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EXAM I

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STT4110/6110 - Spring 2019

Superb!

Name:

1. Suppose that $E(X) = E(Y) = 3, Var(X) = 4, Var(Y) = 9$, and $Cov(X, Y) = 1.5$. Compute

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(a) $Var(X + Y)$

$= Var X + Var Y + 2 Cov(X, Y)$

$= 4 + 9 + 2 \times 1.5$

$= 16$ ✓

(b) $Cov(X + 2Y, X - 2Y)$

$= Cov(X, X) - Cov 2(X, Y) + 2 Cov(Y, X) - 4 Cov(Y, Y)$

$= Var(X) - 4 Var(Y)$

$= 4 - 4 \times 9$

$= -32$.

✓

2. Let Suppose that $Y_t = \beta_0 + \beta_1 t + X_t$, where β_0 and β_1 are constants, $\{X_t\}$ is a stationary process with mean 0 and auto-covariance function γ_k.

(a) Is $\{Y_t\}$ stationary? Explain. No.

5/5 $\because E(Y_t) = E(\beta_0 + \beta_1 t + X_t) = E(\beta_0) + E(\beta_1 t) + E(X_t) = \beta_0 + \beta_1 t + 0$

$E(Y_t)$ depends on t

We conclude that $\{Y_t\}$ is not stationary. ✓

(b) Is $W_t = \nabla Y_t$ stationary? Explain. Yes.

4/7 $W_t = \nabla Y_t = Y_t - Y_{t-1} = \beta_0 + \beta_1 t + X_t - [\beta_0 + \beta_1(t-1) + X_{t-1}]$

$= \beta_1 + X_t - X_{t-1} = \beta_1 + \nabla X_t$

since $\{X_t\}$ is stationary, $\{\nabla X_t\}$ is stationary.

W_t is the simple linear combination of $\{\nabla X_t\}$ and constant β_1

so, $\{W_t\}$ is also stationary ✓

3. Suppose that $\{Y_t\}$ is stationary with auto-covariance function γ_k. Show that $W_t = \nabla Y_t = Y_t - Y_{t-1}$ is stationary and find its auto-covariance function.

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i $E(W_t) = E(Y_t - Y_{t-1}) = E(Y_t) - E(Y_{t-1}) = 0$ is free of t

(since $\{Y_t\}$ is stationary, $E(Y_t) = E(Y_{t-1}) = \mu$)

ii $E(W_t^2) = Var(W_t) = Var(Y_t - Y_{t-1}) = Var(Y_t) + Var(Y_{t-1}) - 2Cov(Y_t, Y_{t-1})$

$= 2\gamma_{(0)} - 2\gamma_{(1)}$ is free of t

iii $Cov(W_t, W_{t-k}) = Cov(Y_t - Y_{t-1}, Y_{t-k} - Y_{t-1-k})$

$= Cov(Y_t, Y_{t-k}) - Cov(Y_t, Y_{t-1-k}) - Cov(Y_{t-1}, Y_{t-k}) + Cov(Y_{t-1}, Y_{t-1-k})$

$= \gamma_k - \gamma_{(1+k)} - \gamma_{(k-1)} + \gamma_{(k)} = 2\gamma_{(k)} - \gamma_{(1+k)} - \gamma_{(k-1)}$ free of t

from i, ii, iii we conclude that $\{W_t\}$ is stationary

auto-covariance function $\gamma_W(k) = \begin{cases} 2\gamma_{(0)} - 2\gamma_{(1)} & k=0 \\ 2\gamma_{(k)} - \gamma_{(1+k)} - \gamma_{(k-1)} & \text{otherwise} \end{cases}$ ✓

4. Let $\{Y_t\}$ be the moving average model specified as below:

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$$Y_t = e_t + 0.4e_{t-1},$$

where $\{e_t\} \sim WN(0, \sigma_e^2)$

(a) Verify that $\{Y_t\}$ is a weak stationary process and compute its auto-covariance function $\gamma(k)$

i $E(Y_t) = E(e_t + 0.4e_{t-1}) = E(e_t) + 0.4E(e_{t-1}) = 0$ (since $\{e_t\} \sim WN(0, \sigma_e^2)$) free of t

ii $E(Y_t^2) = Var(Y_t) = Var(e_t + 0.4e_{t-1}) = Var(e_t) + 0.4^2 Var(e_{t-1}) = \sigma_e^2 + 0.16\sigma_e^2 = 1.16\sigma_e^2$ free of t

iii $Cov(Y_t, Y_{t-k}) = Cov(e_t + 0.4e_{t-1}, e_{t-k} + 0.4e_{t-1-k})$

$= Cov(e_t, e_{t-k}) + 0.4Cov(e_t, e_{t-1-k}) + 0.4Cov(e_{t-1}, e_{t-k}) + 0.16Cov(e_{t-1}, e_{t-1-k})$

case I: $k=1$ $Cov(Y_t, Y_{t-1}) = 0 + 0 + 0.4\sigma_e^2 + 0 = 0.4\sigma_e^2$

case II: $k=-1$ $Cov(Y_t, Y_{t+1}) = 0 + 0.4\sigma_e^2 + 0 + 0 = 0.4\sigma_e^2$

case III: $k=0$ $Cov(Y_t, Y_{t-k}) = Var Y_t = 1.16\sigma_e^2$

otherwise $Cov(Y_t, Y_{t-k}) = 0$

free of t

we conclude $\{Y_t\}$ is stationary.

auto-covariance function

$\gamma(k) = \begin{cases} 1.16\sigma_e^2 & k=0 \\ 0.4\sigma_e^2 & k=\pm1 \\ 0 & \text{otherwise} \end{cases}$ ✓

(b) Derive the variance of the sample mean $\bar{Y}_n$, where n is the sample size.

$$Var(\bar{Y}_n) = Var\left(\frac{1}{n}\sum_{t=1}^{n} Y_t\right) = \frac{1}{n^2} Var\left(\sum_{t=1}^{n} Y_t\right) = \frac{1}{n^2}\left[n \cdot Var(Y_t) + 2\sum_{i=1}^{n}\sum_{j=1}^{i-1} Cov(Y_i, Y_j)\right]$$

$$= \frac{1}{n^2}\left[n\gamma(0) + 2\sum_{k=1}^{n-1}(n-k)\gamma(k)\right] = \frac{1}{n}\left[\gamma(0) + 2\sum_{k=1}^{n-1}\left(1-\frac{k}{n}\right)\gamma(k)\right]$$

since $\gamma(k) = \begin{cases} 1.16\sigma_e^2 & k=0 \\ 0.4\sigma_e^2 & k=\pm 1 \\ 0 & \text{otherwise} \end{cases}$

$$Var(\bar{Y}_n) = \frac{1}{n}\left[1.16\sigma_e^2 + 2\left(1-\frac{1}{n}\right)\times 0.4\sigma_e^2\right]$$

$$= \frac{1}{n}\left[1.16\sigma_e^2 + 0.8\sigma_e^2 - \frac{0.8}{n}\sigma_e^2\right]$$

$$= \frac{1.96n - 0.8}{n^2} \cdot \sigma_e^2$$

5. Suppose that $\{Y_t\}$ is an AR(1) process with $\phi = 0.5$

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(a) Find the auto-covariance function for Y_t.

$$Y_t = \phi Y_{t-1} + e_t = 0.5 Y_{t-1} + e_t$$

using the Equation (4.3.4) $\gamma_k = \phi\gamma_{k-1}$ for $k = 1, 2, 3 \cdots$ for $k>0$.

$Var(Y_t) = Var(\phi Y_{t-1} + e_t) = \phi^2 Var(Y_{t-1}) + Var(e_t)$ since e_t is independent of Y_{t-k}

$$\gamma(0) = \phi^2\gamma(0) + \sigma_e^2 \Rightarrow \gamma(0) = \frac{\sigma_e^2}{1-\phi^2}$$

$\gamma(k) = \begin{cases} \frac{\sigma_e^2}{1-\phi^2} \\ \phi\gamma(k-1) = \phi^k \cdot \gamma(0) \end{cases}$ in this case $\phi = 0.5$ $\gamma(k) = \begin{cases} \frac{4}{3}\sigma_e^2 & k=0 \\ \frac{1}{3\cdot 2^{k-2}} \cdot \sigma_e^2 & k=1, 2, 3 \cdots \end{cases}$

(b) Let $W_t = \nabla Y_t = Y_t - Y_{t-1}$. Find the auto-covariance function for W_t.

$$Cov(W_t, W_{t-k}) = Cov(Y_t - Y_{t-1}, Y_{t-k} - Y_{t-1-k})$$

$$= Cov(Y_t, Y_{t-k}) - Cov(Y_t, Y_{t-1-k}) - Cov(Y_{t-1}, Y_{t-k}) + Cov(Y_{t-1}, Y_{t-1-k})$$

$$= 2\gamma(k) - \gamma(1+k) - \gamma(k-1) = 2\phi^k \cdot \gamma(0) - \phi^{1+k} \cdot \gamma(0) - \phi^{k-1}\gamma(0) = -(1-\phi)^2\phi^{k-1}\gamma(0)$$

$k=0$ $Cov(W_t, W_t) = Var(W_t) = Var(Y_t - Y_{t-1}) = Var(Y_t) + Var(Y_{t-1}) - 2Cov(Y_t, Y_{t-1})$

$$= 2\gamma(0) - 2\gamma(1) = 2\times\frac{4}{3}\sigma_e^2 - 2\times\frac{2}{3}\sigma_e^2 = \frac{4}{3}\sigma_e^2$$

$k \neq 0$ $Cov(W_t, W_{t-k}) = -\left(1-\frac{1}{2}\right)^2 \cdot \left(\frac{1}{2}\right)^{k-1} \cdot \frac{4}{3}\sigma_e^2$

$$= -\left(\frac{1}{2}\right)^k \frac{2}{3}\sigma_e^2 = -\frac{1}{3\cdot 2^{k-1}}\sigma_e^2$$

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6. Consider ARMA(p,q) model $Y_t = 0.5Y_{t-1} + 0.25Y_{t-2} + e_t - 0.6e_{t-1} + 0.09e_{t-2}$.

(a) Identify the values of p and q.

$Y_t = \underbrace{0.5Y_{t-1} + 0.25Y_{t-2}}_{AR(p)} + \underbrace{e_t - 0.6e_{t-1} + 0.09e_{t-2}}_{MA(q)}$

$p=2$, $q=2$ ✓

(b) Give the AR characteristic polynomial and MA characteristic polynomial.

AR: $\begin{cases} \phi_1 = 0.5 \\ \phi_2 = 0.25 \end{cases}$ AR characteristic polynomial. $\phi(x) = 1 - 0.5x - 0.25x^2$ ✓

MA: $\begin{cases} \theta_1 = 0.6 \\ \theta_2 = -0.09 \end{cases}$ MA characteristic polynomial. $\theta(x) = 1 - 0.6x + 0.09x^2$ ✓

(c) Is the process stationary? Explain.

Yes.

$\begin{cases} \phi_1 = 0.5 \\ \phi_2 = 0.25 \end{cases} \Rightarrow \begin{cases} \phi_1 + \phi_2 = 0.75 < 1 \\ \phi_2 - \phi_1 = -0.25 < 1 \\ |\phi_2| = 0.25 < 1 \end{cases}$ ✓

We conclude that the process is stationary.

(d) Is the process invertible? Explain.

Yes.

$\begin{cases} \theta_1 = 0.6 \\ \theta_2 = -0.09 \end{cases} \Rightarrow \begin{cases} \theta_1 + \theta_2 = 0.51 < 1 \\ \theta_2 - \theta_1 = -0.69 < 1 \\ |\theta_2| = 0.09 < 1 \end{cases}$ ✓

So we conclude that process is invertible

7. Identify each of the following as specific ARIMA models. That is, identify the values of p, d, q, and the values of the parameters θ's and ϕ's.

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(a) $Y_t = Y_{t-1} - 0.25e_{t-1}$

$Y_t - Y_{t-1} = -0.25e_{t-1}$

$\nabla Y_t = -0.25 e_{t-1}$

since we do not observe the error terms, there is no way to tell the ~~its~~ ARMA model, with $d=1$, $p=q=0$

ARIMA(0,1,0)

✓

(b) $Y_t = 2Y_{t-1} - Y_{t-2} + e_t + 0.3e_{t-1}$

$\underbrace{Y_t - 2Y_{t-1} + Y_{t-2}}_{\nabla^2 Y} = \underbrace{e_t + 0.3e_{t-1}}_{MA(1)?}$

MA characteristic polynomial

$\theta(x) = 1 + 0.3x$ with $\theta_1 = -0.3$.

Let $\theta(x) = 1 + 0.3x = 0$ $x = -\frac{10}{3}$ $|x| > 1$

the process is invertible.

so $\{Y_t\}$ is stationary and invertible IMA(2,1) model

with $\begin{cases} p=0 \\ d=2 \\ q=1 \\ \theta_1 = -0.3 \end{cases}$

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