

# Dévissage for Algebraic K-theory of Small Stable $\infty$ -categories

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## ABSTRACT

In this article, we extend Quillen's Dévissage Theorem to small stable  $\infty$ -categories. To be precise, we establish sufficient conditions under when an exact functor between stable  $\infty$ -categories induces isomorphisms of non-negative  $K$ -groups.

Keywords: Category Theory K-Theory

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## INTRODUCTION

Algebraic  $K$ -theory provides fundamental tools to investigate the structure of stable  $\infty$ -categories, revealing deep connections between categorical decompositions and invariants arising from homotopy-theoretic contexts. In this article, we establish new results characterizing algebraic  $K$ -groups under conditions generalizing classical dévissage arguments.

Our main contributions focus on exact functors between (idempotent-complete) stable  $\infty$ -categories that satisfy suitable d'évissage conditions. Building upon this framework, we prove that the an exact functor induces isomorphisms on all higher  $K$ -groups  $K_n$  for  $n \geq 1$  whenever it satisfies a weak dévissage

condition and is fillable. Furthermore, if the stronger dévissage condition is met, the isomorphism extends to  $K_0$ . In addition, we establish vanishing results for higher  $K$ -groups of fillable stable  $\infty$ -categories themselves, uncovering new structural constraints.

**Theorem A.** (Theorem 4.12) Let  $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{C} \in \text{Cat}_{\infty}^{\text{perf}}$  be a  $n$ -fillable exact functor between small stable idempotent-complete  $\infty$ -categories. If  $\mathcal{F}$  satisfies the weak dévissage condition (resp. dévissage condition), it induces isomorphisms  $K_i(\mathcal{A}) \xrightarrow{\sim} K_i(\mathcal{C})$  for all  $1 \leq i \leq n-1$ , an epimorphism  $K_n(\mathcal{A}) \twoheadrightarrow K_n(\mathcal{C})$  and a monomorphism (resp. an isomorphism)  $K_0(\mathcal{A}) \hookrightarrow K_0(\mathcal{C})$ .

**Theorem B.** (Theorem 4.15) Let  $\mathcal{C} \in \text{Cat}_{\infty}^{\text{perf}}$  be a small  $n$ -fillable idempotent-complete  $\infty$ -category. Then  $K_i(\mathcal{C}) = 0$  for all  $1 \leq i \leq n$ .

**Theorem C.** (Theorem 4.30) Let  $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{C} \in \text{Cat}_{\infty}^{\text{perf}}$  be a strongly 1-fillable exact functor between small stable idempotent-complete  $\infty$ -categories. If  $\mathcal{F}$  satisfies the dévissage condition, it induces isomorphisms  $K_n(\mathcal{A}) \xrightarrow{\sim} K_n(\mathcal{C})$  for all  $n \geq 0$ .

Overview of this article:

- In Section 1, we introduce basic definitions and properties of stable  $\infty$ -categories.
- In Section 2, we formalize the (weak) Dévissage condition for an exact functor. It concerns whether the objects in the image are enough to generate the whole  $\infty$ -category.
- In Section 3, we develop the categorification of fibers and loops for universal localizing invariant. It is used to describe the higher  $K$ -groups of our target by  $K_0$ -groups of some other  $\infty$ -categories.
- In Section 4, we discuss the fillability of exact functors and present the main results concerning algebraic  $K$ -theory. This condition determines whether the class of cofiber sequences in the image are enough to describe the class of cofiber sequences in the whole  $\infty$ -category.
- In Section 5, we show how to use our results to prove Quillen's Dévissage Theorem.

## 1 PRELIMINARIES ON STABLE $\infty$ -CATEGORY AND EXACT FUNCTORS

### 1.1 Stable $\infty$ -category

Beyond ordinary categories, the landscape of category theory expands with the introduction of higher categories, including the concept of  $\infty$ -categories, which play an important role in  $K$ -theory. Bergner [Ber06] wrote a survey of different models of  $\infty$ -categories. The most widely used model, due to Joyal [And02] and Lurie [Lur09], defines  $\infty$ -categories as *weak Kan complexes* (or *quasi-categories*).

**Definition 1.1.** [Lur17, Definition 1.1.1.4]

A *triangle* in a pointed  $\infty$ -category  $\mathcal{C}$  is a functor  $\Delta^1 \times \Delta^1 \rightarrow \mathcal{C}$

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow g \\ 0 & \longrightarrow & Z \end{array}$$

A *fiber sequence* in  $\mathcal{C}$  is a pullback triangle, and a *cofiber sequence* in  $\mathcal{C}$  is a pushout triangle. We will abuse notation by writing  $X \rightarrow Y \rightarrow Z$  for a triangle.

**Definition 1.2.** [Lur17, Definition 1.1.1.6]

Given a morphism  $g : X \rightarrow Y$  in a pointed  $\infty$ -category  $\mathcal{C}$ . A *fiber* of  $g$  is a fiber sequence

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow g \\ 0 & \longrightarrow & Y \end{array}$$

Dually, a *cofiber* of  $g$  is a cofiber sequence

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z \end{array}$$

We also refer to  $W$  and  $Z$  simply as the fiber and cofiber of  $g$ , respectively, and write  $W = \text{fib}(g)$  and  $Z = \text{cofib}(g)$ .

**Definition 1.3.** [Lur17, Definition 1.1.1.9, Proposition 1.1.3.4]

A *stable*  $\infty$ -category is a pointed  $\infty$ -category satisfying:

- It is finite complete and finite cocomplete.
- A square is a pushout if and only if it is a pullback.

Or, equivalently:

- Every morphism admits both a fiber and a cofiber.
- A triangle is a fiber sequence if and only if it is a cofiber sequence.

**Definition 1.4.** Let  $\mathcal{C}$  be an  $\infty$ -category.

- (1) Two morphisms  $f, g : X \rightarrow Y$  are (*homotopy*) *equivalent*, denote by  $f \simeq g$ , if  $[f] = [g]$  in  $\pi_0 \text{Map}(X, Y)$ .
- (2) Two objects  $X, Y$  are (*homotopy*) *equivalent*, denote by  $X \simeq Y$ , if there are two morphisms  $f : X \rightarrow Y, g : Y \rightarrow X$  such that  $f \circ g \simeq \text{id}_Y$  and  $g \circ f \simeq \text{id}_X$ .  $f, g$  are called (*homotopy*) *equivalences*.

We display some properties of stable  $\infty$ -categories firstly.

**Lemma 1.5.** Suppose  $X' \rightarrow X \rightarrow X''$ ,  $Y' \rightarrow Y \rightarrow Y''$ , and  $Z' \rightarrow Z \rightarrow Z''$  are three cofiber sequences in a stable  $\infty$ -category  $\mathcal{C}$ .

- (1) Given a functor  $f : \Delta^1 \times \Delta^1 \rightarrow \mathcal{C}$  represented by the diagram

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array},$$

there is a unique morphism  $X'' \rightarrow Y''$  extending  $f$  to a functor  $\Delta^2 \times \Delta^1 \rightarrow \mathcal{C}$  of the form

$$\begin{array}{ccccc} X' & \longrightarrow & X & \longrightarrow & X'' \\ \downarrow & & \downarrow & & \downarrow \\ Y' & \longrightarrow & Y & \longrightarrow & Y'' \end{array}.$$

- (2) Given a functor  $f : \Delta^1 \times \Delta^1 \rightarrow \mathcal{C}$  represented by the diagram

$$\begin{array}{ccc} X & \longrightarrow & X'' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y'' \end{array},$$

there is a unique morphism  $X' \rightarrow Y'$  extending  $f$  to a functor  $\Delta^2 \times \Delta^1 \rightarrow \mathcal{C}$  of the form

$$\begin{array}{ccccc} X' & \longrightarrow & X & \longrightarrow & X'' \\ \downarrow & & \downarrow & & \downarrow \\ Y' & \longrightarrow & Y & \longrightarrow & Y'' \end{array}.$$

(3) Given a functor  $f : \Delta^1 \times \Delta^2 \rightarrow \mathcal{C}$  represented by the diagram

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z' & \longrightarrow & Z \end{array}$$

where the two columns are cofiber sequences, there are unique morphisms  $X'' \rightarrow Y''$  and  $Y'' \rightarrow Z''$  extending  $f$  to a functor  $\Delta^2 \times \Delta^2 \rightarrow \mathcal{C}$  of the form

$$\begin{array}{ccccc} X' & \longrightarrow & X & \longrightarrow & X'' \\ \downarrow & & \downarrow & & \downarrow \\ Y' & \longrightarrow & Y & \longrightarrow & Y'' \\ \downarrow & & \downarrow & & \downarrow \\ Z' & \longrightarrow & Z & \longrightarrow & Z'' \end{array}$$

such that  $X'' \rightarrow Y'' \rightarrow Z''$  is a cofiber sequence.

(4) Given a functor  $f : \Delta^1 \times \Delta^2 \rightarrow \mathcal{C}$  represented by the diagram

$$\begin{array}{ccc} X & \longrightarrow & X'' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y'' \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Z'' \end{array}$$

where the two columns are cofiber sequences, there are unique morphisms  $X' \rightarrow Y'$  and  $Y' \rightarrow Z'$  extending  $f$  to a functor  $\Delta^2 \times \Delta^2 \rightarrow \mathcal{C}$  of the form

$$\begin{array}{ccccc} X' & \longrightarrow & X & \longrightarrow & X'' \\ \downarrow & & \downarrow & & \downarrow \\ Y' & \longrightarrow & Y & \longrightarrow & Y'' \\ \downarrow & & \downarrow & & \downarrow \\ Z' & \longrightarrow & Z & \longrightarrow & Z'' \end{array}$$

such that  $X' \rightarrow Y' \rightarrow Z'$  is a cofiber sequence.

*Proof.* (1) and (2) are from the universal property of cofibers. Besides, (1) and (2) give the existence of functors  $\Delta^2 \times \Delta^2 \rightarrow \mathcal{C}$  such that all rows are cofiber sequences. The last column is cofiber sequence is because  $\text{Ho}(\mathcal{C})$  is a triangulated category whose distinguished triangles are cofiber sequences [Lur17, Theorem 1.1.2.14].  $\square$

**Lemma 1.6.** [AGH19, Lemma 2.21] Given a functor  $\Delta^1 \times \Delta^1 \rightarrow \mathcal{C}$  represented by the diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow & & \downarrow \\ P & \xrightarrow{g} & Q \end{array},$$

the induced map  $\text{cofib}(f) \rightarrow \text{cofib}(g)$  is an equivalence if and only if this square is a pushout square.

**Lemma 1.7.** [Lur09, Lemma 4.4.2.1] Let  $\mathcal{C}$  be an  $\infty$ -category and suppose we are given a functor  $\Delta^2 \times \Delta^1 \rightarrow \mathcal{C}$  which we will depict as a diagram

$$\begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' \end{array}$$

Suppose that the left square is a pushout in  $\mathcal{C}$ . Then the right square is a pushout if and only if the outer square is a pushout.

**Definition 1.8.** An *exact* functor between stable  $\infty$ -categories is a functor preserving zero objects and fiber sequences.

**Proposition 1.9.** [Lur17, Proposition 1.1.4.1]

A functor between stable  $\infty$ -categories is exact if and only if it preserves finite limits and colimits.

The collection  $\text{Cat}_\infty^{\text{ex}} \subseteq \text{Cat}_\infty$  of stable  $\infty$ -categories and exact functors is a subcategory.

**Theorem 1.10.** [Lur17, Theorem 1.1.4.4, Proposition 1.1.4.6] The  $\infty$ -category  $\text{Cat}_\infty^{\text{ex}}$  admits all finite limits and filtered colimits.

**Definition 1.11.** An  $\infty$ -functor between  $\infty$ -categories is called *left exact* if it preserves all finite limits.

**Lemma 1.12.** [Lur09, Corollary] An  $\infty$ -functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  out of an  $\infty$ -category  $\mathcal{C}$  that has all finite colimits is left exact if and only if it preserves pullbacks and the terminal object.

An  $\infty$ -category is *idempotent-complete* if the image of this category under the Yoneda embedding is closed with respect to retracts. We denote the category of small idempotent-complete stable  $\infty$ -categories as  $\text{Cat}_\infty^{\text{perf}}$ . The inclusion  $\text{Cat}_\infty^{\text{perf}} \rightarrow \text{Cat}_\infty^{\text{ex}}$  has a left adjoint, referred to as  $\text{Idem} : \text{Cat}_\infty^{\text{ex}} \rightarrow \text{Cat}_\infty^{\text{perf}}$ , which is known as *idempotent completion* [Lur09]. Hence, the inclusion  $\text{Cat}_\infty^{\text{perf}} \rightarrow \text{Cat}_\infty^{\text{ex}}$  is left exact.

## 1.2 Localizing invariants

**Definition 1.13.** [BGT13, Definition 5.12, Proposition 5.15]

An *exact sequence* of small stable idempotent-complete  $\infty$ -categories  $\mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C}$  is a sequence where the composite morphism is zero, the functor  $\mathcal{A} \rightarrow \mathcal{B}$  is fully faithful, and the morphism from  $\mathcal{B}/\mathcal{A}$  to  $\mathcal{C}$  constitutes an equivalence after idempotent-completion.

**Definition 1.14.** [BGT13, Definition 5.18]

A *split-exact* sequence of small stable  $\infty$ -categories is an exact sequence  $\mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{C}$  with exact functors  $i : \mathcal{B} \rightarrow \mathcal{A}$  and  $j : \mathcal{C} \rightarrow \mathcal{B}$ , which serve as right adjoints to  $f$  and  $g$ , respectively, such that  $i \circ f \simeq \text{Id}$  and  $g \circ j \simeq \text{Id}$ .

**Definition 1.15.** [BGT13, Definition 2.14]

Two small stable  $\infty$ -categories are called *Morita equivalent* if their idempotent completions are equivalent.

**Definition 1.16.** A functor  $F : \text{Cat}_\infty^{\text{ex}} \rightarrow \mathcal{D}$  into a stable presentable  $\infty$ -category  $\mathcal{D}$  is a *Morita invariant* if it sends Morita equivalences to homotopy equivalences.

**Definition 1.17.** [BGT13, Definition 6.1]

A functor  $F : \text{Cat}_\infty^{\text{perf}} \rightarrow \mathcal{D}$  into a stable presentable  $\infty$ -category  $\mathcal{D}$  is an *additive invariant* if

1. It preserves zero objects.
2. It commutes with filtered colimits.
3. it sends split-exact sequence

$$\mathcal{A} \xrightleftharpoons[i]{f} \mathcal{B} \xrightleftharpoons[j]{g} \mathcal{C}$$

of small stable  $\infty$ -categories to an equivalence  $F(\mathcal{A}) \vee F(\mathcal{B}) \xrightarrow{\sim} F(\mathcal{C})$  in  $\mathcal{D}$ .

**Definition 1.18.** [BGT13, Definition 8.1]

A functor  $F : \text{Cat}_\infty^{\text{perf}} \rightarrow \mathcal{D}$  into a stable presentable  $\infty$ -category  $\mathcal{D}$  is a *localizing invariant* if

1. It preserves zero objects.
2. It commutes with filtered colimits.
3. it sends exact sequence

$$\mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C}$$

of small stable idempotent-complete  $\infty$ -categories to fiber sequences

$$F(\mathcal{A}) \rightarrow F(\mathcal{B}) \rightarrow F(\mathcal{C})$$

in  $\mathcal{D}$ .

*Remark 1.19.* A localizing invariant is always an additive invariant.

**Example 1.20.** [BGT13] constructed the universal additive invariant and localizing invariant:

$$\mathcal{U}_{\text{add}} : \text{Cat}_\infty^{\text{perf}} \rightarrow \mathcal{M}_{\text{add}}, \mathcal{U}_{\text{loc}} : \text{Cat}_\infty^{\text{perf}} \rightarrow \mathcal{M}_{\text{loc}}.$$

[BGT13, Section 5] construct the connective K-theory  $K^{\text{cn}} : \text{Cat}_\infty^{\text{ex}} \rightarrow \text{Sp}$  of small stable  $\infty$ -categories. Its restriction on  $\text{Cat}_\infty^{\text{perf}}$  is an additive invariant. It is not a Morita invariant. The non-connective K-theory  $K : \text{Cat}_\infty^{\text{ex}} \rightarrow \text{Sp}$  of small stable  $\infty$ -categories, which is constructed by [BGT13, Section 7], is a Morita invariant. Besides, its restriction on  $\text{Cat}_\infty^{\text{perf}}$  is a localizing invariant.

There are many other differential constructions of higher algebraic K-theory. Waldhausen [Wal85] constructed algebraic K-theory for Waldhausen categories. Also, algebraic K-theory are defined for model categories of simplicial categories and dg categories. However, they are very difficult to compute. One of the reason we choose the algebraic K-theory of  $\infty$ -categories is easy. Except for [BGT13], the K-theory of  $\infty$ -categories is also studied by [Bar16]. Other reason is because it very extensive. [RSW24] shows every spectrum is the k-theory of a stable  $\infty$ -category. So we except that it covers any other constructions of algebraic K-theory.

## 2 DÉVISSAGE CONDITION FOR EXACT FUNCTORS

### 2.1 Category of sequences

**Definition 2.1.** [BGT13, Definition 7.1] Let  $\text{Gap}([n], \mathcal{C})$  be the full subcategory of  $\text{Fun}(N(\text{Ar}[n]), \mathcal{C})$  spanned by the functors  $X_{\bullet, \bullet} : N(\text{Ar}[n]) \rightarrow \mathcal{C}$  such that, for each  $i \in I$ ,  $X_{i,i}$  is a zero object of  $\mathcal{C}$ , and for each  $i < j < k$ , the square

$$\begin{array}{ccc} X_{i,j} & \longrightarrow & X_{i,k} \\ \downarrow & & \downarrow \\ X_{j,j} & \longrightarrow & X_{j,k} \end{array}$$

is pushout.

**Lemma 2.2.** [BGT13, Lemma 7.3] Let  $\mathcal{C}$  be a stable  $\infty$ -category. The forgetful functor

$$\text{Gap}([n], \mathcal{C}) \rightarrow \text{Fun}(\Delta^{n-1}, \mathcal{C}), X_{\bullet, \bullet} \mapsto X_{0, \bullet}$$

is an equivalence of  $\infty$ -category.

Let  $F_{\bullet, \bullet} : X_{\bullet, \bullet} \rightarrow Y_{\bullet, \bullet}$  be a morphism in  $\text{Gap}([n], \mathcal{C})$ , then  $\text{fib}(F_{\bullet, \bullet})_{i,j} \simeq \text{fib}(F_{i,j})$  and  $\text{cofib}(F_{\bullet, \bullet})_{i,j} \simeq \text{cofib}(F_{i,j})$ . Consider the functor  $[n+1] \rightarrow [n]$  which sends  $i \mapsto i$  (if  $i \leq n$ ) and  $n+1 \mapsto n$ . This defines a functor:

$$\text{Ar}[n+1] \rightarrow \text{Ar}[n]$$

which induces an exact inclusion functor of stable  $\infty$ -categories

$$i_n : \text{Gap}([n], \mathcal{C}) \hookrightarrow \text{Gap}([n+1], \mathcal{C}).$$

Define

$$\text{Gap}(\mathcal{C}) := \text{colim} \left\{ \mathcal{C} \xrightarrow{i_1} \text{Gap}([2], \mathcal{C}) \xrightarrow{i_2} \cdots \xrightarrow{i_n} \text{Gap}([n+1], \mathcal{C}) \xrightarrow{i_{n+1}} \cdots \right\}.$$

**Definition 2.3.** Assume  $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{C}$  is an exact functor of small stable  $\infty$ -categories. Define  $\text{Gap}([n], \mathcal{F})$  and  $\text{Gap}^w([n], \mathcal{F})$  by the pullback

$$\begin{array}{ccc} \text{Gap}([n], \mathcal{F}) & \longrightarrow & \text{Gap}([n], \mathcal{C}) \\ \downarrow & & \downarrow \\ \mathcal{A}^n & \xrightarrow{\mathcal{F}^n} & \mathcal{C}^n, \end{array}, \quad \begin{array}{ccc} \text{Gap}^w([n], \mathcal{F}) & \longrightarrow & \text{Gap}^w([n], \mathcal{C}) \\ \downarrow & & \downarrow \\ \mathcal{A}^n & \xrightarrow{F^n} & \mathcal{C}^n, \end{array}$$

where  $\text{Gap}([n], \mathcal{C}) \rightarrow \mathcal{C}^n$  is defined by  $X_{\bullet, \bullet} \mapsto (X_{0,1}, \dots, X_{n-1,n})$  and  $\text{Gap}^w([n], \mathcal{C}) \rightarrow \mathcal{C}^n$  is its restriction.

$\text{Gap}([n], \mathcal{F})$  consists of

- Objects are pair  $(A_\bullet, X_{\bullet, \bullet})$ , where  $A_\bullet \in \mathcal{A}^n, X_{\bullet, \bullet} \in \text{Gap}([n], \mathcal{C})$  are objects such that  $\mathcal{F}(A_i) = X_{i-1,i}$ .
- Morphisms are pair  $(f_\bullet, F_{\bullet, \bullet})$ , where  $f_\bullet \in \mathcal{A}^n, F_{\bullet, \bullet} \in \text{Gap}([n], \mathcal{C})$  are morphisms such that  $\mathcal{F}(f_i) = F_{i-1,i}$ .

Thus, the exact inclusion  $i_n : \mathcal{A}^n \rightarrow \mathcal{A}^{n+1}, (A_1, \dots, A_n) \mapsto (A_1, \dots, A_n, 0)$  induces an exact inclusion

$$i_{n, \mathcal{F}} : \text{Gap}([n], \mathcal{F}) \hookrightarrow \text{Gap}([n+1], \mathcal{F})$$

Define

$$\text{Gap}(\mathcal{F}) := \text{colim} \left\{ \mathcal{A} \xrightarrow{i_{1, \mathcal{F}}} \text{Gap}([2], \mathcal{F}) \xrightarrow{i_{2, \mathcal{F}}} \cdots \xrightarrow{i_{n, \mathcal{F}}} \text{Gap}^w([n+1], \mathcal{F}) \xrightarrow{i_{n+1, \mathcal{F}}} \cdots \right\}.$$

$\text{Gap}^w([n], \mathcal{F})$  is naturally a full subcategory of  $\text{Gap}([n], \mathcal{F})$ . Then, the exact inclusion  $i_{n, \mathcal{A}}$  restricts to an exact inclusion

$$i_{n, \mathcal{F}}^w : \text{Gap}^w([n], \mathcal{F}) \hookrightarrow \text{Gap}^w([n+1], \mathcal{F}).$$

Define

$$\text{Gap}^w(\mathcal{F}) := \text{colim} \left\{ 0 \xrightarrow{i_{1, \mathcal{F}}^w} \text{Gap}^w([2], \mathcal{F}) \xrightarrow{i_{2, \mathcal{F}}^w} \cdots \xrightarrow{i_{n, \mathcal{F}}^w} \text{Gap}^w([n+1], \mathcal{F}) \xrightarrow{i_{n+1, \mathcal{F}}^w} \cdots \right\}.$$

$\text{Gap}([n], \mathcal{F}), \text{Gap}^w([n], \mathcal{F})$  and  $\text{Gap}(\mathcal{F}), \text{Gap}^w(\mathcal{F})$  are stable. Furthermore, if  $\mathcal{C}$  and  $\mathcal{A}$  are idempotent-complete, so do them.  $\text{Gap}, \text{Gap}^w : \mathcal{F} \mapsto \text{Gap}(\mathcal{F}), \text{Gap}^w(\mathcal{F})$  can be seen as functors  $\text{Fun}(\Delta^1, \text{Cat}_\infty^{\text{ex}}) \rightarrow \text{Cat}_\infty^{\text{ex}}$ .

## 2.2 Definition of dévissage condition

**Proposition 2.4.** Let  $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{C} \in \text{Cat}_\infty^{\text{ex}}$  be an exact functor between small stable  $\infty$ -categories. Then the following are equivalent:

- (1) For any morphism  $f : X \rightarrow Y$  in  $\mathcal{C}$ , there is a factorization

$$f : X = X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_n = Y$$

such that  $\text{cofib}(X_i \rightarrow X_{i+1}) \in \mathcal{F}(\mathcal{A})$  up to equivalences.

- (2) for any morphism  $f : 0 \rightarrow Y$  in  $\mathcal{C}$ , there is a factorization

$$f : 0 = X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_n = Y$$

such that  $\text{cofib}(X_i \rightarrow X_{i+1}) \in \mathcal{F}(\mathcal{A})$  up to equivalences.

(3) For any morphism  $f : X \rightarrow 0$  in  $\mathcal{C}$ , there is a factorization

$$f : X = X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_n = 0$$

such that  $\text{cofib}(X_i \rightarrow X_{i+1}) \in \mathcal{F}(\mathcal{A})$  up to equivalences.

*Proof.* (1)  $\Rightarrow$  (2), (3) is obvious. Now we prove (2)  $\Rightarrow$  (1) and (3)  $\Rightarrow$  (1).

(2)  $\Rightarrow$  (1). For a morphism  $X \rightarrow Y$ , assume  $Z$  is its cofiber. From (2), there is a sequence

$$0 = Z_0 \rightarrow \cdots \rightarrow Z_n = Z$$

such that  $\text{cofib}(Z_i \rightarrow Z_{i+1}) \in \mathcal{F}(\mathcal{A})$  up to equivalences. Define  $X_i$  by pullback

$$\begin{array}{ccc} X_i & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z_i & \longrightarrow & Z \end{array}$$

Then  $X_n = Y$  and  $X_0 = \text{fib}(Y \rightarrow Z) = X$ . There are also pushout squares in a stable  $\infty$ -category. Hence, we get pushout squares

$$\begin{array}{ccc} X_i & \longrightarrow & X_{i+1} \\ \downarrow & & \downarrow \\ Z_i & \longrightarrow & Z_{i+1} \end{array},$$

by Lemma 1.7. By Lemma 1.6, we get  $\text{cofib}(X_i \rightarrow X_{i+1}) \simeq \text{cofib}(Z_i \rightarrow Z_{i+1}) \in \mathcal{F}(\mathcal{A})$  up to equivalences. Then the sequence

$$X = X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_n = Y$$

is what we desired.

(3)  $\Rightarrow$  (1). For a morphism  $X \rightarrow Y$ , assume  $W$  is its fiber. From (3), there is a sequence

$$W = W_0 \rightarrow \cdots \rightarrow W_n = W$$

such that  $\text{cofib}(W_i \rightarrow W_{i+1}) \in \mathcal{F}(\mathcal{A})$ . Define  $X_i$  by pushout

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow \\ W_i & \longrightarrow & X_i \end{array}$$

Then  $X_0 = X$  and  $X_n = \text{cofib}(W \rightarrow X) = Y$ . Then we get pushout squares

$$\begin{array}{ccc} W_i & \longrightarrow & X_i \\ \downarrow & & \downarrow \\ W_{i+1} & \longrightarrow & X_{i+1} \end{array}.$$

by Lemma 1.7. By Lemma 1.6, we get  $\text{cofib}(X_i \rightarrow X_{i+1}) \simeq \text{cofib}(W_i \rightarrow W_{i+1}) \in \mathcal{F}(\mathcal{A})$  up to equivalences. Then the sequence

$$X = X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_n = Y$$

is what we desired.  $\square$

*Remark 2.5.* We will omit "up to equivalence" in the following.

**Definition 2.6.** We say the exact functor  $\mathcal{F}$  satisfies the *dévissage condition* if the equivalent descriptions in the Proposition 2.4 hold true.

**Example 2.7.** Let  $\mathcal{C}$  be a stable  $\infty$ -category. Then  $\text{id}_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$  and  $0_{\mathcal{C}} : \mathcal{C} \rightarrow *$  satisfy the dévissage condition.

**Example 2.8.** A fully faithful functor satisfies the dévissage condition if and only if it is an equivalence.

Denote by  $E(\mathcal{F})$  the full subcategory of  $\mathcal{C}$  consisting of objects  $X$  such that  $0 \rightarrow X$  admits a factorization as above, i.e. the extension closure of objects in  $\mathcal{F}(\mathcal{A})$ . By [Lur17, Lemma 1.1.3.3],  $E(\mathcal{F})$  is stable full subcategory of  $\mathcal{C}$ . We can naturally get the following results.

**Proposition 2.9.**  $\mathcal{F}$  satisfies the dévissage condition if and only if  $E(\mathcal{F}) = \mathcal{C}$ .

**Remark 2.10.** Ofcourse,  $\mathcal{A} \rightarrow E(\mathcal{F})$  satisfies the dévissage condition.

When we assume  $\mathcal{C}$  and  $\mathcal{A}$  are idempotent-complete, we only need the weaker condition in some cases.

**Definition 2.11.** Assume  $\mathcal{C}$  and  $\mathcal{A}$  are idempotent-complete. We say the  $\mathcal{F}$  satisfies the *weak dévissage condition* if  $\text{Idem}(E(\mathcal{F})) = \mathcal{C}$ .

**Remark 2.12.**  $\text{Idem}(E(\mathcal{F}))$  is just the thick closure of  $\text{Ob}(\mathcal{F}(\mathcal{A}))$  in  $\mathcal{C}$ , i.e. the smallest stable full subcategory of  $\mathcal{C}$  containing  $\mathcal{F}(\mathcal{A})$  which is closed under extensions and direct factors.

There are exact “evaluation” functors:

$$\text{ev}_{n,\mathcal{A}} : \text{Gap}([n], \mathcal{F}) \rightarrow \mathcal{C}, (A_{\bullet}, X_{\bullet,\bullet}) \mapsto X_{0,n}.$$

These are compatible with stabilization along the inclusion functors  $i_{n,\mathcal{F}}$ , so there is also an induced exact functor:

$$\text{ev}_{\mathcal{A}} : \text{Gap}(\mathcal{F}) \rightarrow \mathcal{C}.$$

**Proposition 2.13.** The functor  $\text{ev}_{\mathcal{F}} : \text{Gap}(\mathcal{F}) \rightarrow \mathcal{C}$  induces a well-defined functor

$$\widehat{\text{ev}}_{\mathcal{F}} : \text{Gap}(\mathcal{F})/\text{Gap}^w(\mathcal{F}) \rightarrow E(\mathcal{F}),$$

which is an equivalence.

*Proof.* By [BGT13, Corollary 5.11, Proposition 5.15], we only need to prove

$$\text{Ho}(\text{Gap}(\mathcal{F}))/\text{Ho}(\text{Gap}^w(\mathcal{F})) \rightarrow \text{Ho}(E(\mathcal{F}))$$

is an equivalence of homotopy categories. We prove it by steps.

Step 1: For every morphism  $\text{ev}_{\mathcal{F}}(A_{\bullet}, X_{\bullet,\bullet}) \rightarrow Y$  in  $E(\mathcal{F})$ , there is a morphism  $(A_{\bullet}, X_{\bullet,\bullet}) \rightarrow (A'_{\bullet}, X'_{\bullet,\bullet})$  such that  $\text{ev}_{\mathcal{F}}((A_{\bullet}, X_{\bullet,\bullet}) \rightarrow (A'_{\bullet}, X'_{\bullet,\bullet})) \simeq \text{ev}(A'_{\bullet}, X_{\bullet,\bullet}) \rightarrow Y$ .

Assume  $(A_{\bullet}, X_{\bullet,\bullet}) \in \text{Gap}([n], \mathcal{F})$ . Since  $\mathcal{A} \rightarrow E(\mathcal{F})$  satisfies the dévissage condition, there is a factorization

$$\text{ev}_{\mathcal{F}}(A_{\bullet}, X_{\bullet,\bullet}) = Y_0 \rightarrow Y_1 \rightarrow \cdots \rightarrow Y_m = Y$$

such that  $\text{cofib}(Y_i \rightarrow Y_{i+1}) \in \mathcal{F}(\mathcal{A})$ . Assume  $A'_1, \dots, A'_m \in \mathcal{A}$  such that  $\mathcal{F}(A_i) = \text{cofib}(Y_{i-1} \rightarrow Y_i)$ . Then  $(A_{\bullet}, X_{\bullet,\bullet}) \rightarrow (A'_{\bullet}, X'_{\bullet,\bullet})$  is defined as follows

$$\begin{array}{ccccccccc} X_{\bullet,\bullet} : & 0 & \longrightarrow & X_{0,1} & \longrightarrow & \cdots & \longrightarrow & X_{0,n} & \longrightarrow & X_{0,n} \\ & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ X'_{\bullet,\bullet} : & 0 & \longrightarrow & X_{0,1} & \longrightarrow & \cdots & \longrightarrow & X_{0,n} & \longrightarrow & Y_1 \longrightarrow \cdots \longrightarrow Y_n \end{array}$$
  

$$\begin{array}{ccccccccc} A_{\bullet} : & A_1 & & \cdots & & A_n & & 0 & & 0 \\ & \downarrow \text{id} & & & & \downarrow \text{id} & & \downarrow 0 & & \downarrow 0 \\ A'_{\bullet} : & A_1 & & \cdots & & A_n & & A'_1 & & \cdots & & A'_m \end{array}$$

It is well-defined since  $X_{0,n} = Y_0$ .

Step 2: If  $\text{ev}_{\mathcal{F}}(A_{\bullet}, X_{\bullet, \bullet}) \simeq \text{ev}_{\mathcal{F}}(B_{\bullet}, Y_{\bullet, \bullet})$ ,  $(A_{\bullet}, X_{\bullet, \bullet}) \simeq (B_{\bullet}, Y_{\bullet, \bullet})$  in  $\text{Gap}(\mathcal{F})/\text{Gap}^w(\mathcal{F})$ .

Assume  $(A_{\bullet}, X_{\bullet, \bullet}), (B_{\bullet}, Y_{\bullet, \bullet}) \in \text{Gap}([n], \mathcal{F})$ . Then define an object  $(C_{\bullet}, Z_{\bullet, \bullet})$  and two morphisms  $(C_{\bullet}, Z_{\bullet, \bullet}) \rightarrow (A_{\bullet}, X_{\bullet, \bullet})$  and  $(C_{\bullet}, Z_{\bullet, \bullet}) \rightarrow (B_{\bullet}, Y_{\bullet, \bullet})$  as follows.

$$\begin{array}{ccccccc}
 X_{\bullet, \bullet} : & 0 \longrightarrow \cdots \longrightarrow X_{0,n} \longrightarrow X_{0,n} \longrightarrow \cdots \longrightarrow X_{0,n} \\
 & \uparrow & \uparrow & \uparrow & & \uparrow \\
 Z_{\bullet, \bullet} : & 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow X_{0,1} \longrightarrow \cdots \longrightarrow X_{0,n} \\
 & \downarrow & \downarrow & \downarrow & & \downarrow \\
 Y_{\bullet, \bullet} : & 0 \longrightarrow \cdots \longrightarrow Y_{0,n} \longrightarrow Y_{0,n} \longrightarrow \cdots \longrightarrow Y_{0,n}
 \end{array}$$
  

$$\begin{array}{ccccccc}
 A_{\bullet} : & A_1 & \cdots & A_n & 0 & \cdots & 0 \\
 & \uparrow & & \uparrow & \uparrow & & \uparrow \\
 C_{\bullet} : & 0 & \cdots & 0 & A_1 & \cdots & A_n \\
 & \downarrow & & \downarrow & \downarrow & & \downarrow \\
 B_{\bullet} : & B_1 & \cdots & B_n & 0 & \cdots & 0
 \end{array}$$

There are well-defined since  $X_{0,n} \simeq Y_{0,n}$ . One can checks that there are equivalences in  $\text{Gap}(\mathcal{F})/\text{Gap}^w(\mathcal{F})$ .

Step 3: For two morphisms  $(f_{\bullet}, F_{\bullet, \bullet}), (g_{\bullet}, G_{\bullet, \bullet}) : (A_{\bullet}, X_{\bullet, \bullet}) \rightarrow (B_{\bullet}, Y_{\bullet, \bullet})$ , if  $\text{ev}_{\mathcal{F}}(f_{\bullet}, F_{\bullet, \bullet}) \simeq \text{ev}_{\mathcal{F}}(g_{\bullet}, G_{\bullet, \bullet})$ , then  $(f_{\bullet}, F_{\bullet, \bullet}) \simeq (g_{\bullet}, G_{\bullet, \bullet})$  in  $\text{Gap}(\mathcal{F})/\text{Gap}^w(\mathcal{F})$ .

Assume  $(f_{\bullet}, F_{\bullet, \bullet}), (g_{\bullet}, G_{\bullet, \bullet}) \in \text{Gap}([n], \mathcal{F})$ . Consider the diagram

$$\begin{array}{ccccccc}
 X'_{\bullet, \bullet} : & 0 \longrightarrow \cdots \longrightarrow 0 \xrightarrow{F_{0,1}} X_{0,1} \longrightarrow \cdots \xrightarrow{F_{0,n}} X_{0,n} \\
 Y'_{\bullet, \bullet} : & 0 \xleftarrow{\quad} \cdots \longrightarrow 0 \xleftarrow{\quad} Y_{0,1} \xrightarrow{\quad} \cdots \longrightarrow Y_{0,n} \\
 X_{\bullet, \bullet} : & 0 \longrightarrow \cdots \longrightarrow X_{0,n} \longrightarrow X_{0,n} \longrightarrow \cdots \longrightarrow X_{0,n} \\
 Y_{\bullet, \bullet} : & 0 \longrightarrow \cdots \longrightarrow Y_{0,n} \longrightarrow Y_{0,n} \longrightarrow \cdots \longrightarrow Y_{0,n} \\
 \\
 A'_{\bullet} : & 0 & \cdots & 0 & A_1 & \cdots & A_n \\
 B'_{\bullet} : & 0 \xleftarrow{\quad} \cdots \xleftarrow{\quad} 0 \xleftarrow{\quad} B_1 \xleftarrow{\quad} \cdots \xleftarrow{\quad} B_n \\
 A_{\bullet} : & A_1 & \cdots & A_n & 0 & \cdots & 0 \\
 B_{\bullet} : & B_1 & \cdots & B_n & 0 & \cdots & 0
 \end{array}$$

Here  $(A'_{\bullet}, X'_{\bullet, \bullet}) \rightarrow (A_{\bullet}, X_{\bullet, \bullet})$  and  $(B'_{\bullet}, Y'_{\bullet, \bullet}) \rightarrow (B_{\bullet}, Y_{\bullet, \bullet})$  are just identity in  $\text{Gap}(\mathcal{F})/\text{Gap}^w(\mathcal{F})$  and  $(A_{\bullet}, X_{\bullet, \bullet}) \rightarrow (B_{\bullet}, Y_{\bullet, \bullet})$  can be either  $(f_{\bullet}, F_{\bullet, \bullet})$  or  $(g_{\bullet}, G_{\bullet, \bullet})$ . Hence,  $(f_{\bullet}, F_{\bullet, \bullet}), (g_{\bullet}, G_{\bullet, \bullet})$  are both equivalent to  $(A'_{\bullet}, X'_{\bullet, \bullet}) \rightarrow (B'_{\bullet}, Y'_{\bullet, \bullet})$  in  $\text{Gap}(\mathcal{F})/\text{Gap}^w(\mathcal{F})$ .

Step 4:  $\text{Ho}(\tilde{\text{ev}}_{\mathcal{F}}) : \text{Ho}(\text{Gap}(\mathcal{F}))/\text{Ho}(\text{Gap}^w(\mathcal{F})) \rightarrow \text{Ho}(E(\mathcal{F}))$  is an equivalence.

Step 1 tells us it is essentially surjective. Step 1 and Step 2 tells us it is surjective on morphisms. Step 3 tells us it is injective on morphisms. Hence, it is both essentially surjective and fully faithful.

□

**Corollary 2.14.**  $\mathcal{F}$  satisfies the dévissage condition if and only if  $\text{ev}_{\mathcal{F}}$  is essentially surjective.

### 3 THEOREMS FOR INVARIANTS

#### 3.1 Additivity theorem

Assume  $\mathcal{C} \in \text{Cat}_{\infty}^{\text{perf}}$  is a small idempotent-complete stable  $\infty$ -category and  $\mathcal{F}_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{C}, \mathcal{F}_{\mathcal{B}} : \mathcal{B} \rightarrow \mathcal{C}$  are exact functors between idempotent-complete stable  $\infty$ -categories. Define a category  $\mathcal{S}(\mathcal{F}_{\mathcal{A}}, \mathcal{F}_{\mathcal{B}})$  by

pullback

$$\begin{array}{ccc} \mathcal{S}(\mathcal{F}_A, \mathcal{F}_B) & \longrightarrow & \text{Gap}([2], \mathcal{C}) \\ \downarrow & & \downarrow \\ \mathcal{A} \times \mathcal{B} & \xrightarrow{(\mathcal{F}_A, \mathcal{F}_B)} & \mathcal{C}^2 \end{array}$$

Where  $\text{Gap}([2], \mathcal{C}) \rightarrow \mathcal{C}$  is defined by  $X_{\bullet, \bullet} \mapsto (X_{0,1}, X_{1,2})$ . Then  $\mathcal{S}(\mathcal{F}_A, \mathcal{F}_B)$  is also stable and idempotent-complete.

**Theorem 3.1** ( $\infty$ -categorical Additivity Theorem).

Assume  $\mathcal{C} \in \text{Cat}_{\infty}^{\text{perf}}$  is a small idempotent-complete stable  $\infty$ -category and  $\mathcal{F}_A : \mathcal{A} \rightarrow \mathcal{C}, \mathcal{F}_B : \mathcal{B} \rightarrow \mathcal{C}$  are exact functors between idempotent-complete stable  $\infty$ -categories. The natural projection  $\mathcal{S}(\mathcal{F}_A, \mathcal{F}_B) \rightarrow \mathcal{A} \times \mathcal{B}$  induces a homotopy equivalence  $\mathcal{U}_{\text{add}}(\mathcal{S}(\mathcal{F}_A, \mathcal{F}_B)) \xrightarrow{\sim} \mathcal{U}_{\text{add}}(\mathcal{A}) \times \mathcal{U}_{\text{add}}(\mathcal{B})$ .

*Proof.* There is a split exact sequence

$$\mathcal{A} \xleftarrow[i]{f} \mathcal{S}(\mathcal{A}, \mathcal{C}, \mathcal{B}) \xleftarrow[j]{g} \mathcal{B},$$

where  $i, g$  are projections,  $f, j$  are defined by

$$\begin{aligned} f : \mathcal{A} &\rightarrow \mathcal{A} \times \mathcal{B}, X \mapsto (X, 0) \\ &\mathcal{A} \rightarrow \text{Gap}([2], \mathcal{C}), X \mapsto (0 \rightarrow \mathcal{F}_A(X) \rightarrow \mathcal{F}_A(X)) \\ g : \mathcal{B} &\rightarrow \mathcal{A} \times \mathcal{B}, Y \mapsto (0, Y) \\ &\mathcal{B} \rightarrow \text{Gap}([2], \mathcal{C}), Y \mapsto (0 \rightarrow 0 \rightarrow \mathcal{F}_B(Y)). \end{aligned}$$

Thus, it induces a homotopy equivalence

$$\mathcal{U}_{\text{add}}(\mathcal{S}(\mathcal{F}_A, \mathcal{F}_B)) \xrightarrow{\sim} \mathcal{U}_{\text{add}}(\mathcal{A}) \times \mathcal{U}_{\text{add}}(\mathcal{B}).$$

□

Let  $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{C}$  be an exact functor between small idempotent-complete stable  $\infty$ -categories. Denote by  $q_{n, \mathcal{F}} : \text{Gap}([n], \mathcal{F}) \rightarrow \mathcal{A}^n$  the natural projections.

**Proposition 3.2.** For each  $n \geq 1$ , the exact functor  $q_{n, \mathcal{F}}$  induces a homotopy equivalence:

$$\mathcal{U}_{\text{add}}(\text{Gap}([n], \mathcal{F})) \xrightarrow{\sim} \mathcal{U}_{\text{add}}(\mathcal{A})^n.$$

Furthermore, this also induces a homotopy equivalence  $\mathcal{U}_{\text{add}}(\text{Gap}(\mathcal{F})) \simeq \text{colim}_n \mathcal{U}_{\text{add}}(\mathcal{A}^n)$ .

*Proof.* Define  $\mathcal{F}_1 : \mathcal{A} \rightarrow \text{Gap}([n], \mathcal{F})$  by

$$\begin{aligned} \mathcal{A} &\rightarrow \text{Gap}([n], \mathcal{C}), X \mapsto (0 \rightarrow \mathcal{F}(X) \rightarrow \mathcal{F}(X) \rightarrow \cdots \rightarrow \mathcal{F}(X)) \\ \mathcal{A} &\rightarrow \mathcal{A}^n, X \mapsto (X, 0, \dots, 0). \end{aligned}$$

Define  $\mathcal{F}_2 : \text{Gap}([n-1], \mathcal{F}) \rightarrow \text{Gap}([n], \mathcal{F})$  by

$$\begin{aligned} \text{Gap}([n-1], \mathcal{C}) &\rightarrow \text{Gap}([n], \mathcal{C}), (0 \rightarrow X_{0,1} \rightarrow \cdots \rightarrow X_{0,n-1}) \mapsto (0 \rightarrow 0 \rightarrow X_{0,1} \rightarrow \cdots \rightarrow X_{0,n-1}) \\ \mathcal{A}^{n-1} &\rightarrow \mathcal{A}^n, (A_1, \dots, A_{n-1}) \mapsto (0, A_1, \dots, A_{n-1}) \\ \mathcal{C}^{n-1} &\rightarrow \mathcal{C}^n, (X_1, \dots, X_{n-1}) \mapsto (0, X_1, \dots, X_{n-1}) \end{aligned}$$

These functors admit projections given by the exact functors

$$s_1 : \text{Gap}([n], \mathcal{F}) \rightarrow \mathcal{A}, \quad (A_{\bullet}, X_{\bullet, \bullet}) \mapsto A_1,$$

$$s_2 : \text{Gap}([n], \mathcal{F}) \rightarrow \text{Gap}([n-1], \mathcal{F}), \quad (A_{\bullet}, X_{\bullet, \bullet}) \mapsto (0, A_2, \dots, A_n, 0 \rightarrow 0 \rightarrow X_{1,2} \rightarrow \cdots \rightarrow X_{1,n}),$$

which assemble to an exact functor  $\text{Gap}([n], \mathcal{F}) \rightarrow \mathcal{S}(\mathcal{F}_1, \mathcal{F}_2)$ . By the Additivity Theorem 3.1, the pair of exact functors

$$(s_1, s_2) : \text{Gap}([n], \mathcal{F}) \xleftarrow{\sim} \text{Gap}([n-1], \mathcal{F}) \times \mathcal{A} : \mathcal{F}_2 \vee \mathcal{F}_2$$

induces inverse homotopy equivalences for  $\mathcal{U}_{\text{add}}$ . Inductively, this yields homotopy equivalences

$$\mathcal{U}_{\text{add}}(\text{Gap}([n], \mathcal{F})) \simeq \mathcal{U}_{\text{add}}(\text{Gap}([n-1], \mathcal{F})) \times \mathcal{U}_{\text{add}}(\mathcal{A}) \simeq \cdots \simeq \mathcal{U}_{\text{add}}(\mathcal{A})^n \simeq \mathcal{U}_{\text{add}}(\mathcal{A}^n).$$

Direct inspection shows the composite homotopy equivalence is defined by the exact functor  $q_{n, \mathcal{F}}$ . The map  $\mathcal{U}_{\text{add}}(q_{n, \mathcal{F}})$  admits a homotopy inverse given by the exact functor

$$\mathcal{A}^n \rightarrow \text{Gap}([n], \mathcal{F}), \quad (A_1, \dots, A_n) \mapsto (A_1, \dots, A_n, 0 \rightarrow \mathcal{F}(A_1) \rightarrow \mathcal{F}(A_1) \vee \mathcal{F}(A_2) \rightarrow \cdots \rightarrow \vee_{i=1}^n \mathcal{F}(A_i)).$$

Moreover, we have commutative diagrams of exact functors:

$$\begin{array}{ccc} \text{Gap}([n], \mathcal{F}) & \xrightarrow{i_{n, \mathcal{F}}} & \text{Gap}([n+1], \mathcal{F}) \\ \uparrow \downarrow & & \uparrow \downarrow \\ \mathcal{A}^n & \longrightarrow & \mathcal{A}^{n+1} \end{array}$$

where the bottom functor is the canonical inclusion, sending  $(A_1, \dots, A_{n-1}) \mapsto (A_1, \dots, A_{n-1}, 0)$ . The case  $n = \infty$  follows immediately, as  $\text{colim}_n \mathcal{U}_{\text{add}}(\text{Gap}([n], \mathcal{F})) \xrightarrow{\simeq} \mathcal{U}_{\text{add}}(\text{Gap}(\mathcal{F}))$ .  $\square$

### 3.2 Semiorthogonal decompositions

One can refer [Huy06] for the definition of *semiorthogonal decompositions* of triangulated categories. And, [DKSS24] explains the *semiorthogonal decompositions* of stable  $\infty$ -categories.

**Definition 3.3.** Let  $\mathcal{C}$  be a stable  $\infty$ -category. A *semiorthogonal decomposition* is a sequence of stable full subcategories  $\mathcal{C}_1, \dots, \mathcal{C}_n$  such that

- (1) For  $i > j$  and any  $X_i \in \mathcal{C}_i, X_j \in \mathcal{C}_j$ ,  $\text{Map}_{\mathcal{C}}(X_i, X_j)$  is contractible.
- (2) For any  $X \in \mathcal{C}$ , there is a filtration

$$0 = X_n \rightarrow \cdots \rightarrow X_0 = X$$

such that  $\text{cofib}(X_i \rightarrow X_{i-1}) \in \mathcal{C}_i$  for any  $1 \leq i \leq n$ .

We write  $\mathcal{C} = \langle \mathcal{C}_1, \dots, \mathcal{C}_n \rangle$ .

Let  $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{C}$  be an exact functor. Define

$$\mathfrak{p}_{i, \mathcal{C}} : \mathcal{C} \rightarrow \text{Gap}([n], \mathcal{C})$$

by  $X \mapsto X_{\bullet, \bullet}$  such that  $X_{0,j} = 0$  for  $j < i$  and  $X_{0,j} \rightarrow X_{0,j+1} = X \xrightarrow{\text{id}} X$  for  $j \geq i$ .

**Lemma 3.4.**  $\mathfrak{p}_{i, \mathcal{C}}$  is fully faithful and  $\mathfrak{p}_{n, \mathcal{C}}(\mathcal{C}), \dots, \mathfrak{p}_{1, \mathcal{C}}(\mathcal{C})$  defines a semiorthogonal decomposition of  $\text{Gap}([n], \mathcal{C})$ .

*Proof.* It is easy to see  $\mathfrak{p}_{i, \mathcal{C}}$  is fully faithful.

- (1) For  $i < j$  and  $X, Y \in \mathcal{C}$ ,

$$\text{Map}_{\text{Gap}([n], \mathcal{C})}(\mathfrak{p}_{i, \mathcal{C}}(X), \mathfrak{p}_{j, \mathcal{C}}(Y)) \simeq \text{Map}_{\text{Fun}(\Delta^1, \mathcal{C})}(X \xrightarrow{\text{id}} X, 0 \rightarrow Y) \simeq 0.$$

(2) For  $X_{\bullet,\bullet} \in \text{Gap}([n], \mathcal{C})$ , there is a filtration:

$$\begin{array}{ccccccc}
 X_{\bullet,\bullet}^0 = 0 : & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \cdots \longrightarrow 0 \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 X_{\bullet,\bullet}^1 : & 0 & \longrightarrow & X_{0,1} & \longrightarrow & X_{0,1} & \longrightarrow \cdots \longrightarrow X_{0,1} \longrightarrow X_{0,1} \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & \vdots & & \vdots & & \ddots & \vdots \\
 & \downarrow & & \downarrow & & \downarrow & \\
 X_{\bullet,\bullet}^{n-1} : & 0 & \longrightarrow & X_{0,1} & \longrightarrow & X_{0,2} & \longrightarrow \cdots \longrightarrow X_{0,n-1} \longrightarrow X_{0,n-1} \\
 & \downarrow & & \downarrow & & \downarrow & \\
 X_{\bullet,\bullet}^n = X_{\bullet,\bullet} : & 0 & \longrightarrow & X_{0,1} & \longrightarrow & X_{0,2} & \longrightarrow \cdots \longrightarrow X_{0,n-1} \longrightarrow X_{0,n}
 \end{array}$$

One can checks that  $\text{cofib}(X_{\bullet,\bullet}^i \rightarrow X_{\bullet,\bullet}^{i-1}) = \mathfrak{p}_{i,\mathcal{C}}(X_{i-1,i}) \in \mathfrak{p}_{i,\mathcal{C}}(\mathcal{C})$ .

□

Define

$$\mathfrak{p}_{i,\mathcal{F}} : \mathcal{A} \rightarrow \text{Gap}([n], \mathcal{F})$$

by  $\mathfrak{p}_{i,\mathcal{C}} \circ \mathcal{F} : \mathcal{A} \rightarrow \text{Gap}([n], \mathcal{C})$  and  $\mathcal{A} \rightarrow \mathcal{A}^n, A \mapsto A_\bullet$  such that  $A_i = A, A_j = 0, \forall j \neq i$ .

**Proposition 3.5.**  $\mathfrak{p}_{i,\mathcal{F}}$  is fully faithful and  $\mathfrak{p}_{n,\mathcal{F}}(\mathcal{A}), \dots, \mathfrak{p}_{1,\mathcal{F}}(\mathcal{A})$  defines a semiorthogonal decomposition of  $\text{Gap}([n], \mathcal{F})$ .

*Proof.* For  $X, Y \in \mathcal{A}$ ,

$$\begin{aligned}
 \text{Map}_{\text{Gap}(\mathcal{F})}(\mathfrak{p}_{i,\mathcal{F}}(X), \mathfrak{p}_{i,\mathcal{F}}(Y)) &\simeq \text{Map}_{\text{Gap}(\mathcal{C})}(\mathfrak{p}_{i,\mathcal{C}}(\mathcal{F}(X)), \mathfrak{p}_{i,\mathcal{C}}(\mathcal{F}(Y))) \times_{\text{Map}_{\mathcal{C}}(\mathcal{F}(X), \mathcal{F}(Y))} \text{Map}_{\mathcal{A}}(X, Y) \\
 &\simeq \text{Map}_{\mathcal{C}}(\mathcal{F}(X), \mathcal{F}(Y)) \times_{\text{Map}_{\mathcal{C}}(\mathcal{F}(X), \mathcal{F}(Y))} \text{Map}_{\mathcal{A}}(X, Y) \\
 &\simeq \text{Map}_{\mathcal{A}}(X, Y).
 \end{aligned}$$

Hence,  $\mathfrak{p}_{i,\mathcal{F}}$  is fully faithful.

(1) For  $i < j$  and  $X, Y \in \mathcal{A}$ , by Lemma 3.4,

$$\text{Map}_{\text{Gap}(\mathcal{F})}(\mathfrak{p}_{i,\mathcal{F}}(X), \mathfrak{p}_{j,\mathcal{F}}(Y)) = \text{Map}_{\text{Gap}(\mathcal{C})}(\mathfrak{p}_{i,\mathcal{C}}(\mathcal{F}(X)), \mathfrak{p}_{i,\mathcal{C}}(\mathcal{F}(Y))) \times_0 0 = 0.$$

(2) For  $(A_\bullet, X_{\bullet,\bullet}) \in \text{Gap}([n], \mathcal{F})$ , define  $X_{\bullet,\bullet}^i$  from Lemma 3.4 and  $A_\bullet^i := (A_1, \dots, A_i, 0, \dots, 0)$ . Then we have a sequence

$$(A_\bullet^0, X_{\bullet,\bullet}^0) = 0 \rightarrow \cdots \rightarrow (A_\bullet^n, X_{\bullet,\bullet}^n) = (A_\bullet, X_{\bullet,\bullet})$$

such that  $\text{cofib}((A_\bullet^i, X_{\bullet,\bullet}^i) \rightarrow (A_\bullet^{i-1}, X_{\bullet,\bullet}^{i-1})) = \mathfrak{p}_{i,\mathcal{F}}(A_i) \in \mathfrak{p}_{i,\mathcal{F}}(\mathcal{A})$ .

□

*Remark 3.6.* This is a new proof of the equivalence  $\mathcal{U}_{\text{add}}(\text{Gap}([n], \mathcal{F})) \simeq \mathcal{U}_{\text{add}}(\mathcal{A})^n$ .

### 3.3 Categorization of fiber with dévissage condition

In this subsection, we still assume  $\mathcal{C}$  and  $\mathcal{A}$  are idempotent-complete. For each  $n \geq 1$ , we have an exact functor:

$$q_{n,\mathcal{A}}^w : \text{Gap}^w([n], \mathcal{F}) \rightarrow \mathcal{A}^{n-1}, (A_\bullet, X_{\bullet,\bullet}) \mapsto (A_1, \dots, A_{n-1}),$$

It has an exact section

$$\begin{aligned} j_{n,\mathcal{A}} : \mathcal{A}^{n-1} &\rightarrow \text{Gap}^w([n], \mathcal{F}), \\ (A_1, \dots, A_{n-1}) &\mapsto (A_1, \dots, A_{n-1}, \vee_{i=1}^{n-1} \Sigma A_i, 0 \rightarrow \mathcal{F}(A_1) \rightarrow \mathcal{F}(A_1) \vee \mathcal{F}(A_2) \rightarrow \dots \rightarrow \vee_{i=1}^{n-1} \mathcal{F}(A_i) \rightarrow 0). \end{aligned}$$

Define

$$\begin{aligned} \tau_{n,\mathcal{F}} : \mathcal{A}^{n-1} &\rightarrow \text{Gap}^w([n], \mathcal{F}), \\ (A_1, \dots, A_{n-1}) &\mapsto (A_1, A_2 \vee \Sigma A_1, \dots, A_{n-1} \vee \Sigma A_{n-2}, \Sigma A_{n-1}, 0 \rightarrow \mathcal{F}(A_1) \rightarrow \dots \rightarrow \mathcal{F}(A_{n-1}) \rightarrow 0), \end{aligned}$$

where the morphism  $\mathcal{F}(A_i) \rightarrow \mathcal{F}(A_{i+1})$  is the zero morphism for each  $i$ . The composite functor

$$q_{n,\mathcal{F}}^w \circ \tau_{n,\mathcal{F}} : (A_1, \dots, A_{n-1}) \mapsto (A_1, A_2 \vee \Sigma A_1, \dots, A_{n-1} \vee \Sigma A_{n-2}),$$

induces a homotopy equivalence for any additive invariant.

The diagram

$$\begin{array}{ccc} \text{Gap}^w([n], \mathcal{F}) & \xrightarrow{i_{n,\mathcal{F}}^w} & \text{Gap}^w([n+1], \mathcal{F}) \\ \tau_{n,\mathcal{F}} \uparrow & & \tau_{n+1,\mathcal{F}} \uparrow \\ \mathcal{A}^{n-1} & \longrightarrow & \mathcal{A}^n \end{array}$$

commutes, where the bottom functor is the inclusion  $(A_1, \dots, A_{n-1}) \mapsto (A_1, \dots, A_{n-1}, 0)$ . Thus, we get a functor

$$\tau_{\mathcal{F}} : \text{colim}_n \mathcal{A}^{n-1} \rightarrow \text{Gap}^w(\mathcal{F}).$$

We introduce a result about categorification of cofiber and its proof.

**Proposition 3.7.** [RSW24, Proposition 3.7] Let  $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$  be an exact functor of small stable  $\infty$ -categories. There exists a small stable  $\infty$ -category  $\text{Cone}(\mathcal{F})$  with an exact functor  $\mathcal{D} \rightarrow \text{Cone}(\mathcal{F})$  that induces an equivalence

$$\text{cofib}(\mathcal{U}_{\text{loc}}(\mathcal{C}) \xrightarrow{\mathcal{U}_{\text{loc}}(\mathcal{F})} \mathcal{U}_{\text{loc}}(\mathcal{D})) \simeq \mathcal{U}_{\text{loc}}(\text{Cone}(\mathcal{F}))$$

*Proof.* Define the *lex pullback*  $\mathcal{D} \xrightarrow{\text{Ind}(\mathcal{D})} \text{Ind}(\mathcal{C})$  by the pullback

$$\begin{array}{ccc} \mathcal{D} \xrightarrow{\text{Ind}(\mathcal{D})} \text{Ind}(\mathcal{C}) & \longrightarrow & \text{Fun}(\Delta^1, \text{Ind}(\mathcal{D})) \\ \downarrow & & \downarrow \\ \mathcal{D} \times \text{Ind}(\mathcal{C}) & \longrightarrow & \text{Ind}(\mathcal{D}) \times \text{Ind}(\mathcal{D}). \end{array}$$

Then there is a fully faithful functor  $\mathcal{C} \rightarrow \mathcal{D} \xrightarrow{\text{Ind}(\mathcal{D})} \text{Ind}(\mathcal{C})$ ,  $X \mapsto (\mathcal{F}(X), X, \mathcal{F}(X) \xrightarrow{\text{id}} \mathcal{F}(X))$ . Define  $\text{Cone}(\mathcal{F}) := \text{Idem}(\mathcal{D} \xrightarrow{\text{Ind}(\mathcal{D})} \text{Ind}(\mathcal{C}) / \mathcal{C})$ . From [Tam18, Proposition 10],  $\mathcal{U}_{\text{loc}}(\mathcal{D} \xrightarrow{\text{Ind}(\mathcal{D})} \text{Ind}(\mathcal{C})) \simeq \mathcal{U}_{\text{loc}}(\mathcal{D}) \oplus \mathcal{U}_{\text{loc}}(\text{Ind}(\mathcal{C}))$

$$\text{cofib}(\mathcal{U}_{\text{loc}}(\mathcal{C}) \xrightarrow{\mathcal{U}_{\text{loc}}(\mathcal{F})} \mathcal{U}_{\text{loc}}(\mathcal{D}) \oplus \mathcal{U}_{\text{loc}}(\text{Ind}(\mathcal{C}))) \simeq \mathcal{U}_{\text{loc}}(\text{Cone}(\mathcal{F})).$$

Notice that  $\mathcal{U}_{\text{loc}}(\text{Ind}(\mathcal{C})) \simeq 0$  since  $\text{Ind}(\mathcal{C})$  admits an Eilenberg swindle.  $\square$

*Remark 3.8.* We replace  $\text{Ind}(-)^\omega$  by  $\text{Ind}(-)$ . usually,  $\text{Ind}(\mathcal{C})$  is not small. To define localizing invariants, we need to use bigger universe. For the details of this technology, readers can refer to [Hen17]. Simply put, we choose two universes  $\mathbb{U} \in \mathbb{V}$ . Then we use  $\text{Ind}^{\mathbb{U}}(\mathcal{C})$ , which is  $\mathbb{V}$ -small. Hence, we could define localizing invariants of  $\text{Ind}^{\mathbb{U}}(\mathcal{C})$ . Or, we could use  $\text{Ind}(-)^\kappa$  for some regular cardinal  $\kappa > \omega$  like the construction in the original text [RSW24].

Denote by  $Q(\mathcal{F}) := \text{Cone}(\tau_{\mathcal{F}})$ . Since  $\mathcal{U}_{\text{loc}}(\tau_{\mathcal{F}})$  is the section of  $\mathcal{U}_{\text{loc}}(q_{\mathcal{F}}^w)$ , the cofiber sequence

$$\mathcal{U}_{\text{loc}}(\text{colim}_n \mathcal{A}^{n-1}) \xrightarrow{\mathcal{U}_{\text{loc}}(\tau_{\mathcal{F}})} \mathcal{U}_{\text{loc}}(\text{Gap}^w(\mathcal{F})) \rightarrow \mathcal{U}_{\text{loc}}(Q(\mathcal{F}))$$

split.

**Theorem 3.9.** Let  $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{C} \in \text{Cat}_{\infty}^{\text{perf}}$  be an exact functor between small stable idempotent-complete  $\infty$ -categories. If  $\mathcal{F}$  satisfies the weak dévissage condition, there is a homotopy equivalence

$$\mathcal{U}_{\text{loc}}(Q(\mathcal{F})) \simeq \text{fib}(\mathcal{U}_{\text{loc}}(\mathcal{F}))$$

*Proof.* By Proposition 2.13 and the definition of weak dévissage condition, there is a homotopy fiber sequence

$$\mathcal{U}_{\text{loc}}(\text{Gap}^w(\mathcal{F})) \rightarrow \mathcal{U}_{\text{loc}}(\text{Gap}(\mathcal{F})) \rightarrow \mathcal{U}_{\text{loc}}(\mathcal{C}).$$

There is a homotopy fiber sequence

$$\mathcal{U}_{\text{loc}}(\mathcal{A}^{n-1}) \rightarrow \mathcal{U}_{\text{loc}}(\mathcal{A}^n) \rightarrow \mathcal{U}_{\text{loc}}(\mathcal{A})$$

where the left map is induced from  $(A_1, \dots, A_{n-1}) \mapsto (A_1, A_2 \vee \Sigma A_1, \dots, A_{n-1} \vee \Sigma A_{n-2}, \Sigma A_{n-1})$  and the right map is induced by  $\vee$ . It induces a homotopy fiber sequence

$$\mathcal{U}_{\text{loc}}(\text{colim}_n \mathcal{A}^{n-1}) \rightarrow \mathcal{U}_{\text{loc}}(\text{colim}_n \mathcal{A}^n) \rightarrow \mathcal{U}_{\text{loc}}(\mathcal{A})$$

Consider the diagram

$$\begin{array}{ccccc} \mathcal{U}_{\text{loc}}(\text{Gap}^w(\mathcal{F})) & \longrightarrow & \mathcal{U}_{\text{loc}}(\text{Gap}(\mathcal{F})) & \longrightarrow & \mathcal{U}_{\text{loc}}(\mathcal{C}) \\ \mathcal{U}_{\text{loc}}(\tau_{\mathcal{F}}) \uparrow & & \mathcal{U}_{\text{loc}}(\text{colim}_n q_{n,\mathcal{F}}^{-1}) \uparrow & & \mathcal{U}_{\text{loc}}(\mathcal{F}) \uparrow \\ \mathcal{U}_{\text{loc}}(\text{colim}_n \mathcal{A}^{n-1}) & \longrightarrow & \mathcal{U}_{\text{loc}}(\text{colim}_n \mathcal{A}^n) & \longrightarrow & \mathcal{U}_{\text{loc}}(\mathcal{A}) \end{array}$$

Since  $\mathcal{U}_{\text{loc}}(\text{colim}_n q_{n,\mathcal{F}}^{-1})$  is a homotopy equivalence by Proposition 3.2, we have a homotopy equivalence by Lemma 1.5

$$\mathcal{U}_{\text{loc}}(Q(\mathcal{F})) \simeq \text{cofib}(\mathcal{U}_{\text{loc}}(\tau_{\mathcal{F}})) \simeq \Omega \text{cofib}(\mathcal{U}_{\text{loc}}(\mathcal{F})).$$

Since  $\mathcal{U}_{\text{loc}}$  is into a stable  $\infty$ -category, we get  $\mathcal{U}_{\text{loc}}(Q(\mathcal{F})) \simeq \text{fib}(\mathcal{U}_{\text{loc}}(\mathcal{F}))$ .  $\square$

*Remark 3.10.* Part of ideas of this Theorem from [Rap18].

**Corollary 3.11.** Let  $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{C} \in \text{Cat}_{\infty}^{\text{perf}}$  be an exact functor between small stable idempotent-complete  $\infty$ -categories. Given a localizing invariant  $\mathcal{L}$ , if  $\mathcal{F}$  satisfies the dévissage condition,  $\mathcal{L}(Q(\mathcal{F}))$  is a homotopy equivalence if and only if  $\mathcal{L}(Q(\mathcal{F}))$  is homotopy trivial.

**Corollary 3.12.** Let  $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{C} \in \text{Cat}_{\infty}^{\text{perf}}$  be an exact functor between small stable idempotent-complete  $\infty$ -categories. if  $\mathcal{F}$  satisfies the dévissage condition and  $K_0(\mathcal{F})$  is an isomorphism,  $\mathcal{F}$  induces isomorphisms

$$K_n(\mathcal{F}) : K_n(\mathcal{A}) \xrightarrow{\sim} K_n(\mathcal{C})$$

for all  $n \geq 1$  if and only if  $K_n(Q(\mathcal{F})) \simeq 0$  for all  $n \geq 0$ .

### 3.4 Categorization of loop

Consider the diagonal map  $\delta_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}^2, X \mapsto (X, X)$ . It is injective on both objects and morphisms.

**Lemma 3.13.**  $\delta_{\mathcal{C}}$  satisfies the weak dévissage condition.

*Proof.* For  $(X, Y) \in \mathcal{C}^2$ , it is a direct factor of

$$(X, Y) \oplus (Y, X) \simeq (X \oplus Y, X \oplus Y) \in \delta_{\mathcal{C}}(\mathcal{C}) \subseteq E(\delta_{\mathcal{C}}).$$

Hence,  $(X, Y) \in \text{Idem}(E(\delta_{\mathcal{C}}))$ .  $\square$

Denote  $\theta(\mathcal{C}) = Q(\delta_{\mathcal{C}})$ . Then from Theorem 3.9, we get the loop theorem.

**Theorem 3.14.** Let  $\mathcal{C} \in \text{Cat}_{\infty}^{\text{perf}}$  be a small stable idempotent-complete  $\infty$ -category. Then we have a homotopy equivalence

$$\mathcal{U}_{\text{loc}}(\theta(\mathcal{C})) \simeq \text{fib}(\mathcal{U}_{\text{loc}}(\delta_{\mathcal{C}})) \simeq \Omega \mathcal{U}_{\text{loc}}(\mathcal{C}).$$

Besides,  $\theta$  is functorial.

**Corollary 3.15.** Let  $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$  be an exact functor of small stable  $\infty$ -categories. Then there is a homotopy equivalence

$$\mathcal{U}_{\text{loc}}(\text{Cone}(\theta(\mathcal{F}))) \simeq \text{fib}(\mathcal{U}_{\text{loc}}(\mathcal{F})).$$

Let  $\mathcal{C} \in \text{Cat}_{\infty}^{\text{perf}}$  be a small stable idempotent-complete  $\infty$ -category. Define

$$\text{gr}_n : \text{Gap}([n], \mathcal{C}) \rightarrow \mathcal{C}^n, X_{\bullet, \bullet} \mapsto (X_{0,1}, \dots, X_{n-1,n})$$

and  $\text{gr} : \text{Gap}(\mathcal{C}) \rightarrow \text{colim}_n \mathcal{C}^n$  as the colimit  $\text{colim}_n \text{gr}_n$ . Besides, define some categories of binary sequences by pullbacks

$$\begin{array}{ccc} \text{BGap}([n], \mathcal{C}) & \longrightarrow & \text{Gap}([n], \mathcal{C}) & \text{BGap}^w([n], \mathcal{C}) & \longrightarrow & \text{Gap}^w([n], \mathcal{C}) \\ \downarrow & & \downarrow \text{gr}_n & & \downarrow & \downarrow \text{gr}_n \\ \text{Gap}([n], \mathcal{C}) & \xrightarrow{\text{gr}_n} & \mathcal{C}^n & \text{Gap}^w([n], \mathcal{C}) & \xrightarrow{\text{gr}_n} & \mathcal{C}^n \end{array},$$

and

$$\begin{array}{ccc} \text{BGap}(\mathcal{C}) & \longrightarrow & \text{Gap}([n], \mathcal{C}) & \text{BGap}^w(\mathcal{C}) & \longrightarrow & \text{Gap}(\mathcal{C}) \\ \downarrow & & \downarrow \text{gr} & & \downarrow & \downarrow \text{gr} \\ \text{Gap}(\mathcal{C}) & \xrightarrow{\text{gr}} & \text{colim}_n \mathcal{C}^n & \text{Gap}^w(\mathcal{C}) & \xrightarrow{\text{gr}} & \text{colim}_n \mathcal{C}^n \end{array}.$$

Then the identity define diagonal functors  $\Delta$ .

**Theorem 3.16.** [KW19, Theorem 1.1] Let  $\mathcal{C} \in \text{Cat}_{\infty}^{\text{perf}}$  be a small stable idempotent-complete  $\infty$ -category. There is a homotopy equivalence

$$\text{cofib}(\mathcal{U}_{\text{loc}}(\text{Gap}^w(\mathcal{C}))) \xrightarrow{\mathcal{U}_{\text{loc}}(\Delta)} \mathcal{U}_{\text{loc}}(\text{BGap}^w(\mathcal{C})) \simeq \Omega \mathcal{U}_{\text{loc}}(\mathcal{C}).$$

$\text{BGap}^w(\mathcal{C})$  consisting of

- (1) Objects are 2-tuples  $(X_{\bullet, \bullet}, Y_{\bullet, \bullet})$ , where  $X_{\bullet, \bullet}, Y_{\bullet, \bullet} \in \text{Gap}^w(\mathcal{C})$  and  $X_{i-1,i} = Y_{i-1,i}$ .
- (2) Morphisms between  $(X_{\bullet, \bullet}, Y_{\bullet, \bullet})$  and  $(X'_{\bullet, \bullet}, Y'_{\bullet, \bullet})$  are 2-tuples  $(f_{\bullet, \bullet} : X_{\bullet, \bullet} \rightarrow X'_{\bullet, \bullet}, g_{\bullet, \bullet} : Y_{\bullet, \bullet} \rightarrow Y'_{\bullet, \bullet})$  of morphisms in  $\text{Gap}^w(\mathcal{C})$  such that  $g_{i-1,i} \simeq f_{i-1,i}$  for all  $i \geq 1$ .

And,  $\Delta(X_{\bullet, \bullet}) = (X_{\bullet, \bullet}, X_{\bullet, \bullet})$ .

**Proposition 3.17.**  $\Delta$  satisfies the weak dévissage condition.

*Proof.* For  $(X_{\bullet,\bullet}, Y_{\bullet,\bullet}) \in \text{BGap}^w(\mathcal{C})$ , we have

$$(X_{\bullet,\bullet} \oplus Y_{\bullet,\bullet}, Y_{\bullet,\bullet} \oplus X_{\bullet,\bullet}) \simeq (X_{\bullet,\bullet} \oplus Y_{\bullet,\bullet}, X_{\bullet,\bullet} \oplus Y_{\bullet,\bullet}) \in \text{E}(\Delta),$$

Then  $(X_{\bullet,\bullet}, Y_{\bullet,\bullet})$  is a direct factor of  $(X_{\bullet,\bullet} \oplus Y_{\bullet,\bullet}, Y_{\bullet,\bullet} \oplus X_{\bullet,\bullet})$ , which means it is in  $\text{Idem}(\text{E}(\Delta))$ . Hence,  $\Delta$  satisfies the weak dévissage condition.  $\square$

Denote  $\Theta(\mathcal{C}) := Q(\Delta)$ . This define a functor  $\Theta : \text{Cat}_\infty^{\text{perf}} \rightarrow \text{Cat}_\infty^{\text{perf}}$ . Then we get the square loop theorem.

**Theorem 3.18.** *Let  $\mathcal{C} \in \text{Cat}_\infty^{\text{perf}}$  be a small stable idempotent-complete  $\infty$ -category. Then we have a homotopy equivalence*

$$\mathcal{U}_{\text{loc}}(\Theta(\mathcal{C})) \simeq \Omega^2 \mathcal{U}_{\text{loc}}(\mathcal{C}).$$

Besides,  $\Theta$  is functorial.

*Proof.* By Theorem 3.9 and Proposition 3.17, we have a homotopy equivalence

$$\mathcal{U}_{\text{loc}}(\Theta(\mathcal{C})) \simeq \text{fib}(\mathcal{U}_{\text{loc}}(\text{Gap}^w(\mathcal{C})) \xrightarrow{\mathcal{U}_{\text{loc}}(\Delta)} \mathcal{U}_{\text{loc}}(\text{BGap}^w(\mathcal{C}))).$$

By Theorem 3.16, we have

$$\text{fib}(\mathcal{U}_{\text{loc}}(\text{Gap}^w(\mathcal{C})) \xrightarrow{\mathcal{U}_{\text{loc}}(\Delta)} \mathcal{U}_{\text{loc}}(\text{BGap}^w(\mathcal{C}))) \simeq \Omega \text{cofib}(\mathcal{U}_{\text{loc}}(\text{Gap}^w(\mathcal{C})) \xrightarrow{\mathcal{U}_{\text{loc}}(\Delta)} \mathcal{U}_{\text{loc}}(\text{BGap}^w(\mathcal{C}))) \simeq \Omega^2 \mathcal{U}_{\text{loc}}(\mathcal{C}).$$

Hence, we have a homotopy equivalence

$$\mathcal{U}_{\text{loc}}(\Theta(\mathcal{C})) \simeq \Omega^2 \mathcal{U}_{\text{loc}}(\mathcal{C}).$$

$\square$

*Remark 3.19.* [RSW24] gives a functor  $\Gamma : \text{Cat}_\infty^{\text{perf}} \rightarrow \text{Cat}_\infty^{\text{perf}}$  with  $\mathcal{U}_{\text{loc}}(\Gamma(\mathcal{C})) \simeq \Omega \mathcal{U}_{\text{loc}}(\mathcal{C})$ :  $\Gamma(\mathcal{C}) := \text{Cone}(\Delta)$ . It also use Proposition 3.7 and Theorem 3.16.

Denote by  $\text{Calk}(\mathcal{C}) := \text{Idem}(\text{Ind}(\mathcal{C})^{\omega_1}/\mathcal{C})$  the  $\omega_1$ -small Calkin category of  $\mathcal{C}$ . Then  $\mathcal{U}_{\text{loc}}(\text{Calk}(\mathcal{C})) \simeq \Sigma \mathcal{U}_{\text{loc}}(\mathcal{C})$  since  $\text{Ind}(\mathcal{C})^{\omega_1}$  admits an Eilenberg swindle.

**Corollary 3.20.** Let  $\mathcal{C} \in \text{Cat}_\infty^{\text{perf}}$  be a small stable idempotent-complete  $\infty$ -category. Then

$$\mathcal{U}_{\text{loc}}(\Theta \circ \text{Calk}(\mathcal{C})) \simeq \Omega \mathcal{U}_{\text{loc}}(\mathcal{C}) \simeq \mathcal{U}_{\text{loc}}(\text{Calk} \circ \Theta(\mathcal{C}))$$

*Remark 3.21.* [Hen17] defines  $\text{Tate}(\mathcal{C})$  with  $K(\text{Tate}(\mathcal{C})) \simeq \Sigma K(\mathcal{C})$ . So, we can also use Tate here.

*Remark 3.22.* In fact, we can give a simple proof of Theorem 3.16 now.

*A proof of Theorem 3.16.* Firstly, observation shows that  $\text{BGap}^w(\mathcal{C}) \simeq \text{Gap}^w(\delta_{\mathcal{C}})$ . Then we have a diagram of cofiber sequences

$$\begin{array}{ccccc} \mathcal{U}_{\text{loc}}(\text{Gap}^w(\mathcal{C})) & \longrightarrow & \mathcal{U}_{\text{loc}}(\text{Gap}(\mathcal{C})) & \longrightarrow & \mathcal{U}_{\text{loc}}(\mathcal{C}) \\ \downarrow \mathcal{U}_{\text{loc}}(\Delta) & & \downarrow \mathcal{U}_{\text{loc}}(\Delta) & & \downarrow \mathcal{U}_{\text{loc}}(\Delta) \\ \mathcal{U}_{\text{loc}}(\text{Gap}^w(\delta_{\mathcal{C}})) & \longrightarrow & \mathcal{U}_{\text{loc}}(\text{Gap}(\delta_{\mathcal{C}})) & \longrightarrow & \mathcal{U}_{\text{loc}}(\mathcal{C}^2) \end{array}$$

by Proposition 2.13. The middle  $\mathcal{U}_{\text{loc}}(\Delta)$  is a homotopy equivalence because the second map and entire map in

$$\mathcal{U}_{\text{loc}}(\text{Gap}(\mathcal{C})) \xrightarrow{\mathcal{U}_{\text{loc}}(\Delta)} \mathcal{U}_{\text{loc}}(\text{Gap}(\mathcal{C}^2, \mathcal{C})) \rightarrow \mathcal{U}_{\text{loc}}(\text{colim}_n \mathcal{C}^n)$$

are homotopy equivalences from Proposition 3.2. Then by Lemma 1.5, there is a cofiber sequence

$$\text{cofib}(\mathcal{U}_{\text{loc}}(\text{Gap}^w(\mathcal{C}))) \xrightarrow{\mathcal{U}_{\text{loc}}(\Delta)} \mathcal{U}_{\text{loc}}(\text{BGap}^w(\mathcal{C})) \rightarrow 0 \rightarrow \text{cofib}(\mathcal{U}_{\text{loc}}(\mathcal{C}) \xrightarrow{\mathcal{U}_{\text{loc}}(\Delta)} \mathcal{U}_{\text{loc}}(\mathcal{C}^2)) \simeq \mathcal{U}_{\text{loc}}(\mathcal{C}).$$

Hence,

$$\text{cofib}(\mathcal{U}_{\text{loc}}(\text{Gap}^w(\mathcal{C}))) \xrightarrow{\mathcal{U}_{\text{loc}}(\Delta)} \mathcal{U}_{\text{loc}}(\text{BGap}^w(\mathcal{C})) \simeq \Omega \mathcal{U}_{\text{loc}}(\mathcal{C}).$$

$\square$

## 4 FILLABILITY

### 4.1 1-fillable functors

For a small stable  $\infty$ -categories  $\mathcal{C}$ , its connective  $K_0^{\text{cn}}$ -group is the Grothendieck group of  $\text{Ho}(\mathcal{C})$ :

$$K_0^{\text{cn}}(\mathcal{C}) = \frac{\text{Free abelian group on isomorphism classes in } \text{Ho}(\mathcal{C})}{\langle [X] - [Y] + [Z] : \text{cofiber sequence } X \rightarrow Y \rightarrow Z \rangle}$$

Its non-connective  $K_0$ -group is  $K_0(\mathcal{C}) = K^{\text{cn}}(\text{Idem}(\mathcal{C}))$ .

**Lemma 4.1.** *Let  $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{C} \in \text{Cat}_{\infty}^{\text{perf}}$  be an exact functor between small stable idempotent-complete  $\infty$ -categories. If  $\mathcal{F}$  satisfies the dévissage condition,*

$$K_0(\mathcal{F}) : K_0(\mathcal{A}) \rightarrow K_0(\mathcal{C})$$

*is an epimorphism.*

*Proof.* Just from the definition of dévissage condition and the  $K_0$  groups.  $\square$

Denote by  $\text{Gap}^{\vee}(\mathcal{F})$  the pullback

$$\begin{array}{ccc} \text{Gap}^{\vee}(\mathcal{F}) & \longrightarrow & \text{Gap}(\mathcal{F}) \\ \downarrow & & \downarrow \text{ev}_{\mathcal{F}} \\ \text{Gap}(\mathcal{F}) & \xrightarrow{\text{ev}_{\mathcal{F}}} & \mathcal{C} \end{array}$$

Hence, it is stable. Denote  $\text{Gap}^{\vee}(\mathcal{C}) := \text{Gap}^{\vee}(\text{id}_{\mathcal{C}})$ .

Define an exact functor  $\text{Gap}_{\mathcal{F}} : \text{Gap}(\mathcal{A}) \rightarrow \text{Gap}(\mathcal{F})$  by  $\text{Gap}(\mathcal{A}) \rightarrow \text{Gap}(\mathcal{C}), X_{\bullet,\bullet} \mapsto \mathcal{F}(X_{\bullet,\bullet})$  and  $\text{gr} : \text{Gap}(\mathcal{A}) \rightarrow \text{colim}_n \mathcal{A}^n$  in last section. Then it induces an exact functor  $\text{Gap}_{\mathcal{F}}^{\vee} : \text{Gap}^{\vee}(\mathcal{A}) \rightarrow \text{Gap}^{\vee}(\mathcal{F})$ .

**Definition 4.2.** Let  $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{C} \in \text{Cat}_{\infty}^{\text{perf}}$  be an exact functor between small stable idempotent-complete  $\infty$ -categories. We call  $\mathcal{F}$  1-fillable if  $\text{Gap}_{\mathcal{F}}^{\vee}$  satisfies the dévissage condition.

Define exact functors

$$\tau_{\perp} : \text{Gap}^w(\mathcal{F}) \rightarrow \text{Gap}^{\vee}(\mathcal{F}), (A_{\bullet}, X_{\bullet,\bullet}) \mapsto ((A_{\bullet}, X_{\bullet,\bullet}), (0_{\bullet}, 0_{\bullet,\bullet}))$$

and

$$p_{\dashv} : \text{Gap}^{\vee}(\mathcal{F}) \rightarrow \text{Gap}(\mathcal{F}), ((A_{\bullet}, X_{\bullet,\bullet}), (B_{\bullet}, Y_{\bullet,\bullet})) \mapsto (B_{\bullet}, Y_{\bullet,\bullet}).$$

### Proposition 4.3.

$$\text{Gap}^w(\mathcal{F}) \xrightarrow{\tau_{\perp}} \text{Gap}^{\vee}(\mathcal{F}) \xrightarrow{p_{\dashv}} \text{Gap}(\mathcal{F})$$

is an exact sequence. Besides, the cofiber sequence

$$K(\text{Gap}^w(\mathcal{F})) \xrightarrow{K(\tau_{\perp})} K(\text{Gap}^{\vee}(\mathcal{F})) \xrightarrow{K(p_{\dashv})} K(\text{Gap}(\mathcal{F}))$$

split.

*Proof.* By [BGT13, Proposition 5.15], we only need to prove the exact sequence in the sense of homotopy categories.

For two objects  $(A_{\bullet}, X_{\bullet,\bullet}), (B_{\bullet}, Y_{\bullet,\bullet}) \in \text{Gap}^w(\mathcal{F})$ , we have

$$\begin{aligned} \text{Map}_{\text{Gap}^w(\mathcal{F})}(\tau_{\perp}(A_{\bullet}, X_{\bullet,\bullet}), \tau_{\perp}(B_{\bullet}, Y_{\bullet,\bullet})) &\simeq \text{Map}_{\text{Gap}(\mathcal{F})}((A_{\bullet}, X_{\bullet,\bullet}), (B_{\bullet}, Y_{\bullet,\bullet})) \times_{\text{Map}_{\mathcal{C}}(0,0)} \text{Map}_{\text{Gap}(\mathcal{F})}(0,0) \\ &\simeq \text{Map}_{\text{Gap}(\mathcal{F})}((A_{\bullet}, X_{\bullet,\bullet}), (B_{\bullet}, Y_{\bullet,\bullet})) \\ &\simeq \text{Map}_{\text{Gap}^w(\mathcal{F})}((A_{\bullet}, X_{\bullet,\bullet}), (B_{\bullet}, Y_{\bullet,\bullet})) \end{aligned}$$

Hence,  $\tau_{\perp}$  is fully faithful. Since  $p_{+}$  admits a section

$$\Delta : \text{Gap}(\mathcal{F}) \rightarrow \text{Gap}^{\vee}(\mathcal{F}), (A_{\bullet}, X_{\bullet, \bullet}) \mapsto ((A_{\bullet}, X_{\bullet, \bullet}), (A_{\bullet}, X_{\bullet, \bullet})),$$

$p_{+}$  is essentially surjective on both objects and morphisms. Let  $((A_{\bullet}, X_{\bullet, \bullet}), (B_{\bullet}, Y_{\bullet, \bullet})) \in \text{Gap}^{\vee}(\mathcal{F})$ , there are two morphisms  $(A_{\bullet}, X_{\bullet, \bullet}) \rightarrow (C_{\bullet}, Z_{\bullet, \bullet}), (B_{\bullet}, Y_{\bullet, \bullet}) \rightarrow (C_{\bullet}, Z_{\bullet, \bullet})$  which are stable at identities from Proposition 2.13. Hence, there are two morphisms

$$((A_{\bullet}, X_{\bullet, \bullet}), (B_{\bullet}, Y_{\bullet, \bullet})) \rightarrow ((C_{\bullet}, Z_{\bullet, \bullet}), (B_{\bullet}, Y_{\bullet, \bullet})), ((B_{\bullet}, Y_{\bullet, \bullet}), (B_{\bullet}, Y_{\bullet, \bullet})) \rightarrow ((C_{\bullet}, Z_{\bullet, \bullet}), (B_{\bullet}, Y_{\bullet, \bullet}))$$

whose cofibers are in  $\tau_{\perp}(\text{Gap}^w(\mathcal{F}))$ . Hence,

$$\text{Gap}^{\vee}(\mathcal{F}) / \text{Gap}^w(\mathcal{F}) \rightarrow \text{Gap}(\mathcal{F})$$

is injective on objects. Similarly, from Proposition 2.13, it is also injective on morphisms. Hence, it is an equivalence in the sense of homotopy categories.

Then we have a cofiber sequence

$$K(\text{Gap}^w(\mathcal{F})) \xrightarrow{K(\tau_{\perp})} K(\text{Gap}^{\vee}(\mathcal{F})) \xrightarrow{K(p_{+})} K(\text{Gap}(\mathcal{F})).$$

Since  $K(p_{+})$  admits a section  $K(\Delta)$ , it is split.  $\square$

**Theorem 4.4.** *Let  $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{C} \in \text{Cat}_{\infty}^{\text{perf}}$  be a 1-fillable exact functor between small stable idempotent-complete  $\infty$ -categories. Then  $K_0(Q(\mathcal{F})) = 0$ . Furthermore, if  $\mathcal{F}$  satisfies the dévissage condition,  $K_0(\mathcal{F}) : K_0(\mathcal{A}) \xrightarrow{\sim} K_0(\mathcal{C})$  is an isomorphism.*

*Proof.*  $\text{Gap}_{\mathcal{F}}$  restricts to a functor  $\text{Gap}_{\mathcal{F}}^w : \text{Gap}^w(\mathcal{A}) \rightarrow \text{Gap}^w(\mathcal{F})$ . Consider the diagram

$$\begin{array}{ccccc} \text{Gap}^w(\mathcal{A}) & \longrightarrow & \text{Gap}^{\vee}(\mathcal{A}) & \longrightarrow & \text{Gap}(\mathcal{A}) \\ \downarrow \text{Gap}_{\mathcal{F}}^w & & \downarrow \text{Gap}_{\mathcal{F}}^{\vee} & & \downarrow \text{Gap}_{\mathcal{F}} \\ \text{Gap}^w(\mathcal{F}) & \longrightarrow & \text{Gap}^{\vee}(\mathcal{F}) & \longrightarrow & \text{Gap}(\mathcal{F}) \end{array}$$

By Proposition 4.3,  $K_0(\text{Gap}_{\mathcal{F}}^{\vee})$  is just the direct sum of  $K_0(\text{Gap}_{\mathcal{F}}^w)$  and  $K_0(\text{Gap}_{\mathcal{F}})$ . By Lemma 4.1,  $K_0(\text{Gap}_{\mathcal{F}}^w)$  is an epimorphism. Consider the diagram of two split cofiber sequences

$$\begin{array}{ccccc} K(\text{colim}_n \mathcal{A}^n) & \xrightarrow{K(\tau_{\text{id}, \mathcal{A}})} & K(\text{Gap}^w(\mathcal{A})) & \longrightarrow & K(Q(\text{id}, \mathcal{A})) \\ \downarrow \text{id} & & \downarrow K(\text{Gap}_{\mathcal{F}}^w) & & \downarrow \\ K(\text{colim}_n \mathcal{A}^n) & \xrightarrow{K(\tau_{\mathcal{F}})} & K(\text{Gap}^w(\mathcal{F})) & \longrightarrow & K(Q(\mathcal{F})) \end{array}$$

Then  $K_0(Q(\text{id}, \mathcal{A})) \rightarrow K_0(Q(\mathcal{F}))$  is an epimorphism. Since  $K(Q(\text{id}, \mathcal{A})) \simeq \text{fib}(K(\text{id}, \mathcal{A})) = 0$  by Theorem 3.9,  $K_0(Q(\mathcal{F})) = 0$ .

If  $\mathcal{F}$  satisfies the dévissage condition, by Lemma 4.1,  $K_0(\mathcal{F})$  is an epimorphism. By Theorem 3.9, there is a cofiber sequence

$$K(Q(\mathcal{F})) \rightarrow K(\mathcal{A}) \xrightarrow{K(\mathcal{F})} K(\mathcal{C}).$$

Since  $K_0(Q(\mathcal{F})) = 0$ ,  $K_0(\mathcal{F})$  is a monomorphism. Hence,  $K_0(\mathcal{F})$  is an isomorphism.  $\square$

**Example 4.5.**  $\text{id}_{\mathcal{C}}$  is always 1-fillable:  $\text{Gap}^{\vee}(\text{id}_{\mathcal{C}}) \simeq \text{Gap}^{\vee}(\mathcal{C})$  and  $\text{Gap}_{\text{id}_{\mathcal{C}}}^{\vee} \simeq \text{id}_{\text{Gap}^{\vee}(\mathcal{C})}$  is also the identity map.

$\text{Gap}([n], \text{Gap}([n], \mathcal{C}))$  is a full subcategory of  $\text{Fun}(\text{N}(\text{Ar}[n] \times \text{Ar}[n]), \mathcal{C})$ , whose objects are denoted by  $X_{\bullet, \bullet}^{*, *}$ , where for every  $i, j$ ,  $X_{i,j}^{*, *}, X_{\bullet, \bullet}^{i,j}$  are objects in  $\text{Gap}(\mathcal{C})$ . By [BGT13, Lemma 7.3], an object  $X_{\bullet, \bullet}^{*, *}$  is decided by  $X_{0,0}^{0,0}$ , the 0-the grid. Hence, an object  $X_{\bullet, \bullet}^{*, *}$  in  $\text{Gap}(\text{Gap}(\mathcal{C}))$  has two sides and one vertex at infinity, which is an object in  $\text{Gap}^{\vee}(\mathcal{C})$ . This actually means that, whether we look at it from rows or

columns, it is an object within  $\text{Gap}(\text{Gap}(\mathcal{C}))$ . From this, we stipulate the evaluation functor is in the sense of columns:

$$\text{ev}_{\text{id}_{\text{Gap}(\mathcal{C})}} : \text{Gap}(\text{Gap}(\mathcal{C})) \rightarrow \text{Gap}(\mathcal{C}), X_{\bullet,\bullet}^{\bullet,\bullet} \mapsto \left( 0 \rightarrow \text{colim}_n X_{0,n}^{0,1} \rightarrow \cdots \rightarrow \text{colim}_n X_{0,n}^{0,m} \rightarrow \cdots \right)$$

And, we have a "transpose" functor, which swaps rows and columns:

$$T_{\mathcal{C}} : \text{Gap}(\text{Gap}(\mathcal{C})) \rightarrow \text{Gap}(\text{Gap}(\mathcal{C})), T_{\mathcal{C}}(X_{\bullet,\bullet}^{\bullet,\bullet})_{i,j}^{k,l} = X_{k,l}^{i,j},$$

which induces a transpose evaluation:  $\text{ev}_{\text{id}_{\text{Gap}(\mathcal{C})}}^T := \text{ev}_{\text{id}_{\text{Gap}(\mathcal{C})}} \circ T_{\mathcal{C}}$ .

Objects  $\text{Gap}(\text{Gap}_{\mathcal{F}})$  are three tuples  $(A_{\bullet,\bullet}^{\bullet,\bullet}, X_{\bullet,\bullet}^{\bullet,\bullet}, A_{\bullet,\bullet}^{\bullet,\bullet})$ , where  $A_{\bullet,\bullet}^{\bullet,\bullet}, A_{\bullet,\bullet}^{\bullet,\bullet} \in \text{colim}_n (\text{Gap}(\mathcal{A}))^n$  and  $X_{\bullet,\bullet}^{\bullet,\bullet} \in \text{Gap}(\text{Gap}(\mathcal{C}))$  such that for each  $i, j$ ,  $(A_{i,j}^{\bullet,\bullet}, X_{i,j}^{\bullet,\bullet}), (A_{\bullet,i}^{i,j}, X_{\bullet,i}^{i,j}) \in \text{Gap}(\mathcal{F})$ . We also stipulate the evaluation, transpose and the transpose evaluation functors as follows:

$$\begin{aligned} \text{ev}_{\text{Gap}_{\mathcal{F}}} : \text{Gap}(\text{Gap}_{\mathcal{F}}) &\rightarrow \text{Gap}(\mathcal{F}), (A_{\bullet,\bullet}^{\bullet,\bullet}, X_{\bullet,\bullet}^{\bullet,\bullet}, A_{\bullet,\bullet}^{\bullet,\bullet}) \mapsto \left( \text{colim}_n A_{0,n}^{\bullet}, \text{ev}_{\text{id}_{\text{Gap}(\mathcal{C})}}(X_{\bullet,\bullet}^{\bullet,\bullet}) \right) \\ T_{\mathcal{F}} : \text{Gap}(\text{Gap}_{\mathcal{F}}) &\rightarrow \text{Gap}(\text{Gap}_{\mathcal{F}}), (A_{\bullet,\bullet}^{\bullet,\bullet}, X_{\bullet,\bullet}^{\bullet,\bullet}, A_{\bullet,\bullet}^{\bullet,\bullet}) \mapsto (A_{\bullet,\bullet}^{\bullet,\bullet}, T_{\mathcal{C}}(X_{\bullet,\bullet}^{\bullet,\bullet}), A_{\bullet,\bullet}^{\bullet,\bullet}) \\ \text{ev}_{\text{Gap}_{\mathcal{F}}}^T &:= \text{ev}_{\text{Gap}_{\mathcal{F}}} \circ T_{\mathcal{F}} \end{aligned}$$

Then we can define a "forgetful" functor  $U_{\mathcal{F}} : \text{Gap}(\text{Gap}_{\mathcal{F}}) \rightarrow \text{Gap}^{\vee}(\mathcal{F})$  by  $(\text{ev}_{\text{Gap}_{\mathcal{F}}}^T, \text{ev}_{\text{Gap}_{\mathcal{F}}})$ .

the composition

$$\text{Gap}(\text{Gap}_{\mathcal{F}}) \rightarrow \text{Gap}(\text{Gap}_{\mathcal{F}}^{\vee}) \xrightarrow{\text{ev}_{\text{Gap}_{\mathcal{F}}^{\vee}}} \text{Gap}^{\vee}(\mathcal{F}).$$

**Proposition 4.6.** The following are equivalent:

- (1)  $\mathcal{F}$  is 1-fillable.
- (2)  $U_{\mathcal{F}}$  is essentially surjective.

*Proof.* By Corollary 2.14,  $\mathcal{F}$  is 1-fillable if and only if  $\text{ev}_{\text{Gap}_{\mathcal{F}}^{\vee}}$  is essentially surjective.

$U_{\mathcal{F}}$  is the composition

$$\text{Gap}(\text{Gap}_{\mathcal{F}}) \rightarrow \text{Gap}(\text{Gap}_{\mathcal{F}}^{\vee}) \simeq \text{Gap}^{\vee}(\text{Gap}_{\mathcal{F}}) \xrightarrow{\text{ev}_{\text{Gap}_{\mathcal{F}}^{\vee}}} \text{Gap}^{\vee}(\mathcal{F}),$$

where the first map  $\text{Gap}(\text{Gap}_{\mathcal{F}}) \rightarrow \text{Gap}^{\vee}(\text{Gap}_{\mathcal{F}})$  is defined by  $(T_{\mathcal{F}}, \text{id})$ .

Hence, (2)  $\Rightarrow$  (1) is obvious. Now consider (1)  $\Rightarrow$  (2).

For  $((A_{\bullet,\bullet}, X_{\bullet,\bullet}^{\bullet,\bullet}), (B_{\bullet,\bullet}, Y_{\bullet,\bullet}^{\bullet,\bullet})) \in \text{Gap}^{\vee}(\mathcal{F})$ , assume there is an object

$$((A_{\bullet,\bullet}^{\bullet,\bullet}, X_{\bullet,\bullet}^{\bullet,\bullet}, A_{\bullet,\bullet}^{\bullet,\bullet}), (B_{\bullet,\bullet}^{\bullet,\bullet}, Y_{\bullet,\bullet}^{\bullet,\bullet}, B_{\bullet,\bullet}^{\bullet,\bullet})) \in \text{Gap}([n], \text{Gap}_{\mathcal{F}}^{\vee})$$

such that  $(A_{0,n}^{\bullet}, X_{0,n}^{\bullet,\bullet}) = (A_{\bullet,\bullet}, X_{\bullet,\bullet}^{\bullet,\bullet}), (B_{0,n}^{\bullet}, Y_{0,n}^{\bullet,\bullet}) = (B_{\bullet,\bullet}, Y_{\bullet,\bullet}^{\bullet,\bullet})$ . Also, we can assume  $X_{0,i}^{\bullet,\bullet}, Y_{0,i}^{\bullet,\bullet} \in \text{Gap}([m], \mathcal{F})$

for all  $1 \leq i \leq n$ . It means  $X_{0,i}^{0,m} = Y_{0,i}^{0,m}$  for all  $1 \leq i \leq n$ . Then define  $Z_{\bullet,\bullet}^{\bullet,\bullet} \in \text{Gap}(\text{Gap}(\mathcal{C}))$  by

$$\begin{array}{ccccccccccccc}
& 0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & & \downarrow \\
0 & \longrightarrow & \tilde{X}_{0,1}^{0,1} & \longrightarrow & \cdots & \longrightarrow & \tilde{X}_{0,1}^{0,m} = Y_{0,1}^{0,1} & \longrightarrow & Y_{0,2}^{0,1} & \longrightarrow & \cdots & \longrightarrow & Y_{0,n}^{0,1} \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & & \downarrow \\
\vdots & & \vdots & & \ddots & & \vdots & & \vdots & & \ddots & & & \vdots \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow & & & & & \downarrow \\
0 & \longrightarrow & \tilde{X}_{0,m}^{0,1} = X_{0,1}^{0,1} & \longrightarrow & \cdots & \longrightarrow & \tilde{X}_{0,m}^{0,m} = X_{0,1}^{0,m} & \longrightarrow & X_{0,2}^{0,m} & \longrightarrow & \cdots & \longrightarrow & X_{0,n}^{0,m} \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow & & & & & \downarrow \\
0 & \longrightarrow & X_{0,2}^{0,1} & \longrightarrow & \cdots & \longrightarrow & X_{0,2}^{0,m} & \longrightarrow & X_{0,2}^{0,m} & \longrightarrow & \cdots & \longrightarrow & X_{0,n}^{0,m} \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow & & & & & \downarrow \\
\vdots & & \vdots & & \ddots & & \vdots & & \vdots & & \ddots & & & \vdots \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow & & & & & \downarrow \\
0 & \longrightarrow & X_{0,n}^{0,1} & \longrightarrow & \cdots & \longrightarrow & X_{0,n}^{0,m} & \longrightarrow & X_{0,n}^{0,m} & \longrightarrow & \cdots & \longrightarrow & X_{0,n}^{0,m}
\end{array}$$

where for  $1 \leq i, j \leq m$ ,  $\tilde{X}_{0,j}^{0,i} := X_{0,1}^{0,i} \times_{X_{0,1}^{0,m}} Y_{0,1}^{0,j}$ . Besides, define  $C_{\bullet,\bullet}^{\bullet}, C_{\bullet}^{\bullet,\bullet}$  by:

$$C_{0,j}^i = \begin{cases} A_{0,1}^i, & 1 \leq i, j \leq m \\ B_{0,j-m+1}^i, & 1 \leq i \leq m, m < j \leq m+n-1 \\ A_{j}^{0,i-m+1}, & 1 \leq j \leq m, m < i \leq m+n-1 \\ A_{0,i-m+1}^{j-m+1}, & m < i, j \leq m+n-1 \\ 0, & \text{otherwise} \end{cases}$$

and

$$C_i^{0,j} = \begin{cases} B_i^{0,1}, & 1 \leq i, j \leq m \\ A_i^{0,j-m+1}, & 1 \leq i \leq m, m < j \leq m+n-1 \\ B_{0,i-m+1}^j, & 1 \leq j \leq m, m < i \leq m+n-1 \\ B_{0,i-m+1}^{j-m+1}, & m < i, j \leq m+n-1 \\ 0, & \text{otherwise} \end{cases}$$

The image of  $(C_{\bullet,\bullet}^{\bullet}, X_{\bullet,\bullet}^{\bullet}, C_{\bullet}^{\bullet,\bullet})$  under  $U_{\mathcal{F}}$  is just  $((A_{\bullet}, X_{\bullet,\bullet}), (B_{\bullet}, Y_{\bullet,\bullet}))$ .  $\square$

*Remark 4.7.* The essentially surjection of  $U_{\mathcal{F}}$  just means that given two infinity sides with the same infinity vertex in  $\text{Gap}^{\vee}(\mathcal{F})$ , we can "fill" it into a whole grid in  $\text{Gap}(\text{Gap}_{\mathcal{F}})$ . This is the origin of the name "fillability".

#### 4.2 Higher fillability for higher K-groups

Let  $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{C} \in \text{Cat}_{\infty}^{\text{perf}}$  be an exact functor between small stable idempotent-complete  $\infty$ -categories. Define exact functors  $\mathcal{F}_n : \mathcal{A}_n \rightarrow \mathcal{C}_n$  by

- $\mathcal{C}_1 := \mathcal{C}, \mathcal{A}_1 := \mathcal{A}, \mathcal{F}_1 := \mathcal{F}$ .
- For  $n \geq 2$ ,  $\mathcal{C}_n := \text{Gap}^{\vee}(\mathcal{F}_{n-1}), \mathcal{A}_n := \text{Gap}^{\vee}(\mathcal{A}_{n-1})$  and  $\mathcal{F}_n := \text{Gap}_{\mathcal{F}_{n-1}}^{\vee}$ .

**Definition 4.8.**  $\mathcal{F}$  is called  $n$ -fillable if for every  $1 \leq i \leq n$ ,  $\mathcal{F}_i$  is 1-fifiable.  $\mathcal{F}$  is called fillable for every  $i \geq 1$ ,  $\mathcal{F}_i$  is 1-fifiable.

**Proposition 4.9.** Let  $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{C} \in \text{Cat}_{\infty}^{\text{perf}}$  be an exact functor between small stable idempotent-complete  $\infty$ -categories.

- (1)  $\mathcal{F}$  is  $n$ -fillable if and only if  $\mathcal{F}_i$  satisfies the dévissage condition for all  $2 \leq i \leq n+1$ .
- (2)  $\mathcal{F}$  is fillable if and only if  $\mathcal{F}_n$  is fillable for all  $n \geq 1$ .

*Proof.* Just definition. □

**Lemma 4.10.** Let  $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{C} \in \text{Cat}_{\infty}^{\text{perf}}$  be an exact functor between small stable idempotent-complete  $\infty$ -categories. If  $\mathcal{F}$  is  $n$ -fillable,  $\mathcal{U}_{\text{loc}}(Q(\mathcal{F}_{k+1})) \simeq \Omega \mathcal{U}_{\text{loc}}(Q(\mathcal{F}_k))$  for all  $1 \leq k \leq n$ . As a consequence,  $\mathcal{U}_{\text{loc}}(Q(\mathcal{F}_{k+1})) \simeq \Omega^k \mathcal{U}_{\text{loc}}(Q(\mathcal{F}))$ .

*Proof.* From Proposition 4.9,  $\mathcal{F}_{k+1}$  satisfies the dévissage condition for all  $1 \leq k \leq n$ . By Theorem 3.9, there is a cofiber sequence

$$\mathcal{U}_{\text{loc}}(Q(\mathcal{F}_{k+1})) \simeq \text{fib}(\mathcal{U}_{\text{loc}}(\mathcal{F}_{k+1})).$$

By Proposition 4.3,

$$\mathcal{U}_{\text{loc}}(\mathcal{A}_{k+1}) \simeq \mathcal{U}_{\text{loc}}(\text{Gap}^w(\mathcal{A}_k)) \oplus \mathcal{U}_{\text{loc}}(\text{Gap}(\mathcal{A}_k)), \mathcal{U}_{\text{loc}}(\mathcal{C}_{k+1}) \simeq \mathcal{U}_{\text{loc}}(\text{Gap}^w(\mathcal{F}_k)) \oplus \mathcal{U}_{\text{loc}}(\text{Gap}(\mathcal{F}_k)).$$

Hence,

$$\begin{aligned} & \text{fib}(\mathcal{U}_{\text{loc}}(\mathcal{F}_{k+1})) \\ & \simeq \text{fib}(\mathcal{U}_{\text{loc}}(\text{Gap}_{\mathcal{F}_k}^w)) \oplus \text{fib}(\mathcal{U}_{\text{loc}}(\text{Gap}_{\mathcal{F}_k})). \end{aligned}$$

The first part  $\text{fib}(\mathcal{U}_{\text{loc}}(\text{Gap}_{\mathcal{F}_k}^w))$  is just  $\Omega \mathcal{U}_{\text{loc}}(Q(\mathcal{F}_k))$ . And since  $\mathcal{U}_{\text{loc}}(\text{Gap}(\mathcal{F}_k))$  is an equivalence from Proposition 3.2,  $\text{fib}(\mathcal{U}_{\text{loc}}(\text{Gap}_{\mathcal{F}_k})) \simeq 0$ . Thus,

$$\mathcal{U}_{\text{loc}}(Q(\mathcal{F}_{k+1})) \simeq \Omega \mathcal{U}_{\text{loc}}(Q(\mathcal{F}_k)).$$

□

**Theorem 4.11.** Let  $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{C} \in \text{Cat}_{\infty}^{\text{perf}}$  be an exact functor between small stable idempotent-complete  $\infty$ -categories. If  $\mathcal{F}$  is  $n$ -fillable,  $K_i(Q(\mathcal{F})) = 0$  for all  $0 \leq i \leq n-1$ .

*Proof.* By Theorem 4.4,  $K_0(Q(\mathcal{F}_i)) = 0$  for all  $1 \leq i \leq n$ . By Lemma 4.10,  $K_i(Q(\mathcal{F})) = 0$  for all  $0 \leq i \leq n-1$ . □

**Theorem 4.12.** (Theorem A) Let  $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{C} \in \text{Cat}_{\infty}^{\text{perf}}$  be a  $n$ -fillable exact functor between small stable idempotent-complete  $\infty$ -categories. If  $\mathcal{F}$  satisfies the weak dévissage condition (resp. dévissage condition), it induces isomorphisms  $K_i(\mathcal{A}) \xrightarrow{\sim} K_i(\mathcal{C})$  for all  $1 \leq i \leq n-1$ , an epimorphism  $K_n(\mathcal{A}) \twoheadrightarrow K_n(\mathcal{C})$  and a monomorphism (resp. an isomorphism)  $K_0(\mathcal{A}) \hookrightarrow K_0(\mathcal{C})$ .

*Proof.* Just from Theorem 4.11 and Theorem 3.9. □

**Corollary 4.13.** Let  $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{C} \in \text{Cat}_{\infty}^{\text{perf}}$  be a fillable exact functor between small stable idempotent-complete  $\infty$ -categories. If  $\mathcal{F}$  satisfies the weak dévissage condition (resp. dévissage condition), it induces isomorphisms  $K_n(\mathcal{A}) \xrightarrow{\sim} K_n(\mathcal{C})$  for all  $n \geq 1$ , and a monomorphism (resp. an isomorphism)  $K_0(\mathcal{A}) \hookrightarrow K_0(\mathcal{C})$ .

**Definition 4.14.** Let  $\mathcal{C}$  be a small stable  $\infty$ -category.  $\mathcal{C}$  is called  $n$ -fillable if the delta map  $\delta_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}^2$  is  $n$ -fillable.  $\mathcal{C}$  is called fillable if  $\delta_{\mathcal{C}}$  is fillable.

**Theorem 4.15.** (Theorem B) Let  $\mathcal{C}$  be a small idempotent-complete  $n$ -fillable  $\infty$ -category. Then  $K_i(\mathcal{C}) = 0$  for all  $1 \leq i \leq n$ .

*Proof.* By Theorem 4.12 and Lemma 3.13,  $K_i(\mathcal{C}) \rightarrow K_i(\mathcal{C}^2)$  are surjective for all  $1 \leq i \leq n$ , which forces them to be trivial. □

**Corollary 4.16.** Let  $\mathcal{C}$  be a small idempotent-complete fillable  $\infty$ -category. Then  $K_n(\mathcal{C}) = 0$  for all  $n \geq 1$ .

### 4.3 Strongly 1-fillability

Let  $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{C} \in \text{Cat}_{\infty}^{\text{perf}}$  be an exact functor between small stable idempotent-complete  $\infty$ -categories.

**Lemma 4.17.** *Let  $J$  be a finite  $\infty$ -category.  $\text{Fun}(J, -) : \text{Cat}_{\infty}^{\text{ex}} \rightarrow \text{Cat}_{\infty}^{\text{ex}}$  satisfies:*

- (1) *It preserves all filtered colimits.*
- (2) *If  $\mathcal{C}$  is a full subcategory of  $\mathcal{D}$ , then  $\text{Fun}(J, \mathcal{C})$  is a full subcategory of  $\text{Fun}(J, \mathcal{D})$ . Besides,  $\text{Fun}(J, \mathcal{D}/\mathcal{C}) \simeq \text{Fun}(J, \mathcal{D})/\text{Fun}(J, \mathcal{C})$ .*

*Proof.*

- (1) It is because  $J$  is a compact object in  $\text{Cat}_{\infty}$  and the inclusion  $\text{Cat}_{\infty}^{\text{ex}} \rightarrow \text{Cat}_{\infty}$  preserves filtered colimits [Lur17, Proposition 1.1.4.6].
- (2) Mapping spaces in a functor category are computed as limits of objectwise mapping spaces:

$$\text{Map}_{\text{Fun}(J, \mathcal{D})}(F, G) \simeq \lim_{j \in J} \text{Map}_{\mathcal{D}}(F(j), G(j)).$$

If  $F, G$  land in  $\mathcal{C}$  and  $\mathcal{C} \subseteq \mathcal{D}$  is fully faithful, then

$$\text{Map}_{\mathcal{C}}(F(j), G(j)) \xrightarrow{\sim} \text{Map}_{\mathcal{D}}(F(j), G(j))$$

is an equivalence for each  $j$ , hence taking the finite limit over  $j \in J$  gives

$$\text{Map}_{\text{Fun}(J, \mathcal{C})}(F, G) \simeq \text{Map}_{\text{Fun}(J, \mathcal{D})}(F, G).$$

So the inclusion is fully faithful.

The map  $\mathcal{D} \rightarrow \mathcal{D}/\mathcal{C}$  induces a map

$$\Phi : \text{Fun}(J, \mathcal{D}) \rightarrow \text{Fun}(J, \mathcal{D}/\mathcal{C}).$$

One can check that  $\Phi(\text{Fun}(J, \mathcal{C})) = \{0\}$ . Hence, it induces a functor

$$\tilde{\Phi} : \text{Fun}(J, \mathcal{D})/\text{Fun}(J, \mathcal{C}) \rightarrow \text{Fun}(J, \mathcal{D}/\mathcal{C}).$$

It is essentially surjective since  $\Phi$  is. Denote

$$W := \{\eta : F \rightarrow F' \mid \text{the cofiber cofib}(\eta) \in \text{Fun}(J, \mathcal{C})\}$$

For  $F, G \in \text{Fun}(J, \mathcal{D})$ ,

$$\begin{aligned} \text{Map}_{\text{Fun}(J, \mathcal{D})/\text{Fun}(\Delta^n, \mathcal{C})}(F, G) &\simeq \text{colim}_{\left(F' \xrightarrow{\eta} F\right) \in W/F} \left( \lim_{j \in J} \text{Map}_{\mathcal{D}}(F'(j), G(j)) \right) \\ &\simeq \lim_{j \in J} \left( \text{colim}_{\left(F' \xrightarrow{\eta} F\right) \in W/F} \text{Map}_{\mathcal{D}}(F'(j), G(j)) \right) \\ &\simeq \lim_{j \in J} \text{Map}_{\mathcal{D}/\mathcal{C}}(\tilde{\Phi}(F)(j), \tilde{\Phi}(G)(j)) \\ &\simeq \text{Map}_{\text{Fun}(J, \mathcal{D}/\mathcal{C})}(\tilde{\Phi}(F), \tilde{\Phi}(G)) \end{aligned}$$

It means  $\tilde{\Phi}$  is fully faithful; hence, it is an equivalence. Here, we use that finite limits commute with filtered colimits in the  $\infty$ -category of Kan complexes.

□

**Lemma 4.18.** *If  $\mathcal{C}$  is a full subcategory of  $\mathcal{D}$ , then  $\text{Gap}^w(\mathcal{C})$  is a full subcategory of  $\text{Gap}^w(\mathcal{D})$  and  $\text{Gap}^w(\mathcal{D})/\text{Gap}^w(\mathcal{C}) \rightarrow \text{Gap}^w(\mathcal{D}/\mathcal{C})$  is an equivalence.*

*Proof.* Notice that  $\text{Gap}^w \simeq \text{colim}_n \text{Fun}(\Delta^{n-2}, -)$ . This is from Lemma 4.17 naturally. □

**Lemma 4.19.**  $\text{BGap}^w : \text{Cat}_{\infty}^{\text{ex}} \rightarrow \text{Cat}_{\infty}^{\text{ex}}, \mathcal{C} \mapsto \text{BGap}^w(\mathcal{C})$  is left exact.

*Proof.* Since  $\text{BGap}^w(*) \simeq *$ , from Lemma 1.12, we only need to verify it preserves pullbacks. Notice that  $\text{Gap}^w \simeq \text{colim}_n \text{Fun}(\Delta^{n-2}, -)$  is left exact. For a pullback  $\mathcal{A} \simeq \mathcal{B} \times_{\mathcal{D}} \mathcal{C}$  in  $\text{Cat}_{\infty}^{\text{ex}}$ , we have

$$\begin{aligned}\text{BGap}^w(\mathcal{A}) &\simeq \text{Gap}^w(\mathcal{A}) \times_{\text{colim}_n \mathcal{A}^n} \text{Gap}^w(\mathcal{A}) \\ &\simeq \text{Gap}^w(\mathcal{B} \times_{\mathcal{D}} \mathcal{C}) \times_{\text{colim}_n (\mathcal{B} \times_{\mathcal{D}} \mathcal{C})^n} \text{Gap}^w(\mathcal{B} \times_{\mathcal{D}} \mathcal{C}) \\ &\simeq \left( \text{Gap}^w(\mathcal{B}) \times_{\text{Gap}^w(\mathcal{D})} \text{Gap}^w(\mathcal{C}) \right) \times_{\text{colim}_n (\mathcal{B} \times_{\mathcal{D}} \mathcal{C})^n} \left( \text{Gap}^w(\mathcal{B}) \times_{\text{Gap}^w(\mathcal{D})} \text{Gap}^w(\mathcal{C}) \right) \\ &\simeq \text{BGap}^w(\mathcal{B}) \times_{\text{BGap}^w(\mathcal{D})} \text{BGap}^w(\mathcal{C})\end{aligned}$$

□

**Lemma 4.20.** *Let  $J$  be a finite  $\infty$ -category. Then  $\text{Fun}(J, -) \circ \text{BGap}^w \simeq \text{BGap}^w \circ \text{Fun}(J, -)$ .*

*Proof.* Notice that  $\text{Gap}^w([n], -) \simeq \text{Fun}(\Delta^{n-2}, -)$ . We have  $\text{Fun}(J, -) \circ \text{Gap}^w([n], -) \simeq \text{Gap}^w([n], -) \circ \text{Fun}(J, -)$ . Since  $\text{Fun}(J, -)$  is left exact, we have

$$\begin{aligned}\text{Fun}(J, \text{BGap}^w([n], \mathcal{C})) &\simeq \text{Fun}(J, \text{Gap}^w([n], \mathcal{C})) \times_{\text{Fun}(J, \mathcal{C}^n)} \text{Fun}(J, \text{Gap}^w([n], \mathcal{C})) \\ &\simeq \text{Gap}^w([n], \text{Fun}(J, \mathcal{C})) \times_{\text{Fun}(J, \mathcal{C})^n} \text{Gap}^w([n], \text{Fun}(J, \mathcal{C})) \\ &\simeq \text{BGap}^w([n], \text{Fun}(J, \mathcal{C}))\end{aligned}$$

Hence,  $\text{Fun}(J, -) \circ \text{BGap}^w([n], -) \simeq \text{BGap}^w([n], -) \circ \text{Fun}(J, -)$ . Since  $\text{Fun}(J, -)$  preserves filtered colimits by Lemma 4.17, we have  $\text{Fun}(J, -) \circ \text{BGap}^w \simeq \text{BGap}^w \circ \text{Fun}(J, -)$ . □

**Lemma 4.21.** *If  $\mathcal{C}$  is a full subcategory of  $\mathcal{D}$ , then  $\text{BGap}^w(\mathcal{C})$  is a full subcategory of  $\text{BGap}^w(\mathcal{D})$  and  $\text{BGap}^w(\mathcal{D})/\text{BGap}^w(\mathcal{C}) \rightarrow \text{BGap}^w(\mathcal{D}/\mathcal{C})$  is fully faithful. Furthermore, if  $K_0^{\text{cn}}(\mathcal{C}) \rightarrow K_0^{\text{cn}}(\mathcal{D})$  is injective,  $\text{BGap}^w(\mathcal{D})/\text{BGap}^w(\mathcal{C}) \rightarrow \text{BGap}^w(\mathcal{D}/\mathcal{C})$  is an equivalence.*

*Proof.* From Lemma 4.17, we know  $\text{Gap}^w(\mathcal{C})$  is a full subcategory of  $\text{Gap}^w(\mathcal{D})$ . For two objects  $(X_{\bullet, \bullet}, Y_{\bullet, \bullet}), (X'_{\bullet, \bullet}, Y'_{\bullet, \bullet}) \in \text{BGap}^w(\mathcal{C})$ , we have

$$\begin{aligned}\text{Map}_{\text{BGap}^w(\mathcal{C})}((X_{\bullet, \bullet}, Y_{\bullet, \bullet}), (X'_{\bullet, \bullet}, Y'_{\bullet, \bullet})) &\simeq \text{Map}_{\text{Gap}^w(\mathcal{C})}(X_{\bullet, \bullet}, X'_{\bullet, \bullet}) \times_{\text{Map}_{\text{colim}_n \mathcal{C}^n}(X_{\bullet, \bullet+1}, X'_{\bullet, \bullet+1})} \text{Map}_{\text{Gap}^w(\mathcal{C})}(Y_{\bullet, \bullet}, Y'_{\bullet, \bullet}) \\ &\simeq \text{Map}_{\text{Gap}^w(\mathcal{D})}(X_{\bullet, \bullet}, X'_{\bullet, \bullet}) \times_{\text{Map}_{\text{colim}_n \mathcal{D}^n}(X_{\bullet, \bullet+1}, X'_{\bullet, \bullet+1})} \text{Map}_{\text{Gap}^w(\mathcal{D})}(Y_{\bullet, \bullet}, Y'_{\bullet, \bullet}) \\ &\simeq \text{Map}_{\text{BGap}^w(\mathcal{D})}((X_{\bullet, \bullet}, Y_{\bullet, \bullet}), (X'_{\bullet, \bullet}, Y'_{\bullet, \bullet}))\end{aligned}$$

Hence,  $\text{BGap}^w(\mathcal{C})$  is a full subcategory of  $\text{BGap}^w(\mathcal{D})$ . Now, the functor

$$\Phi : \text{BGap}^w(\mathcal{D}) \rightarrow \text{BGap}^w(\mathcal{D}/\mathcal{C})$$

induces a functor

$$\tilde{\Phi} : \text{BGap}^w(\mathcal{D})/\text{BGap}^w(\mathcal{C}) \rightarrow \text{BGap}^w(\mathcal{D}/\mathcal{C})$$

We need to prove it is an equivalence.

Denote by  $W$  the collection of morphisms in  $\text{BGap}^w(\mathcal{D})$  whose cofibers are in  $\text{BGap}^w(\mathcal{C})$ . For two objects  $(X_{\bullet, \bullet}, Y_{\bullet, \bullet}), (X'_{\bullet, \bullet}, Y'_{\bullet, \bullet}) \in \text{BGap}^w(\mathcal{D})$ ,

$$\begin{aligned}\text{Map}_{\text{BGap}^w(\mathcal{D})/\text{BGap}^w(\mathcal{C})}((X_{\bullet, \bullet}, Y_{\bullet, \bullet}), (X'_{\bullet, \bullet}, Y'_{\bullet, \bullet})) \\ \simeq \text{colim}_{((X''_{\bullet, \bullet}, Y''_{\bullet, \bullet}) \rightarrow (X_{\bullet, \bullet}, Y_{\bullet, \bullet})) \in W/(X_{\bullet, \bullet}, Y_{\bullet, \bullet})} \text{Map}_{\text{BGap}^w(\mathcal{D})}((X''_{\bullet, \bullet}, Y''_{\bullet, \bullet}), (X'_{\bullet, \bullet}, Y'_{\bullet, \bullet})).\end{aligned}$$

Notice that

$$\text{Map}_{\text{BGap}^w(\mathcal{D})}((X''_{\bullet, \bullet}, Y''_{\bullet, \bullet}), (X'_{\bullet, \bullet}, Y'_{\bullet, \bullet})) \simeq \text{Map}_{\text{Gap}^w(\mathcal{D})}(X''_{\bullet, \bullet}, X'_{\bullet, \bullet}) \times_{\text{Map}_{\text{colim}_n \mathcal{D}^n}(X''_{\bullet, \bullet+1}, X'_{\bullet, \bullet+1})} \text{Map}_{\text{Gap}^w(\mathcal{D})}(Y''_{\bullet, \bullet}, Y'_{\bullet, \bullet})$$

From Lemma 4.18, we know  $\tilde{\phi} : \text{Gap}^w(\mathcal{D})/\text{Gap}^w(\mathcal{C}) \rightarrow \text{Gap}^w(\mathcal{D}/\mathcal{C})$  is an equivalence. Hence,

$$\text{colim}_{((X''_{\bullet, \bullet}, Y''_{\bullet, \bullet}) \rightarrow (X_{\bullet, \bullet}, Y_{\bullet, \bullet})) \in W/(X_{\bullet, \bullet}, Y_{\bullet, \bullet})} \text{Map}_{\text{Gap}^w(\mathcal{D})}(X''_{\bullet, \bullet}, X'_{\bullet, \bullet}) \simeq \text{Map}_{\text{Gap}^w(\mathcal{D}/\mathcal{C})}(\tilde{\phi}(X_{\bullet, \bullet}), \tilde{\phi}(X'_{\bullet, \bullet})),$$

$$\text{colim}_{((X''_{\bullet, \bullet}, Y''_{\bullet, \bullet}) \rightarrow (X_{\bullet, \bullet}, Y_{\bullet, \bullet})) \in W/(X_{\bullet, \bullet}, Y_{\bullet, \bullet})} \text{Map}_{\text{Gap}^w(\mathcal{D})}(Y''_{\bullet, \bullet}, Y'_{\bullet, \bullet}) \simeq \text{Map}_{\text{Gap}^w(\mathcal{D}/\mathcal{C})}(\tilde{\phi}(Y_{\bullet, \bullet}), \tilde{\phi}(Y'_{\bullet, \bullet})),$$

$$\text{colim}_{((X''_{\bullet, \bullet}, Y''_{\bullet, \bullet}) \rightarrow (X_{\bullet, \bullet}, Y_{\bullet, \bullet})) \in W/(X_{\bullet, \bullet}, Y_{\bullet, \bullet})} \text{Map}_{\text{colim}_n \mathcal{D}^n}(X''_{\bullet, \bullet+1}, X'_{\bullet, \bullet+1}) \simeq \text{Map}_{\text{colim}_n (\mathcal{D}/\mathcal{C})^n}(X_{\bullet, \bullet+1}, X'_{\bullet, \bullet+1})$$

It means

$$\begin{aligned} & \text{Map}_{\text{BGap}^w(\mathcal{D})/\text{BGap}^w(\mathcal{C})}((X_{\bullet,\bullet}, Y_{\bullet,\bullet}), (X'_{\bullet,\bullet}, Y'_{\bullet,\bullet})) \\ & \simeq \text{Map}_{\text{Gap}^w(\mathcal{D}/\mathcal{C})}(\tilde{\phi}(X_{\bullet,\bullet}), \tilde{\phi}(X'_{\bullet,\bullet})) \times_{\text{Map}_{\text{colim}_n(\mathcal{D}/\mathcal{C})^n}(X_{\bullet,\bullet+1}, X'_{\bullet,\bullet+1})} \text{Map}_{\text{Gap}^w(\mathcal{D}/\mathcal{C})}(\tilde{\phi}(Y_{\bullet,\bullet}), \tilde{\phi}(Y'_{\bullet,\bullet})), \\ & \simeq \text{Map}_{\text{BGap}^w(\mathcal{D}/\mathcal{C})}(\tilde{\Phi}(X_{\bullet,\bullet}, Y_{\bullet,\bullet}), \tilde{\Phi}(X'_{\bullet,\bullet}, Y'_{\bullet,\bullet})) \end{aligned}$$

i.e.  $\tilde{\Phi}$  is fully faithful.

Now we assume  $K_0^{\text{cn}}(\mathcal{C}) \rightarrow K_0^{\text{cn}}(\mathcal{D})$  is injective. Recall that  $\phi : \text{Gap}^w(\mathcal{D}) \rightarrow \text{Gap}^w(\mathcal{D}/\mathcal{C})$  is essentially surjective. For  $(F_{\bullet,\bullet}, G_{\bullet,\bullet}) \in \text{BGap}^w(\mathcal{D}/\mathcal{C})$ , there are  $X_{\bullet,\bullet}, Y_{\bullet,\bullet} \in \text{Gap}^w(\mathcal{D})$  such that  $\phi(X_{\bullet,\bullet}) \simeq F_{\bullet,\bullet}, \phi(Y_{\bullet,\bullet}) \simeq G_{\bullet,\bullet}$ .

Assume  $X_{0,i} = Y_{0,i} = 0$  for  $i > n$ . Define two sequences

$$\begin{aligned} 0 \rightarrow X'_{0,1} & \rightarrow \cdots \rightarrow X'_{0,n} \rightarrow X'_{0,n+1} \\ 0 \rightarrow Y'_{0,1} & \rightarrow \cdots \rightarrow Y'_{0,n} \rightarrow Y'_{0,n+1} \end{aligned}$$

as follows:

- Since  $X_{0,1} \simeq Y_{0,1}$  in  $\mathcal{D}/\mathcal{C}$ , there are two morphisms  $Z_1 \rightarrow X_{0,1}, Z_1 \rightarrow Y_{0,1}$  whose cofibers are in  $\mathcal{C}$ . Define  $X'_{0,1} = Y'_{0,1} := Z_{0,1}$ .
- Consider the map  $Z_1 \rightarrow X_{0,1} \rightarrow X_{0,2}$ . Since  $\text{cofib}(Z_1 \rightarrow X_{0,1}) \in \mathcal{C}$ ,  $\text{cofib}(Z_1 \rightarrow X_{0,2}) \simeq X_{1,2}$  in  $\mathcal{D}/\mathcal{C}$ . Hence,

$$\text{cofib}(Z_1 \rightarrow X_{0,2}) \simeq X_{1,2} \simeq Y_{1,2} \simeq \text{cofib}(Z_1 \rightarrow Y_{0,2}) \in \mathcal{D}/\mathcal{C}.$$

There are two morphisms  $Z_2 \rightarrow \text{cofib}(Z_1 \rightarrow X_{0,2}), Z_2 \rightarrow \text{cofib}(Z_1 \rightarrow Y_{0,2})$  whose cofibers are in  $\mathcal{C}$ . Define  $X'_{0,2} := Z_2 \times_{\text{cofib}(Z_1 \rightarrow X_{0,2})} X_{0,2}$  and  $Y'_{0,2} := Z_2 \times_{\text{cofib}(Z_1 \rightarrow Y_{0,2})} Y_{0,2}$ . Then by Lemma 1.6,

$$\begin{aligned} \text{fib}(X'_{0,2} \rightarrow Z_2) & \simeq \text{fib}(X_{0,2} \rightarrow \text{cofib}(Z_1 \rightarrow X_{0,2})) \simeq Z_1 \\ \text{fib}(Y'_{0,2} \rightarrow Z_2) & \simeq \text{fib}(Y_{0,2} \rightarrow \text{cofib}(Z_1 \rightarrow Y_{0,2})) \simeq Z_1 \end{aligned}$$

Then we get  $X'_{0,1} = Z_1 \rightarrow X'_{0,2}$  and  $Y'_{0,1} = Z_1 \rightarrow Y'_{0,2}$  whose cofibers are both  $Z_2$ .

- For  $2 \leq i \leq n$ , consider the map  $X'_{0,i} \rightarrow X_{0,i} \rightarrow X_{0,i+1}$ . By Lemma 1.6,  $\text{cofib}(X'_{0,i} \rightarrow X_{0,i}) \simeq \text{cofib}(Z_i \rightarrow \text{cofib}(X'_{0,i-1} \rightarrow X_{0,i}))$  is in  $\mathcal{C}$ . We have

$$\text{cofib}(X'_{0,i} \rightarrow X_{0,i+1}) \simeq X_{i,i+1} \simeq Y_{i,i+1} \simeq \text{cofib}(Y'_{0,i} \rightarrow Y_{0,i+1}) \in \mathcal{D}/\mathcal{C}.$$

There are two morphisms  $Z_{i+1} \rightarrow \text{cofib}(X'_{0,i} \rightarrow X_{0,i+1}), Z_{i+1} \rightarrow \text{cofib}(Y'_{0,i} \rightarrow Y_{0,i+1})$  whose cofibers are in  $\mathcal{C}$ . Define  $X'_{0,i+1} := Z_{i+1} \times_{\text{cofib}(X'_{0,i} \rightarrow X_{0,i+1})} X_{0,i+1}$  and  $Y'_{0,i+1} := Z_{i+1} \times_{\text{cofib}(Y'_{0,i} \rightarrow Y_{0,i+1})} Y_{0,i+1}$ . Then by Lemma 1.6,

$$\begin{aligned} \text{fib}(X'_{0,i+1} \rightarrow Z_{i+1}) & \simeq \text{fib}(X_{0,i+1} \rightarrow \text{cofib}(X'_{0,i} \rightarrow X_{0,i+1})) \simeq X'_{0,i} \\ \text{fib}(Y'_{0,i+1} \rightarrow Z_{i+1}) & \simeq \text{fib}(Y_{0,i+1} \rightarrow \text{cofib}(Y'_{0,i} \rightarrow Y_{0,i+1})) \simeq Y'_{0,i} \end{aligned}$$

Then we get  $X'_{0,i} \rightarrow X'_{0,i+1}$  and  $Y'_{0,i} \rightarrow Y'_{0,i+1}$  whose cofibers are both  $Z_{i+1}$ .

The same as above,  $\text{cofib}(X'_{0,n+1} \rightarrow X_{0,n+1} = 0)$  is in  $\mathcal{C}$ . Hence,  $X'_{0,n+1} \in \mathcal{C}$ . Also,  $Y'_{0,n+1} \in \mathcal{C}$ .

Since  $[X'_{0,n+1}] = [Z_1] + [Z_2] + \cdots + [Z_{n+1}] = [Y'_{0,n+1}]$  in  $K_0^{\text{cn}}(\mathcal{D})$ , we also have  $[X'_{0,n+1}] = [Y'_{0,n+1}]$  in  $K_0^{\text{cn}}(\mathcal{C})$  since  $K_0^{\text{cn}}(\mathcal{C}) \rightarrow K_0^{\text{cn}}(\mathcal{D})$  is injective. Hence, there are two factorizations of  $X'_{0,n+1} \rightarrow 0$  and  $Y'_{0,n+1} \rightarrow 0$  in  $\mathcal{C}$

$$\begin{aligned} X'_{0,n+1} & \rightarrow X'_{0,n+2} \rightarrow \cdots \rightarrow X'_{0,n+m} = 0 \\ Y'_{0,n+1} & \rightarrow Y'_{0,n+2} \rightarrow \cdots \rightarrow Y'_{0,n+m} = 0 \end{aligned}$$

such that  $\text{cofib}(X'_{0,n+i} \rightarrow X'_{0,n+i+1}) \simeq \text{cofib}(Y'_{0,n+i} \rightarrow Y'_{0,n+i+1})$  for some  $m > 0$ .

Now we get two objects in  $\text{Gap}^w(\mathcal{D})$ :

$$\begin{aligned} 0 \rightarrow X'_{0,1} & \rightarrow \cdots \rightarrow X'_{0,n} \rightarrow X'_{0,n+1} \rightarrow \cdots \rightarrow X'_{0,n+m} \rightarrow 0 \rightarrow \cdots \\ 0 \rightarrow Y'_{0,1} & \rightarrow \cdots \rightarrow Y'_{0,n} \rightarrow Y'_{0,n+1} \rightarrow \cdots \rightarrow Y'_{0,n+m} \rightarrow 0 \rightarrow \cdots \end{aligned}$$

Above discussion tells us  $(X'_{\bullet,\bullet}, Y'_{\bullet,\bullet}) \in \text{BGap}^w(\mathcal{D})$ .

There are natural morphisms  $X'_{0,i} \rightarrow X_{0,i}$  (when  $i > n$ , they are just  $X'_{0,i} \rightarrow 0$ ), and their cofibers are in  $\mathcal{C}$ . The same for  $Y'_{0,i} \rightarrow Y_{0,i}$ . They give morphisms

$$X'_{\bullet,\bullet} \rightarrow X_{\bullet,\bullet}, Y'_{\bullet,\bullet} \rightarrow Y_{\bullet,\bullet}$$

whose cofibers are in  $\text{Gap}^w(\mathcal{C})$ . Hence,  $\phi(X'_{\bullet,\bullet}) \simeq \phi(X_{\bullet,\bullet}) \simeq F_{\bullet,\bullet}$  and  $\phi(Y'_{\bullet,\bullet}) \simeq \phi(Y_{\bullet,\bullet}) \simeq G_{\bullet,\bullet}$ . Thus,

$$\Phi(X'_{\bullet,\bullet}, Y'_{\bullet,\bullet}) \simeq (\phi(X'_{\bullet,\bullet}), \phi(Y'_{\bullet,\bullet})) \simeq (F_{\bullet,\bullet}, G_{\bullet,\bullet}).$$

It tells us  $\Phi$  is essential surjective, i.e. so is  $\tilde{\Phi}$ .

In conclusion,  $\tilde{\Phi}$  is an equivalence.  $\square$

**Lemma 4.22.** *Let  $\mathcal{F}: \mathcal{A} \rightarrow \mathcal{C} \in \text{Cat}_{\infty}^{\text{perf}}$  be an exact functor between small stable idempotent-complete  $\infty$ -categories. Then*

$$\text{Gap}([n], \text{BGap}^w(\mathcal{F})) \simeq \text{BGap}^w(\text{Gap}([n], \mathcal{F}))$$

$$\text{Gap}^w([n], \text{BGap}^w(\mathcal{F})) \simeq \text{BGap}^w(\text{Gap}^w([n], \mathcal{F}))'$$

where  $\text{BGap}^w(\mathcal{F})$  is the functor  $\text{BGap}^w(\mathcal{A}) \rightarrow \text{BGap}^w(\mathcal{C})$  induced by  $\mathcal{C}$ .

*Proof.* Recall  $\text{BGap}^w$  is left exact from Lemma 4.19. Then

$$\begin{aligned} \text{BGap}^w(\text{Gap}([n], \mathcal{F})) &\simeq \text{BGap}^w(\text{Gap}([n], \mathcal{C}) \times_{\mathcal{C}^n} \mathcal{A}^n) \\ &\simeq \text{BGap}^w(\text{Gap}([n], \mathcal{C})) \times_{\text{BGap}^w(\mathcal{C})^n} \text{BGap}^w(\mathcal{A})^n \end{aligned}$$

From Lemma 4.20,  $\text{BGap}^w(\text{Gap}([n], \mathcal{C})) \simeq \text{Gap}([n], \text{BGap}^w(\mathcal{C}))$ . Hence,

$$\begin{aligned} \text{BGap}^w(\text{Gap}([n], \mathcal{F})) &\simeq \text{Gap}([n], \text{BGap}^w(\mathcal{C})) \times_{\text{BGap}^w(\mathcal{C})^n} \text{BGap}^w(\mathcal{A})^n \\ &\simeq \text{Gap}([n], \text{BGap}^w(\mathcal{F})). \end{aligned}$$

Similar, we have

$$\text{Gap}^w([n], \text{BGap}^w(\mathcal{F})) \simeq \text{BGap}^w(\text{Gap}^w([n], \mathcal{F})).$$

$\square$

**Lemma 4.23.** *Let  $\mathcal{F}: \mathcal{A} \rightarrow \mathcal{C} \in \text{Cat}_{\infty}^{\text{perf}}$  be an exact functor between small stable idempotent-complete  $\infty$ -categories. Then  $K_0(\text{Gap}^w(\mathcal{F})) \rightarrow K_0(\text{Gap}(\mathcal{F}))$  is injective if  $K_0(Q(\mathcal{F})) = 0$ .*

*Proof.* By Proposition 3.2, Theorem 3.9 and the split cofiber sequence

$$\mathcal{U}_{\text{loc}}(\text{colim}_n \mathcal{A}^{n-1}) \xrightarrow{\mathcal{U}_{\text{loc}}(\tau_{\mathcal{A}})} \mathcal{U}_{\text{loc}}(\text{Gap}^w(\mathcal{F})) \rightarrow \mathcal{U}_{\text{loc}}(Q(\mathcal{F})),$$

we know  $K_0(\text{Gap}^w(\mathcal{F})) \rightarrow K_0(\text{Gap}(\mathcal{F}))$  is just

$$\text{colim}_n K_0(\mathcal{A})^{n-1} \oplus K_0(Q(\mathcal{F})) \rightarrow \text{colim}_n K_0(\mathcal{A})^{n-1} \oplus K_0(\mathcal{A}).$$

Hence, if  $K_0(Q(\mathcal{F})) = 0$ ,  $K_0(\text{Gap}^w(\mathcal{F})) \rightarrow K_0(\text{Gap}(\mathcal{F}))$  is injective.  $\square$

**Proposition 4.24.** Let  $\mathcal{F}: \mathcal{A} \rightarrow \mathcal{C} \in \text{Cat}_{\infty}^{\text{perf}}$  be an exact functor between small stable idempotent-complete  $\infty$ -categories. If  $\mathcal{F}$  satisfies the dévissage condition and  $K_0(Q(\mathcal{F})) = 0$ ,  $\text{BGap}^w(\mathcal{F})$  satisfies the dévissage condition.

*Proof.* From Proposition 2.9, we have  $E(\mathcal{F}) = \mathcal{C}$  and we only need to prove

$$E(\text{BGap}^w(\mathcal{F})) \simeq \text{BGap}^w(E(\mathcal{F})).$$

By Lemma 4.21 and Lemma 4.22, we have

$$\begin{aligned} E(\text{BGap}^w(\mathcal{F})) &\simeq \text{Gap}(\text{BGap}^w(\mathcal{F}))/\text{Gap}^w(\text{BGap}^w(\mathcal{F})) \\ &\simeq \text{BGap}^w(\text{Gap}(\mathcal{F}))/\text{BGap}^w(\text{Gap}^w(\mathcal{F})) \end{aligned}$$

Since  $\text{BGap}^w(\text{Gap}(\mathcal{F})/\text{Gap}^w(\mathcal{F})) \simeq \text{BGap}^w(\text{E}(\mathcal{F}))$ , to prove  $\text{E}(\text{BGap}^w(\mathcal{F})) \simeq \text{BGap}^w(\text{E}(\mathcal{F}))$ , we need to prove

$$\text{BGap}^w(\text{Gap}(\mathcal{F}))/\text{BGap}^w(\text{Gap}^w(\mathcal{F})) \rightarrow \text{BGap}^w(\text{Gap}(\mathcal{F})/\text{Gap}^w(\mathcal{F}))$$

is an equivalence. From Lemma 4.21 and Lemma 4.23, we just need  $K_0(\text{Q}(\mathcal{F})) = 0$ .  $\square$

Proposition 4.24 and Theorem 4.4 tells us:

**Corollary 4.25.** Let  $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{C} \in \text{Cat}_{\infty}^{\text{perf}}$  be a 1-fifiable exact functor between small stable idempotent-complete  $\infty$ -categories. If  $\mathcal{F}$  satisfies the dévissage condition,  $\text{BGap}(\mathcal{F})$  satisfies the dévissage condition.

**Lemma 4.26.**

$$\text{BGap}^w(\text{Gap}^{\vee}(\mathcal{F})) \simeq \text{Gap}^{\vee}(\text{BGap}^w(\mathcal{F})).$$

*Proof.* By Lemma 4.19 and Lemma 4.22,

$$\begin{aligned} \text{BGap}^w(\text{Gap}^{\vee}(\mathcal{F})) &\simeq \text{BGap}^w(\text{Gap}(\mathcal{F})) \times_{\text{BGap}^w(\mathcal{C})} \text{BGap}^w(\text{Gap}(\mathcal{F})) \\ &\simeq \text{Gap}(\text{BGap}^w(\mathcal{F})) \times_{\text{BGap}^w(\mathcal{C})} \text{Gap}(\text{BGap}^w(\mathcal{F})) \\ &\simeq \text{Gap}^{\vee}(\text{BGap}^w(\mathcal{F})). \end{aligned}$$

$\square$

**Definition 4.27.** Let  $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{C} \in \text{Cat}_{\infty}^{\text{perf}}$  be an exact functor between small stable idempotent-complete  $\infty$ -categories.  $\mathcal{F}$  is called *strongly 1-fifiable* if there is an exact functor  $\text{fill} : \text{Gap}^{\vee}(\mathcal{F}) \rightarrow \text{Gap}(\text{Gap}_{\mathcal{F}}^{\vee})$  such that  $\text{ev}_{\text{Gap}_{\mathcal{F}}^{\vee}} \circ \text{fill} \simeq \text{id}$ .

**Remark 4.28.** Notice that  $\mathcal{F}$  is 1-fifiable if and only if  $\text{ev}_{\text{Gap}_{\mathcal{F}}^{\vee}} : \text{Gap}(\text{Gap}_{\mathcal{F}}^{\vee}) \rightarrow \text{Gap}^{\vee}(\mathcal{F})$  is essentially surjective (Corollary 2.14). Hence, this condition is naturally stronger than 1-fifiability. Hence, the above results of 1-fifiability are also true.

**Theorem 4.29.** Let  $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{C} \in \text{Cat}_{\infty}^{\text{perf}}$  be an exact functor between small stable idempotent-complete  $\infty$ -categories. If  $\mathcal{F}$  is strongly 1-fifiable,  $\text{BGap}^w(\mathcal{F})$  is also strongly 1-fifiable.

*Proof.* Lemma 4.22 and Lemma 4.26 tells us

$$\text{Gap}(\text{Gap}_{\text{BGap}^w(\mathcal{F})}^{\vee}) \simeq \text{BGap}^w(\text{Gap}(\text{Gap}_{\mathcal{F}}^{\vee})).$$

Hence,

$$\text{BGap}^w(\text{fill})\text{Gap}^{\vee}(\text{BGap}^w(\mathcal{F})) \rightarrow \text{Gap}(\text{Gap}_{\text{BGap}^w(\mathcal{F})}^{\vee})$$

satisfies  $\text{BGap}^w(\text{ev}_{\text{Gap}_{\mathcal{F}}^{\vee}}) \circ \text{BGap}^w(\text{fill}) \simeq \text{id}$ . Notice that  $\text{BGap}^w(\text{ev}_{\text{Gap}_{\mathcal{F}}^{\vee}}) \simeq \text{ev}_{\text{Gap}_{\text{BGap}^w(\mathcal{F})}^{\vee}}$  naturally, we have  $\text{ev}_{\text{Gap}_{\text{BGap}^w(\mathcal{F})}^{\vee}} \circ \text{BGap}^w(\text{fill}) \simeq \text{id}$ . Thus,  $\text{BGap}^w(\mathcal{F})$  is a strongly 1-fifiable.  $\square$

**Theorem 4.30. (Theorem C)** Let  $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{C} \in \text{Cat}_{\infty}^{\text{perf}}$  be a strongly 1-fifiable exact functor between small stable idempotent-complete  $\infty$ -categories. If  $\mathcal{F}$  satisfies the dévissage condition, it induces isomorphisms  $K_n(\mathcal{A}) \xrightarrow{\sim} K_n(\mathcal{C})$  for all  $n \geq 0$ .

*Proof.* We prove it by induction on  $n$ .

- $n = 0$ . It is Theorem 4.4.
- Assume it holds for  $n - 1$ . From Lemma 4.24, Corollary 4.25 and assumption,  $K_{n-1}(\text{BGap}^w(\mathcal{A})) \rightarrow K_{n-1}(\text{BGap}^w(\mathcal{C}))$  is an isomorphism. In the last section, we know that the cofiber sequence

$$K(\text{colim}_n \mathcal{A}^{n-1}) \rightarrow K(\text{BGap}^w(\mathcal{A})) \rightarrow K(\theta(\mathcal{A}))$$

split. Hence,  $K_{n-1}(\text{BGap}^w(\mathcal{A})) \simeq K_{n-1}(\text{colim}_n \mathcal{A}^{n-1}) \oplus K_{n-1}(\theta(\mathcal{A}))$ . Since  $K_{n-1}(\text{BGap}^w(\mathcal{A})) \rightarrow K_{n-1}(\text{BGap}^w(\mathcal{C}))$  is injective, both  $K_{n-1}(\text{colim}_n \mathcal{A}^{n-1}) \rightarrow K_{n-1}(\text{colim}_n \mathcal{C}^{n-1})$  and  $K_{n-1}(\theta(\mathcal{A})) \rightarrow K_{n-1}(\theta(\mathcal{C}))$  is an isomorphism. By Theorem 3.14, it means  $K_n(\mathcal{A}) \rightarrow K_n(\mathcal{C})$  is isomorphism.  $\square$

## 5 ON QUILLEN'S DÉVISSAGE THEOREM

Recall Quillen's Dévissage Theorem:

**Theorem 5.1.** [Qui73, Theorem 4] Let  $\mathcal{E}$  be a small abelian category, and  $\mathcal{E}_0$  be a full subcategory of  $\mathcal{E}$  that is closed under taking subobjects and subquotients. Suppose that each object of  $\mathcal{E}$  has a finite filtration with subquotients in  $\mathcal{E}_0$ . Then we have isomorphisms  $K_n(\mathcal{E}_0) \xrightarrow{\sim} K_n(\mathcal{E})$  for  $n \geq 0$ .

**Remark 5.2.** If  $(\mathcal{E}, \mathcal{E}_0)$  satisfies the condition of Theorem 5.1, we say  $(\mathcal{E}, \mathcal{E}_0)$  satisfies Quillen's dévissage condition.

Let  $\mathcal{E}$  be a small abelian category. A classical result is  $K(\mathcal{E}) \simeq K(\mathcal{D}^b(\mathcal{E}))$ . It is from the equivalence of differential constructions of algebraic K-theory [Bar15][Bar16][BGT13]. For the details of  $\mathcal{D}^b(\mathcal{E})$ , there are some good references [Lur16, Appendix C][Kra14].

### 5.1 Quasi-abelian categories

We briefly review the main notions on quasi-abelian categories and their derived categories, which is from [Sch99] and [KS06]. Simply put, a *quasi-abelian* category is an additive category which admits kernels and cokernels, the class of strict monomorphisms is closed under pushouts and the class of strict epimorphisms is closed under pullbacks. Here a morphism  $f$  is called *strict* if  $\text{coim } f \rightarrow \text{im } f$  is an isomorphism. In particular, a quasi-abelian category  $\mathcal{E}$  has an exact structure, for which the admissible short exact sequences are of the form

$$0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0,$$

where  $f = \ker(g)$  and  $g = \text{coker}(f)$ . In particular, we have the derived category  $\mathcal{D}(\mathcal{E})$   $\mathcal{D}^*(\mathcal{E})$  for  $* = +, -, b$ .

Clearly, any abelian category is quasi-abelian. An additive functor between quasi-abelian categories is called exact if it preserves admissible short exact sequences.

**Proposition 5.3.** [Sch99] Let  $\mathcal{E}$  be a small quasi-abelian category.

- (1) The category  $\mathcal{D}^b(\mathcal{E})$  has a bounded  $t$ -structure, whose heart consists of objects of the form  $\text{Cone}(X \xrightarrow{f} Y)$ , where  $f$  is a monomorphism (but not necessarily a strict monomorphism). This heart is denoted by  $\mathcal{LH}(\mathcal{E})$ . The inclusion functor  $\mathcal{E} \rightarrow \mathcal{LH}(\mathcal{E})$ ,  $X \mapsto X$ , is exact.
- (2) The functor  $\mathcal{D}^b(\mathcal{E}) \rightarrow \mathcal{D}^b(\mathcal{LH}(\mathcal{E}))$  is an equivalence.

**Proposition 5.4.** [Sch99] Let  $\mathcal{E}$  be a small quasi-abelian category and let  $\mathcal{A}$  be a small abelian category. Suppose that we have a fully faithful functor  $F : \mathcal{E} \rightarrow \mathcal{A}$  such that the following conditions hold.

- (1) The essential image  $F(\mathcal{E}) \subset \mathcal{A}$  is closed under taking subobjects.
- (2) For every object  $X \in \mathcal{A}$  there exists an object  $Y \in \mathcal{E}$  and an epimorphism  $F(Y) \rightarrow X$ .

Then  $F$  extends to an equivalence of categories  $\mathcal{LH}(\mathcal{E}) \xrightarrow{\sim} \mathcal{A}$ . Besides, the converse is also correct.

### 5.2 Abelian categories of functors

Let  $\mathcal{E}$  be a small abelian category. Then  $\text{Seq}_n(\mathcal{E}) := \text{Fun}([n], \mathcal{E})$  is also abelian. Consider the functor  $[n+1] \rightarrow [n]$  which sends  $i \mapsto i$  (if  $i \leq n$ ) and  $n+1 \mapsto n$ . This defines a functor:

$$i_n : \text{Seq}_n(\mathcal{E}) \rightarrow \text{Seq}_{n+1}(\mathcal{E}).$$

Define  $\text{Seq}(\mathcal{E}) := \text{colim}_n \text{Seq}_n(\mathcal{E})$ .

Denote by  $\text{Fil}_n(\mathcal{E})$  the full subcategory of  $\text{Seq}_n(\mathcal{E})$  consisting of objects

$$X_0 \rightarrowtail \cdots \rightarrowtail X_n,$$

where  $X_i \rightarrowtail X_{i+1}$  are monomorphisms. Then we can define  $\text{Fil}(\mathcal{E}) := \text{colim}_n \text{Fil}_n(\mathcal{E})$

**Proposition 5.5.** [SS13, Theorem 3.9, Theorem 3.16]  $\text{Fil}_n(\mathcal{E})$  is quasi-abelian and the inclusion  $\text{Fil}_n(\mathcal{E}) \hookrightarrow \text{Seq}_n(\mathcal{E})$  induces an equivalence

$$\mathcal{L}\mathcal{H}(\text{Fil}_n(\mathcal{E})) \xrightarrow{\sim} \text{Seq}_n(\mathcal{E}).$$

**Proposition 5.6.** [SS13, Corollary 3.6] Let  $f_\bullet : X_\bullet \rightarrow Y_\bullet$  be a morphism in  $\text{Fil}_n(\mathcal{E})$ .

- (1)  $(\ker f_\bullet)_i = \ker f_i$ .
- (2)  $(\text{coker } f_\bullet)_i = \text{im}(Y_i \rightarrow \text{coker } f_n)$ .

Define exact functors

$$\text{ev} : \text{Seq}_n(\mathcal{E}) \rightarrow \mathcal{E}, (X_0 \rightarrow \cdots \rightarrow X_n) \mapsto X_n.$$

It defines exact functor  $\text{ev} : \text{Seq}(\mathcal{E}) \rightarrow \mathcal{E}$ . Define a subcategory  $\text{Seq}^\vee(\mathcal{E})$  of  $\text{Seq}(\mathcal{E})^2$  as follows:

- Objects are pairs  $(X_\bullet, Y_\bullet)$  with  $\text{ev}(X_\bullet) = \text{ev}(Y_\bullet)$ .
- Morphisms are pairs  $(f_\bullet, g_\bullet) : (X_\bullet, Y_\bullet) \rightarrow (X'_\bullet, Y'_\bullet)$  with  $\text{ev}(f_\bullet) = \text{ev}(g_\bullet)$ .

**Lemma 5.7.**  $\text{Seq}^\vee(\mathcal{E})$  is abelian.

*Proof.* It is just the pullback  $\text{Seq}(\mathcal{E}) \times_{\mathcal{E}} \text{Seq}(\mathcal{E})$ . □

Denote  $\text{Fil}^\vee(\mathcal{E}) := \text{Seq}^\vee(\mathcal{E}) \cap \text{Fil}(\mathcal{E})^2$ . Then we naturally have  $\text{Fil}^\vee(\mathcal{E})$  is quasi-abelian.

**Lemma 5.8.**  $\mathcal{L}\mathcal{H}(\text{Fil}^\vee(\mathcal{E})) = \text{Seq}^\vee(\mathcal{E})$ .

*Proof.* Let  $(Z_\bullet, Z'_\bullet) \in \text{Seq}^\vee(\mathcal{E})$ . Assume  $Z_\bullet, Z'_\bullet \in \text{Seq}_n(\mathcal{E})$ . Then  $Z_n = Z'_n$  by definition. Define  $X_k, Y_k, X'_k, Y'_k$  by

$$\begin{aligned} X_k &= (\bigoplus_{i=0}^{k-1} Z_i) \oplus (\bigoplus_{i=0}^n Z'_i), Y_k = (\bigoplus_{i=0}^k Z_k) \oplus (\bigoplus_{i=0}^n Z'_i) \\ X'_k &= (\bigoplus_{i=0}^{k-1} Z'_i) \oplus (\bigoplus_{i=0}^n Z_i), Y'_k = (\bigoplus_{i=0}^k Z'_k) \oplus (\bigoplus_{i=0}^n Z_i) \end{aligned}$$

for  $0 \leq k \leq n$ . Define

$$\begin{aligned} X_k \rightarrow X_{k+1} &: (z_1, \dots, z_{k-1}, z'_1, \dots, z'_n) \mapsto (z_1, \dots, z_{k-1}, 0, z'_1, \dots, z'_n) \\ Y_k \rightarrow Y_{k+1} &: (z_1, \dots, z_k, z'_1, \dots, z'_n) \mapsto (z_1, \dots, z_k, f_k(z_k), z'_1, \dots, z'_n) \\ X'_k \rightarrow X'_{k+1} &: (z'_1, \dots, z'_{k-1}, z_1, \dots, z_n) \mapsto (z'_1, \dots, z'_{k-1}, 0, x_1, \dots, x_n) \\ Y'_k \rightarrow Y'_{k+1} &: (z'_1, \dots, z'_k, z_1, \dots, z_n) \mapsto (z'_1, \dots, z'_k, f'_k(z_k), z_1, \dots, z_n) \end{aligned}$$

where  $f_k, f'_k$  are morphisms  $Z_k \rightarrow Z_{k+1}, Z'_k \rightarrow Z'_{k+1}$  in  $Z_\bullet, Z'_\bullet$ . Then we define  $X_\bullet, X'_\bullet, Y_\bullet, Y'_\bullet \in \text{Fil}_n(\mathcal{E})$ . One can check that  $(X_\bullet, X'_\bullet), (Y_\bullet, Y'_\bullet) \in \text{Fil}^\vee(\mathcal{E})$ .

Define

$$\begin{aligned} X_k \rightarrow Y_k &: (z_1, \dots, z_{k-1}, z'_1, \dots, z'_n) \mapsto (z_1, \dots, z_{k-1}, 0, z'_1, \dots, z'_n) \\ X'_k \rightarrow Y'_k &: (z'_1, \dots, z'_{k-1}, z_1, \dots, z_n) \mapsto (z'_1, \dots, z'_{k-1}, 0, x_1, \dots, x_n) \\ Y_k \rightarrow Z_k &: (z_1, \dots, z_k, z'_1, \dots, z'_n) \mapsto z_k \\ Y'_k \rightarrow Z'_k &: (z'_1, \dots, z'_k, z_1, \dots, z_n) \mapsto z'_k \end{aligned}$$

Then we define an exact sequence

$$(X_\bullet, X'_\bullet) \rightarrow (Y_\bullet, Y'_\bullet) \rightarrow (Z_\bullet, Z'_\bullet)$$

in  $\text{Seq}^\vee(\mathcal{E})$ .

It is easy to show  $\text{Fil}^\vee(\mathcal{E})$  is closed under subobjects in  $\text{Seq}^\vee(\mathcal{E})$ . By Proposition 5.4,  $\mathcal{L}\mathcal{H}(\text{Fil}^\vee(\mathcal{E})) = \text{Seq}^\vee(\mathcal{E})$ . □

**Lemma 5.9.** Let  $J$  be a finite  $\infty$ -category. Then there is a natural equivalence

$$\mathcal{D}^b(\text{Fun}(J, \mathcal{E})) \simeq \text{Fun}(J, \mathcal{D}^b(\mathcal{E})).$$

*Proof.* There is a natural functor

$$\mathcal{D}^b(\mathrm{Fun}(J, \mathcal{E})) \rightarrow \mathrm{Fun}(J, \mathcal{D}^b(\mathcal{E})).$$

We only need to prove it induces equivalence

$$\mathrm{Fun}^{\mathrm{ex}}(\mathcal{D}^b(\mathrm{Fun}(J, \mathcal{E})), \mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{ex}}(\mathrm{Fun}(J, \mathcal{D}^b(\mathcal{E})), \mathcal{C})$$

for any stable  $\infty$ -categories  $\mathcal{C}$ . By [Kle22, Theorem 1], we know

$$\begin{aligned} \mathrm{Fun}^{\mathrm{ex}}(\mathcal{D}^b(\mathrm{Fun}(J, \mathcal{E})), \mathcal{C}) &\simeq \mathrm{Fun}^{\mathrm{ex}}(\mathrm{Fun}(J, \mathcal{E}), \mathcal{C}) \\ &\simeq \mathrm{Fun}(J, \mathrm{Fun}^{\mathrm{ex}}(\mathcal{E}, \mathcal{C})) \\ &\simeq \mathrm{Fun}(J, \mathrm{Fun}^{\mathrm{ex}}(\mathcal{D}^b(\mathcal{E}), \mathcal{C})) \\ &\simeq \mathrm{Fun}^{\mathrm{ex}}(\mathrm{Fun}(J, \mathcal{D}^b(\mathcal{E})), \mathcal{C}). \end{aligned}$$

□

Hence, there is a natural equivalence

$$\mathcal{D}^b(\mathrm{Seq}_n(\mathcal{E})) \simeq \mathrm{Fun}([n], \mathcal{D}^b(\mathcal{E})),$$

which induces equivalences

$$\mathcal{D}^b(\mathrm{Fil}(\mathcal{E})) \simeq \mathcal{D}^b(\mathrm{Seq}(\mathcal{E})) \simeq \mathrm{Gap}(\mathcal{D}^b(\mathcal{E})).$$

Let  $\mathcal{E}_0$  be a full abelian subcategory of  $\mathcal{E}$  that is closed under taking subobjects and subquotients.

Denote by  $\mathrm{Fil}_n(\mathcal{E}, \mathcal{E}_0)$  the full subcategory of  $\mathrm{Fil}_n(\mathcal{E})$  consisting of objects

$$X_0 \rightarrowtail \cdots \rightarrowtail X_n,$$

where  $X_{i+1}/X_i$  are in  $\mathcal{E}_0$ . Define  $\mathrm{Fil}(\mathcal{E}, \mathcal{E}_0) := \mathrm{colim}_n \mathrm{Fil}_n(\mathcal{E}, \mathcal{E}_0)$ .

**Lemma 5.10.**  $\mathrm{Fil}_n(\mathcal{E}, \mathcal{E}_0)$  is quasi-abelian.

*Proof.* It contains zero objects and finite direct sums in  $\mathrm{Fil}_n(\mathcal{E})$ . Consider a morphism  $f_\bullet : X_\bullet \rightarrow Y_\bullet$  in  $\mathrm{Fil}_n(\mathcal{E}, \mathcal{E}_0)$ . Then

$$(\ker f_\bullet)_{i+1}/(\ker f_\bullet)_i = \ker f_{i+1}/\ker f_i \subset X_{i+1}/X_i$$

which is also in  $\mathcal{E}_0$  since  $\mathcal{E}_0$  is closed under subobjects. Besides,

$$(\mathrm{coker} f_\bullet)_{i+1}/(\mathrm{coker} f_\bullet)_i = \mathrm{im}(Y_{i+1} \rightarrow \mathrm{coker} f_n)/\mathrm{im}(Y_i \rightarrow \mathrm{coker} f_n)$$

is a subquotient of  $Y_{i+1}/Y_i$ . Hence, it is also in  $\mathcal{E}_0$  since  $\mathcal{E}_0$  is closed under subquotients.

Similarly,  $\mathrm{Fil}_n(\mathcal{E}, \mathcal{E}_0)$  preserves pushouts and pullbacks of  $\mathrm{Fil}_n(\mathcal{E})$ . Then it is also quasi-abelian. □

Denote  $\mathrm{Seq}_n(\mathcal{E}, \mathcal{E}_0) := \mathcal{L}\mathcal{H}(\mathrm{Fil}_n(\mathcal{E}, \mathcal{E}_0))$  and  $\mathrm{Seq}(\mathcal{E}, \mathcal{E}_0) := \mathcal{L}\mathcal{H}(\mathrm{Fil}(\mathcal{E}, \mathcal{E}_0))$ . Notice that  $\mathrm{Seq}(\mathcal{E}, \mathcal{E}_0)$  is the full abelian subcategory of  $\mathrm{Seq}(\mathcal{E})$ .

Denote  $\mathrm{Fil}^\vee(\mathcal{E}, \mathcal{E}_0) := \mathrm{Seq}^\vee(\mathcal{E}) \cap \mathrm{Fil}(\mathcal{E}, \mathcal{E}_0)^2$ . Then we naturally have  $\mathrm{Fil}^\vee(\mathcal{E}, \mathcal{E}_0)$  is quasi-abelian. Define

$$\mathrm{Seq}^\vee(\mathcal{E}, \mathcal{E}_0) := \mathcal{L}\mathcal{H}(\mathrm{Fil}^\vee(\mathcal{E}, \mathcal{E}_0)).$$

**Lemma 5.11.** Each object of  $\mathrm{Fil}^\vee(\mathcal{E}, \mathcal{E}_0)$  has a finite filtration of strict monomorphisms with subquotients in  $\mathrm{Fil}^\vee(\mathcal{E}_0)$ .

*Proof.* For  $(X_\bullet, X'_\bullet) \in \text{Fil}^\vee(\mathcal{E}, \mathcal{E}_0)$ , assume  $X_\bullet, X'_\bullet \in \text{Fil}_n(\mathcal{E}, \mathcal{E}_0)$ . Since  $X_n = X'_n$ , define  $X_{i,j} = X_i \times_{X_n} X'_j = X_i \cap X'_j$ . Define two filtrations

$$\begin{array}{ccccc}
0 & \longrightarrow & \cdots & \longrightarrow & 0 \\
\downarrow & & & & \downarrow \\
X_{0,0} & \longrightarrow & \cdots & \longrightarrow & X_{n,0} \\
\downarrow & & & & \downarrow \\
\vdots & \ddots & \vdots & , & \vdots \\
\downarrow & & \downarrow & & \downarrow \\
X_{0,n-1} & \longrightarrow & \cdots & \longrightarrow & X_{n,n-1} \\
\downarrow & & & & \downarrow \\
X_0 & \longrightarrow & \cdots & \longrightarrow & X_n
\end{array}
\quad
\begin{array}{ccccc}
0 & \longrightarrow & \cdots & \longrightarrow & 0 \\
\downarrow & & & & \downarrow \\
X_{0,0} & \longrightarrow & \cdots & \longrightarrow & X_{0,n} \\
\downarrow & & & & \downarrow \\
\vdots & \ddots & \vdots & , & \vdots \\
\downarrow & & \downarrow & & \downarrow \\
X_{n-1,0} & \longrightarrow & \cdots & \longrightarrow & X_{n-1,n} \\
\downarrow & & & & \downarrow \\
X'_0 & \longrightarrow & \cdots & \longrightarrow & X'_n
\end{array}$$

Since  $\mathcal{E}_0$  is closed under subobjects and subquotients in  $\mathcal{E}$ , they define a filtration of  $(X_\bullet, X'_\bullet)$  in  $\text{Fil}^\vee(\mathcal{E}, \mathcal{E}_0)$  with subquotients in  $\text{Fil}^\vee(\mathcal{E}_0)$ .  $\square$

**Proposition 5.12.**  $\text{Seq}^\vee(\mathcal{E}_0)$  is full abelian category of  $\text{Seq}^\vee(\mathcal{E}, \mathcal{E}_0)$ . Besides,  $(\text{Seq}^\vee(\mathcal{E}, \mathcal{E}_0), \text{Seq}^\vee(\mathcal{E}_0))$  satisfies Quillen's dévissage condition.

*Proof.* From Lemma 5.11 and Proposition 5.4, each object of  $\text{Seq}^\vee(\mathcal{E}, \mathcal{E}_0)$  has a finite filtration with subquotients in  $\text{Seq}^\vee(\mathcal{E}_0)$ . Since  $\mathcal{E}_0$  is closed under subobjects and subquotients in  $\mathcal{E}$ , the same as  $\text{Seq}^\vee(\mathcal{E}_0)$  in  $\text{Seq}^\vee(\mathcal{E}, \mathcal{E}_0)$ .  $\square$

### 5.3 A new proof

**Lemma 5.13.** If  $(\mathcal{E}, \mathcal{E}_0)$  satisfies Quillen's dévissage condition,  $\mathcal{D}^b(\mathcal{E}_0) \rightarrow \mathcal{D}^b(\mathcal{E}_0)$  satisfies the dévissage condition.

*Proof.* It is obvious that  $\mathcal{E} \subseteq \text{E}(\mathcal{D}^b(\mathcal{E}_0) \rightarrow \mathcal{D}^b(\mathcal{E}_0))$ , which forces  $\text{E}(\mathcal{D}^b(\mathcal{E}_0) \rightarrow \mathcal{D}^b(\mathcal{E}_0)) = \mathcal{D}^b(\mathcal{E})$  by the boundedness. By Proposition 2.9,  $\mathcal{D}^b(\mathcal{E}_0) \rightarrow \mathcal{D}^b(\mathcal{E}_0)$  satisfies the dévissage condition.  $\square$

**Lemma 5.14.** Let  $J$  be a finite  $\infty$ -category. Then there is a natural equivalence

$$\mathcal{D}^b(\text{Fun}(J, \mathcal{E})) \simeq \text{Fun}(J, \mathcal{D}^b(\mathcal{E})).$$

*Proof.* There is a natural functor

$$\mathcal{D}^b(\text{Fun}(J, \mathcal{E})) \rightarrow \text{Fun}(J, \mathcal{D}^b(\mathcal{E})).$$

We only need to prove it induces equivalence

$$\text{Fun}^{\text{ex}}(\mathcal{D}^b(\text{Fun}(J, \mathcal{E})), \mathcal{C}) \simeq \text{Fun}^{\text{ex}}(\text{Fun}(J, \mathcal{D}^b(\mathcal{E})), \mathcal{C})$$

for any stable  $\infty$ -categories  $\mathcal{C}$ . By [Kle22, Theorem 1], we know

$$\begin{aligned}
\text{Fun}^{\text{ex}}(\mathcal{D}^b(\text{Fun}(J, \mathcal{E})), \mathcal{C}) &\simeq \text{Fun}^{\text{ex}}(\text{Fun}(J, \mathcal{E}), \mathcal{C}) \\
&\simeq \text{Fun}(J, \text{Fun}^{\text{ex}}(\mathcal{E}, \mathcal{C})) \\
&\simeq \text{Fun}(J, \text{Fun}^{\text{ex}}(\mathcal{D}^b(\mathcal{E}), \mathcal{C})) \\
&\simeq \text{Fun}^{\text{ex}}(\text{Fun}(J, \mathcal{D}^b(\mathcal{E})), \mathcal{C}).
\end{aligned}$$

$\square$

Hence, there is a natural equivalence

$$\mathcal{D}^b(\text{Seq}_n(\mathcal{E})) \simeq \text{Fun}([n], \mathcal{D}^b(\mathcal{E})),$$

which induces equivalences

$$\mathcal{D}^b(\text{Fil}(\mathcal{E})) \simeq \mathcal{D}^b(\text{Seq}(\mathcal{E})) \simeq \text{Gap}(\mathcal{D}^b(\mathcal{E})).$$

**Lemma 5.15.**  $\mathcal{D}^b(\text{Seq}(\mathcal{E}, \mathcal{E}_0)) \simeq \mathcal{D}^b(\text{Fil}(\mathcal{E}, \mathcal{E}_0)) \simeq \text{Gap}(\mathcal{D}^b(\mathcal{E}_0) \rightarrow \mathcal{D}^b(\mathcal{E}_0))$ .

*Proof.* An object in  $\text{Gap}(\mathcal{D}^b(\mathcal{E}_0) \rightarrow \mathcal{D}^b(\mathcal{E}_0))$  is just a filtration of complexes

$$0 \rightarrow X_\bullet^1 \rightarrow X_\bullet^2 \rightarrow \dots$$

where all  $X_\bullet^i \rightarrow X_\bullet^{i+1}$  are degreewise monomorphisms such that  $X_\bullet^{i+1}/X_\bullet^i \in \text{Ch}^b(\mathcal{E}_0)$ . It is equivalent to an object in  $\mathcal{D}^b(\text{Fil}(\mathcal{E}, \mathcal{E}_0))$ . Their morphisms are just all objects in

$$\text{Gap}(\text{Fun}([1], \mathcal{D}^b(\mathcal{E}_0)) \rightarrow \text{Fun}([1], \mathcal{D}^b(\mathcal{E})), \mathcal{D}^b(\text{Fil}(\text{Seq}_l(\mathcal{E}), \text{Seq}_l(\mathcal{E}_0))))$$

respectively. Similarly, their morphisms are the same.  $\square$

**Proposition 5.16.**  $\mathcal{D}^b(\text{Seq}^\vee(\mathcal{E}, \mathcal{E}_0)) \simeq \mathcal{D}^b(\text{Fil}^\vee(\mathcal{E}, \mathcal{E}_0)) \simeq \text{Gap}^\vee(\mathcal{D}^b(\mathcal{E}_0) \rightarrow \mathcal{D}^b(\mathcal{E}_0))$ .

*Proof.* By Lemma 5.15, their objects and morphisms are both subsets of these of  $\text{Gap}(\mathcal{D}^b(\mathcal{E}_0) \rightarrow \mathcal{D}^b(\mathcal{E}_0))^2 \simeq \mathcal{D}^b(\text{Fil}(\mathcal{E}, \mathcal{E}_0))^2$ . It is easy to check they have the same objects. For morphisms, one can checks that their morphisms are all objects of

$$\mathcal{D}^b(\text{Fil}^\vee(\text{Seq}_l(\mathcal{E}), \text{Seq}_l(\mathcal{E}_0))), \text{Gap}^\vee(\text{Fun}(\Delta^1, \mathcal{D}^b(\mathcal{E}_0)) \rightarrow \text{Fun}(\Delta^1, \mathcal{D}^b(\mathcal{E}))),$$

respectively. Similarly, their morphisms are the same.  $\square$

Define exact functors  $\mathcal{F}_n : \mathcal{A}_n \rightarrow \mathcal{C}_n$  by

- $\mathcal{C}_1 = \mathcal{D}^b(\mathcal{E}), \mathcal{A}_1 = \mathcal{D}^b(\mathcal{E}_0), \mathcal{F}_1 = \mathcal{D}^b(\mathcal{E}_0) \rightarrow \mathcal{D}^b(\mathcal{E})$ .
- For  $n \geq 2$ ,  $\mathcal{C}_n = \text{Gap}^\vee(\mathcal{F}_{n-1}), \mathcal{A}_n = \text{Gap}^\vee(\mathcal{A}_{n-1}), \mathcal{F}_n = \text{Gap}_{\mathcal{F}_{n-1}}^\vee$ .

Define abelian categories  $\mathcal{E}_n, \mathcal{E}_{0,n}$  by

- $\mathcal{E}_1 = \mathcal{E}, \mathcal{E}_{0,1} = \mathcal{E}_0$ .
- For  $n \geq 2$ ,  $\mathcal{E}_n = \text{Seq}^\vee(\mathcal{E}_{n-1}, \mathcal{E}_{0,n-1}), \mathcal{E}_{0,n} = \text{Seq}^\vee(\mathcal{E}_{0,n-1})$ .

**Theorem 5.17.**  $\mathcal{D}^b(\mathcal{E}_0) \rightarrow \mathcal{D}^b(\mathcal{E})$  is fillable. Furthermore, if  $(\mathcal{E}, \mathcal{E}_0)$  satisfies Quillen's dévissage condition, there are isomorphisms

$$K_n(\mathcal{D}^b(\mathcal{E}_0)) \xrightarrow{\sim} K_n(\mathcal{D}^b(\mathcal{E}))$$

for all  $n \geq 0$ .

*Proof.* By Proposition 5.16, we have  $\mathcal{A}_n = \mathcal{D}^b(\mathcal{E}_{0,n})$  and  $\mathcal{C}_n = \mathcal{D}^b(\mathcal{E}_n)$  and  $\mathcal{F}_n = \mathcal{D}^b(\mathcal{E}_{0,n}) \rightarrow \mathcal{D}^b(\mathcal{E}_n)$ . By Proposition 5.12 and Lemma 5.13,  $\mathcal{F}_n$  satisfies the dévissage condition for all  $n \geq 2$ . Hence,  $\mathcal{D}^b(\mathcal{E}_0) \rightarrow \mathcal{D}^b(\mathcal{E})$  is fillable.

If  $(\mathcal{E}, \mathcal{E}_0)$  satisfies Quillen's dévissage condition, by Lemma 5.13 and Corollary 4.13, there are isomorphisms

$$K_n(\mathcal{D}^b(\mathcal{E}_0)) \xrightarrow{\sim} K_n(\mathcal{D}^b(\mathcal{E}))$$

for all  $n \geq 0$ .  $\square$

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