HW7 Chunhua Yu.

7.8 7.10, 7.17 . 7.28

7.29. 8.7

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7.8 Consider an ARMA(1,1) model with $\phi=0.5$ and $\theta=0.45$

(a) For $n=48$. evaluate the variances and correlation of the maximum likelihood estimators of $\phi$ and $\theta$ using Equations (7.4.13) on Page 161.

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$$Var(\hat{\phi}) \approx [\frac{1-\phi^2}{n}][\frac{1-\phi\theta}{\phi-\theta}]^2 = (\frac{1-0.5^2}{48})\cdot(\frac{1-0.5\times0.45}{0.5-0.45})^2 = 3.75$$

$$\sqrt{Var(\hat{\phi})} = \sqrt{3.75} = 1.94$$

$$Var(\hat{\theta}) \approx [\frac{1-\theta^2}{n}][\frac{1-\phi\theta}{\phi-\theta}]^2 = (\frac{1-0.45^2}{48})(\frac{1-0.5\times0.45}{0.5-0.45})^2 = 3.99$$

$$\sqrt{Var(\hat{\theta})} = 1.997$$

$$corr(\hat{\phi},\hat{\theta}) \approx \frac{\sqrt{(1-\phi^2)(1-\theta^2)}}{1-\phi\theta} = \frac{\sqrt{(1-0.5^2)(1-0.45^2)}}{1-0.5\times0.4} = 0.998$$

Comments: (1) since $\phi$ and $\theta$ are very close, (nearly equal) the variability in the estimators of $\theta$ and $\phi$ is extremly large

(2). $corr(\hat{\phi},\hat{\theta})=0.998$, The estimates are very highly correlated. ✓

(b) Repeat part(a) with $n=120$.

$$Var(\hat{\phi}) \approx (\frac{1-0.5^2}{120})(\frac{1-0.5\times0.45}{0.5\times0.45})^2 = 1.50 \quad \sqrt{Var(\hat{\phi})} = 1.22.$$

$$Var(\hat{\theta}) \approx (\frac{1-0.45^2}{120})(\frac{1-0.5\times0.45}{0.5\times0.45})^2 = 1.60 \quad \sqrt{Var(\hat{\theta})} = 1.26$$

$Corr(\hat{\phi},\hat{\theta}) \approx 0.998$ since correlation is not related to $n$.

Comments: with $n$ going larger ($48 \to 120$), the standard errors are slightly decreasing.

7.10 Simulate an MA(1) series with θ = -0.6 and n = 36.

```
> set.seed(110)
> data.1=arima.sim(n=36,list(ma=0.6))
```

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(a) Find the method-of-moments estimate of θ.

```
> estimate.ma1.mom=function(x){r=acf(x,plot = F)$acf[1];if(abs(r)<0.5) return((-1+sqrt(1-4*r^2))/(2*r))
else return(NA)}
> estimate.ma1.mom(data.2)
[1] -0.4507127
```

$\hat{\theta}_{mom} = -0.4507127$

(b) Find the conditional least squares estimate of θ and compare it with part (a).

```
> arima(data.1,order=c(0,0,1),method = 'CSS')
Call:
arima(x = data.1, order = c(0, 0, 1), method = "CSS")
Coefficients:
      ma1  intercept
   0.5214   0.3248
s.e. 0.1591   0.2688
sigma^2 estimated as 1.15:  part log likelihood = -53.59
```

$\hat{\theta}_{LSE} = -0.5214$, $\hat{\theta}_{LSE}$ is better than $\hat{\theta}_{mom}$

(c) Find the maximum likelihood estimate of θ and compare it with parts (a) and (b).

```
> arima(data.1,order=c(0,0,1),method = 'ML')
Call:
arima(x = data.1, order = c(0, 0, 1), method = "ML")
Coefficients:
      ma1  intercept
   0.5255   0.3099
s.e. 0.1639   0.2689
sigma^2 estimated as 1.139:  log likelihood = -53.59,  aic = 111.19
```

$\hat{\theta}_{MLE} = -0.5255$, $\hat{\theta}_{MLE}$ is better than both of $\hat{\theta}_{LSE}$ and $\hat{\theta}_{mom}$. and $\hat{\theta}_{MLE}$ and $\hat{\theta}_{LSE}$ are nearly equal.

(d) Repeat parts (a), (b), and (c) with a new simulated series using the same parameters and same sample size. Compare your results with your results from the first simulation.

```
> set.seed(1112)
> data.n=arima.sim(n=36,list(ma=0.6))
```

Find the method-of-moments estimate of θ

```
> estimate.ma1.mom(data.n)
[1] -0.9031791
```

$\hat{\theta}_{mom} = -0.903$

Find the conditional least squares estimate of θ and compare it with part (a).

```
> arima(data.n,order=c(0,0,1),method = 'CSS')
Call:
arima(x = data.n, order = c(0, 0, 1), method = "CSS")
Coefficients:
      ma1  intercept
   0.6617   0.0664
s.e. 0.1096   0.2797
sigma^2 estimated as 1.059:  part log likelihood = -52.11
```

$\hat{\theta}_{LSE} = -0.6617$  $\hat{\theta}_{LSE}$ is better than $\hat{\theta}_{mom}$.

Find the maximum likelihood estimate of θ and compare it with parts (a) and (b).

```
> arima(data.n,order=c(0,0,1),method = 'ML')
Call:
arima(x = data.n, order = c(0, 0, 1), method = "ML")
Coefficients:
      ma1  intercept
   0.6650   0.0452
s.e. 0.1151   0.2813
sigma^2 estimated as 1.05:  log likelihood = -52.26,  aic = 108.51
```

$\hat{\theta}_{MLE} = -0.6650$  $\hat{\theta}_{MLE}$ and $\hat{\theta}_{LSE}$ are nearly equal.

in this case, $\hat{\theta}_{LSE}$ is the best.

7.17 Simulate an ARMA(1,1) series with $\phi = 0.7$, $\theta = 0.4$, and $n = 72$.

```
> set.seed(1113)
> data.2=arima.sim(list(order=c(1,0,1),ar=0.7,ma=-0.4),n=72)
```

$$r_1 = \frac{(1-\theta\hat{\phi})(\hat{\phi}-\theta)}{1-2\theta\hat{\phi}+\theta^2}$$

$$0.4433 = \frac{(1-\theta\, 0.818)(0.818-\theta)}{1-2\theta \cdot 0.818+\theta^2}$$

(a) Find the method-of-moments estimates of $\phi$ and $\theta$.

```
> r=acf(data.2)$acf
> c(r[1],r[2])
[1] 0.4433428 0.3625078
> r[2]/r[1]
[1] 0.8176693
```

$\hat{\phi}_{mom} = \frac{r_2}{r_1} = 0.818$.

Solving this equation, I found $\theta$ is not exist in the real number

$\hat{\theta}_{mom}$ is not exist

(b) Find the conditional least squares estimates of $\phi$ and $\theta$ and compare them with part (a).

```
> arima(data.2,order=c(1,0,1),method = 'CSS')
Call:
arima(x = data.2, order = c(1, 0, 1), method = "CSS")
Coefficients:
        ar1     ma1  intercept
     0.6390 -0.2380     0.1919
s.e. 0.1485  0.1699     0.2392
sigma^2 estimated as 0.9169:  part log likelihood = -99.04
```

$\hat{\phi}_{LSE} = 0.639$

$\hat{\theta}_{LSE} = 0.238$

conditional least squares estimates are better.

(c) Find the maximum likelihood estimates of $\phi$ and $\theta$ and compare them with parts (a) and (b).

```
> arima(data.2,order=c(1,0,1),method = 'ML')
Call:
arima(x = data.2, order = c(1, 0, 1), method = "ML")
Coefficients:
        ar1     ma1  intercept
     0.6482 -0.2488     0.1347
s.e. 0.1475  0.1709     0.2364
sigma^2 estimated as 0.9151:  log likelihood = -99.1,  aic = 204.2
```

$\hat{\theta}_{MLE} = 0.2488$

$\hat{\phi}_{MLE} = 0.6482$.

in this case maximum likelihood estimates are the best.

(d) Repeat parts (a), (b), and (c) with a new simulated series using the same parameters and same sample size. Compare your new results with your results from the first simulation.

```
> set.seed(1234)
> data.w=arima.sim(list(order=c(1,0,1),ar=0.7,ma=-0.4),n=72)
```

Find the method-of-moments estimates of $\phi$ and $\theta$

```
> c(acf(data.w)$acf[1],acf(data.w)$acf[2])
[1] 0.5685294 0.4630916
> acf(data.w)$acf[2]/acf(data.w)$acf[1]
[1] 0.814543
```

$\hat{\phi}_{mom} = \frac{r_2}{r_1} = 0.8145$

$\hat{\theta}$ is not exist in the real number.

$\hat{\theta}_{mom}$ is not exist.

Find the conditional least squares estimates of $\phi$ and $\theta$ and compare them with part (a).

```
> arima(data.w,order=c(1,0,1),method = 'CSS')
Call:
arima(x = data.w, order = c(1, 0, 1), method = "CSS")
Coefficients:
        ar1     ma1  intercept
     0.7899 -0.3530    -0.5408
s.e. 0.1091  0.1705     0.3525
sigma^2 estimated as 0.9006:  part log likelihood = -98.39
```

$\hat{\phi}_{LSE} = 0.7899$

$\hat{\theta}_{LSE} = 0.353$

the least squares estimates are better.

Find the maximum likelihood estimates of $\phi$ and $\theta$ and compare them with parts (a) and (b).

```
> arima(data.w,order=c(1,0,1),method = 'ML')
Call:
arima(x = data.w, order = c(1, 0, 1), method = "ML")
Coefficients:
        ar1     ma1  intercept
     0.8527 -0.4408    -0.3135
s.e. 0.0955  0.1747     0.4045
sigma^2 estimated as 0.9174:  log likelihood = -99.34,  aic = 204.69
```

$\hat{\phi}_{MLE} = 0.8527$.

$\hat{\theta}_{MLE} = 0.4408$.

$\hat{\phi}_{LSE}$ is the best estimate of $\phi$

$\hat{\theta}_{MLE}$ is the best estimate of $\theta$.

7.28 The data file named deere3 contains 57 consecutive values from a complex machine tool at Deere & Co. The values given are deviations from a target value in units of ten millionths of an inch. The process employs a control mechanism that resets some of the parameters of the machine tool depending on the magni- tude of deviation from target of the last item produced.

(a) Estimate the parameters of an AR(1) model for this series.

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```
> data("deere3")
> arima(deere3,order=c(1,0,0))
Call:
arima(x = deere3, order = c(1, 0, 0))
Coefficients:
      ar1  intercept
   0.5255  124.3832
s.e. 0.1108  394.2067
sigma^2 estimated as 2069355:  log likelihood = -495.51,  aic = 995.02
```

$\hat{\phi}_{MLE} = 0.1522$

(b) Estimate the parameters of an AR(2) model for this series and compare the results with those in part (a).

```
> arima(deere3,order=c(2,0,0))
Call:
arima(x = deere3, order = c(2, 0, 0))
Coefficients:
      ar1     ar2  intercept
   0.5211  0.0083   123.2979
s.e. 0.1310  0.1315   397.6134
sigma^2 estimated as 2069208:  log likelihood = -495.51,  aic = 997.01
```

$\hat{\phi}_{1\,MLE} = 0.5211 \quad \hat{\phi}_{2\,MLE} = 0.0083$

Since. $\hat{\phi}_{2\,MLE} \pm 2Se\{\hat{\phi}_{2\,MLE}\} = (-0.2547,\ 0.2713)$

I conclude that $\phi_2$ is insignificant

Finding: AR(1) is the better model.

7.29 The data file named robot contains a time series obtained from an industrial robot. The robot was put through a sequence of maneuvers, and the distance from a desired ending point was recorded in inches. This was repeated 324 times to form the time series.

(a) Estimate the parameters of an AR(1) model for these data.

```
> data("robot")
> arima(robot,order=c(1,0,0))
Call:
arima(x = robot, order = c(1, 0, 0))
Coefficients:
      ar1  intercept
   0.3074    0.0015
s.e. 0.0528    0.0002
sigma^2 estimated as 6.482e-06:  log likelihood = 1475.54,  aic = -2947.08
```

$\hat{\phi}_1 = 0.3074$. intercept : 0.0015

Since. $\begin{cases} \text{intercept} : 0.0015 \pm 2 \times 0.0002 = (0.0011,\ 0.0019) \\ \hat{\phi}_1 : 0.3074 \pm 2 \times 0.0528 = (+0.2018,\ 0.413) \end{cases}$

both $\hat{\phi}_1$ and intercept are significant.

model: $Y_t = 0.0015 + 0.3074 Y_{t-1} + e_t$

(b) Estimate the parameters of an IMA(1,1) model for these data.

```
> arima(robot,order=c(0,1,1))
Call:
arima(x = robot, order = c(0, 1, 1))
Coefficients:
       ma1
   -0.8713
s.e. 0.0389
sigma^2 estimated as 6.069e-06:  log likelihood = 1480.95,  aic = -2959.9
```

$\hat{\theta} = 0.8713$. since 0 is not within $(0.8713 \pm 2(0.0389))$

$\hat{\theta}$ is significant.

model: $Y_t - Y_{t-1} = e_t - 0.8713 e_{t-1}$

(c) Compare the results from parts (a) and (b) in terms of AIC.

AIC values in IMA(1,1) and AR(1) are very close to each other.

AIC in IMA(1,1) model is smaller than AIC in AR(1).

7.5 Given the data $Y_1=10$, $Y_2=9$ and $Y_3=9.5$, we wish to fit an IMA(1,1) model without constant term.

5/5 (a) Find the conditional least square estimate of $\theta$.

Solution: The model IMA(1,1) without constant term

$$Y_t - Y_{t-1} = e_t - \theta e_{t-1}$$

We assume $Y_0 = Y_1 = 10$

$W_t = Y_t - Y_{t-1} \sim MA(1)$ model. $\quad W_t = e_t - \theta e_{t-1}$

$W_1 = Y_1 - Y_0 = 10 - 10 = 0 \quad \Rightarrow \quad e_1 = W_1 = 0$

$W_2 = Y_2 - Y_1 = 9 - 10 = -1 \quad e_2 = W_2 + \theta e_1 = -1$

$W_3 = Y_3 - Y_2 = 9.5 - 9 = 0.5 \quad e_3 = W_3 + \theta e_2 = 0.5 - \theta$

$$S_c(\theta) = \sum (e_t)^2$$
$$= 0 + 1 + (0.5-\theta)^2$$

let $\frac{\partial S_c(\theta)}{\partial \theta} = 0 \Rightarrow 2(0.5-\theta)\times(-1) = 0$

$\theta = 0.5$

thus, the conditional least square $\hat{\theta} = 0.5$ ✓

(b) Estimate $\sigma_e^2$.

$$\hat{\sigma}_e^2 = \frac{S_c(\theta)}{n-1} = \frac{1}{2}\times[0+1+0] = \frac{1}{2} ✓$$

7.3. If $\{Y_t\}$ statisfies an AR(1) Model with $\phi$ of about 0.7, how long

5|15 of a series do we need to estimate $\phi = \rho_1$ with 95% confidence that out estimation error is no more than $\pm 0.1$?

Solution: Using formula (6.1.5) $Var(r_1) \approx \frac{1-\phi^2}{n}$

the standard error of $\hat{\phi} = r_1$ is $\sqrt{Var(r_1)} = \sqrt{\frac{1-\hat{\phi}^2}{n}}$

95% confidence implied $2\sqrt{Var(r_1)} = 0.1$

$$2^2 \cdot \frac{1-\hat{\phi}^2}{n} = 0.01$$

$$n = 4 \times (1-\hat{\phi}^2) \times 100 = 4 \times (1-0.7^2) \times 100$$

$$= 204 \checkmark$$