Some of the slides from

- Thomas Opsahl, UNIK

Homography Estimation

Estimating the homography

- The homography has 8 degrees of freedom, but it is custom to treat all 9 entries of the matrix as unknowns instead of setting one of the entries to 1 which excludes all potential solutions where this entry is 0

- Let us solve the equation $H\tilde{\boldsymbol{u}} = \tilde{\boldsymbol{u}}'$ for the entries of the homography matrix

$$H\tilde{\boldsymbol{u}} = \tilde{\boldsymbol{u}}'$$

$$\begin{bmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{bmatrix} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} u' \\ v' \\ 1 \end{bmatrix}$$

More specifically

Assume N (≥ 4) point correspondences are given: $\{(u_i, v_i), (u'_i, v'_i\}|i = 1 \cdots N\}$,
λ_i is a point-dependent scale factor.

$$\lambda_i \begin{bmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{bmatrix} \begin{bmatrix} u_i \\ v_i \\ 1 \end{bmatrix} = \begin{bmatrix} u'_i \\ v'_i \\ 1 \end{bmatrix}$$

$$\lambda_i(h_1 u_i + h_2 v_i + h_3) = u'_i \qquad (1)$$
$$\lambda_i(h_4 u_i + h_5 v_i + h_6) = v'_i \qquad (2)$$
$$\lambda_i(h_7 u_i + h_8 v_i + h_9) = 1 \qquad (3)$$

More specifically

Assume N (≥ 4) point correspondences are given: $\{(u_i, v_i), (u_i', v_i'\}|i = 1 \cdots N\}$,
λ_i is a point-dependent scale factor.

$$\lambda_i \begin{bmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{bmatrix} \begin{bmatrix} u_i \\ v_i \\ 1 \end{bmatrix} = \begin{bmatrix} u'_i \\ v'_i \\ 1 \end{bmatrix}$$

$$\lambda_i(h_1 u_i + h_2 v_i + h_3) = u_i' \quad (1)$$
$$\lambda_i(h_4 u_i + h_5 v_i + h_6) = v_i' \quad (2)$$
$$\lambda_i(h_7 u_i + h_8 v_i + h_9) = 1 \quad (3)$$

From (3), the scale factor λ_i is

$$\lambda_i = 1/(h_7 u_i + h_8 v_i + h_9) \quad (4)$$

Substituting λ_i into (1) and (2),

$$h_1 u_i + h_2 v_i + h_3 = u_i'(h_7 u_i + h_8 v_i + h_9) \quad (5)$$
$$h_4 u_i + h_5 v_i + h_6 = v_i'(h_7 u_i + h_8 v_i + h_9) \quad (6)$$

Two equations

Express the unknowns as a 9x1 vector

In matrix form,

$$\begin{bmatrix} u_i & v_i & 1 & 0 & 0 & 0 & -u_i' u_i & -u_i' v_i & -u_i' \\ 0 & 0 & 0 & u_i & v_i & 1 & -v'_i u_i & -v_i' v_i & -v_i' \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \\ h_7 \\ h_8 \\ h_9 \end{bmatrix} = 0 \quad (7)$$

Express the unknowns as a 9x1 vector $\boldsymbol{h}$

In matrix form,

$$\begin{bmatrix} u_i & v_i & 1 & 0 & 0 & 0 & -u_i'u_i & -u_i'v_i & -u_i' \\ 0 & 0 & 0 & u_i & v_i & 1 & -v'{}_i u_i & -v_i'v_i & -v_i' \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \\ h_7 \\ h_8 \\ h_9 \end{bmatrix} = 0 \qquad (7)$$

N points: $2N$ equations

$$\begin{bmatrix} u_1 & v_1 & 1 & 0 & 0 & 0 & -u_1'u_1 & -u_1'v_1 & -u_1' \\ 0 & 0 & 0 & u_1 & v_1 & 1 & -v_1'u_1 & -v_1'v_i & -v_1' \\ u_2 & v_2 & 1 & 0 & 0 & 0 & -u_2'u_2 & -u_2'v_2 & -u_2' \\ 0 & 0 & 0 & u_2 & v_2 & 1 & -v_2'u_2 & -v_2'v_2 & -v_2' \\ & & & & & \vdots & & & \\ & & & & & \vdots & & & \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \\ h_7 \\ h_8 \\ h_9 \end{bmatrix} = 0 \qquad (8)$$

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$$A\boldsymbol{h} = 0 \quad (9)$$

Solving $\boldsymbol{h}$

Because $\boldsymbol{h}$ is homogeneous, the solutions of $\boldsymbol{h}$ and $\alpha\boldsymbol{h}$ make no difference for any $\alpha \neq 0$.
In addition, (9) have $2N$ equations with 9 unknowns, exact solution cannot always exist.

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We thus solve the least-squared-error solution:

$$\min_{\boldsymbol{h}} \|A\boldsymbol{h}\|^2, \quad \text{s.t. } \|\boldsymbol{h}\|^2 = 1 \qquad (10)$$

The solution is equivalent to

$$\operatorname*{argmin}_{\boldsymbol{h}} \frac{\|A\boldsymbol{h}\|^2}{\|\boldsymbol{h}\|^2} = \frac{\boldsymbol{h}^T A^T A\boldsymbol{h}}{\|\boldsymbol{h}\|^2} \qquad (11)$$

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According to the principle of Rayleigh quotient, the solution of (11) is the eigenvector corresponding to the smallest eigenvalue of matrix $A^T A$

Solving $\boldsymbol{h}$

Thus, the solution can be obtained by performing the eigen-decomposition, $A^T A = PDP^T$, where $D \in R^{N \times N}$ is a diagonal matrix consisting of the eigenvalues (w.l.o.g., from large to small) of the matrix $A^T A$. The solution is then the last column of P (i.e., the eigenvector corresponding to the smallest eigenvalue of matrix $A^T A$)

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In linear algebra, the eigenvector corresponding to the smallest eigenvalue of matrix $A^T A$ can also be computed as performing the singular value decomposition (SVD) at first, $A = U\Sigma V^T$ (the singular values are from large to small in the matrix Σ). The solution is then the last column of V.

Homography Estimation

Basic homography estimation

- Since H (and thus $\boldsymbol{h}$) is homogeneous, we only need the matrix A to have rank 8 in order to determine $\boldsymbol{h}$ up to scale
- It is sufficient with 4 point correspondences where no 3 points are collinear
- We can calculate the non-trivial solution to the equation $A\boldsymbol{h} = \boldsymbol{0}$ by SVD
$$\mathrm{svd}(A) = USV^T$$
- The solution is given by the right singular vector without a singular value which is the last column of V, i.e. $\boldsymbol{h} = \boldsymbol{v}_9$

$$\begin{bmatrix} 0 & 0 & 0 & -u_1 & -v_1 & -1 & v_1'u_1 & v_1'v_1 & v_1' \\ u_1 & v_1 & 1 & 0 & 0 & 0 & -u_1'u_1 & -u_1'v_1 & -u_1' \\ 0 & 0 & 0 & -u_2 & -v_2 & -1 & v_2'u_2 & v_2'v_2 & v_2' \\ u_2 & v_2 & 1 & 0 & 0 & 0 & -u_2'u_2 & -u_2'v_2 & -u_2' \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \\ h_7 \\ h_8 \\ h_9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

$$A\mathbf{h} = \mathbf{0}$$

Summary from Thomas Opsahl's slides. Note that the form of matrix A is different from the one defined in ours (9), but the resulted equations are equivalent.

SVD Review

SVD

Singular Value Decomposition

The singular value decomposition of a real $m \times n$ matrix A is a factorization $A = USV^T$

Here U is a orthogonal $m \times m$ matrix, V is a orthogonal $n \times n$ matrix and S is a real positive diagonal $m \times n$ matrix

The diagonal entries of $S = \mathrm{diag}\left(s_1, \dots, s_{\min(m,n)}\right)$ are known as the singular values of A and the columns of $U = [\boldsymbol{u}_1, \dots, \boldsymbol{u}_m]$ and $V = [\boldsymbol{v}_1, \dots, \boldsymbol{v}_n]$are known as the left and right singular vectors of A respectively

The nullspace of A is the span of the right singular vectors $\boldsymbol{v}_i$ that corresponds to a zero singular value s_i (or does not have a corresponding singular value)

SVD Review

SVD

Example

$$A\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From `[U,S,V] = svd(A);` we get

$$U = \begin{bmatrix} -0.3863 & -0.9224 \\ -0.9224 & 0.3863 \end{bmatrix} \quad S = \begin{bmatrix} 9.5080 & 0 & 0 \\ 0 & 0.7729 & 0 \end{bmatrix}$$

$$V = \begin{bmatrix} -0.4287 & 0.8060 & 0.4082 \\ -0.5663 & 0.1124 & -0.8165 \\ -0.7039 & -0.5812 & 0.4082 \end{bmatrix}$$

From this we see that A has:

- 2 left singular vectors

$$\mathbf{u}_1 = \begin{bmatrix} -0.3863 \\ -0.9224 \end{bmatrix} \quad \mathbf{u}_2 = \begin{bmatrix} -0.9224 \\ 0.3863 \end{bmatrix}$$

- 2 nonzero singular values

$$s_1 = 9.5080 \quad s_2 = 0.7729$$

- 3 right singular vectors

$$\mathbf{v}_1 = \begin{bmatrix} -0.4287 \\ -0.5663 \\ -0.7039 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 0.8060 \\ 0.1124 \\ -0.5812 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 0.4082 \\ -0.8165 \\ 0.4082 \end{bmatrix}$$

Since $\boldsymbol{v}_3$ does not have a corresponding singular value, $\boldsymbol{x} = \boldsymbol{v}_3$ is a non-trivial solution to $A\boldsymbol{x} = 0$ and $\boldsymbol{x} = k \cdot \boldsymbol{v}_3;\ k \in \mathbb{R}\backslash\{0\}$ is the family of all non-trivial solutions

Homography Estimation - DLT method

Basic homography estimation

1. **A basic homography estimation method for n point-correspondences**
2. A way to determine which of the point correspondences that are inliers for a given homography

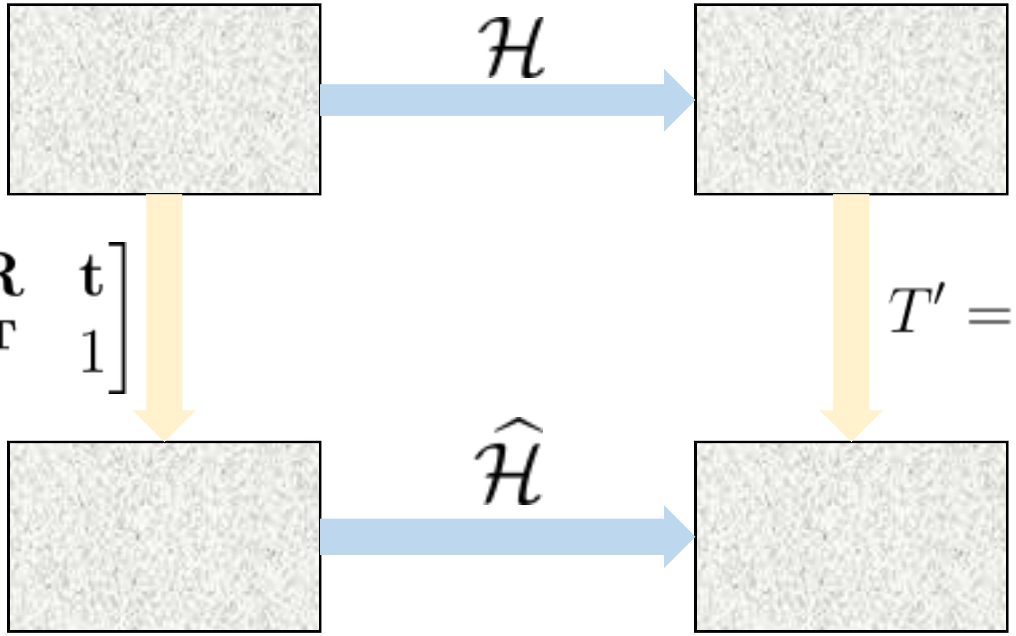
Direct Linear Transform

$$A = \begin{bmatrix} 0 & 0 & 0 & -u_1 & -v_1 & -1 & v'_1 u_1 & v'_1 v_1 & v'_1 \\ u_1 & v_1 & 1 & 0 & 0 & 0 & -u'_1 u_1 & -u'_1 v_1 & -u'_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

1. Build the matrix A from at least 4 point-correspondences $(u_i, v_i) \leftrightarrow (u'_i, v'_i)$
2. Obtain the SVD of A: $A = USV^T$
3. If S is diagonal with positive values in descending order along the main diagonal, then $\boldsymbol{h}$ equals the last column of V
4. Reconstruct H from $\boldsymbol{h}$

Normalized DLT method

The result of the DLT algorithm for computing homographies depends on the coordinate frame in which points are expressed. In fact the result is not invariant to similarity transformations of the image. This suggests the question whether some coordinate systems are in some way better than others for computing a homography.



$$T = \begin{bmatrix} s\mathbf{R} & \mathbf{t} \\ \mathbf{0}^{\mathbf{T}} & 1 \end{bmatrix}$$

$$T' = \begin{bmatrix} s'\mathbf{R}' & \mathbf{t}' \\ \mathbf{0}^{\mathbf{T}} & 1 \end{bmatrix}$$

$$\mathcal{H} = T'^{-1}\widehat{\mathcal{H}}T$$

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As a first step of normalization, the coordinates in each image are translated (by a different translation for each image) so as to bring the centroid of the set of all points to the origin.

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As a first step of normalization, the coordinates in each image are translated (by a different translation for each image) so as to bring the centroid of the set of all points to the origin.

The coordinates are also scaled so that on the average a point is of the form $\tilde{u} = (u, v, w)^T$, with each of u, v and w having the same average magnitude.

To this end, we choose to scale the coordinates so that the average distance of a point $(u, v)^T$ from the origin is equal to $\sqrt{2}$. This means that the "average" point is equal to $(1,1,1)^T$.

Normalized DLT method

In summary the transformation is as follows:

(i) The points are translated so that their centroid is at the origin.
(ii) The points are then scaled so that the average distance from the origin is equal to $\sqrt{2}$.
(iii) This transformation is applied to each of the two images independently.

Shortage without normalization. Without normalization typical image are of the order $(u, v, w)^T = (100,100,1)^T$, i.e., u, v are much larger than w. In A the entries uu' , uv', $u'u$, $u'v$ will be of order 10^4 , entries uw' , vw' etc. of order 10^2 , and entries ww' will be unity. This results in an ill-conditioning number of matrix A and numerically unstable solutions.

Homography Estimation – Normalized DLT method

Basic homography estimation

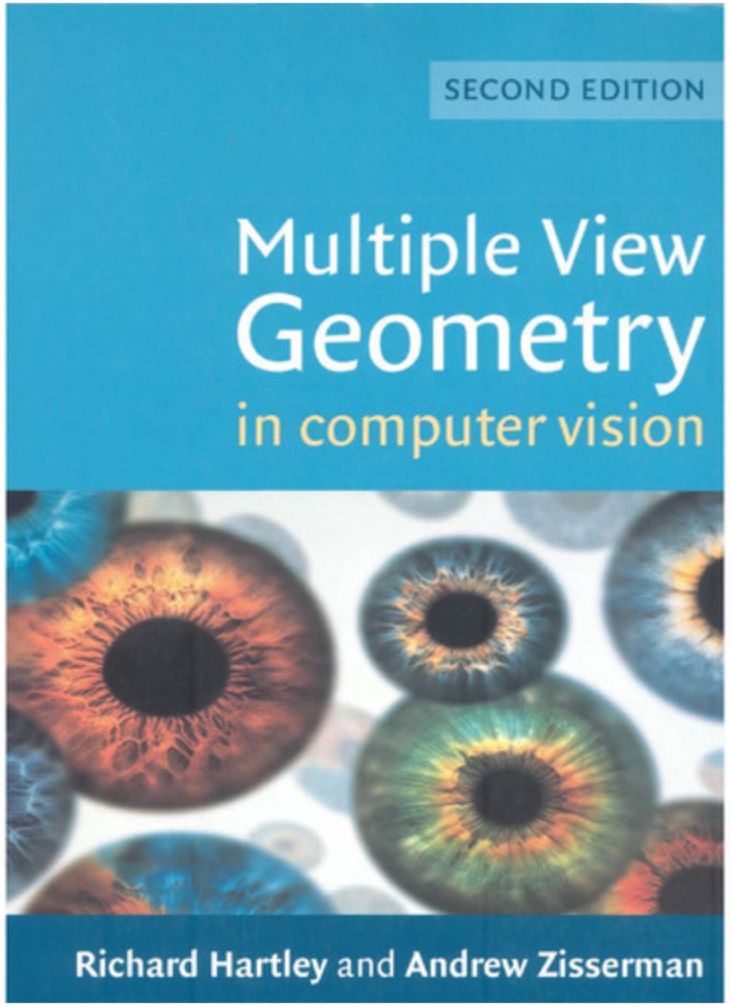
- The basic DLT algorithm is never used with more than 4 point-correspondences
- This is because the algorithm performs better when all the terms of A has a similar scale
  - Note that some of the terms will always be of scale 1
- To achieve this, it is common to extend the algorithm with a normalization and a denormalization step

Normalized Direct Linear Transform

1. Normalize the set of points $\boldsymbol{u}_i = [u_i, v_i]^T$ by computing a similarity transform T that translates the centroid to the origin and scales such that the average distance from the origin is $\sqrt{2}$
2. In the same way normalize the set of points $\boldsymbol{u'}_i = [u'_i, v'_i]^T$ by computing a similarity transform T'
3. Apply the basic DLT algorithm on the normalized points to obtain a homography $\widehat{H}$
4. Denormalize to get the homography: $H = T'^{-1}\widehat{H}T$

More details can be found in

- R. Hartley & A. Zisserman's book



Example of applications – image mosaicing

Image mosaicing


- Find key points and represent by descriptors

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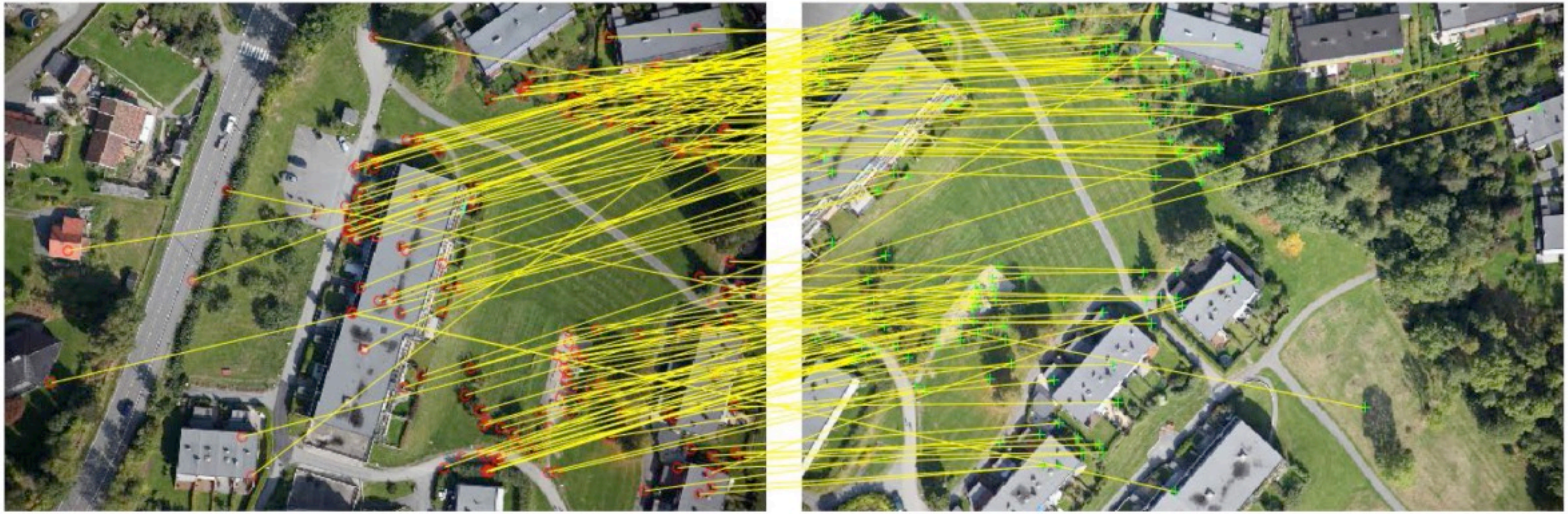

- Establish point-correspondences by matching descriptors
- Several wrong correspondences. Suppose that we have removed them.

Example of applications – image mosaicing

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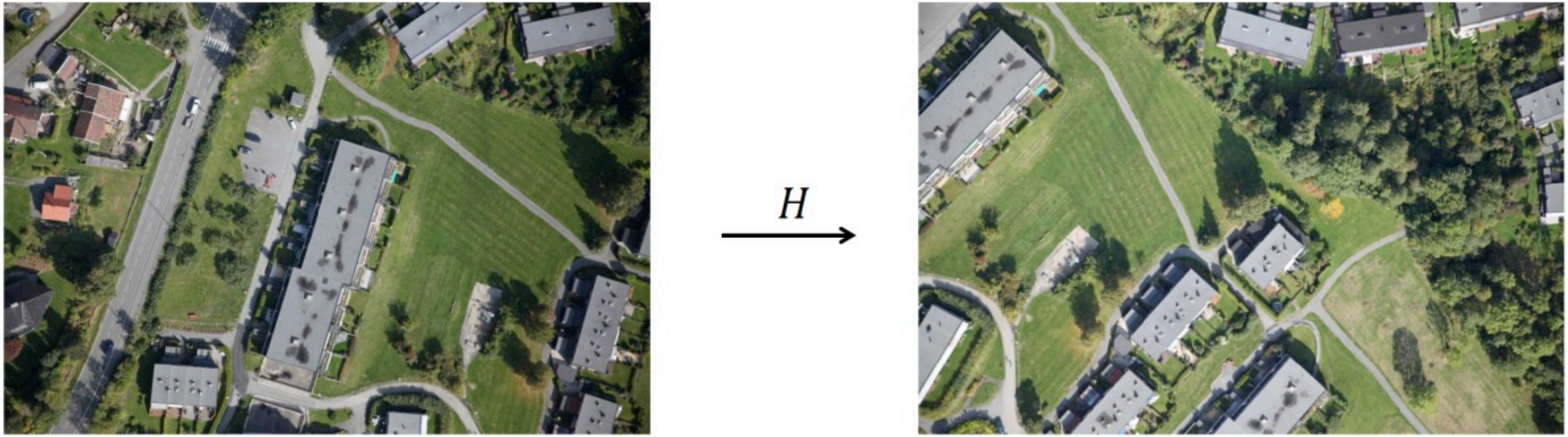

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