective plane
Transformations

- One characteristic of homographies is that they preserve lines, in fact any invertible transformation of $\mathbb{P}^2$ that preserves lines is a homography

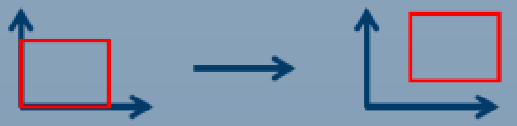
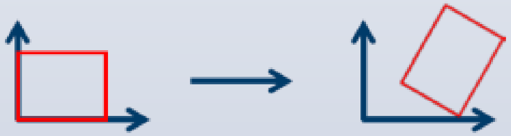
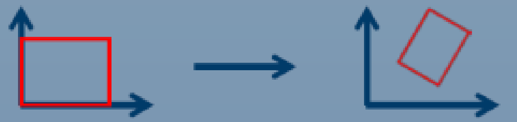
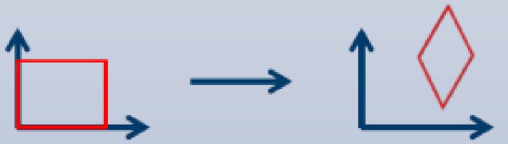
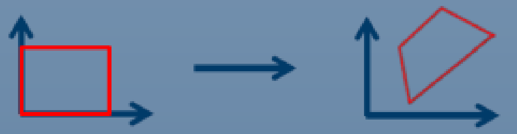
- Examples
  - Central projection from one plane to another is a homography
  - Two images, captured by perspective cameras, of the same planar scene is related by a homography

- One can show that the product of two homographies also must be a homography
  We say that the homographies constitute a group – the projective linear group $PL(3)$

- Within this group there are several more specialized subgroups

Basic projective geometry

Transformations of the projective plane

Transformation of $\mathbb{P}^2$	Matrix	#DoF	Preserves	Visualization
Translation	$\begin{bmatrix} I & \boldsymbol{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$	2	Orientation + all below	
Euclidean	$\begin{bmatrix} R & \boldsymbol{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$	3	Lengths + all below	
Similarity	$\begin{bmatrix} sR & \boldsymbol{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$	4	Angles + all below	
Affine	$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix}$	6	Parallelism, line at infinity + all below	
Homography /projective	$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$	8	Straight lines	

Basic projective geometry - space

The projective space

- The relationship between the Euclidean space $\mathbb{R}^3$ and the projective space $\mathbb{P}^3$ is much like the relationship between $\mathbb{R}^2$ and $\mathbb{P}^2$

- In the projective space
  - We represent points in homogeneous coordinates

$$\tilde{\boldsymbol{x}} = \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \\ \tilde{w} \end{bmatrix} = \begin{bmatrix} \lambda\tilde{x} \\ \lambda\tilde{y} \\ \lambda\tilde{z} \\ \lambda\tilde{w} \end{bmatrix} \forall\lambda \in \mathbb{R}\backslash\{0\}$$

  - Points at infinity have last homogeneous coordinate equal to zero
  - Planes and points are dual objects

$$\tilde{\Pi} = \{\tilde{\boldsymbol{x}} \in \mathbb{P}^3 \mid \tilde{\boldsymbol{\pi}}^T\tilde{\boldsymbol{x}} = 0\}$$

  - The plane at infinity are made up of all points at infinity

$$\mathbb{R}^3 \ni \boldsymbol{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \mapsto \quad \tilde{\boldsymbol{x}} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \in \mathbb{P}^3$$

$$\mathbb{P}^3 \ni \tilde{\boldsymbol{x}} = \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \\ \tilde{w} \end{bmatrix} \quad \mapsto \quad \boldsymbol{x} = \begin{bmatrix} \tilde{x}/\tilde{w} \\ \tilde{y}/\tilde{w} \\ \tilde{z}/\tilde{w} \end{bmatrix} \in \mathbb{R}^3$$

Basic projective geometry - space

Transformations of the projective space

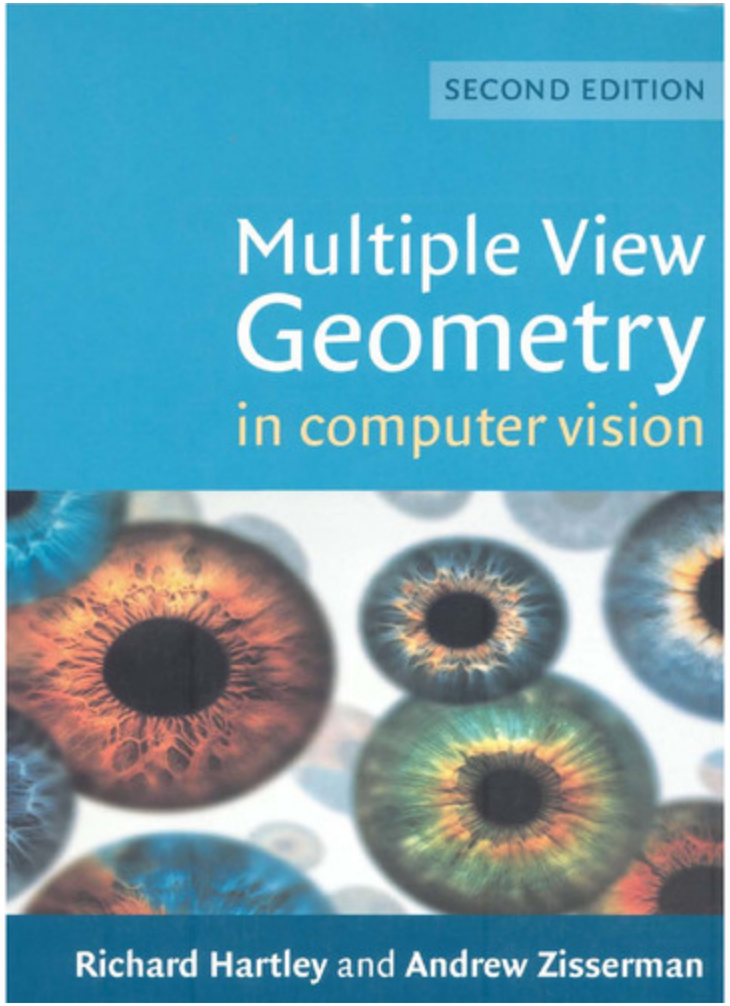
Transformation of $\mathbb{P}^3$	Matrix	#DoF	Preserves
Translation	$\begin{bmatrix} I & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$	3	Orientation + all below
Euclidean	$\begin{bmatrix} R & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$	6	Volumes, volume ratios, lengths + all below
Similarity	$\begin{bmatrix} sR & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$	7	Angles + all below
Affine	$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$	12	Parallelism of planes, The plane at infinity + all below
Homography /projective	$\begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ h_{31} & h_{32} & h_{33} & h_{34} \\ h_{41} & h_{42} & h_{43} & h_{44} \end{bmatrix}$	15	Intersection and tangency of surfaces in contact, straight lines

Basic projective geometry (brief summary)

Summary

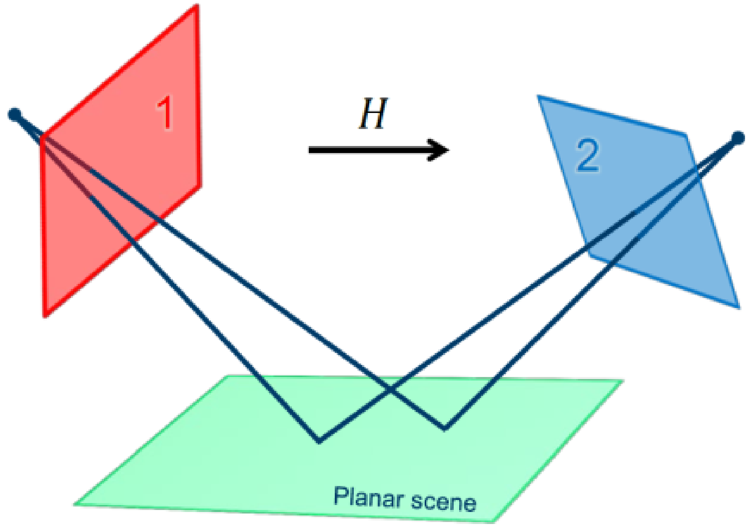
- The projective plane $\mathbb{P}^2$
  - Homogeneous coordinates
  - Line at infinity
  - Points & lines are dual
- The projective space $\mathbb{P}^3$
  - Homogeneous coordinates
  - Plane at infinity
  - Points & planes are dual
- Linear transformations of $\mathbb{P}^2$ and $\mathbb{P}^3$
  - Represented by homogeneous matrices
  - Homographies $\supset$ Affine $\supset$ Similarities $\supset$ Euclidean $\supset$ Translations

More details of projective geometry in CV in the book



Homography

- Homography induced by central projecton and two-view homography are of the same form.

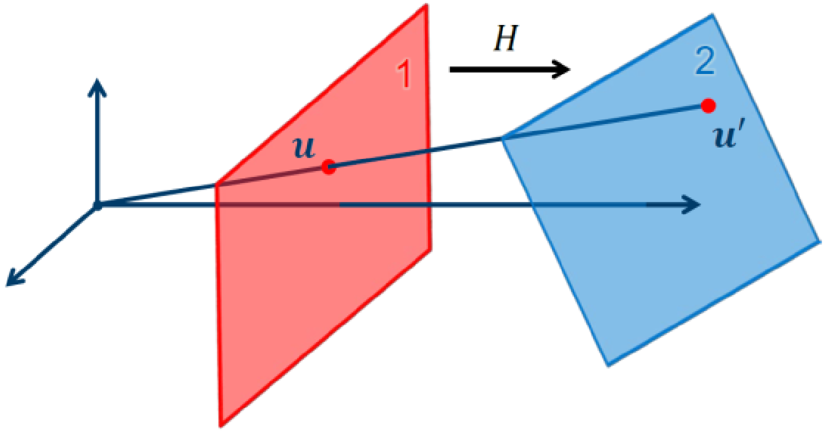


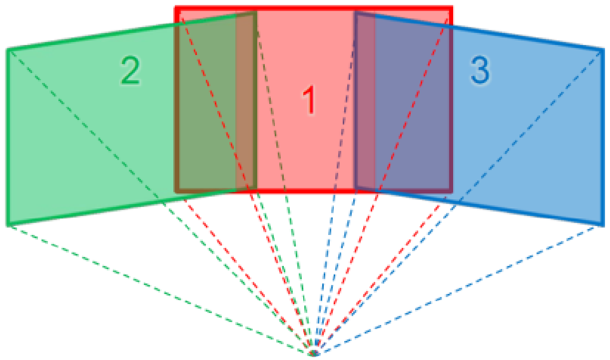
Two-view homography

Homography $H\tilde{\mathbf{u}} = \tilde{\mathbf{u}}'$

$$H = \begin{bmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{bmatrix}$$

Up to a scale factor

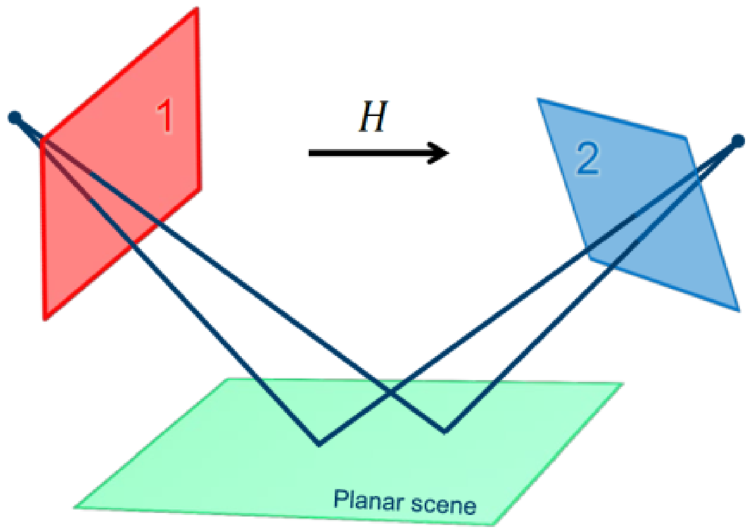




Homography induced by central projection

Homography

- Planar homography and infinite homography are both representable as 3x3 matrices H.

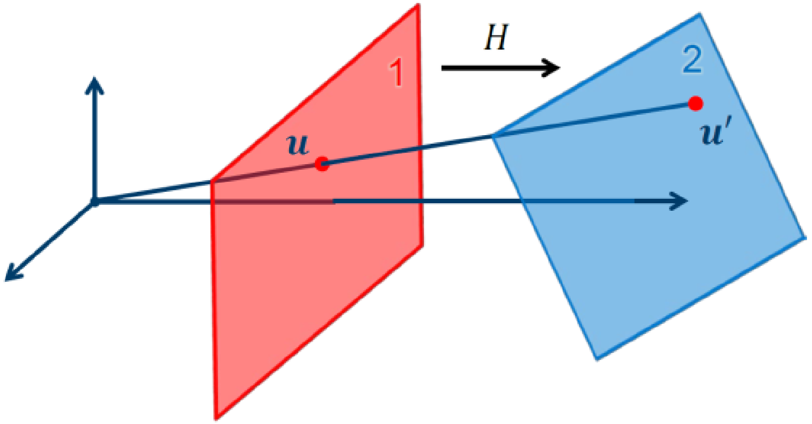


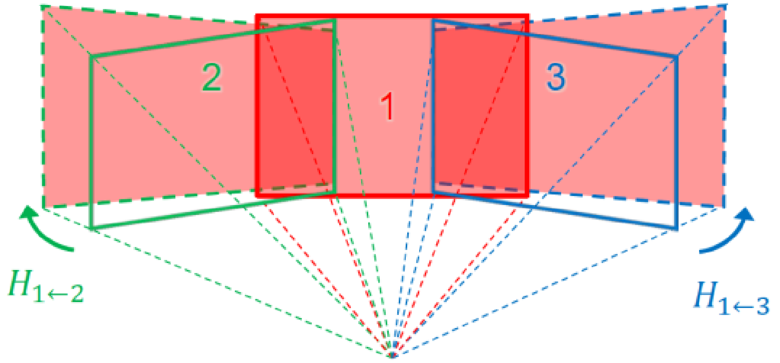
planar homography

Homography $H\tilde{\boldsymbol{u}} = \tilde{\boldsymbol{u}}'$

$$H = \begin{bmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{bmatrix}$$

Up to a scale factor





Homography induced by central projection

Preliminary of homography estimation

Projective transformations (rigorous definition)

Definition:

A *projectivity* is an invertible mapping h from P^2 to itself such that three points x_1, x_2, x_3 lie on the same line if and only if $h(x_1), h(x_2), h(x_3)$ do.

Theorem:

A mapping $h: P^2 \rightarrow P^2$ is a projectivity if and only if there exist a non-singular 3x3 matrix **H** such that for any point in P^2 represented by a vector x it is true that $h(x) = \mathbf{H}x$

Definition: Projective transformation

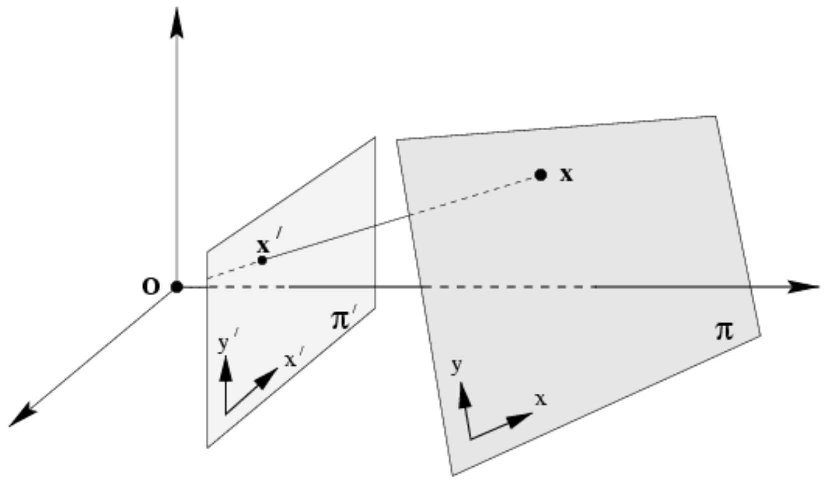
$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{or} \quad \mathrm{x}' = \mathbf{H}\,\mathrm{x}$$

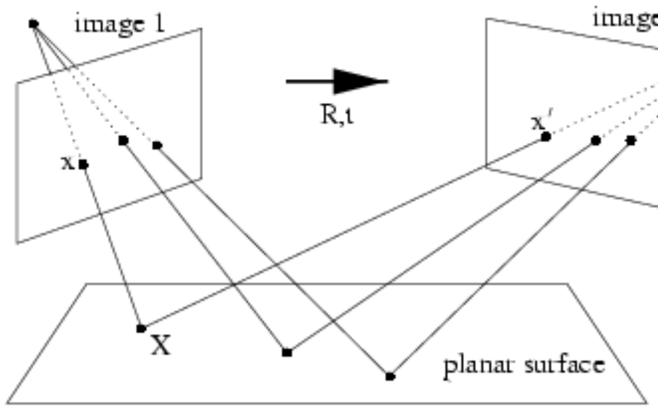
8DOF

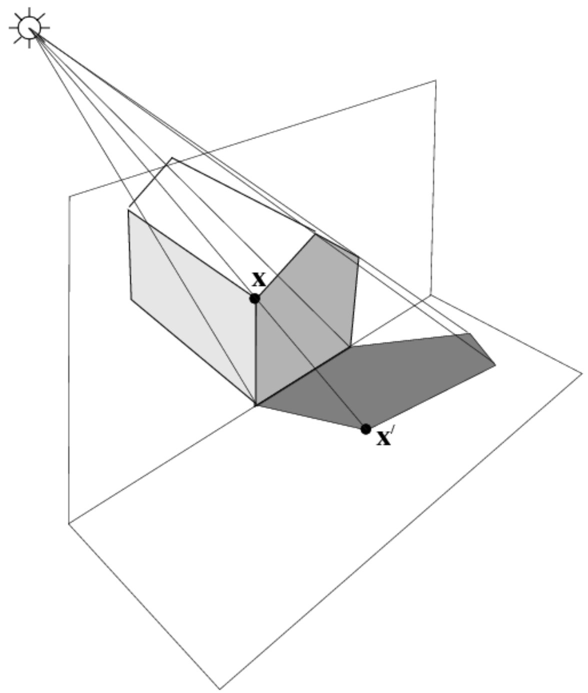
projectivity=collineation=projective transformation=homography

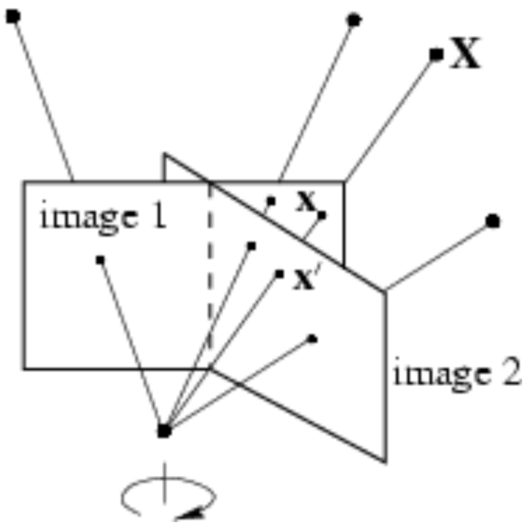
Estimation of homography (8 degree-of-freedom)

Mapping-between-planes applications









central projection may be expressed by
x'=Hx (up to a scale factor)

(application of theorem)

H has 9 parameters, but up to a scale factor.
Its degree-of-freedom (DOF) is 8

More examples

Estimation of homography (with 4 points)

Removing projective distortion

select four points in a plane with know coordinates in the two views (referred to as point correspondences)

$$x' = \frac{x'_1}{x'_3} = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + h_{33}} \qquad y' = \frac{x'_2}{x'_3} = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + h_{33}}$$

$$x'\left(h_{31}x + h_{32}y + h_{33}\right) = h_{11}x + h_{12}y + h_{13}$$
$$y'\left(h_{31}x + h_{32}y + h_{33}\right) = h_{21}x + h_{22}y + h_{23}$$

(linear in h_{ij})

(one point provides 2 constraints/point, 8DOF $\Rightarrow$ 4 point correspondences needed)

Remark: no calibration at all necessary, better ways to compute.

Transformation of lines with homography

For a point transformation

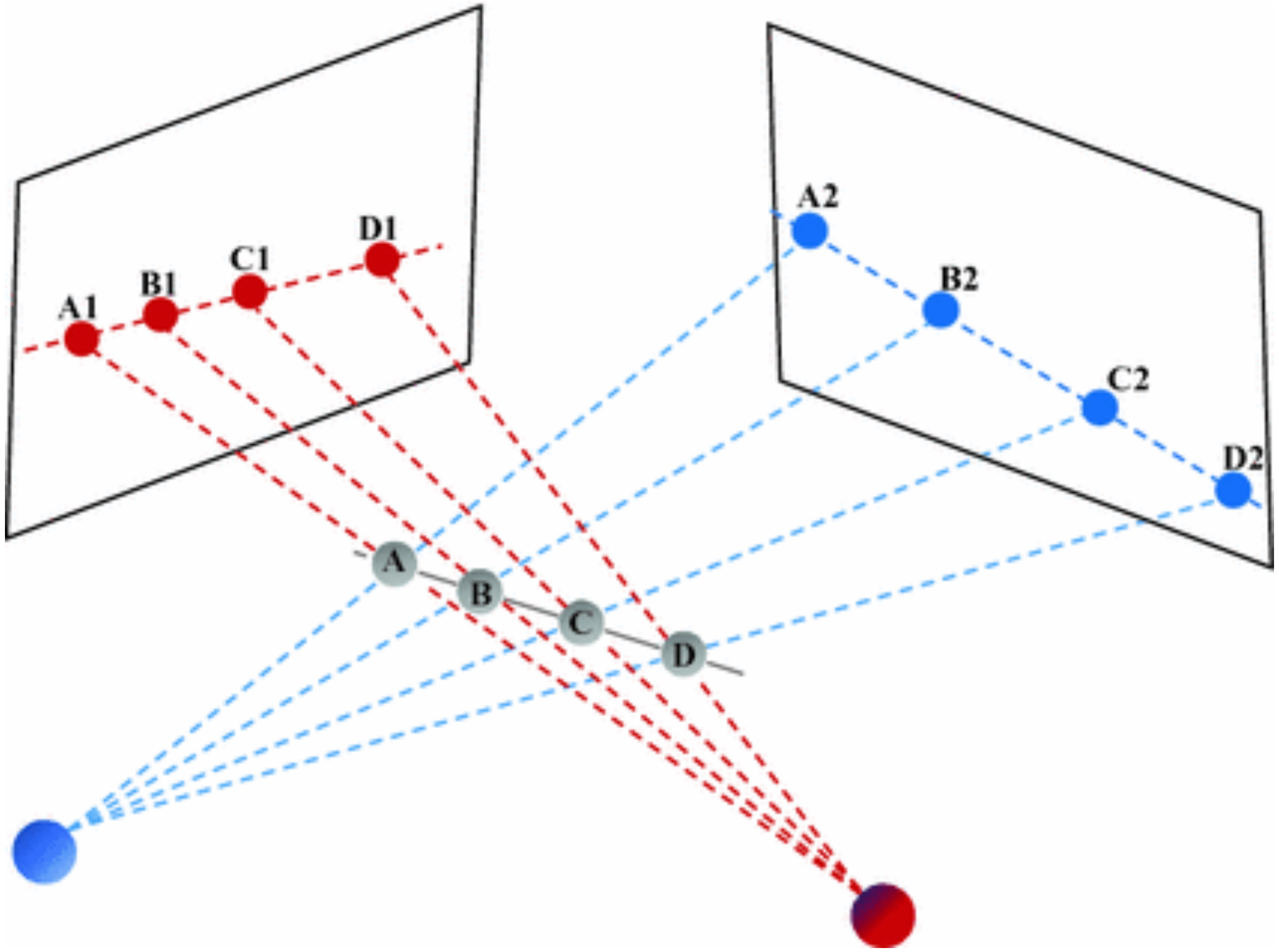
$$\mathrm{x}' = \mathbf{H}\,\mathrm{x}$$

Transformation for lines

$$\mathrm{l}' = \mathbf{H}^{-T}\,\mathrm{l}$$

Hence, similarly, we could also use line correspondences to estimate homography.

1D Projective Geometry and the Cross-ratio



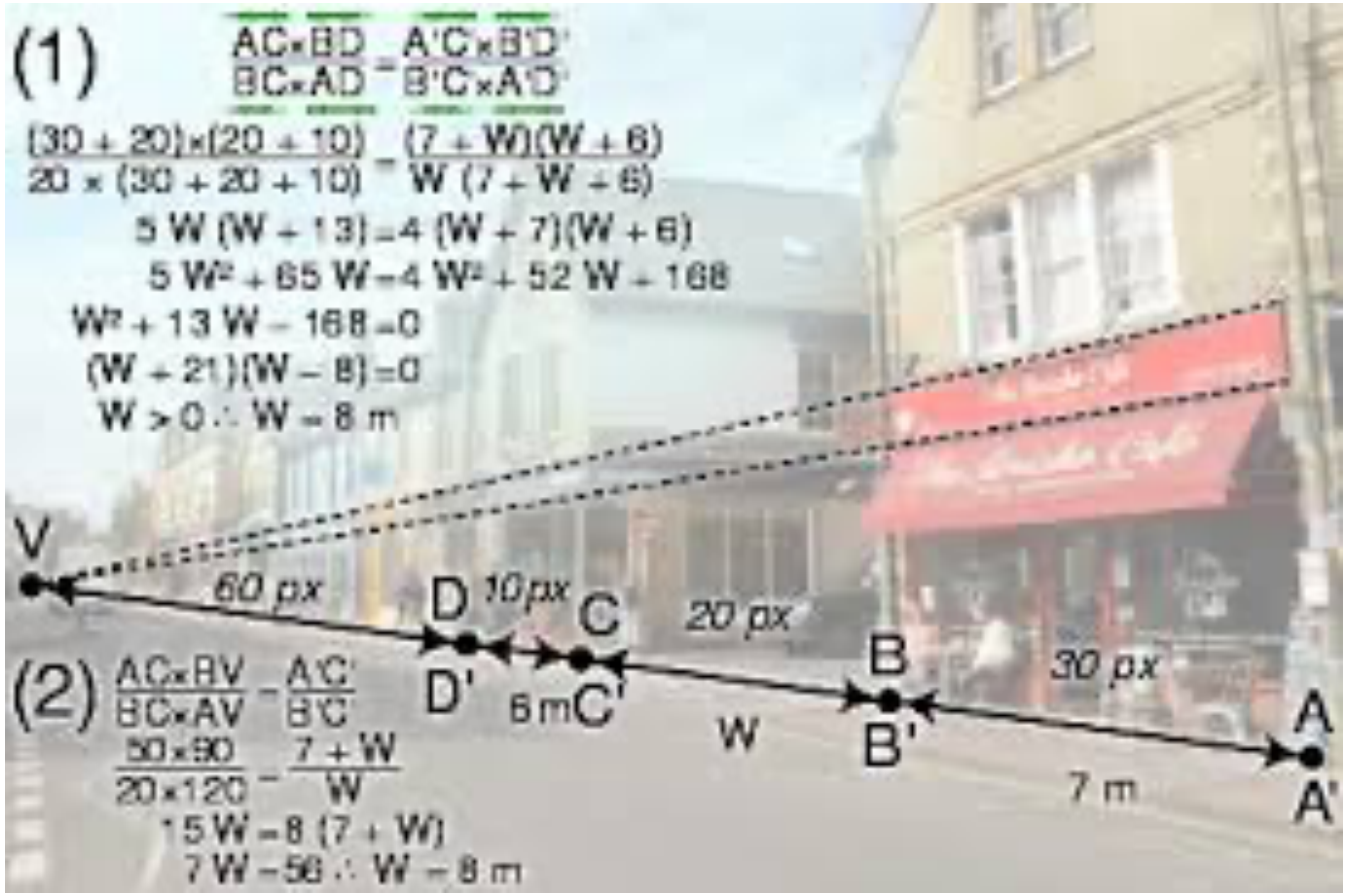
- **Cross ratio of collinear points**

$$CR(A,B,C,D) = \frac{\overline{AC}/\overline{BC}}{\overline{AD}/\overline{BD}}$$

$$= \frac{\overline{AC}\cdot\overline{BD}}{\overline{AD}\cdot\overline{BC}}$$

$$CR(A,B,C,D) = CR(A1,B1,C1,D1)$$
$$= CR(A2,B2,C2,D2)$$

1D Projective Geometry and the Cross-ratio



From Wikipedia

• Use of **cross-ratios** in projective geometry to measure real-world dimensions of features depicted in a perspective projection. A, B, C, D and V are points on the image, their separation given in pixels; A', B', C' and D' are in the real world, their separation in metres.

• In (1), the width of the side street, W is computed from the known widths of the adjacent shops.

• In (2), the width of only one shop is needed because a vanishing point, V is visible.

Complementary: conics in projective space

Conics

Curve described by 2nd-degree equation in the plane

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

or *homogenized* $x \mapsto x_1/x_3,\ y \mapsto x_2/x_3$

$$ax_1^2 + bx_1x_2 + cx_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$$

or in matrix form

$$\mathrm{x}^\mathsf{T}\mathbf{C}\,\mathrm{x} = 0 \quad \text{with} \quad \mathbf{C} = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}$$

5DOF: $\{a:b:c:d:e:f\}$

Complementary: conics in projective space

Five points define a conic

For each point the conic passes through

$$ax_i^2 + bx_iy_i + cy_i^2 + dx_i + ey_i + f = 0$$

or

$$\left(x_i^2, x_iy_i, y_i^2, x_i, y_i, f\right)\mathbf{c} = 0 \qquad \mathbf{c} = \left(a, b, c, d, e, f\right)^\mathsf{T}$$

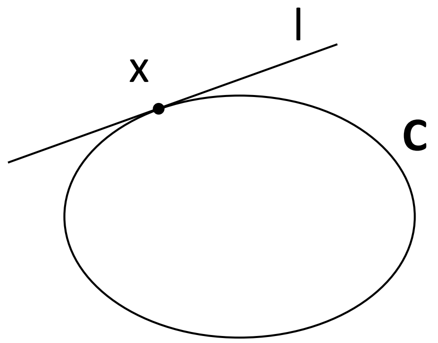
stacking constraints yields

$$\begin{bmatrix} x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 & 1 \end{bmatrix}\mathbf{c} = 0$$

Complementary: conics in projective space

Tangent lines to conics

The line l tangent to **C** at point x on **C** is given by l=**C**x



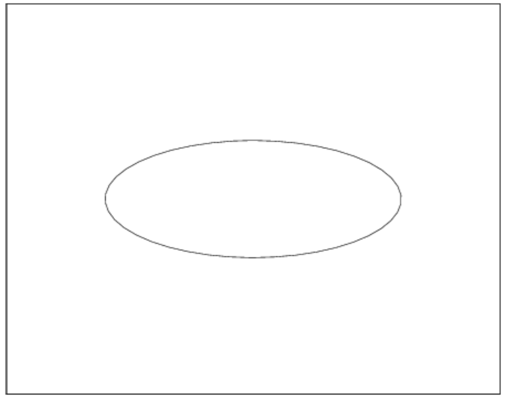
Complementary: conics in projective space

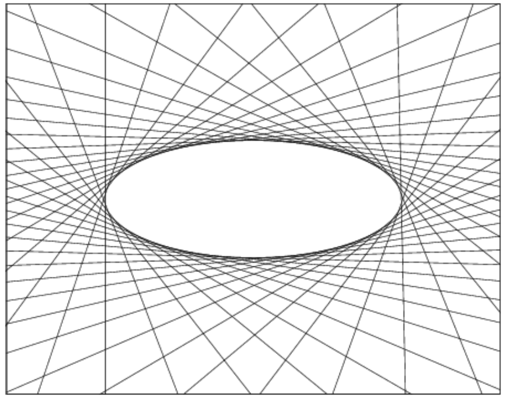
Dual Conics

A line tangent to the conic **C** satisfies $\mathbf{l}^{\mathsf{T}} \mathbf{C}^{*} \mathbf{l} = 0$

In general (**C** full rank): $\mathbf{C}^{*} = \mathbf{C}^{-1}$

Dual conics = line conics = conic envelopes





Transformation of conics with homography

For a point transformation

$$\mathbf{x}' = \mathbf{H}\,\mathbf{x}$$

Transformation for lines

$$\mathbf{l}' = \mathbf{H}^{-T}\,\mathbf{l}$$

Transformation for conics

$$\mathbf{C}' = \mathbf{H}^{-T}\mathbf{C}\mathbf{H}^{-1}$$

Transformation for dual conics

$$\mathbf{C}'^{*} = \mathbf{H}\mathbf{C}^{*}\mathbf{H}^{T}$$