6.5 Verify Equation (6.1.9) and Equation (6.1.10) for the MA(1) process.

$$\begin{cases} C_{11} = 1-3\rho_1^2+4\rho_1^4 \\ C_{kk} = 1+2\rho_1^2 \text{ for } k>1 \end{cases} \quad (6.1.9)$$

$$C_{12} = 2\rho_1(1-\rho_1^2) \quad (6.1.10)$$

Solution: In general

$$C_{ij} = \sum_{k=-\infty}^{\infty} (\rho_{k+i}\rho_{k+j} + \rho_{k-i}\rho_{k+j} - 2\rho_i\rho_k\rho_{k+j} - 2\rho_j\rho_k\rho_{k+i} + 2\rho_i\rho_j\rho_k^2)$$

$$C_{11} = \sum_{k=-\infty}^{\infty} \underbrace{(\rho_{k+1}^2 + \rho_{k+1}\rho_{k-1} - 2\rho_1\rho_k\rho_{k+1} - 2\rho_1\rho_k\rho_{k+1} + 2\rho_1^2\rho_k^2)}_{\text{denoted by } Y(k)} = \sum_{k=-\infty}^{\infty} Y(k)$$

for MA(1) process $\rho(k)=0$ if $k>1$ and $\rho_0=1$ and $\rho_k=\rho_{-k}$ for all k

$$C_{11} = \sum_{k=-\infty}^{\infty} Y(k) = Y(-2) + Y(-1) + Y(0) + Y(1) + 0$$

$$= \rho_{-1}^2 + (\rho_0^2 - 4\rho_1\rho_{-1}\rho_0 + 2\rho_1^2\rho_{-1}^2) + (\rho_1^2 + \rho_1\rho_{-1} - 4\rho_1^2\rho_0 + 2\rho_1^2\rho_0^2) + 2\rho_1^4$$

$$= 1-3\rho_1^2+4\rho_1^4$$

$$C_{kk} = \sum_{t=-\infty}^{\infty} (\rho_{t+k}\rho_{t+k} + \rho_{t-k}\rho_{t+k} - 2\rho_k\rho_t\rho_{t+k} - 2\rho_k\rho_t\rho_{t+k} + 2\rho_k\rho_k\rho_t^2) \quad \text{where } k>1$$

$$= \sum_{t=-\infty}^{\infty} (\rho_{t+k}^2 + \rho_{t-k}\rho_{t+k} + 0) = \sum_{t=-\infty}^{\infty} \rho_{t+k}^2 = \rho_{-1}^2 + \rho_0^2 + \rho_1^2 = 1+2\rho_1^2$$

Equation 6.1.9 is verified

$$C_{12} = \sum_{k=-\infty}^{\infty} (\rho_{k+1}\rho_{k+2} + \rho_{k-1}\rho_{k+2} - 2\rho_1\rho_k\rho_{k+2} - 2\rho_2\rho_k\rho_{k+1} + 2\rho_1\rho_2\rho_k^2)$$

$$= \sum_{k=-\infty}^{\infty} (\rho_{k+1}\rho_{k+2} + \rho_{k-1}\rho_{k+2} - 2\rho_1\rho_k\rho_{k+2} + 0)$$

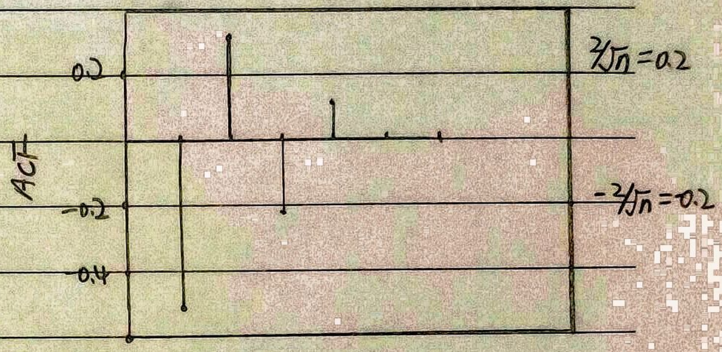
$$= (\rho_1\rho_0 + \rho_0\rho_1) + 0 - 2\rho_1\rho_{-1}\rho_1 = \rho_1 + \rho_1 - 2\rho_1^3 = 2\rho_1(1-\rho_1^2)$$

Equation 6.1.10 is verifred.

6.12 From a time series of 100 observations, we calculate $r_1 = -0.49$, $r_2 = 0.31$, $r_3 = -0.21$, $r_4 = 0.11$ and $|r_k| < 0.09$ for $k > 4$. On this basis alone, what ARIMA model would we tentatively specify for the series?

10/15

Solution:



From the correlogram

r_k for $k \geq 4$ are placed within $\pm 2/\sqrt{n}$ lines.

for $r_3 = -0.21 \approx -\frac{2}{\sqrt{n}} = -0.2$

We consider MA(2) or MA(3) model to do further test.

By 6.1.11 $Var(r_3) = \frac{1}{100}[1+2(r_1^2+r_2^2)]$

$= \frac{1}{100}[1+2(-0.49)^2+2(0.31)^2] = 0.0167$

$r_1 \pm \frac{2}{\sqrt{n}}$

$r_2 \pm \frac{2}{\sqrt{n}}$

thus the standard error of $r_3 = \sqrt{0.0167} = 0.1292$.

Since r_3 is within $(-2\times 0.1292, 2\times 0.1292) = (-0.258, 0.258)$ 10

MA(2) is not rejected.

6.13 A stationary time series of length 121 produced sample partial autocorrelation of $\hat{\phi}_{11} = 0.8$, $\hat{\phi}_{22} = -0.6$, $\hat{\phi}_{33} = 0.08$ and $\hat{\phi}_{44} = 0.00$. Based on this information alone, what model would we tentatively specify for the series?

Solution:

Using $2/\sqrt{n} = 2/\sqrt{121} = 0.181$ as a guide.

Clearly $\hat{\phi}_{22}$ is out of $(-0.181, 0.181)$ } $k=3 \Rightarrow p=2$

$\hat{\phi}_{33}$ is within $(-0.181, 0.181)$

AR(2) model is able to be applied.

HW6. Chunhua Yu. 30/30 6.17 , 6.30
7.3 7.5

10/10 6.17 Consider AR(1) series of length 100, with $\phi=0.7$

(a) Would you be surprised if $r_1=0.6$

For an AR(1) model with $\phi=0.7$ and $n=100$ (large Sample)

Using Exhibit 6.1 Table

$\sqrt{Var(r_1)} = 0.71/\sqrt{n} = 0.71/10 = 0.071$

$0.7 \pm 2\sqrt{Var(r_1)} = 0.7 \pm 2\times0.071 = [0.558, 0.842]$

if $r_1=0.6$, r_1 within $[0.558, 0.842]$

We conclude $r_1=0.6$ is reasonable, so It is not surprising. ✓

(b) Would $r_{10}=-0.15$ be unusual?

Based on Exhibit 6.1 Table

$\sqrt{Var(r_{10})} = 1.70/\sqrt{n} = 1.70/10 = 0.170$

$\rho_{10} = (\phi)^{10} = (0.7)^{10} = 0.028$

$0.028 \pm 2\sqrt{Var(r_{10})} = 0.028 \pm 2\times0.170 = [-0.312, 0.368]$

if $r_{10}=-0.15$, r_{10} within $[-0.312, 0.368]$

We conclude $r_{10}=-0.15$, is reasonable., it's usual ✓

7.3 If $\{Y_t\}$ statisfies an AR(1) Model with ϕ of about 0.7, how long

15/15 of a series do we need to estimate $\phi=\rho_1$ with 95% confidence

that out estimation error is no more than ± 0.1?

Solution: Using formula (6.1.5) $Var(r_1) \approx \frac{1-\phi^2}{n}$

the standard error of $\hat{\phi}=r_1$ is $\sqrt{Var(r_1)} = \sqrt{\frac{1-\hat{\phi}^2}{n}}$

95% confidence implied $2\sqrt{Var(r_1)} = 0.1$

$2^2 \cdot \frac{1-\hat{\phi}^2}{n} = 0.01$

$n = 4\times(1-\hat{\phi}^2)\times100 = 4\times(1-0.7^2)\times100$

$=204$ ✓

7.5 Given the data $Y_1=10$, $Y_2=9$ and $Y_3=9.5$, we wish to fit an IMA(1,1) model without constant term.

5/15 (a) Find the conditional least square estimate of θ.

Solution: The model IMA(1,1) without constant term

$$Y_t - Y_{t-1} = e_t - \theta e_{t-1}$$

We assume $Y_0 = Y_1 = 10$

$W_t = Y_t - Y_{t-1} \sim MA(1)$ model. $\quad W_t = e_t - \theta e_{t-1}$

$W_1 = Y_1 - Y_0 = 10 - 10 = 0 \quad \Rightarrow \quad e_1 = W_1 = 0$

$W_2 = Y_2 - Y_1 = 9 - 10 = -1 \quad e_2 = W_2 + \theta e_1 = -1$

$W_3 = Y_3 - Y_2 = 9.5 - 9 = 0.5 \quad e_3 = W_3 + \theta e_2 = 0.5 - \theta$

$$S_c(\theta) = \Sigma (e_t)^2$$
$$= 0 + 1 + (0.5-\theta)^2$$

let $\frac{\partial S_c(\theta)}{\partial \theta} = 0 \Rightarrow 2(0.5-\theta)\times(-1) = 0$

$\theta = 0.5$

thus, the conditional least square $\hat{\theta} = 0.5$ ✓

(b) Estimate σ_e^2

$$\hat{\sigma}_e^2 = \frac{S_c(\theta)}{n-1} = \frac{1}{2}\times\{0+1+0\} = \frac{1}{2}$$ ✓

Q 6.30:

Simulate a Mixed ARMA(1,1) model of length n=100 with ϕ=0.8 and θ=0.4.

10/10

```
> set.seed(2)
> data1=arima.sim(n=100,list(ar=0.8,ma=-0.4))
```

(a) Calculate and plot the theoretical autocorrelation function for this model. Plot sufficient lags until the correlations are negligible.

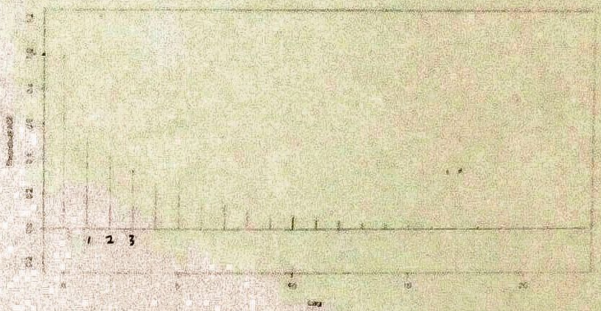
```
> data1.tacf=ARMAacf(ar=0.8,ma=-0.4,lag.max = 22)
> plot(data1.tacf,x=0:22,xlab = 'Lag',ylab = 'Theoretical ACF',type = 'h',ylim = c(-0.2,1.2))
> abline(h=0)
```

$$\rho_k = \frac{(1-\theta\phi)(\phi-\theta)}{1-2\theta\phi+\theta^2}\cdot\phi^{k-1} \quad k\geq 1$$

$$= \frac{(1-0.8\times0.4)(0.8-0.4)}{1-2\times0.8\times0.4+0.4^2}\cdot(0.8)^{k-1}$$

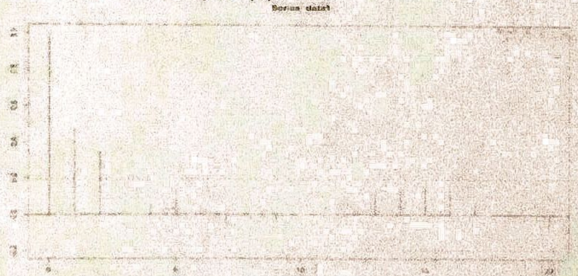
$$= \frac{0.68\times0.4}{1-0.64+0.16}\times 0.8^{k-1}$$

$$= 0.523\times0.8^{k-1}$$



$$\begin{cases} \rho_0 = 1 \\ \rho_1 = 0.523 \\ \rho_2 = 0.4184 \\ \rho_3 = 0.33472 \\ \cdots \end{cases}$$

(b) Calculate and plot the sample ACF for your simulated series. How well do the Values and patterns match the theoretical ACF from part (a)?



$$r_k = \frac{\sum_{t=k+1}^{n}(Y_t-\bar{Y})(Y_{t-k}-\bar{Y})}{\sum_{t=1}^{n}(Y_t-\bar{Y})} \quad \text{for } k=1,2,\cdots$$

```
> data1
Time Series:
Start = 1   End = 100    Frequency = 1
 [1] 0.5752552 1.0832881 0.8900154 1.6605925 0.6138506 -0.1719317 -0.4225358 -1.8257442 -1.6727879 -1.5362585 -1.2518946 -
1.2864969 -2.8348662 -2.3259567 0.3794641 0.1646462 1.8736398 0.3970600 0.3489972 0.1313740 -1.0200040
 [22] -1.1747832 1.4617896 -0.2193359 1.3251456 -0.4977423 -1.9450431 -1.0926542 0.1909276 0.9176269 1.9500283 -0.8968671
2.0290458 0.1075953 0.5254987 0.8633679 -0.3317947 -1.9362847 -1.2287816 -0.7071283 -1.4948612 -1.7589700
 [43] -0.7082162 -0.8404136 -0.1808188 -0.3723168 -1.1834747 0.7195766 0.8260462 1.4046466 -0.7073313 0.9941349 -1.2988508 -
0.8941468 -1.8742381 -3.1584025 0.1785685 -1.2393876 -1.0148340 -1.0849443 -0.3264806 1.1845284 1.9886213
 [64] -0.2651713 -1.0971518 -1.8470093 -2.1256440 0.7600838 -0.1680299 -0.9800975 -1.0481920 0.4763698 0.2119100 -0.2489832 -
0.8231078 -1.2208326 1.4162536 1.2537068 2.6356845 0.8836992 0.5246744 -0.4673073 -0.2134127 0.4013369
 [85] 1.4912656 1.2136051 1.2011968 2.0095190 2.2623706 1.4576398 0.3401560 1.8266513 1.1036995 2.5380478 0.9155446 -
0.1395375 0.8436414 -0.2097047 0.7388762 -1.3884111
```

in this sample :

$\bar{Y} = -0.03468$

$\sum_{t=1}^{n}(Y_t-\bar{Y})^2 = 158.7295$

```
> ybar=mean(data1)
> ybar
[1] -0.03467907
> st=data1-ybar
> st2=(data1-ybar)^2
> sts=sum(st2)
> sts
[1] 158.7295
> st.2=st[2:100]
> st.21=st[1:99]
> r1=sum(st.2*st.21)/sts
> r1
[1] 0.4889166
```

if $k=1$ $\sum_{t=2}^{n}(Y_t-\bar{Y})(Y_{t-1}-\bar{Y}) \approx 77.6$

$$r_1 = \frac{\sum_{t=2}^{n}(Y_t-\bar{Y})(Y_{t-1}-\bar{Y})}{\sum_{t=1}^{n}(Y_t-\bar{Y})^2} \approx 0.4889$$

likewise, using $r_k = \frac{\sum_{t=k+1}^{n}(Y_t-\bar{Y})(Y_{t-k}-\bar{Y})}{\sum_{t=1}^{n}(Y_t-\bar{Y})^2}$ calucate other r_k.

(c) Calculate and interpret the sample EACF for this series. Does the EACF help you specify the correct orders for the model?

```
> eacf(data1)
AR/MA 0 1 2 3 4 5 6 7 8 9 10 11 12 13
    0 x x o o o o o o o o o  o  o  o
    1 x x o o o o o o o o o  o  o  o
    2 x o o o o o o o o o o  o  o  o
    3 o x o o o o o o o o o  o  o  o
    4 o x o o o o o o o o o  o  o  o
    5 x x o o o o o o o o o  o  o  o
    6 x x x o o o o o o o o  o  o  o
    7 x x o o o o o o o o o  o  o  o
```

o^*

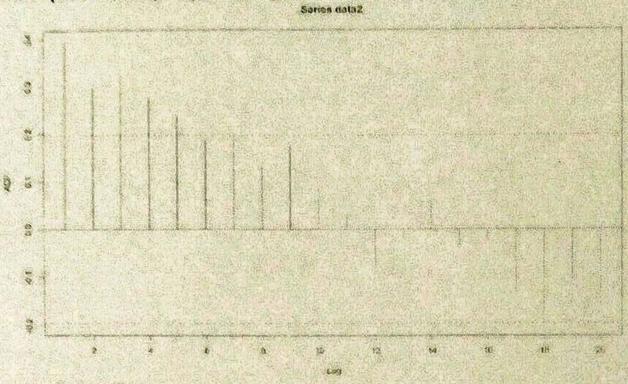
Since o^* is located $(2,1)$.

This sample fit ARMA(2,1) rather than ARMA(1, 1)

Finding: [redacted]ll.

(d) Repeat parts (b) and (c) with a new simulation using the same parameter values and sample size.

```
> data2=arima.sim(n=100,list(ar=0.8,ma=-0.4))
> acf(data2,xaxp=c(0,22,11))
```



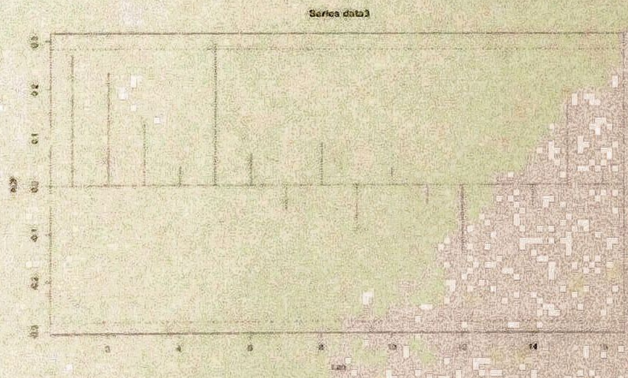

```
> eacf(data2)
AR/MA 0 1 2 3 4 5 6 7 8 9 10 11 12 13
    0  x x x x x o o o o o o o o o
O*  1  x o o o o o o o o o o o o o
    2  x x o o o o o o o o o o o o
    3  x o o o o o o o o o o o o o
    4  x o o o o o o o o o o o o o
    5  x o o o x o o o o o o o o o
    6  o x o o o o o o o o o o o o
    7  x x x o o o o o o o o o o o
```

O* is located (1, 1)

Finding: From the output of EACF, the sample model fit ARMA(1,1)model very well

(e) Repeat parts (b) and (c) with a new simulation using the same parameter values but sample size n = 48.

```
> data3=arima.sim(n=48,list(ar=0.8,ma=-0.4))
> acf(data3,xaxp=c(0,22,11))
```

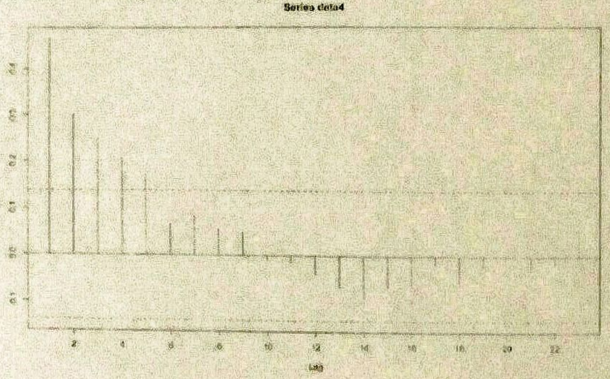



```
> eacf(data3)
AR/MA 0 1 2 3 4 5 6 7 8 9 10 11 12 13
    0  o o o o o o o o o o o o o o
    1  x o o o o x o o o o o o o o
O*  2  o o o o o o o o o o o o o o
    3  x o o o o o o o o o o o o o
    4  o o x o o o o o o o o o o o
    5  o x o o o o o o o o o o o o
    6  o x o o o o o o o o o o o o
    7  x o x o o o o o o o o o o o
```

finding: This sample fits ARMA(2,1) rather than ARMA(1,1), since O* is located (2, 1).

(f) Repeat parts (b) and (c) with a new simulation using the same parameter values but sample size n = 200.

```
> data4=arima.sim(n=200,list(ar=0.8,ma=-0.4))
> acf(data4,xaxp=c(0,22,11))
```




```
> eacf(data4)
AR/MA0 1 2 3 4 5 6 7 8 9 10 11 12 13
   0   x x x x x o o o o o o  o  o  o
   1   x o o o o o o o o o o  o  o  o
   2   x o o o o o o o o o o  o  o  o
   3   x o o o o o o o o o o  o  o  o
   4   x o o x o o o o o o o  o  o  o
   5   x x o o x o o o o o o  o  o  o
   6   x x x o o o o o o o o  o  o  o
   7   x x o o x o o o o o o  o  o  o
```

finding: The sample fits ARMA(1, 1) model very well.