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Solutions to $SU(n+1)$ Toda system generated by spherical metricsYiqian Shi^a, Chunhui Wei^{b,*}, Bin Xu^c^a School of Mathematical Sciences, University of Science and Technology of China, Hefei, 230026, People's Republic of China^b School of Gifted Young, University of Science and Technology of China, Hefei, 230026, People's Republic of China^c School of Mathematical Sciences, University of Science and Technology of China, Hefei, 230026, People's Republic of China

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ABSTRACT

Following A.B. Givental (1989) [5], we refer to an n -tuple $(\omega_1, \dots, \omega_n)$ of Kähler forms on a Riemann surface S as a *solution to the $SU(n+1)$ Toda system* if and only if

$$(\text{Ric}(\omega_1), \dots, \text{Ric}(\omega_n)) = (2\omega_1, \dots, 2\omega_n)C_n,$$

where C_n is the Cartan matrix of type A_n . In particular, when $n = 1$, this solution corresponds to a spherical metric. Using the correspondence between solutions and totally unramified unitary curves, we show that a spherical metric ω generates a family of solutions, including $(i(n+1-i)\omega)_{i=1}^n$. Moreover, we characterize this family in terms of the monodromy group of the spherical metric. As a consequence, we obtain a new solution class to the $SU(n+1)$ Toda system with cone

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singularities on compact Riemann surfaces, complementing the existence results of Lin et al. (2020) [9].

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1. Introduction

We present a natural and precise method for generating solutions to the $SU(n+1)$ Toda system on Riemann surfaces using spherical metrics (Theorems 1.1 and 1.3). As a consequence, we identify a new class of solutions to the $SU(n+1)$ Toda system with cone singularities on compact Riemann surfaces (Corollary 1.5), which complements the results in [9, Theorems 1.8 and 1.9]. To obtain these results, we employ the complex differential-geometric framework for solutions to the $SU(n+1)$ Toda system with cone singularities, as established in [5] and [11, Subsections 1.1 and 1.2]. For further details, interested readers may refer to [11, Section 1] for the latest developments in this field.

Let S be a Riemann surface, not necessarily compact, and let n be a positive integer. An n -tuple $\vec{\omega} = (\omega_1, \dots, \omega_n)$ of Kähler forms is called a *solution to the $SU(n+1)$ Toda system* ([11, Definition 1]) on S if and only if

$$\text{Ric}(\vec{\omega}) = 2\vec{\omega}C_n, \quad (1.1)$$

where $\text{Ric}(\vec{\omega}) = (\text{Ric}(\omega_1), \dots, \text{Ric}(\omega_n))$ is the n -tuple of Ricci forms, and

$$C_n = \begin{pmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 2 \end{pmatrix}_{n \times n}$$

is the Cartan matrix of type A_n . In particular, a solution ω_1 to the $SU(2)$ Toda system coincides with a conformal spherical metric on S .

In 2022, we made the simple observation that if ω is a solution to the $SU(2)$ Toda system, then $(i(n+1-i)\omega)_{i=1}^n$ solves the $SU(n+1)$ Toda system on S . In this paper, we will develop this strategy in detail using the *basic correspondence between solutions to the $SU(n+1)$ Toda system on S and the totally unramified unitary curves from S to the complex projective space \mathbb{P}^n of dimension n* . The definition of a totally unramified unitary curve and the proof of this correspondence can be found in [11, Subsection 1.2 and Section 2]. Simply put, a totally unramified unitary curve $f : S \rightarrow \mathbb{P}^n$ is a multi-valued holomorphic map whose monodromy group resides within $\text{PSU}(n+1)$ and any local germs are totally unramified. We also refer to a unitary curve corresponding to a solution as an *associated curve of the solution*. Any two associated curves of a solution

differ by a rigid motion of \mathbb{P}^n endowed with the Fubini-Study metric ω_{FS} ([6, (4.12)]). In particular, an associated curve of the solution ω to the $\text{SU}(2)$ Toda system coincides with the developing map of the spherical metric ω on S ([1, Section 2]). First, we characterize unitary curves $S \rightarrow \mathbb{P}^n$ associated with the solution $(i(n+1-i)\omega)_{i=1}^n$ in terms of a unitary curve $S \rightarrow \mathbb{P}^1$ associated with ω .

Theorem 1.1. *Let ω be a solution to the $\text{SU}(2)$ Toda system on S , and let $v : S \rightarrow \mathbb{P}^1$ be a curve associated with ω . Let $r_n : \mathbb{P}^1 \rightarrow \mathbb{P}^n$ be the rational normal map defined by*

$$r_n : [z_0, z_1] \mapsto \left[\sqrt{\frac{1}{n!}} z_0^n, \sqrt{\frac{1}{(n-1)!1!}} z_0^{n-1} z_1, \dots, \sqrt{\frac{1}{n!}} z_1^n \right].$$

Then $(i(n+1-i)\omega)_{i=1}^n$ solves the $\text{SU}(n+1)$ Toda system on S . Moreover, the set

$$\{U \circ r_n \circ v : S \rightarrow \mathbb{P}^n \mid U \in \text{PSU}(n+1)\}$$

consists of all the associated curves of this solution.

Given a basis of \mathbb{C}^{n+1} endowed with the standard Hermitian inner product $\langle \cdot, \cdot \rangle$, the Gram-Schmidt procedure provides a new orthonormal basis of $(\mathbb{C}^{n+1}, \langle \cdot, \cdot \rangle)$. Then, we obtain the Iwasawa decomposition of $\text{SL}(n+1, \mathbb{C})$ in the form

$$\text{SL}(n+1, \mathbb{C}) = \text{SU}(n+1) \Delta_{n+1},$$

where $\text{SU}(n+1)$ is the group of special unitary transformations and Δ_{n+1} is the group of linear transformations by left multiplication of upper triangular matrices with positive diagonal entries in $\text{SL}(n+1, \mathbb{C})$. Hence, an automorphism $\varphi \in \text{PSL}(n+1, \mathbb{C})$ of \mathbb{P}^n has the decomposition $\varphi = U \circ \delta$, where $U \in \text{PSU}(n+1)$ and $\delta \in \Delta_{n+1}$. Based on this and Theorem 1.1, we introduce the following definition:

Definition 1.2. We call a solution $\vec{\omega}$ to the $\text{SU}(n+1)$ Toda system *reduced* if and only if it is generated by another solution ω_1 to the $\text{SU}(2)$ Toda system, i.e., a conformal spherical metric, on S in the following sense: there exists a linear transformation $\delta \in \Delta_{n+1}$, an associated curve $f : S \rightarrow \mathbb{P}^n$ of $\vec{\omega}$, and an associated curve $v : S \rightarrow \mathbb{P}^1$ of ω_1 such that

$$f = \delta \circ r_n \circ v. \quad (1.2)$$

Notably, the curve f should have monodromy in $\text{PSU}(n+1)$, which imposes a constraint on the variety of such δ 's (Theorem 1.3).

Given a solution ω_1 to the $\text{SU}(2)$ Toda system, we can characterize all the reduced solutions to the $\text{SU}(n+1)$ Toda system generated by it in the following theorem:

Theorem 1.3. We use the notions in Theorem 1.1. Let

$$M_v = \{\varphi \in \mathrm{PSL}(n+1, \mathbb{C}) \mid \varphi \circ r_n \circ v : S \rightarrow \mathbb{P}^n \text{ is a unitary curve}\}.$$

M_v can be decomposed into $M_v = \mathrm{PSU}(n+1)\Delta_v$, where $\Delta_v \subset \Delta_{n+1}$. It is determined by the closure $\overline{G_v}$ in $\mathrm{PSU}(2)$ of the monodromy group G_v of v . Consider the classification of closed subgroups of $\mathrm{SU}(2)$ ([4, Chapter 1]):

$$\begin{aligned} \mathrm{O}(2) &= \left\langle \mathrm{U}(1), \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle, \quad \mathrm{U}(1), \\ C_k &= \left\langle \begin{pmatrix} e^{2\pi\sqrt{-1}/k} & 0 \\ 0 & e^{-2\pi\sqrt{-1}/k} \end{pmatrix} \right\rangle, k \in \mathbb{Z}_{>0}, \\ D_k &= \left\langle C_{2k}, \begin{pmatrix} 0 & -\sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix} \right\rangle, k \in \mathbb{Z}_{>0}, \\ E_6 &= \left\langle \begin{pmatrix} \frac{1+\sqrt{-1}}{2} & \frac{1+\sqrt{-1}}{2} \\ \frac{\sqrt{-1}-1}{2} & \frac{1-\sqrt{-1}}{2} \end{pmatrix}, \begin{pmatrix} 0 & -\sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix} \right\rangle, \\ E_7 &= \left\langle \begin{pmatrix} \frac{1+\sqrt{-1}}{2} & \frac{1+\sqrt{-1}}{2} \\ \frac{\sqrt{-1}-1}{2} & \frac{1-\sqrt{-1}}{2} \end{pmatrix}, \begin{pmatrix} 0 & \frac{\sqrt{2}(1+\sqrt{-1})}{2} \\ \frac{\sqrt{2}(\sqrt{-1}-1)}{2} & 0 \end{pmatrix} \right\rangle, \\ E_8 &= \left\langle \begin{pmatrix} \frac{\sqrt{5}-1}{4} + \frac{\frac{1}{2}}{\frac{\sqrt{5}+1}{4}}\sqrt{-1} & -\frac{\sqrt{5}-1}{4} + \frac{\sqrt{5}+1}{4}\sqrt{-1} \\ \frac{\sqrt{5}-1}{4} & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 & -\sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix} \right\rangle. \end{aligned}$$

Let $p : \mathrm{SU}(2) \rightarrow \mathrm{PSU}(2)$ be the quotient map. Then there hold the following statements:

- (1) When $\overline{G_v} = \mathrm{PSU}(2)$, $\Delta_v = \{I_{n+1}\}$;
- (2) When $\overline{G_v} = \mathrm{PU}(1)$, $\Delta_v = \{\mathrm{diag}(a_0, \dots, a_n) \in \Delta_{n+1}\}$ with $\dim_{\mathbb{R}} \Delta_v = n$;
- (3) When $\overline{G_v} = \mathrm{PO}(2)$, $\Delta_v = \{\mathrm{diag}(a_0, \dots, a_n) \in \Delta_{n+1} \mid a_i = a_{n-i}\}$ with $\dim_{\mathbb{R}} \Delta_v = \lfloor n/2 \rfloor$;
- (4) When $\overline{G_v} = p(C_k)$, $\Delta_v = \{(a_{ij})_{0 \leq i, j \leq n} \in \Delta_{n+1} \mid a_{i,j} = 0 \text{ if } \frac{k}{\gcd(k,2)} \nmid (i-j)\}$ with

$$\dim_{\mathbb{R}} \Delta_v = -\frac{k}{\gcd(k,2)} \lfloor \frac{n \gcd(k,2)}{k} \rfloor^2 + (2n+2 - \frac{k}{\gcd(k,2)}) \lfloor \frac{n \gcd(k,2)}{k} \rfloor + n;$$

- (5) When $\overline{G_v} = p(D_k)$,

$$\Delta_v = \left\{ (a_{i,j})_{0 \leq i, j \leq n} \in \Delta_{n+1} \mid \begin{cases} \sum_{l=0}^n \bar{a}_{l,i} a_{l,j} = 0 \text{ if } k \nmid i-j \\ \sum_{l=0}^n \bar{a}_{l,i} a_{l,j} = (-\sqrt{-1})^{i-j} \sum_{l=0}^n \bar{a}_{l,n-i} a_{l,n-j} \end{cases} \right\}$$

$$\text{with } \dim_{\mathbb{R}} \Delta_v = -\frac{k}{2} \lfloor \frac{n}{k} \rfloor^2 + (n+1 - \frac{k}{2}) \lfloor \frac{n}{k} \rfloor + \lfloor \frac{n}{2} \rfloor;$$

(6) When $\overline{G_v} = p(E_6)$,

$$\dim_{\mathbb{R}} \Delta_v = \begin{cases} \dim_{\mathbb{R}} \Delta_v = \frac{n^2}{12} + \frac{2c^2}{3} + \frac{n}{6} - \frac{11}{12} & \text{if } n \text{ is odd} \\ \dim_{\mathbb{R}} \Delta_v = \frac{n^2}{12} + \frac{2c^2}{3} + \frac{n}{6} - \frac{2}{3} & \text{if } n \text{ is even} \end{cases},$$

where $c = \cos \frac{n\pi}{3} + \frac{\sqrt{3}}{3} \sin \frac{n\pi}{3}$;

(7) When $\overline{G_v} = p(E_7)$,

$$\dim_{\mathbb{R}} \Delta_v = \begin{cases} \frac{n^2}{24} + \frac{n}{12} + \frac{c_1^2}{3} + \frac{c_2^2}{4} - \frac{23}{24} & \text{if } n \text{ is odd} \\ \frac{n^2}{24} + \frac{n}{12} + \frac{c_1^2}{3} + \frac{c_2^2}{4} - \frac{7}{12} & \text{if } n \text{ is even} \end{cases},$$

where $c_1 = \cos \frac{n\pi}{3} + \frac{\sqrt{3}}{3} \sin \frac{n\pi}{3}$ and $c_2 = \cos \frac{n\pi}{4} + \sin \frac{n\pi}{4}$;

(8) When $\overline{G_v} = p(E_8)$,

$$\dim_{\mathbb{R}} \Delta_v = \begin{cases} \frac{n^2}{60} + \frac{n}{30} + \frac{c_1^2}{3} + \frac{c_2^2}{5} + \frac{c_3^2}{5} - \frac{59}{60} & \text{if } n \text{ is odd} \\ \frac{n^2}{60} + \frac{n}{30} + \frac{c_1^2}{3} + \frac{c_2^2}{5} + \frac{c_3^2}{5} - \frac{13}{15} & \text{if } n \text{ is even} \end{cases},$$

where $c_1 = \cos \frac{n\pi}{3} + \frac{\sqrt{3}}{3} \sin \frac{n\pi}{3}$, $c_2 = \sqrt{1 + \frac{2}{\sqrt{5}}} \sin \frac{n\pi}{5} + \cos \frac{n\pi}{5}$ and $c_3 = \sqrt{1 - \frac{2}{\sqrt{5}}} \sin \frac{2n\pi}{5} + \cos \frac{2n\pi}{5}$.

Remark 1.4. Notice that all the cases are possible: for any closed subgroup of $\mathrm{PSU}(2)$, there exists a multi-valued meromorphic function such that the closure of its monodromy group is the given subgroup. The dimensions of cases (6)–(8) arise from norms of characters of finite-dimensional representations of finite groups; hence, they are naturally integers, although their forms appear complicated.

As an application of Theorems 1.1 and 1.3, we identify a novel class of solvable $\mathrm{SU}(n+1)$ Toda systems with cone singularities on compact Riemann surfaces as follows:

Corollary 1.5. We adopt the notions introduced in [11, Subsection 1.1]. Suppose that there exists a cone spherical metric that represents the real divisor $D = \sum_{j=1}^n \gamma_j [P_j]$, where $0 \neq \gamma_j > -1$ for all $1 \leq j \leq n$, on a compact Riemann surface X . Then, for each positive integer $n > 1$, the $\mathrm{SU}(n+1)$ Toda system on X with cone singularities

$$\underbrace{(D, D, \dots, D)}_{n \text{ divisors}}$$

has a family of reduced solutions, including $(i(n+1-i)\omega)_{i=1}^n$ and is characterized in Theorem 1.3.

We organize the remainder of this paper as follows. In Section 2, we prove Theorem 1.1 using the infinitesimal Plücker formula ([7, p. 269]) and the symmetric product

representation of $SU(2)$ [3]. We classify all the reduced solutions generated by a spherical metric in terms of its monodromy in $PSU(2)$, and then prove Theorem 1.3 in Section 3 by using the characters of some symmetric product representations of E_6, E_7 and E_8 . In the final section, we present new solvable $SU(n+1)$ Toda systems with cone singularities on both the Riemann sphere and compact Riemann surfaces of positive genus.

2. Existence of reduced solutions

In this section, we prove Theorem 1.1. In particular, we first perform some preliminary calculations on the Wronskian of curves in \mathbb{C}^{n+1} , followed by proving the theorem in a local coordinate system. Finally, we apply representation theory and complete the proof on the entire Riemann surface.

2.1. Computation of Wronskian

Assume that U is a domain of \mathbb{C} .

Lemma 2.1. *Let $f = (f_0, \dots, f_n) : U \rightarrow \mathbb{C}^{n+1}$ be a holomorphic curve and $v : U \rightarrow \mathbb{C}$ be a meromorphic function. Then the curve $v \cdot f := (vf_0, \dots, vf_n)$ satisfies*

$$\Lambda_n(v \cdot f) = v^{n+1} \Lambda_n(f).$$

Proof. Omitted. \square

Lemma 2.2. *Let $v : U \rightarrow \mathbb{C}$ be a non-degenerate meromorphic function and $f = (1, \frac{1}{1!}v, \dots, \frac{1}{n!}v^n) : U \rightarrow \mathbb{C}^{n+1}$. Then*

$$\Lambda_n(f) = (v')^{\frac{n(n+1)}{2}}.$$

Proof. We prove it by induction.

- (1) Case $n = 1$ is easy.
- (2) Suppose that $n \geq 2$ and for all $1 \leq k \leq n-1$, we have

$$\Lambda_k \left(1, \frac{1}{1!}v, \dots, \frac{1}{k!}v^k \right) = (v')^{\frac{k(k+1)}{2}}.$$

Then

$$\begin{aligned} \Lambda_n(f) &= \Lambda_n \left(1, \frac{1}{1!}v, \dots, \frac{1}{n!}v^n \right) \\ &= \Lambda_{n-1} \left(\frac{1}{1!}v', \dots, \frac{1}{n!}nv^{n-1}v' \right) \end{aligned}$$

$$\begin{aligned}
&= (v')^{n+1} \Lambda_{n-1} \left(\frac{1}{1!}, \dots, \frac{1}{(n-1)!} v^{n-1} \right) \text{ (by Lemma 2.1)} \\
&= (v')^{n+1} (v')^{\frac{n(n-1)}{2}} \\
&= (v')^{\frac{n(n+1)}{2}} \quad \square
\end{aligned}$$

Lemma 2.3. Let $v_0, v_1 : U \rightarrow \mathbb{C}$ be holomorphic functions such that

$$v_0(z)v_1'(z) - v_1(z)v_0'(z) \equiv 1 \quad \text{on } U.$$

Then the canonical lifting

$$f = \left(\sqrt{\frac{1}{n!}} v_0^n, \sqrt{\frac{1}{(n-1)!1!}} v_0^{n-1} v_1, \dots, \sqrt{\frac{1}{n!}} v_1^n \right) : U \rightarrow \mathbb{C}^{n+1}$$

has Wronskian $\equiv 1$.

Proof. Let $v = v_1/v_0$. Then we have $v' = \frac{1}{v_0^2}$ and

$$\begin{aligned}
\Lambda_n(f) &= \Lambda_n \left(\sqrt{\frac{1}{n!}} v_0^n : \sqrt{\frac{1}{(n-1)!1!}} v_0^{n-1} v_1 : \dots : \sqrt{\frac{1}{n!}} v_1^n \right) \\
&= \Lambda_n \left(\sqrt{\frac{1}{n!}} (v')^{-\frac{n}{2}} : \sqrt{\frac{1}{(n-1)!1!}} (v')^{-\frac{n}{2}} v : \dots : \sqrt{\frac{1}{n!}} (v')^{-\frac{n}{2}} v^n \right) \\
&= (v')^{-\frac{n(n+1)}{2}} \Lambda_n \left(\sqrt{\frac{1}{n!}} : \sqrt{\frac{1}{(n-1)!1!}} v : \dots : \sqrt{\frac{1}{n!}} v^n \right) \text{ (by Lemma 2.1)} \\
&= (v')^{-\frac{n(n+1)}{2}} (v')^{\frac{n(n+1)}{2}} \text{ (by Lemma 2.2)} \\
&= 1. \quad \square
\end{aligned}$$

2.2. Reduced solutions on a chart

Let $\{U, z\}$ be a complex coordinate chart of S . Assume that $\vec{\omega} = (\omega_1 = \frac{\sqrt{-1}}{2} e^{u_1} dz \wedge d\bar{z}, \dots, \omega_n = \frac{\sqrt{-1}}{2} e^{u_n} dz \wedge d\bar{z})$ in U . Then the $SU(n+1)$ Toda system (1.1) takes the following form:

$$\left(\frac{\partial^2 u_1}{\partial z \partial \bar{z}}, \dots, \frac{\partial^2 u_n}{\partial z \partial \bar{z}} \right) = -(e^{u_1}, \dots, e^{u_n}) C_n. \quad (2.1)$$

Thus, we also call (u_1, \dots, u_n) a solution to the $SU(n+1)$ Toda system on U . We now prove the existence of reduced solutions on U .

Lemma 2.4. Let $\omega = \frac{\sqrt{-1}}{2} e^u dz \wedge d\bar{z}$ be a solution to the $SU(2)$ Toda system on U , and let $v : U \rightarrow \mathbb{P}^1$ be a curve associated with ω . Let $r_n : \mathbb{P}^1 \rightarrow \mathbb{P}^n$ be the rational normal map defined by

$$r_n : [z_0, z_1] \mapsto \left[\sqrt{\frac{1}{n!}} z_0^n, \sqrt{\frac{1}{(n-1)!1!}} z_0^{n-1} z_1, \dots, \sqrt{\frac{1}{n!}} z_1^n \right].$$

Then $(i(n+1-i)\omega = \frac{\sqrt{-1}}{2} e^{u+\ln(i(n+1-i))} dz \wedge d\bar{z})_{i=1}^n$ solves the $SU(n+1)$ Toda system on U . Moreover, the set

$$\{U \circ r_n \circ v : U \rightarrow \mathbb{P}^n \mid U \in \text{PSU}(n+1)\}$$

consists of all the associated curves of this solution.

Proof. A direct computation shows that $(u + \ln(i(n+1-i)))_{i=1}^n$ solves (2.1). Denote by $v = [v_0 : v_1]$ the curve $v : U \rightarrow \mathbb{P}^1$ associated to u such that $v_0(z)v_1'(z) - v_1(z)v_0'(z) \equiv 1$ on U . By Lemma 2.3, the canonical lifting

$$\hat{f} = \left(\sqrt{\frac{1}{n!}} v_0^n : \sqrt{\frac{1}{(n-1)!1!}} v_0^{n-1} v_1 : \dots : \sqrt{\frac{1}{n!}} v_1^n \right) : U \rightarrow \mathbb{C}^{n+1}$$

of the curve

$$f = r_n \circ v = \left[\sqrt{\frac{1}{n!}} v_0^n : \sqrt{\frac{1}{(n-1)!1!}} v_0^{n-1} v_1 : \dots : \sqrt{\frac{1}{n!}} v_1^n \right] : U \rightarrow \mathbb{P}^n$$

has Wronskian $\equiv 1$ i.e. $\hat{f} \wedge \hat{f}' \wedge \dots \wedge \hat{f}^{(n)} \equiv e_0 \wedge \dots \wedge e_n$ on U (It also means that f is totally unramified).

It suffices to check that $u + \ln n$ equals the first component u_1 of solution (u_1, \dots, u_n) of (2.1) from the lifting \hat{f} of the curve f . We have

$$\begin{aligned} u_1 &= \log \left(\frac{\|\Lambda_1(\hat{f})\|^2}{\|\hat{f}\|^4} \right) \\ &= \log \left(\frac{\partial^2}{\partial z \partial \bar{z}} \log \|\hat{f}\|^2 \right) \\ &= \log \left(\frac{\partial^2}{\partial z \partial \bar{z}} \log \left(\frac{1}{n!} (|v_0|^2 + |v_1|^2)^n \right) \right) \\ &= \log \left(\frac{\partial^2}{\partial z \partial \bar{z}} \log (|v_0|^2 + |v_1|^2) \right) + \ln n \\ &= u + \ln n, \end{aligned} \tag{2.2}$$

where we use the infinitesimal Plücker formula ([7, p.269]) in the second equality. \square

2.3. Reduced solutions on Riemann surface

Then we achieve the global result considering the monodromy. Firstly, let us recall some facts about the symmetric product space.

Definition 2.5. [2, p.50] Let V be a vector space over \mathbb{C} . The k -th **symmetric product** of V , denoted $\text{Sym}^k(V)$, is the subspace of the k -fold tensor product space $V^{\otimes k}$ consisting of all tensors that are invariant under the action of the symmetric group S_k . Formally, $\text{Sym}^k(V) = \{T \in V^{\otimes k} \mid \sigma(T) = T, \forall \sigma \in S_k\}$, where σ acts on $V^{\otimes k}$ by permuting arguments $\sigma(v_1 \otimes v_2 \otimes \cdots \otimes v_k) = v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(k)}$ for any $v_1, \dots, v_k \in V$.

Definition 2.6. [2, Definition 2.5] The **symmetrization operator** is a map that projects any tensor $T \in V^{\otimes k}$ onto its symmetric part. It is defined as

$$S^k(T) = \frac{1}{k!} \sum_{\sigma \in S_k} \sigma(T).$$

Proposition 2.7. [2, Theorem 2.2]

(1) If $\{e_1, e_2, \dots, e_n\}$ is a basis of V , then a basis of $\text{Sym}^k(V)$ consists of

$$\{S^k(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}) \mid i_1 \leq i_2 \leq \cdots \leq i_k\}.$$

(2) The dimension of $\text{Sym}^k(V)$ is $\binom{n+k-1}{k}$ with $n = \dim V$.

Definition 2.8. [3] Let G be a group, and let V be a finite-dimensional vector space over \mathbb{C} , equipped with a representation of G :

$$\rho : G \rightarrow \text{GL}(V),$$

where $\text{GL}(V)$ is the general linear group of V . The k -th **symmetric product representation** of G , denoted $\text{Sym}^k(V)$, is defined as the natural induced representation of G on the k -th symmetric product space $\text{Sym}^k(V)$, which is a subspace of $V^{\otimes k}$. The action of G on $\text{Sym}^k(V)$ is given by:

$$g \cdot S^k(v_1 \otimes v_2 \otimes \cdots \otimes v_k) = S^k((g \cdot v_1) \otimes (g \cdot v_2) \otimes \cdots \otimes (g \cdot v_k)),$$

for all $g \in G$, $v_1, v_2, \dots, v_k \in V$.

Definition 2.9. Let V be a vector space over \mathbb{C} equipped with a Hermitian inner product $\langle \cdot, \cdot \rangle_V$. Then, for tensors $v_1 \otimes v_2 \otimes \cdots \otimes v_n$ and $w_1 \otimes w_2 \otimes \cdots \otimes w_n$ in $V^{\otimes n}$, the Hermitian inner product on the tensor product space $V^{\otimes n}$ is defined as

$$\langle v_1 \otimes v_2 \otimes \cdots \otimes v_n, w_1 \otimes w_2 \otimes \cdots \otimes w_n \rangle_{V^{\otimes n}} = \prod_{i=1}^n \langle v_i, w_i \rangle_V,$$

which induces a Hermitian inner product on the subspace $\text{Sym}^n(V)$ of $V^{\otimes n}$.

Lemma 2.10. *If $\{e_1, \dots, e_n\}$ is an orthonormal basis of V , then an orthonormal basis of $\text{Sym}^k(V)$ consists of $\left\{ \sqrt{\frac{i_1! \cdots i_k!}{k!}} S^k(e_1^{\otimes i_1} \otimes \cdots \otimes e_n^{\otimes i_n}) \right\}$, where $0 \leq i_1, \dots, i_n \leq k$ and $i_1 + i_2 + \cdots + i_n = k$. Then we obtain the corresponding homogeneous coordinates on both $\mathbb{P}(V)$ and $\mathbb{P}(\text{Sym}^k(V))$.*

Proof. Of course $\{S^k(e_1^{\otimes i_1} \otimes \cdots \otimes e_n^{\otimes i_n})\}$ forms a basis of $\text{Sym}^k(V)$ and $\langle S^k(e_1^{\otimes i_1} \otimes \cdots \otimes e_n^{\otimes i_n}), S^k(e_1^{\otimes j_1} \otimes \cdots \otimes e_n^{\otimes j_n}) \rangle \neq 0$ if and only if $i_1 = j_1, \dots, i_n = j_n$. In addition, $\langle S^k(e_1^{\otimes i_1} \otimes \cdots \otimes e_n^{\otimes i_n}), S^k(e_1^{\otimes i_1} \otimes \cdots \otimes e_n^{\otimes i_n}) \rangle = \binom{k}{i_1, \dots, i_n} = \frac{k!}{i_1! \cdots i_n!}$. Thus, $\left\{ \sqrt{\frac{i_1! \cdots i_n!}{k!}} S^k(e_1^{\otimes i_1} \otimes \cdots \otimes e_n^{\otimes i_n}) \right\}$ forms an orthonormal basis. \square

Definition 2.11. For a projective space $\mathbb{P}(V)$, the Fubini-Study metric can be described as follows:

- (1) In homogeneous coordinates $[u] \in \mathbb{P}(V)$, the Fubini-Study distance between two points $[u], [v] \in \mathbb{P}(V)$ is given by $d_{\text{FS}}([u], [v]) = \arccos\left(\frac{|\langle u, v \rangle|^2}{\langle u, u \rangle \langle v, v \rangle}\right)$, where $\langle u, v \rangle$ is the Hermitian inner product on V .
- (2) The associated Kähler form ω_{FS} is given by $\omega_{\text{FS}} = \sqrt{-1} \partial \bar{\partial} \log \langle u, u \rangle$, where $\langle u, u \rangle$ is the norm square of the vector $u \in V$.

Lemma 2.12. *Let V be a \mathbb{C} -Hermitian space of dimension 2. The rational normal map $r_n : \mathbb{P}(V) \rightarrow \mathbb{P}(\text{Sym}^n(V))$, $[u] \mapsto [u^{\otimes n}]$ induces a Lie group monomorphism $\sigma : \text{PSU}(V) \rightarrow \text{PSU}(\text{Sym}^n(V))$ such that $r_n \circ U = \sigma(U) \circ r_n$ for any $U \in \text{PSU}(V)$.*

Proof. Since

$$\begin{aligned} r_n^* \omega_{\text{FS}, \mathbb{P}(\text{Sym}^n(V))} &= \sqrt{-1} \partial \bar{\partial} \log \langle u^{\otimes n}, u^{\otimes n} \rangle \\ &= \sqrt{-1} \partial \bar{\partial} \log \langle u, u \rangle^n \\ &= n \omega_{\text{FS}, \mathbb{P}(V)}, \end{aligned}$$

we have $(r_n \circ U)^* g_{\text{FS}, \mathbb{P}(\text{Sym}^n(V))} = n U^* g_{\text{FS}, \mathbb{P}(V)} = n g_{\text{FS}, \mathbb{P}(V)} = r_n^* g_{\text{FS}, \mathbb{P}(\text{Sym}^n(V))}$ for any $U \in \text{PSU}(V)$. By the rigidity theorem [7, (4.12)], there exists a unique $U' \in \text{PSU}(\text{Sym}^n(V))$ such that $r_n \circ U = U' \circ r_n$. Defining $\sigma(U) = U'$, we are done.

In addition, let e_0, e_1 be an orthonormal basis of V , and let $[u] = [z_0 e_0 + z_1 e_1]$. Then, we have $[u^{\otimes n}] = [\sum_{i=0}^n z_0^{n-i} z_1^i S^n(e_0^{\otimes n-i} \otimes e_1^{\otimes i})]$. Notice that the orthonormal basis for the

symmetric powers is given by $\left\{ \sqrt{\frac{i!(n-i)!}{n!}} S^n(e_0^{\otimes n-i} \otimes e_1^{\otimes i}) \right\}$. Thus, this map corresponds to the rational normal map

$$[z_0 : z_1] \mapsto \left[\sqrt{\frac{1}{n!}} z_0^n : \sqrt{\frac{1}{1!(n-1)!}} z_0^{n-1} z_1 : \cdots : \sqrt{\frac{1}{n!}} z_1^n \right]. \quad \square$$

Proof of Theorem 1.1. Since ω solves the $SU(2)$ Toda system on S , it is straightforward for us to verify that $(i(n+1-i)\omega)_{i=1}^n$ is a solution to the $SU(n+1)$ Toda system on S .

Consider a chart U with a branch v_0 of v on U . Then, by Lemma 2.4, $r_n \circ v_0$ is an associated curve of this solution restricted to U . Furthermore, $r_n \circ v_0$ is a branch of $r_n \circ v$ on U . Therefore, we need to prove that $r_n \circ v$ is a unitary curve.

For $z \in S$, since the monodromy of v belongs to the group $PSU(2)$, there exists a special unitary representation $\rho : \pi_1(S, z) \rightarrow PSU(V, H)$, where V is the natural representation space \mathbb{C}^2 of $PSU(2)$, and H is the Hermitian inner product on V (with v being viewed as a map $v : S \rightarrow \mathbb{P}(V)$). There is a symmetric product representation $\rho' = \sigma \circ \rho : \pi_1(S, z) \rightarrow PSU(\text{Sym}^n(V), H')$, where $\sigma : PSU(V, H) \rightarrow PSU(\text{Sym}^n(V), H')$ is the embedding induced by r_n (Lemma 2.12), and H' is the Hermitian inner product on $\text{Sym}^n(V)$ induced from H (Definition 2.9).

Assume that the monodromy representation of $r_n \circ v$ is $\varrho : \pi_1(S, z) \rightarrow PSL(\text{Sym}^n(V))$. For $\gamma \in \pi_1(S, z)$ and a branch v_0 of v near z , if we extend v_0 analytically along γ , we get $\rho(\gamma) \circ v_0$. Thus, for a branch $r_n \circ v_0$ of $r_n \circ v$, if we extend $r_n \circ v_0$ analytically along γ , we get both $\varrho(\gamma) \circ r_n \circ v_0$ and $r_n \circ \rho(\gamma) \circ v_0$, which means $\varrho(\gamma) \circ r_n = r_n \circ \rho(\gamma) = \rho'(\gamma) \circ r_n$. Since r_n is non-degenerate, $\varrho = \rho'$ is a unitary representation. So $r_n \circ v$ is a unitary curve. \square

3. Classification of reduced solutions

In this section, we prove Theorem 1.3. Firstly, we describe M_v by the closure of the monodromy group. Then, we achieve the classification from the classification of the closure. Finally, based on the classification, we compute the real dimension of M_v . Moreover, we also use the characters of the symmetric product representations of the natural two-dimensional representations of E_6, E_7 and E_8 . Denote by $C(S)$ the centralizer of a subset $S \subset PSL(n+1, \mathbb{C})$. Recall that $\sigma : PSU(2) \rightarrow PSU(n+1)$ is a monomorphism of the Lie group induced by r_n (Lemma 2.12).

Lemma 3.1. Denote by $G_v \subset PSU(2)$ the monodromy group of a unitary curve $v : S \rightarrow \mathbb{P}^1$. Then

$$\begin{aligned} M_v &= \{ \varphi \in PSL(n+1, \mathbb{C}) \mid \varphi \sigma(G_v) \varphi^{-1} \subset PSU(n+1) \} \\ &= \{ \varphi \in PSL(n+1, \mathbb{C}) \mid \varphi^* \varphi \in C(\sigma(G_v)) \}. \end{aligned}$$

Furthermore, for any unitary curve $v, v_1, v_2 : S \rightarrow \mathbb{P}^1$, the following properties hold:

- (1) $M_{U \circ v} = \{\sigma(U) \circ \varphi \mid \varphi \in M_v\}$ for $U \in \text{PSU}(2)$,
 (2) $M_{v_1} = M_{v_2}$ if $\overline{G}_{v_1} = \overline{G}_{v_2}$.

Proof. For $z \in S$, let $\varrho : \pi_1(S, z) \rightarrow \text{PSU}(n+1)$ be the monodromy representation of $r_n \circ v$. From Section 2, we know that $\text{Im} \varrho = \sigma(G_v)$. For $\gamma \in \pi_1(S, z)$, if we extend a branch $\varphi \circ r_n \circ v_0$ of $\varphi \circ r_n \circ v$ along γ , we obtain the curve $\varphi \circ \varrho(\gamma) \circ r_n \circ v_0 = (\varphi \varrho(\gamma) \varphi^{-1}) \circ \varphi \circ r_n \circ v_0$. Therefore, the monodromy group is given by $\varphi \sigma(G_v) \varphi^{-1}$.

Since we need $\varphi \circ r_n \circ v$ to be a unitary curve, it follows that $\varphi U \varphi^{-1} \in \text{PSU}(n+1)$ for all $U \in \sigma(G_v)$. This implies that $U \varphi^* \varphi = \varphi^* \varphi U$ for all $U \in \sigma(G_v)$. Thus, M_v is given by

$$M_v = \{\varphi \in \text{PSL}(n+1, \mathbb{C}) \mid \varphi^* \varphi \in C(\sigma(G_v))\}.$$

It is straightforward to verify (1). Moreover, (2) follows from the continuity of the left and right multiplications of the Lie group $\text{PSL}(n+1, \mathbb{C})$. \square

Lemma 3.2. M_v has a decomposition of the form $M_v = \text{PSU}(n+1) \Delta_v$, where $\Delta_v = \{\delta \in \Delta_{n+1} \mid \delta^* \delta \in C(\sigma(G_v))\}$ is a subset of Δ_{n+1} .

Proof. For $\varphi \in \text{PSL}(n+1, \mathbb{C})$, assume $\varphi = U \circ \delta$, where $\delta \in \Delta_{n+1}$ and $U \in \text{PSU}(n+1)$. Then we have $\varphi^* \varphi = \delta^* \delta$, which implies that $\varphi \in M_v$ if and only if $\delta \in M_v$. Therefore, we obtain the decomposition $M_v = \text{PSU}(n+1) \Delta_v$. \square

Lemma 3.3 (Cholesky factorization). [8, Corollary 7.2.9]

The map $\Delta_{n+1} \rightarrow \text{Herm}_{n+1}^+(1)$, $\delta \mapsto \delta^* \delta$ is a bijection, where

$$\text{Herm}_{n+1}^+(1) = \{H \in \text{SL}(n+1, \mathbb{C}) \mid H \text{ is positive definite Hermitian}\}.$$

Lemma 3.4 (algorithm for Cholesky factorization). Let $M = (m_{i,j})_{0 \leq i,j \leq n} \in \text{Herm}_{n+1}^+(1)$ be a positive semi-definite Hermitian matrix. Define an upper triangular matrix $\delta = (a_{i,j})$ (i.e., $a_{i,j} = 0$ for $i > j$) by:

- (1) For diagonal entries $j = i$:

$$a_{j,j} = \sqrt{m_{j,j} - \sum_{s=0}^{j-1} |a_{s,j}|^2},$$

- (2) For upper triangular entries $i < j$:

$$a_{i,j} = \frac{1}{a_{i,i}} \left(m_{i,j} - \sum_{s=0}^{i-1} \overline{a_{s,i}} a_{s,j} \right).$$

Then $M = \delta^* \delta$, where δ^* denotes the conjugate transpose of δ .

This is a well-known classic algorithm for Cholesky factorization, which is easy to check, while I'm not sure what the initial article of it is.

Lemma 3.5. *Inherit the notation of the previous lemma. Given $k \in \mathbb{Z}_{>0}$. Then $a_{i,j} = 0$ whenever $k \nmid i - j$ if and only if $m_{i,j} = 0$ whenever $k \nmid i - j$.*

Proof. We prove both directions of the equivalence.

Direction (\Rightarrow): Assume $a_{i,j} = 0$ whenever $k \nmid (i - j)$. Then $M = \delta^* \delta$ satisfies:

$$m_{i,j} = \sum_{s=0}^n \overline{a_{s,i}} a_{s,j}.$$

Fix i, j such that $k \nmid (i - j)$. For the term $\overline{a_{s,i}} a_{s,j}$ to be nonzero, we must have both $a_{s,i} \neq 0$ and $a_{s,j} \neq 0$. By the sparsity of δ , this requires $k \mid (s - i)$ and $k \mid (s - j)$. Consequently:

$$k \mid ((s - i) - (s - j)) = j - i \implies k \mid (i - j),$$

contradicting $k \nmid (i - j)$. Thus $\overline{a_{s,i}} a_{s,j} = 0$ for all s , so $m_{i,j} = 0$.

Direction (\Leftarrow): Assume $m_{i,j} = 0$ whenever $k \nmid (i - j)$. We prove by induction on j (from 0 to n) and on i (from 0 to j) that $a_{i,j} = 0$ for $k \nmid (i - j)$.

- *Base case ($j = 0$):* Trivial (no off-diagonal entries).
- *Inductive step ($j \geq 1$):* Assume the claim holds for all columns $< j$. Then $a_{0,j} = m_{0,j}$ holds. For $j > i \geq 1$, assume the claim holds for all rows $< i$ when column = j .

$$a_{i,j} = \frac{1}{a_{i,i}} \left(m_{i,j} - \sum_{s=0}^{i-1} \overline{a_{s,i}} a_{s,j} \right).$$

If $k \nmid (i - j)$, then $m_{i,j} = 0$. For each $s \in \{0, \dots, i - 1\}$:

- If $k \nmid (s - i)$, then $a_{s,i} = 0$ (by induction on column $i < j$).
- If $k \nmid (s - j)$, then $a_{s,j} = 0$ (by induction on row $s < i$).
- If both nonzero, then $k \mid (s - i)$ and $k \mid (s - j)$, implying $k \mid (i - j)$ (contradiction).

Thus $\overline{a_{s,i}} a_{s,j} = 0$, so $a_{i,j} = 0$. Diagonal entries $a_{j,j}$ have $i - j = 0$ (always divisible by k). \square

Lemma 3.6. *For any unitary curve $v : S \rightarrow \mathbb{P}^1$, let us define a subspace*

$$V_v = \{A \in \text{Mat}_{n+1}(\mathbb{C}) \mid AU = UA, \forall U \in \sigma(G_v)\}$$

of the complex vector space $\text{Mat}_{n+1}(\mathbb{C})$ formed by all $n + 1$ -order matrices. Then we have $\dim_{\mathbb{R}}(\Delta_v) = \dim_{\mathbb{C}}(V_v) - 1$.

Proof. We divide the proof into the following three steps.

- Let $\text{Herm}_{n+1}(1)$ denote the set of $(n+1)$ by $(n+1)$ Hermitian matrices with determinant 1. Since the map $\Delta_{n+1} \rightarrow \text{Herm}_{n+1}^+(1)$, $\delta \mapsto \delta^* \delta$ is a bijection (Lemma 3.3), it induces a bijection $\Delta_v \rightarrow \text{Herm}_{n+1}^+(1) \cap C(\sigma(G_v))$ by restricting the domain to Δ_v . Therefore, $\dim_{\mathbb{R}}(\Delta_v) = \dim_{\mathbb{R}}(\text{Herm}_{n+1}^+(1) \cap C(\sigma(G_v)))$. Since $\text{Herm}_{n+1}^+(1)$ is an open subset of $\text{Herm}_{n+1}(1)$ and $\text{Herm}_{n+1}^+(1) \cap C(\sigma(G)) \neq \emptyset$, we conclude that

$$\dim_{\mathbb{R}}(\Delta_v) = \dim_{\mathbb{R}}(\text{Herm}_{n+1}^+(1) \cap C(\sigma(G_v))) = \dim_{\mathbb{R}}(\text{Herm}_{n+1}(1) \cap C(\sigma(G_v))).$$

- Let Herm_{n+1} denote the set of $(n+1)$ by $(n+1)$ Hermitian matrices, and let \mathcal{H}_{n+1} be its projection in $\mathbb{P}(\text{Mat}_{n+1}(\mathbb{C}))$. Since $\text{Herm}_{n+1}(1)$ is an open dense subset of \mathcal{H}_{n+1} , and $\text{PSL}(n+1, \mathbb{C})$ is an open dense subset of $\mathbb{P}(\text{Mat}_{n+1}(\mathbb{C}))$, it follows that $\text{Herm}_{n+1}(1) \cap C(\sigma(G_v)) = \text{Herm}_{n+1}(1) \cap (\text{PSL}(n+1, \mathbb{C}) \cap \mathbb{P}(V_v))$ is also a non-empty open subset of $\mathcal{H}_{n+1} \cap \mathbb{P}(V_v)$. Therefore, we conclude that

$$\dim_{\mathbb{R}}(\Delta_v) = \dim_{\mathbb{R}}(\text{Herm}_{n+1}(1) \cap C(\sigma(G_v))) = \dim_{\mathbb{R}}(\mathcal{H}_{n+1} \cap \mathbb{P}(V_v)).$$

- Since $\text{Mat}_{n+1}(\mathbb{C}) = \text{Herm}_{n+1} \otimes_{\mathbb{R}} \mathbb{C}$, any matrix $A \in \text{Mat}_{n+1}(\mathbb{C})$ can be expressed uniquely as $A = H_1 + \sqrt{-1}H_2$, where $H_1, H_2 \in \text{Herm}_{n+1}$. For any $U \in \text{SU}(n+1)$, the matrix $U^* H U$ remains Hermitian for any $H \in \text{Herm}_{n+1}$. Hence, $A \in V_v$ if and only if $H_1, H_2 \in \text{Herm}_{n+1} \cap V_v$. Thus, we can write $V_v = (\text{Herm}_{n+1} \cap V_v) \otimes_{\mathbb{R}} \mathbb{C}$, which implies $\dim_{\mathbb{R}}(\text{Herm}_{n+1} \cap V_v) = \dim_{\mathbb{C}}(V_v)$. Since $\mathcal{H}_{n+1} \cap \mathbb{P}(V_v)$ is the projection of $\text{Herm}_{n+1} \cap V_v$ in $\mathbb{P}(\text{Mat}_{n+1}(\mathbb{C}))$, we conclude:

$$\dim_{\mathbb{R}}(\Delta_v) = \dim_{\mathbb{R}}(\mathcal{H}_{n+1} \cap \mathbb{P}(V_v)) = \dim_{\mathbb{R}}(\text{Herm}_{n+1} \cap V_v) - 1 = \dim_{\mathbb{C}}(V_v) - 1. \quad \square$$

Remark 3.7. Recall that $p : \text{SU}(2) \rightarrow \text{PSU}(2)$ is the projection and V is the natural representation of $\text{SU}(2)$. Because $p^{-1}(G_v)$ is a subgroup of $\text{SU}(V)$, we could see $\text{Sym}^n(V)$ as a representation space of $p^{-1}(G_v)$. Then V_v is just the space $\text{End}_{p^{-1}(G_v)}(\text{Sym}^n(V))$ of module homomorphisms.

Let us recall some results of the representation theory.

Lemma 3.8. [3] Let G be a group. If $V = V_1^{\oplus a_1} \oplus \cdots \oplus V_n^{\oplus a_n}$ is a complex representation of G , where all $V_i, i = 1, \dots, n$ are distinct irreducible representation spaces, then $\dim_{\mathbb{C}} \text{End}_G(V) = a_1^2 + \cdots + a_n^2$. In particular, when G is a finite group, $\dim_{\mathbb{C}} \text{End}_G(V) = \frac{1}{|G|} \sum_{g \in G} |\chi_V(g)|^2$, where $\chi_V : G \rightarrow \mathbb{C}$ is the character of V .

Proof of Theorem 1.3. (1) $G_v = \text{PSU}(2)$.

Notice $\sigma : \text{PSU}(2) = \text{PSU}(V) \rightarrow \text{PSU}(n+1) = \text{PSU}(\text{Sym}^n(V))$ (Lemma 2.12) is a irreducible representation of $\text{PSU}(2)$. It is irreducible because symmetric product representation $\text{Sym}^n(V)$ is an irreducible representation of $\text{SL}(V)$ [3, Section 11.1].

Thus $C(\text{Im } \sigma) = \{I\}$ by Schur's Lemma [3, Lemma 1.7]. Therefore, $\delta \in \delta_v$ if and only if $\delta^* \delta = I_{n+1}$, which means $\delta = I_{n+1}$. Consequently, we conclude that $\Delta_v = \{I_{n+1}\}$.

(2) $G_v = \text{PU}(1)$.

We have $\sigma(G_v) = \{\text{diag}(c^n, c^{n-2}, \dots, c^{-n}) \mid |c| = 1\}$. Then $C(\sigma(\text{PU}(1))) = \{\text{all diagonal matrices}\}$. Thus, $\delta \in \Delta_v$ if and only if $\delta^* \delta$ is diagonal. Since δ is induced by an upper triangular matrix with positive diagonal entries, it must be diagonal with positive entries, which implies

$$\Delta_v = \{\text{diag}(a_0, \dots, a_n) \in \Delta_{n+1}\}.$$

It is obviously $\dim_{\mathbb{R}} \Delta_v = n$.

(3) $G_v = \text{PO}(2)$

Let $g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Since $C(\sigma(g)) \cap C(\sigma(\text{PU}(1))) = \{\text{diag}(\lambda_1, \dots, \lambda_{n+1}) \mid \lambda_i = \lambda_{n+1-i}\}$, it follows that $C(\sigma(G_v)) = \{\text{diag}(\lambda_1, \dots, \lambda_{n+1}) \mid \lambda_i = \lambda_{n+1-i}\}$. Then $\delta \in \Delta_v$ if and only if $\delta^* \delta \in C(\sigma(G_v))$. Since δ is an upper triangular matrix with positive diagonal entries, it implies

$$\Delta_v = \{\text{diag}(a_0, \dots, a_n) \in \Delta_{n+1} \mid a_i = a_{n-i}\}.$$

It is obviously $\dim_{\mathbb{R}} \Delta_v = \lfloor \frac{n}{2} \rfloor$.

(4) $G_v = p(C_k)$.

The group $\sigma(G)$ is generated by $\text{diag}(\xi_k^n, \xi_k^{n-2}, \dots, \xi_k^{-n})$, where ξ_k is a primitive k -th root of unity. Then, the centralizer of $\sigma(G_v)$ in $\text{PSL}(n+1, \mathbb{C})$ is $C(\sigma(G_v)) = \{(z_{ij})_{0 \leq i, j \leq n} \mid z_{ij} = 0 \text{ if } k \nmid 2(i-j)\} = \{(z_{ij})_{0 \leq i, j \leq n} \mid z_{ij} = 0 \text{ if } \frac{k}{\gcd(k, 2)} \nmid (i-j)\}$. Then, $\delta \in \Delta_v$ if and only if $\delta^* \delta \in C(\sigma(G_v))$. Hence, by Lemma 3.5

$$\Delta_v = \{(a_{ij})_{0 \leq i, j \leq n} \in \Delta_{n+1} \mid a_{ij} = 0 \text{ if } \frac{k}{\gcd(k, 2)} \nmid (i-j)\}.$$

Thus, the number of independent equations given by $C(\sigma(G_v))$ are $|\{(i, j) \mid 0 \leq i, j \leq n, \frac{k}{\gcd(k, 2)} \nmid (i-j)\}|$. It means $\dim_{\mathbb{R}} \Delta_v = |\{(i, j) \mid 0 \leq i, j \leq n, \frac{k}{\gcd(k, 2)} \mid (i-j)\}| - 1$. The number can be a sum by row:

$$\begin{aligned} \dim_{\mathbb{R}} \Delta_v &= 2 \sum_{i=0}^n \lfloor \frac{i \gcd(k, 2)}{k} \rfloor + (n+1) - 1 \\ &= -\frac{k}{\gcd(k, 2)} \lfloor \frac{n \gcd(k, 2)}{k} \rfloor^2 + (2n+2 - \frac{k}{\gcd(k, 2)}) \lfloor \frac{n \gcd(k, 2)}{k} \rfloor + n \end{aligned}$$

(5) $G_v = p(D_k)$.

In this case, the centralizer $C(\sigma(G_v))$ can be expressed as $C(\sigma(G_v)) = \{(z_{i,j})_{0 \leq i, j \leq n} \in C(\sigma(p(C_{2k}))) \mid z_{i,j} = (-\sqrt{-1})^{i-j} z_{n-i, n-j}\}$. Then $\delta \in \Delta_v$ if and only if $\delta^* \delta \in C(\sigma(G_v))$. Hence, by Lemma 3.5,

$$\Delta_v = \left\{ (a_{i,j})_{0 \leq i,j \leq n} \in \Delta_{n+1} \left| \begin{array}{l} a_{i,j} = 0 \text{ if } k \nmid i-j \\ \sum_{l=0}^n \bar{a}_{l,i} a_{l,j} = (-\sqrt{-1})^{i-j} \sum_{l=0}^n \bar{a}_{l,n-i} a_{l,n-j} \end{array} \right. \right\}.$$

The number of independent equations given by $C(\sigma(G_v))$ are $|\{(i,j)|0 \leq i,j \leq n, k \nmid i-j\}| + |\{(i,j)|0 \leq i,j \leq n, k \mid i-j\}|/2$. Thus

$$\begin{aligned} \dim_{\mathbb{R}} \Delta_v &= n^2 + 2n - |\{(i,j)|0 \leq i,j \leq n, k \nmid i-j\}| \\ &\quad - |\{(i,j)|0 \leq i,j \leq n, k \mid i-j\}|/2 \\ &= |\{(i,j)|k \mid i-j\}|/2 \\ &= -\frac{k}{2} \lfloor \frac{n}{k} \rfloor^2 + (n+1 - \frac{k}{2}) \lfloor \frac{n}{k} \rfloor + \lfloor \frac{n}{2} \rfloor \end{aligned}$$

(6) $G_v = p(E_6)$.

Denote $g_1 = \begin{pmatrix} 0 & -\sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix}$, $g_2 = \begin{pmatrix} \frac{1+\sqrt{-1}}{2} & \frac{1+\sqrt{-1}}{2} \\ \frac{\sqrt{-1}-1}{2} & \frac{1-\sqrt{-1}}{2} \end{pmatrix}$. Let V be the natural representation of $SU(2)$ and $\rho : E_6 \rightarrow SU(V)$ be the given embedding. Then $V_v = \text{End}_{E_6}(\text{Sym}^n(V))$. Since E_6 is a finite group, direct computation shows that the conjugacy classes of E_6 are listed in the following table (which is also a well-known result of tetrahedral group):

conjugacy classes	I	$-I$	g_1	g_2	g_2^2	g_2^4	g_2^5
their cardinality	1	1	6	4	4	4	4

Notice that $\chi_{\text{Sym}^n(V)}(g) = \sum_{i=0}^n \lambda_1^i \lambda_2^{n-i}$, where λ_1, λ_2 are two eigenvalues of $\rho(g)$. The character of $\text{Sym}^n(V)$ is as the following table:

$\text{Sym}^n(V)$	I	$-I$	g_1	g_2	g_2^2	g_2^4	g_2^5
$n \equiv 1 \pmod{2}$	$n+1$	$-n-1$	0	c	$-c$	$-c$	c
$n \equiv 0 \pmod{2}$	$n+1$	$n+1$	$(-1)^{n/2}$	c	c	c	c

where $c = \cos \frac{n\pi}{3} + \frac{\sqrt{3}}{3} \sin \frac{n\pi}{3}$. Thus,

- i. When n is odd, by Lemma 3.8, we have $\dim_{\mathbb{C}} V_v = \frac{n^2}{12} + \frac{2c^2}{3} + \frac{n}{6} + \frac{1}{12}$. Thus, $\dim_{\mathbb{R}} \Delta_v = \frac{n^2}{12} + \frac{2c^2}{3} + \frac{n}{6} - \frac{11}{12}$.
- ii. When n is even, by Lemma 3.8, we have $\dim_{\mathbb{C}} V_v = \frac{n^2}{12} + \frac{n}{6} + \frac{2c^2}{3} + \frac{1}{3}$. Thus, $\dim_{\mathbb{R}} \Delta_v = \frac{n^2}{12} + \frac{n}{6} + \frac{2c^2}{3} - \frac{2}{3}$.

(7) $G_v = p(E_7)$. Let $g_1 = \begin{pmatrix} 0 & \frac{\sqrt{2}(1+\sqrt{-1})}{2} \\ \frac{\sqrt{2}(\sqrt{-1}-1)}{2} & 0 \end{pmatrix}$ and $g_2 = \begin{pmatrix} \frac{1+\sqrt{-1}}{2} & \frac{1+\sqrt{-1}}{2} \\ \frac{\sqrt{-1}-1}{2} & \frac{1-\sqrt{-1}}{2} \end{pmatrix}$. Let V be the natural representation of $SU(2)$ and $\rho :$

$E_7 \rightarrow \mathrm{SU}(V)$ be the given embedding. Then $V_v = \mathrm{End}_{E_7}(\mathrm{Sym}^n(V))$. The conjugacy classes of E_7 are listed as (which is also a well-known result of octahedral group)

conjugacy classes	I_2	$-I_2$	$(g_1 g_2)^2$	g_2	g_2^2	$g_1 g_2$	$(g_1 g_2)^3$	g_1
their cardinality	1	1	6	8	8	6	6	12

Similar to E_6 , the character of $\mathrm{Sym}^n(V)$ of E_7 is expressed in the following table:

$\mathrm{Sym}^n(V)$	I_2	$-I_2$	$(g_1 g_2)^2$	g_2	g_2^2	$g_1 g_2$	$(g_1 g_2)^3$	g_1
$n \equiv 1 \pmod{2}$	$n+1$	$-n-1$	0	c_1	$-c_1$	$-c_2$	c_2	0
$n \equiv 0 \pmod{2}$	$n+1$	$n+1$	$(-1)^{n/2}$	c_1	c_1	c_2	c_2	$(-1)^{n/2}$

where $c_1 = \cos \frac{n\pi}{3} + \frac{\sqrt{3}}{3} \sin \frac{n\pi}{3}$ and $c_2 = \cos \frac{n\pi}{4} + \sin \frac{n\pi}{4}$. Thus,

- i. When n is odd, by Lemma 3.8, we have $\dim_{\mathbb{C}} V_v = \frac{n^2}{24} + \frac{n}{12} + \frac{c_1^2}{3} + \frac{c_2^2}{4} + \frac{1}{24}$. Thus, $\dim_{\mathbb{R}} \Delta_v = \frac{n^2}{24} + \frac{n}{12} + \frac{c_1^2}{3} + \frac{c_2^2}{4} - \frac{23}{24}$.
- ii. When n is even, by Lemma 3.8, we have $\dim_{\mathbb{C}} V_v = \frac{n^2}{24} + \frac{n}{12} + \frac{c_1^2}{3} + \frac{c_2^2}{4} + \frac{5}{12}$. Thus, $\dim_{\mathbb{R}} \Delta_v = \frac{n^2}{24} + \frac{n}{12} + \frac{c_1^2}{3} + \frac{c_2^2}{4} - \frac{7}{12}$.

(8) $G_v = p(E_8)$. Let

$$g_1 = \begin{pmatrix} 0 & -\sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix} \quad \text{and} \quad g_2 = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{5}-1}{4} + \frac{\sqrt{5}+1}{4}\sqrt{-1} \\ \frac{\sqrt{5}-1}{4} + \frac{\sqrt{5}+1}{4}\sqrt{-1} & \frac{1}{2} \end{pmatrix}.$$

Let V be the natural representation of $\mathrm{SU}(2)$ and $\rho : E_8 \rightarrow \mathrm{SU}(V)$ be the given embedding. Then $V_v = \mathrm{End}_{E_8}(\mathrm{Sym}^n(V))$. The conjugacy classes of E_8 are listed as (which is also a well-known result of icosahedral group):

conjugacy classes	I	$-I$	g_2^2	g_1	$(g_1 g_2)^2$	$(g_1 g_2)^4$	g_2	$g_1 g_2$	$(g_1 g_2)^3$
their cardinality	1	1	20	30	12	12	20	12	12

Similar to E_6 , the character of $\mathrm{Sym}^n(V)$ is expressed in the following table:

$\mathrm{Sym}^n(V)$	I	$-I$	g_2^2	g_1	$(g_1 g_2)^2$	$(g_1 g_2)^4$	g_2	$g_1 g_2$	$(g_1 g_2)^3$
$n \equiv 1 \pmod{1}$	$n+1$	$-n-1$	$-c_1$	0	c_3	$-c_2$	c_1	c_2	$-c_3$
$n \equiv 0 \pmod{2}$	$n+1$	$n+1$	c_1	$(-1)^{n/2}$	c_3	c_2	c_1	c_2	c_3

where $c_1 = \cos \frac{n\pi}{3} + \frac{\sqrt{3}}{3} \sin \frac{n\pi}{3}$, $c_2 = \sqrt{1 + \frac{2}{\sqrt{5}}} \sin \frac{n\pi}{5} + \cos \frac{n\pi}{5}$ and $c_3 = \sqrt{1 - \frac{2}{\sqrt{5}}} \sin \frac{2n\pi}{5} + \cos \frac{2n\pi}{5}$. Thus,

- i. When n is odd, by Lemma 3.8, we have $\dim_{\mathbb{C}} V_v = \frac{n^2}{60} + \frac{n}{30} + \frac{c_1^2}{3} + \frac{c_2^2}{5} + \frac{c_3^2}{5} + \frac{1}{60}$.
Thus, $\dim_{\mathbb{R}} \Delta_v = \frac{n^2}{60} + \frac{n}{30} + \frac{c_1^2}{3} + \frac{c_2^2}{5} + \frac{c_3^2}{5} - \frac{59}{60}$.
- ii. When n is even, by Lemma 3.8, we have $\dim_{\mathbb{C}} V_v = \frac{n^2}{60} + \frac{n}{30} + \frac{c_1^2}{3} + \frac{c_2^2}{5} + \frac{c_3^2}{5} + \frac{2}{15}$.
Thus, $\dim_{\mathbb{R}} \Delta_v = \frac{n^2}{60} + \frac{n}{30} + \frac{c_1^2}{3} + \frac{c_2^2}{5} + \frac{c_3^2}{5} - \frac{13}{15}$. \square

4. Examples

Let us recall the result of the spherical metric.

Theorem 4.1. [10, Theorem A] Let $g > 0$ be an integer. Assume $\beta_1, \dots, \beta_m > 0$ satisfy

$$\beta_1 + \dots + \beta_m > 2g - 2 + m,$$

then there exists a compact orientable Riemann surface X of genus g with a spherical metric ω on X that represents $D = \sum_{j=1}^m (\beta_j - 1)[P_j]$ for some distinct points $P_1, \dots, P_m \in X$.

Then there will be a natural corollary.

Corollary 4.2. Let X , ω and D be the same as above. Then, for each positive integer $n > 1$, the $SU(n+1)$ Toda system on X with cone singularities

$$\underbrace{(D, D, \dots, D)}_{n \text{ divisors}}$$

has a family of reduced solutions, including $(i(n+1-i)\omega)_{i=1}^n$ and is characterized in Theorem 1.3.

Remark 4.3. Consider the $SU(n+1)$ -Toda system with cone singularities

$$\text{Ric}(\vec{\omega}) = 2\vec{\omega}C_n + (\delta_{P_1}, \dots, \delta_{P_m})\Gamma,$$

where δ_P denotes the Dirac measure at P and $\Gamma = (\gamma_{j,i})_{m \times n}$ is a real matrix with $\gamma_{j,i} > -1$. The solution $\vec{\omega}$ represents an n -tuple of divisors $(D_i = \sum_{j=1}^m \gamma_{j,i}[P_j])_{i=1}^n$. The readers may find the detail of this framework of Toda system with cone singularities in [11, Section 1]. When $\gamma_{j,i} = \beta_j - 1$ for all i and j , this corollary shows that the system with cone singularities is solvable. It should be noted that Lin, Yang and Zhong [9, Theorem 1.9] provide a sufficient condition for the solvability of the Toda system with cone singularities. Our corollary, however, offers a different sufficient condition. These conditions are not equivalent. For example, in the case $n > 1, \beta_i \in \mathbb{Z}_{>1}, g > 0$, which does not satisfy the condition in [9, Theorem 1.9], our corollary demonstrates that the system is solvable.

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Data availability

No data was used for the research described in the article.

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