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### Regular Article

## Solutions to $SU(n+1)$ Toda system generated by spherical metrics



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### ABSTRACT

Following A.B. Givental (1989) [5], we refer to an  $n$ -tuple  $(\omega_1, \dots, \omega_n)$  of Kähler forms on a Riemann surface  $S$  as a solution to the  $SU(n+1)$  Toda system if and only if

$$(\text{Ric}(\omega_1), \dots, \text{Ric}(\omega_n)) = (2\omega_1, \dots, 2\omega_n)C_n,$$

where  $C_n$  is the Cartan matrix of type  $A_n$ . In particular, when  $n = 1$ , this solution corresponds to a spherical metric. Using the correspondence between solutions and totally unramified unitary curves, we show that a spherical metric  $\omega$  generates a family of solutions, including  $(i(n+1-i)\omega)_{i=1}^n$ . Moreover, we characterize this family in terms of the monodromy group of the spherical metric. As a consequence, we obtain a new solution class to the  $SU(n+1)$  Toda system with cone

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singularities on compact Riemann surfaces, complementing the existence results of Lin et al. (2020) [9].

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## 1. Introduction

We present a natural and precise method for generating solutions to the  $SU(n+1)$  Toda system on Riemann surfaces using spherical metrics (Theorems 1.1 and 1.3). As a consequence, we identify a new class of solutions to the  $SU(n+1)$  Toda system with cone singularities on compact Riemann surfaces (Corollary 1.5), which complements the results in [9, Theorems 1.8 and 1.9]. To obtain these results, we employ the complex differential-geometric framework for solutions to the  $SU(n+1)$  Toda system with cone singularities, as established in [5] and [11, Subsections 1.1 and 1.2]. For further details, interested readers may refer to [11, Section 1] for the latest developments in this field.

Let  $S$  be a Riemann surface, not necessarily compact, and let  $n$  be a positive integer. An  $n$ -tuple  $\vec{\omega} = (\omega_1, \dots, \omega_n)$  of Kähler forms is called a *solution to the  $SU(n+1)$  Toda system* ([11, Definition 1]) on  $S$  if and only if

$$\text{Ric}(\vec{\omega}) = 2\vec{\omega}C_n, \quad (1.1)$$

where  $\text{Ric}(\vec{\omega}) = (\text{Ric}(\omega_1), \dots, \text{Ric}(\omega_n))$  is the  $n$ -tuple of Ricci forms, and

$$C_n = \begin{pmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 2 \end{pmatrix}_{n \times n}$$

is the Cartan matrix of type  $A_n$ . In particular, a solution  $\omega_1$  to the  $SU(2)$  Toda system coincides with a conformal spherical metric on  $S$ .

In 2022, we made the simple observation that if  $\omega$  is a solution to the  $SU(2)$  Toda system, then  $(i(n+1-i)\omega)_{i=1}^n$  solves the  $SU(n+1)$  Toda system on  $S$ . In this paper, we will develop this strategy in detail using the *basic correspondence between solutions to the  $SU(n+1)$  Toda system on  $S$  and the totally unramified unitary curves from  $S$  to the complex projective space  $\mathbb{P}^n$  of dimension  $n$* . The definition of a totally unramified unitary curve and the proof of this correspondence can be found in [11, Subsection 1.2 and Section 2]. Simply put, a totally unramified unitary curve  $f : S \rightarrow \mathbb{P}^n$  is a multi-valued holomorphic map whose monodromy group resides within  $\text{PSU}(n+1)$  and any local germs are totally unramified. We also refer to a unitary curve corresponding to a solution as an *associated curve of the solution*. Any two associated curves of a solution

differ by a rigid motion of  $\mathbb{P}^n$  endowed with the Fubini-Study metric  $\omega_{\text{FS}}$  ([6, (4.12)]). In particular, an associated curve of the solution  $\omega$  to the SU(2) Toda system coincides with the developing map of the spherical metric  $\omega$  on  $S$  ([1, Section 2]). First, we characterize unitary curves  $S \rightarrow \mathbb{P}^n$  associated with the solution  $(i(n+1-i)\omega)_{i=1}^n$  in terms of a unitary curve  $S \rightarrow \mathbb{P}^1$  associated with  $\omega$ .

**Theorem 1.1.** *Let  $\omega$  be a solution to the SU(2) Toda system on  $S$ , and let  $v : S \rightarrow \mathbb{P}^1$  be a curve associated with  $\omega$ . Let  $r_n : \mathbb{P}^1 \rightarrow \mathbb{P}^n$  be the rational normal map defined by*

$$r_n : [z_0, z_1] \mapsto \left[ \sqrt{\frac{1}{n!}} z_0^n, \sqrt{\frac{1}{(n-1)!1!}} z_0^{n-1} z_1, \dots, \sqrt{\frac{1}{1!}} z_1^n \right].$$

*Then  $(i(n+1-i)\omega)_{i=1}^n$  solves the SU( $n+1$ ) Toda system on  $S$ . Moreover, the set*

$$\{U \circ r_n \circ v : S \rightarrow \mathbb{P}^n \mid U \in \text{PSU}(n+1)\}$$

*consists of all the associated curves of this solution.*

Given a basis of  $\mathbb{C}^{n+1}$  endowed with the standard Hermitian inner product  $\langle \cdot, \cdot \rangle$ , the Gram-Schmidt procedure provides a new orthonormal basis of  $(\mathbb{C}^{n+1}, \langle \cdot, \cdot \rangle)$ . Then, we obtain the Iwasawa decomposition of  $\text{SL}(n+1, \mathbb{C})$  in the form

$$\text{SL}(n+1, \mathbb{C}) = \text{SU}(n+1) \Delta_{n+1},$$

where  $\text{SU}(n+1)$  is the group of special unitary transformations and  $\Delta_{n+1}$  is the group of linear transformations by left multiplication of upper triangular matrices with positive diagonal entries in  $\text{SL}(n+1, \mathbb{C})$ . Hence, an automorphism  $\varphi \in \text{PSL}(n+1, \mathbb{C})$  of  $\mathbb{P}^n$  has the decomposition  $\varphi = U \circ \delta$ , where  $U \in \text{PSU}(n+1)$  and  $\delta \in \Delta_{n+1}$ . Based on this and Theorem 1.1, we introduce the following definition:

**Definition 1.2.** We call a solution  $\vec{\omega}$  to the SU( $n+1$ ) Toda system *reduced* if and only if it is generated by another solution  $\omega_1$  to the SU(2) Toda system, i.e., a conformal spherical metric, on  $S$  in the following sense: there exists a linear transformation  $\delta \in \Delta_{n+1}$ , an associated curve  $f : S \rightarrow \mathbb{P}^n$  of  $\vec{\omega}$ , and an associated curve  $v : S \rightarrow \mathbb{P}^1$  of  $\omega_1$  such that

$$f = \delta \circ r_n \circ v. \tag{1.2}$$

Notably, the curve  $f$  should have monodromy in  $\text{PSU}(n+1)$ , which imposes a constraint on the variety of such  $\delta$ 's (Theorem 1.3).

Given a solution  $\omega_1$  to the SU(2) Toda system, we can characterize all the reduced solutions to the SU( $n+1$ ) Toda system generated by it in the following theorem:

**Theorem 1.3.** *We use the notions in Theorem 1.1. Let*

$$M_v = \{\varphi \in \mathrm{PSL}(n+1, \mathbb{C}) | \varphi \circ r_n \circ v : S \rightarrow \mathbb{P}^n \text{ is a unitary curve}\}.$$

$M_v$  can be decomposed into  $M_v = \mathrm{PSU}(n+1)\Delta_v$ , where  $\Delta_v \subset \Delta_{n+1}$ . It is determined by the closure  $\overline{G_v}$  in  $\mathrm{PSU}(2)$  of the monodromy group  $G_v$  of  $v$ . Consider the classification of closed subgroups of  $\mathrm{SU}(2)$  ([4, Chapter 1]):

$$\begin{aligned} \mathrm{O}(2) &= \left\langle \mathrm{U}(1), \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle, \quad \mathrm{U}(1), \\ C_k &= \left\langle \begin{pmatrix} e^{2\pi\sqrt{-1}/k} & 0 \\ 0 & e^{-2\pi\sqrt{-1}/k} \end{pmatrix} \right\rangle, k \in \mathbb{Z}_{>0}, \\ D_k &= \left\langle C_{2k}, \begin{pmatrix} 0 & -\sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix} \right\rangle, k \in \mathbb{Z}_{>0}, \\ E_6 &= \left\langle \begin{pmatrix} \frac{1+\sqrt{-1}}{2} & \frac{1+\sqrt{-1}}{2} \\ \frac{\sqrt{-1}-1}{2} & \frac{1-\sqrt{-1}}{2} \end{pmatrix}, \begin{pmatrix} 0 & -\sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix} \right\rangle, \\ E_7 &= \left\langle \begin{pmatrix} \frac{1+\sqrt{-1}}{2} & \frac{1+\sqrt{-1}}{2} \\ \frac{\sqrt{-1}-1}{2} & \frac{1-\sqrt{-1}}{2} \end{pmatrix}, \begin{pmatrix} 0 & \frac{\sqrt{2}(1+\sqrt{-1})}{2} \\ \frac{\sqrt{2}(\sqrt{-1}-1)}{2} & 0 \end{pmatrix} \right\rangle, \\ E_8 &= \left\langle \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{5}-1}{4} + \frac{\sqrt{5}+1}{4}\sqrt{-1} \\ \frac{\sqrt{5}-1}{4} + \frac{\sqrt{5}+1}{4}\sqrt{-1} & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 & -\sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix} \right\rangle. \end{aligned}$$

Let  $p : \mathrm{SU}(2) \rightarrow \mathrm{PSU}(2)$  be the quotient map. Then there hold the following statements:

- (1) When  $\overline{G_v} = \mathrm{PSU}(2)$ ,  $\Delta_v = \{I_{n+1}\}$ ;
- (2) When  $\overline{G_v} = \mathrm{PU}(1)$ ,  $\Delta_v = \{\mathrm{diag}(a_0, \dots, a_n) \in \Delta_{n+1}\}$  with  $\dim_{\mathbb{R}} \Delta_v = n$ ;
- (3) When  $\overline{G_v} = \mathrm{PO}(2)$ ,  $\Delta_v = \{\mathrm{diag}(a_0, \dots, a_n) \in \Delta_{n+1} | a_i = a_{n-i}\}$  with  $\dim_{\mathbb{R}} \Delta_v = \lfloor n/2 \rfloor$ ;
- (4) When  $\overline{G_v} = p(C_k)$ ,  $\Delta_v = \{(a_{ij})_{0 \leq i,j \leq n} \in \Delta_{n+1} | a_{i,j} = 0 \text{ if } \frac{k}{\gcd(k,2)} \nmid (i-j)\}$  with

$$\dim_{\mathbb{R}} \Delta_v = -\frac{k}{\gcd(k,2)} \lfloor \frac{n \gcd(k,2)}{k} \rfloor^2 + (2n+2 - \frac{k}{\gcd(k,2)}) \lfloor \frac{n \gcd(k,2)}{k} \rfloor + n;$$

- (5) When  $\overline{G_v} = p(D_k)$ ,

$$\Delta_v = \left\{ (a_{i,j})_{0 \leq i,j \leq n} \in \Delta_{n+1} \left| \begin{array}{l} \sum_{l=0}^n \bar{a}_{l,i} a_{l,j} = 0 \text{ if } k \nmid i-j \\ \sum_{l=0}^n \bar{a}_{l,i} a_{l,j} = (-\sqrt{-1})^{i-j} \sum_{l=0}^n \bar{a}_{l,n-i} a_{l,n-j} \end{array} \right. \right\}$$

with  $\dim_{\mathbb{R}} \Delta_v = -\frac{k}{2} \lfloor \frac{n}{k} \rfloor^2 + (n+1 - \frac{k}{2}) \lfloor \frac{n}{k} \rfloor + \lfloor \frac{n}{2} \rfloor$ ;

(6) When  $\overline{G_v} = p(E_6)$ ,

$$\dim_{\mathbb{R}} \Delta_v = \begin{cases} \dim_{\mathbb{R}} \Delta_v = \frac{n^2}{12} + \frac{2c^2}{3} + \frac{n}{6} - \frac{11}{12} & \text{if } n \text{ is odd} \\ \dim_{\mathbb{R}} \Delta_v = \frac{n^2}{12} + \frac{2c^2}{3} + \frac{n}{6} - \frac{2}{3} & \text{if } n \text{ is even} \end{cases},$$

where  $c = \cos \frac{n\pi}{3} + \frac{\sqrt{3}}{3} \sin \frac{n\pi}{3}$ ;

(7) When  $\overline{G_v} = p(E_7)$ ,

$$\dim_{\mathbb{R}} \Delta_v = \begin{cases} \frac{n^2}{24} + \frac{n}{12} + \frac{c_1^2}{3} + \frac{c_2^2}{4} - \frac{23}{24} & \text{if } n \text{ is odd} \\ \frac{n^2}{24} + \frac{n}{12} + \frac{c_1^2}{3} + \frac{c_2^2}{4} - \frac{7}{12} & \text{if } n \text{ is even} \end{cases},$$

where  $c_1 = \cos \frac{n\pi}{3} + \frac{\sqrt{3}}{3} \sin \frac{n\pi}{3}$  and  $c_2 = \cos \frac{n\pi}{4} + \sin \frac{n\pi}{4}$ ;

(8) When  $\overline{G_v} = p(E_8)$ ,

$$\dim_{\mathbb{R}} \Delta_v = \begin{cases} \frac{n^2}{60} + \frac{n}{30} + \frac{c_1^2}{3} + \frac{c_2^2}{5} + \frac{c_3^2}{5} - \frac{59}{60} & \text{if } n \text{ is odd} \\ \frac{n^2}{60} + \frac{n}{30} + \frac{c_1^2}{3} + \frac{c_2^2}{5} + \frac{c_3^2}{5} - \frac{13}{15} & \text{if } n \text{ is even} \end{cases},$$

where  $c_1 = \cos \frac{n\pi}{3} + \frac{\sqrt{3}}{3} \sin \frac{n\pi}{3}$ ,  $c_2 = \sqrt{1 + \frac{2}{\sqrt{5}}} \sin \frac{n\pi}{5} + \cos \frac{n\pi}{5}$  and  $c_3 = \sqrt{1 - \frac{2}{\sqrt{5}}} \sin \frac{2n\pi}{5} + \cos \frac{2n\pi}{5}$ .

**Remark 1.4.** Notice that all the cases are possible: for any closed subgroup of  $\mathrm{PSU}(2)$ , there exists a multi-valued meromorphic function such that the closure of its monodromy group is the given subgroup. The dimensions of cases (6)–(8) arise from norms of characters of finite-dimensional representations of finite groups; hence, they are naturally integers, although their forms appear complicated.

As an application of Theorems 1.1 and 1.3, we identify a novel class of solvable  $\mathrm{SU}(n+1)$  Toda systems with cone singularities on compact Riemann surfaces as follows:

**Corollary 1.5.** We adopt the notions introduced in [11, Subsection 1.1]. Suppose that there exists a cone spherical metric that represents the real divisor  $D = \sum_{j=1}^n \gamma_j [P_j]$ , where  $0 \neq \gamma_j > -1$  for all  $1 \leq j \leq n$ , on a compact Riemann surface  $X$ . Then, for each positive integer  $n > 1$ , the  $\mathrm{SU}(n+1)$  Toda system on  $X$  with cone singularities

$$\underbrace{(D, D, \dots, D)}_{n \text{ divisors}}$$

has a family of reduced solutions, including  $(i(n+1-i)\omega)_{i=1}^n$  and is characterized in Theorem 1.3.

We organize the remainder of this paper as follows. In Section 2, we prove Theorem 1.1 using the infinitesimal Plücker formula ([7, p. 269]) and the symmetric product

representation of  $SU(2)$  [3]. We classify all the reduced solutions generated by a spherical metric in terms of its monodromy in  $PSU(2)$ , and then prove Theorem 1.3 in Section 3 by using the characters of some symmetric product representations of  $E_6, E_7$  and  $E_8$ . In the final section, we present new solvable  $SU(n+1)$  Toda systems with cone singularities on both the Riemann sphere and compact Riemann surfaces of positive genus.

## 2. Existence of reduced solutions

In this section, we prove Theorem 1.1. In particular, we first perform some preliminary calculations on the Wronskian of curves in  $\mathbb{C}^{n+1}$ , followed by proving the theorem in a local coordinate system. Finally, we apply representation theory and complete the proof on the entire Riemann surface.

### 2.1. Computation of Wronskian

Assume that  $U$  is a domain of  $\mathbb{C}$ .

**Lemma 2.1.** *Let  $f = (f_0, \dots, f_n) : U \rightarrow \mathbb{C}^{n+1}$  be a holomorphic curve and  $v : U \rightarrow \mathbb{C}$  be a meromorphic function. Then the curve  $v \cdot f := (vf_0, \dots, vf_n)$  satisfies*

$$\Lambda_n(v \cdot f) = v^{n+1} \Lambda_n(f).$$

**Proof.** Omitted.  $\square$

**Lemma 2.2.** *Let  $v : U \rightarrow \mathbb{C}$  be a non-degenerate meromorphic function and  $f = (1, \frac{1}{1!}v, \dots, \frac{1}{n!}v^n) : U \rightarrow \mathbb{C}^{n+1}$ . Then*

$$\Lambda_n(f) = (v')^{\frac{n(n+1)}{2}}.$$

**Proof.** We prove it by induction.

- (1) Case  $n = 1$  is easy.
- (2) Suppose that  $n \geq 2$  and for all  $1 \leq k \leq n - 1$ , we have

$$\Lambda_k \left( 1, \frac{1}{1!}v, \dots, \frac{1}{k!}v^k \right) = (v')^{\frac{k(k+1)}{2}}.$$

Then

$$\begin{aligned} \Lambda_n(f) &= \Lambda_n \left( 1, \frac{1}{1!}v, \dots, \frac{1}{n!}v^n \right) \\ &= \Lambda_{n-1} \left( \frac{1}{1!}v', \dots, \frac{1}{n!}nv^{n-1}v' \right) \end{aligned}$$

$$\begin{aligned}
&= (v')^{n+1} \Lambda_{n-1} \left( \frac{1}{1!}, \dots, \frac{1}{(n-1)!} v^{n-1} \right) \text{ (by Lemma 2.1)} \\
&= (v')^{n+1} (v')^{\frac{n(n-1)}{2}} \\
&= (v')^{\frac{n(n+1)}{2}} \quad \square
\end{aligned}$$

**Lemma 2.3.** Let  $v_0, v_1 : U \rightarrow \mathbb{C}$  be holomorphic functions such that

$$v_0(z)v'_1(z) - v_1(z)v'_0(z) \equiv 1 \quad \text{on } U.$$

Then the canonical lifting

$$f = \left( \sqrt{\frac{1}{n!}} v_0^n, \sqrt{\frac{1}{(n-1)!1!}} v_0^{n-1} v_1, \dots, \sqrt{\frac{1}{n!}} v_1^n \right) : U \rightarrow \mathbb{C}^{n+1}$$

has Wronskian  $\equiv 1$ .

**Proof.** Let  $v = v_1/v_0$ . Then we have  $v' = \frac{1}{v_0^2}$  and

$$\begin{aligned}
\Lambda_n(f) &= \Lambda_n \left( \sqrt{\frac{1}{n!}} v_0^n : \sqrt{\frac{1}{(n-1)!1!}} v_0^{n-1} v_1 : \dots : \sqrt{\frac{1}{n!}} v_1^n \right) \\
&= \Lambda_n \left( \sqrt{\frac{1}{n!}} (v')^{-\frac{n}{2}} : \sqrt{\frac{1}{(n-1)!1!}} (v')^{-\frac{n}{2}} v : \dots : \sqrt{\frac{1}{n!}} (v')^{-\frac{n}{2}} v^n \right) \\
&= (v')^{-\frac{n(n+1)}{2}} \Lambda_n \left( \sqrt{\frac{1}{n!}} : \sqrt{\frac{1}{(n-1)!1!}} v : \dots : \sqrt{\frac{1}{n!}} v^n \right) \text{ (by Lemma 2.1)} \\
&= (v')^{-\frac{n(n+1)}{2}} (v')^{\frac{n(n+1)}{2}} \text{ (by Lemma 2.2)} \\
&= 1. \quad \square
\end{aligned}$$

## 2.2. Reduced solutions on a chart

Let  $\{U, z\}$  be a complex coordinate chart of  $S$ . Assume that  $\vec{\omega} = (\omega_1 = \frac{\sqrt{-1}}{2} e^{u_1} dz \wedge d\bar{z}, \dots, \omega_n = \frac{\sqrt{-1}}{2} e^{u_n} dz \wedge d\bar{z})$  in  $U$ . Then the  $SU(n+1)$  Toda system (1.1) takes the following form:

$$\left( \frac{\partial^2 u_1}{\partial z \partial \bar{z}}, \dots, \frac{\partial^2 u_n}{\partial z \partial \bar{z}} \right) = -(e^{u_1}, \dots, e^{u_n}) C_n. \quad (2.1)$$

Thus, we also call  $(u_1, \dots, u_n)$  a solution to the  $SU(n+1)$  Toda system on  $U$ . We now prove the existence of reduced solutions on  $U$ .

**Lemma 2.4.** Let  $\omega = \frac{\sqrt{-1}}{2}e^u dz \wedge d\bar{z}$  be a solution to the SU(2) Toda system on  $U$ , and let  $v : U \rightarrow \mathbb{P}^1$  be a curve associated with  $\omega$ . Let  $r_n : \mathbb{P}^1 \rightarrow \mathbb{P}^n$  be the rational normal map defined by

$$r_n : [z_0, z_1] \mapsto \left[ \sqrt{\frac{1}{n!}} z_0^n, \sqrt{\frac{1}{(n-1)!1!}} z_0^{n-1} z_1, \dots, \sqrt{\frac{1}{n!}} z_1^n \right].$$

Then  $(i(n+1-i)\omega = \frac{\sqrt{-1}}{2}e^{u+\ln(i(n+1-i))} dz \wedge d\bar{z})_{i=1}^n$  solves the SU( $n+1$ ) Toda system on  $U$ . Moreover, the set

$$\{U \circ r_n \circ v : U \rightarrow \mathbb{P}^n \mid U \in \mathrm{PSU}(n+1)\}$$

consists of all the associated curves of this solution.

**Proof.** A direct computation shows that  $(u + \ln(i(n+1-i)))_{i=1}^n$  solves (2.1). Denote by  $v = [v_0 : v_1]$  the curve  $v : U \rightarrow \mathbb{P}^1$  associated to  $u$  such that  $v_0(z)v'_1(z) - v_1(z)v'_0(z) \equiv 1$  on  $U$ . By Lemma 2.3, the canonical lifting

$$\hat{f} = \left( \sqrt{\frac{1}{n!}} v_0^n : \sqrt{\frac{1}{(n-1)!1!}} v_0^{n-1} v_1 : \dots : \sqrt{\frac{1}{n!}} v_1^n \right) : U \rightarrow \mathbb{C}^{n+1}$$

of the curve

$$f = r_n \circ v = \left[ \sqrt{\frac{1}{n!}} v_0^n : \sqrt{\frac{1}{(n-1)!1!}} v_0^{n-1} v_1 : \dots : \sqrt{\frac{1}{n!}} v_1^n \right] : U \rightarrow \mathbb{P}^n$$

has Wronskian  $\equiv 1$  i.e.  $\hat{f} \wedge \hat{f}' \wedge \dots \wedge \hat{f}^{(n)} \equiv e_0 \wedge \dots \wedge e_n$  on  $U$  (It also means that  $f$  is totally unramified).

It suffices to check that  $u + \ln n$  equals the first component  $u_1$  of solution  $(u_1, \dots, u_n)$  of (2.1) from the lifting  $\hat{f}$  of the curve  $f$ . We have

$$\begin{aligned} u_1 &= \log \left( \frac{\|\Lambda_1(\hat{f})\|^2}{\|\hat{f}\|^4} \right) \\ &= \log \left( \frac{\partial^2}{\partial z \partial \bar{z}} \log \|\hat{f}\|^2 \right) \\ &= \log \left( \frac{\partial^2}{\partial z \partial \bar{z}} \log \left( \frac{1}{n!} (|v_0|^2 + |v_1|^2)^n \right) \right) \\ &= \log \left( \frac{\partial^2}{\partial z \partial \bar{z}} \log(|v_0|^2 + |v_1|^2) \right) + \ln n \\ &= u + \ln n, \end{aligned} \tag{2.2}$$

where we use the infinitesimal Plücker formula ([7, p.269]) in the second equality.  $\square$

### 2.3. Reduced solutions on Riemann surface

Then we achieve the global result considering the monodromy. Firstly, let us recall some facts about the symmetric product space.

**Definition 2.5.** [2, p.50] Let  $V$  be a vector space over  $\mathbb{C}$ . The  $k$ -th **symmetric product** of  $V$ , denoted  $\text{Sym}^k(V)$ , is the subspace of the  $k$ -fold tensor product space  $V^{\otimes k}$  consisting of all tensors that are invariant under the action of the symmetric group  $S_k$ . Formally,  $\text{Sym}^k(V) = \{T \in V^{\otimes k} \mid \sigma(T) = T, \forall \sigma \in S_k\}$ , where  $\sigma$  acts on  $V^{\otimes k}$  by permuting arguments  $\sigma(v_1 \otimes v_2 \otimes \cdots \otimes v_k) = v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(k)}$  for any  $v_1, \dots, v_k \in V$ .

**Definition 2.6.** [2, Definition 2.5] The **symmetrization operator** is a map that projects any tensor  $T \in V^{\otimes k}$  onto its symmetric part. It is defined as

$$S^k(T) = \frac{1}{k!} \sum_{\sigma \in S_k} \sigma(T).$$

**Proposition 2.7.** [2, Theorem 2.2]

(1) If  $\{e_1, e_2, \dots, e_n\}$  is a basis of  $V$ , then a basis of  $\text{Sym}^k(V)$  consists of

$$\{S^k(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}) \mid i_1 \leq i_2 \leq \cdots \leq i_k\}.$$

(2) The dimension of  $\text{Sym}^k(V)$  is  $\binom{n+k-1}{k}$  with  $n = \dim V$ .

**Definition 2.8.** [3] Let  $G$  be a group, and let  $V$  be a finite-dimensional vector space over  $\mathbb{C}$ , equipped with a representation of  $G$ :

$$\rho : G \rightarrow \text{GL}(V),$$

where  $\text{GL}(V)$  is the general linear group of  $V$ . The  $k$ -th **symmetric product representation** of  $G$ , denoted  $\text{Sym}^k(V)$ , is defined as the natural induced representation of  $G$  on the  $k$ -th symmetric product space  $\text{Sym}^k(V)$ , which is a subspace of  $V^{\otimes k}$ . The action of  $G$  on  $\text{Sym}^k(V)$  is given by:

$$g \cdot S^k(v_1 \otimes v_2 \otimes \cdots \otimes v_k) = S^k((g \cdot v_1) \otimes (g \cdot v_2) \otimes \cdots \otimes (g \cdot v_k)),$$

for all  $g \in G, v_1, v_2, \dots, v_k \in V$ .

**Definition 2.9.** Let  $V$  be a vector space over  $\mathbb{C}$  equipped with a Hermitian inner product  $\langle \cdot, \cdot \rangle_V$ . Then, for tensors  $v_1 \otimes v_2 \otimes \cdots \otimes v_n$  and  $w_1 \otimes w_2 \otimes \cdots \otimes w_n$  in  $V^{\otimes n}$ , the Hermitian inner product on the tensor product space  $V^{\otimes n}$  is defined as

$$\langle v_1 \otimes v_2 \otimes \cdots \otimes v_n, w_1 \otimes w_2 \otimes \cdots \otimes w_n \rangle_{V^{\otimes n}} = \prod_{i=1}^n \langle v_i, w_i \rangle_V,$$

which induces a Hermitian inner product on the subspace  $\text{Sym}^n(V)$  of  $V^{\otimes n}$ .

**Lemma 2.10.** *If  $\{e_1, \dots, e_n\}$  is an orthonormal basis of  $V$ , then an orthonormal basis of  $\text{Sym}^k(V)$  consists of  $\left\{ \sqrt{\frac{i_1! \cdots i_k!}{k!}} S^k(e_1^{\otimes i_1} \otimes \cdots \otimes e_n^{\otimes i_n}) \right\}$ , where  $0 \leq i_1, \dots, i_n \leq k$  and  $i_1 + i_2 + \cdots + i_n = k$ . Then we obtain the corresponding homogeneous coordinates on both  $\mathbb{P}(V)$  and  $\mathbb{P}(\text{Sym}^k(V))$ .*

**Proof.** Of course  $\{S^k(e_1^{\otimes i_1} \otimes \cdots \otimes e_n^{\otimes i_n})\}$  forms a basis of  $\text{Sym}^k(V)$  and  $\langle S^k(e_1^{\otimes i_1} \otimes \cdots \otimes e_n^{\otimes i_n}), S^k(e_1^{\otimes j_1} \otimes \cdots \otimes e_n^{\otimes j_n}) \rangle \neq 0$  if and only if  $i_1 = j_1, \dots, i_n = j_n$ . In addition,  $\langle S^k(e_1^{\otimes i_1} \otimes \cdots \otimes e_n^{\otimes i_n}), S^k(e_1^{\otimes i_1} \otimes \cdots \otimes e_n^{\otimes i_n}) \rangle = \binom{k}{i_1, \dots, i_n} = \frac{k!}{i_1! \cdots i_n!}$ . Thus,  $\left\{ \sqrt{\frac{i_1! \cdots i_k!}{k!}} S^k(e_1^{\otimes i_1} \otimes \cdots \otimes e_n^{\otimes i_n}) \right\}$  forms an orthonormal basis.  $\square$

**Definition 2.11.** For a projective space  $\mathbb{P}(V)$ , the Fubini-Study metric can be described as follows:

- (1) In homogeneous coordinates  $[u] \in \mathbb{P}(V)$ , the Fubini-Study distance between two points  $[u], [v] \in \mathbb{P}(V)$  is given by  $d_{\text{FS}}([u], [v]) = \arccos \left( \frac{|\langle u, v \rangle|^2}{\langle u, u \rangle \langle v, v \rangle} \right)$ , where  $\langle u, v \rangle$  is the Hermitian inner product on  $V$ .
- (2) The associated Kähler form  $\omega_{\text{FS}}$  is given by  $\omega_{\text{FS}} = \sqrt{-1} \partial \bar{\partial} \log \langle u, u \rangle$ , where  $\langle u, u \rangle$  is the norm square of the vector  $u \in V$ .

**Lemma 2.12.** *Let  $V$  be a  $\mathbb{C}$ -Hermitian space of dimension 2. The rational normal map  $r_n : \mathbb{P}(V) \rightarrow \mathbb{P}(\text{Sym}^n(V))$ ,  $[u] \mapsto [u^{\otimes n}]$  induces a Lie group monomorphism  $\sigma : \text{PSU}(V) \rightarrow \text{PSU}(\text{Sym}^n(V))$  such that  $r_n \circ U = \sigma(U) \circ r_n$  for any  $U \in \text{PSU}(V)$ .*

**Proof.** Since

$$\begin{aligned} r_n^* \omega_{\text{FS}, \mathbb{P}(\text{Sym}^n(V))} &= \sqrt{-1} \partial \bar{\partial} \log \langle u^{\otimes n}, u^{\otimes n} \rangle \\ &= \sqrt{-1} \partial \bar{\partial} \log \langle u, u \rangle^n \\ &= n \omega_{\text{FS}, \mathbb{P}(V)}, \end{aligned}$$

we have  $(r_n \circ U)^* g_{\text{FS}, \mathbb{P}(\text{Sym}^n(V))} = n U^* g_{\text{FS}, \mathbb{P}(V)} = n g_{\text{FS}, \mathbb{P}(V)} = r_n^* g_{\text{FS}, \mathbb{P}(\text{Sym}^n(V))}$  for any  $U \in \text{PSU}(V)$ . By the rigidity theorem [7, (4.12)], there exists a unique  $U' \in \text{PSU}(\text{Sym}^n(V))$  such that  $r_n \circ U = U' \circ r_n$ . Defining  $\sigma(U) = U'$ , we are done.

In addition, let  $e_0, e_1$  be an orthonormal basis of  $V$ , and let  $[u] = [z_0 e_0 + z_1 e_1]$ . Then, we have  $[u^{\otimes n}] = [\sum_{i=0}^n z_0^{n-i} z_1^i S^n(e_0^{\otimes n-i} \otimes e_1^{\otimes i})]$ . Notice that the orthonormal basis for the

symmetric powers is given by  $\left\{ \sqrt{\frac{i!(n-i)!}{n!}} S^n(e_0^{\otimes n-i} \otimes e_1^{\otimes i}) \right\}$ . Thus, this map corresponds to the rational normal map

$$[z_0 : z_1] \mapsto \left[ \sqrt{\frac{1}{n!}} z_0^n : \sqrt{\frac{1}{1!(n-1)!}} z_0^{n-1} z_1 : \cdots : \sqrt{\frac{1}{n!}} z_1^n \right]. \quad \square$$

**Proof of Theorem 1.1.** Since  $\omega$  solves the  $SU(2)$  Toda system on  $S$ , it is straightforward for us to verify that  $(i(n+1-i)\omega)_{i=1}^n$  is a solution to the  $SU(n+1)$  Toda system on  $S$ .

Consider a chart  $U$  with a branch  $v_0$  of  $v$  on  $U$ . Then, by Lemma 2.4,  $r_n \circ v_0$  is an associated curve of this solution restricted to  $U$ . Furthermore,  $r_n \circ v_0$  is a branch of  $r_n \circ v$  on  $U$ . Therefore, we need to prove that  $r_n \circ v$  is a unitary curve.

For  $z \in S$ , since the monodromy of  $v$  belongs to the group  $PSU(2)$ , there exists a special unitary representation  $\rho : \pi_1(S, z) \rightarrow PSU(V, H)$ , where  $V$  is the natural representation space  $\mathbb{C}^2$  of  $PSU(2)$ , and  $H$  is the Hermitian inner product on  $V$  (with  $v$  being viewed as a map  $v : S \rightarrow \mathbb{P}(V)$ ). There is a symmetric product representation  $\rho' = \sigma \circ \rho : \pi_1(S, z) \rightarrow PSU(\text{Sym}^n(V), H')$ , where  $\sigma : PSU(V, H) \rightarrow PSU(\text{Sym}^n(V), H')$  is the embedding induced by  $r_n$  (Lemma 2.12), and  $H'$  is the Hermitian inner product on  $\text{Sym}^n(V)$  induced from  $H$  (Definition 2.9).

Assume that the monodromy representation of  $r_n \circ v$  is  $\varrho : \pi_1(S, z) \rightarrow PSL(\text{Sym}^n(V))$ . For  $\gamma \in \pi_1(S, z)$  and a branch  $v_0$  of  $v$  near  $z$ , if we extend  $v_0$  analytically along  $\gamma$ , we get  $\rho(\gamma) \circ v_0$ . Thus, for a branch  $r_n \circ v_0$  of  $r_n \circ v$ , if we extend  $r_n \circ v_0$  analytically along  $\gamma$ , we get both  $\varrho(\gamma) \circ r_n \circ v_0$  and  $r_n \circ \rho(\gamma) \circ v_0$ , which means  $\varrho(\gamma) \circ r_n = r_n \circ \rho(\gamma) = \rho'(\gamma) \circ r_n$ . Since  $r_n$  is non-degenerate,  $\varrho = \rho'$  is a unitary representation. So  $r_n \circ v$  is a unitary curve.  $\square$

### 3. Classification of reduced solutions

In this section, we prove Theorem 1.3. Firstly, we describe  $M_v$  by the closure of the monodromy group. Then, we achieve the classification from the classification of the closure. Finally, based on the classification, we compute the real dimension of  $M_v$ . Moreover, we also use the characters of the symmetric product representations of the natural two-dimensional representations of  $E_6$ ,  $E_7$  and  $E_8$ . Denote by  $C(S)$  the centralizer of a subset  $S \subset PSL(n+1, \mathbb{C})$ . Recall that  $\sigma : PSU(2) \rightarrow PSU(n+1)$  is a monomorphism of the Lie group induced by  $r_n$  (Lemma 2.12).

**Lemma 3.1.** Denote by  $G_v \subset PSU(2)$  the monodromy group of a unitary curve  $v : S \rightarrow \mathbb{P}^1$ . Then

$$\begin{aligned} M_v &= \{\varphi \in PSL(n+1, \mathbb{C}) \mid \varphi \sigma(G_v) \varphi^{-1} \subset PSU(n+1)\} \\ &= \{\varphi \in PSL(n+1, \mathbb{C}) \mid \varphi^* \varphi \in C(\sigma(G_v))\}. \end{aligned}$$

Furthermore, for any unitary curve  $v, v_1, v_2 : S \rightarrow \mathbb{P}^1$ , the following properties hold:

- (1)  $M_{U \circ v} = \{\sigma(U) \circ \varphi \mid \varphi \in M_v\}$  for  $U \in \mathrm{PSU}(2)$ ,
- (2)  $M_{v_1} = M_{v_2}$  if  $\overline{G}_{v_1} = \overline{G}_{v_2}$ .

**Proof.** For  $z \in S$ , let  $\varrho : \pi_1(S, z) \rightarrow \mathrm{PSU}(n+1)$  be the monodromy representation of  $r_n \circ v$ . From Section 2, we know that  $\mathrm{Im} \varrho = \sigma(G_v)$ . For  $\gamma \in \pi_1(S, z)$ , if we extend a branch  $\varphi \circ r_n \circ v_0$  of  $\varphi \circ r_n \circ v$  along  $\gamma$ , we obtain the curve  $\varphi \circ \varrho(\gamma) \circ r_n \circ v_0 = (\varphi \varrho(\gamma) \varphi^{-1}) \circ \varphi \circ r_n \circ v_0$ . Therefore, the monodromy group is given by  $\varphi \sigma(G_v) \varphi^{-1}$ .

Since we need  $\varphi \circ r_n \circ v$  to be a unitary curve, it follows that  $\varphi U \varphi^{-1} \in \mathrm{PSU}(n+1)$  for all  $U \in \sigma(G_v)$ . This implies that  $U \varphi^* \varphi = \varphi^* \varphi U$  for all  $U \in \sigma(G_v)$ . Thus,  $M_v$  is given by

$$M_v = \{\varphi \in \mathrm{PSL}(n+1, \mathbb{C}) \mid \varphi^* \varphi \in C(\sigma(G_v))\}.$$

It is straightforward to verify (1). Moreover, (2) follows from the continuity of the left and right multiplications of the Lie group  $\mathrm{PSL}(n+1, \mathbb{C})$ .  $\square$

**Lemma 3.2.**  $M_v$  has a decomposition of the form  $M_v = \mathrm{PSU}(n+1)\Delta_v$ , where  $\Delta_v = \{\delta \in \Delta_{n+1} \mid \delta^* \delta \in C(\sigma(G_v))\}$  is a subset of  $\Delta_{n+1}$ .

**Proof.** For  $\varphi \in \mathrm{PSL}(n+1, \mathbb{C})$ , assume  $\varphi = U \circ \delta$ , where  $\delta \in \Delta_{n+1}$  and  $U \in \mathrm{PSU}(n+1)$ . Then we have  $\varphi^* \varphi = \delta^* \delta$ , which implies that  $\varphi \in M_v$  if and only if  $\delta \in M_v$ . Therefore, we obtain the decomposition  $M_v = \mathrm{PSU}(n+1)\Delta_v$ .  $\square$

**Lemma 3.3** (Cholesky factorization). [8, Corollary 7.2.9]

The map  $\Delta_{n+1} \rightarrow \mathrm{Herm}_{n+1}^+(1)$ ,  $\delta \mapsto \delta^* \delta$  is a bijection, where

$$\mathrm{Herm}_{n+1}^+(1) = \{H \in \mathrm{SL}(n+1, \mathbb{C}) \mid H \text{ is positive definite Hermitian}\}.$$

**Lemma 3.4** (algorithm for Cholesky factorization). Let  $M = (m_{i,j})_{0 \leq i,j \leq n} \in \mathrm{Herm}_{n+1}^+(1)$  be a positive semi-definite Hermitian matrix. Define an upper triangular matrix  $\delta = (a_{i,j})$  (i.e.,  $a_{i,j} = 0$  for  $i > j$ ) by:

- (1) For diagonal entries  $j = i$ :

$$a_{j,j} = \sqrt{m_{j,j} - \sum_{s=0}^{j-1} |a_{s,j}|^2},$$

- (2) For upper triangular entries  $i < j$ :

$$a_{i,j} = \frac{1}{a_{i,i}} \left( m_{i,j} - \sum_{s=0}^{i-1} \overline{a_{s,i}} a_{s,j} \right).$$

Then  $M = \delta^* \delta$ , where  $\delta^*$  denotes the conjugate transpose of  $\delta$ .

This is a well-known classic algorithm for Cholesky factorization, which is easy to check, while I'm not sure what the initial article of it is.

**Lemma 3.5.** *Inherit the notation of the previous lemma. Given  $k \in \mathbb{Z}_{>0}$ . Then  $a_{i,j} = 0$  whenever  $k \nmid i - j$  if and only if  $m_{i,j} = 0$  whenever  $k \nmid i - j$ .*

**Proof.** We prove both directions of the equivalence.

**Direction ( $\Rightarrow$ ):** Assume  $a_{i,j} = 0$  whenever  $k \nmid i - j$ . Then  $M = \delta^* \delta$  satisfies:

$$m_{i,j} = \sum_{s=0}^n \overline{a_{s,i}} a_{s,j}.$$

Fix  $i, j$  such that  $k \nmid (i - j)$ . For the term  $\overline{a_{s,i}} a_{s,j}$  to be nonzero, we must have both  $a_{s,i} \neq 0$  and  $a_{s,j} \neq 0$ . By the sparsity of  $\delta$ , this requires  $k \mid (s - i)$  and  $k \mid (s - j)$ . Consequently:

$$k \mid ((s - i) - (s - j)) = j - i \implies k \mid (i - j),$$

contradicting  $k \nmid (i - j)$ . Thus  $\overline{a_{s,i}} a_{s,j} = 0$  for all  $s$ , so  $m_{i,j} = 0$ .

**Direction ( $\Leftarrow$ ):** Assume  $m_{i,j} = 0$  whenever  $k \nmid (i - j)$ . We prove by induction on  $j$  (from 0 to  $n$ ) and on  $i$  (from 0 to  $j$ ) that  $a_{i,j} = 0$  for  $k \nmid (i - j)$ .

- *Base case ( $j = 0$ ):* Trivial (no off-diagonal entries).
- *Inductive step ( $j \geq 1$ ):* Assume the claim holds for all columns  $< j$ . Then  $a_{0,j} = m_{0,j}$  holds. For  $j > i \geq 1$ , assume the claim holds for all rows  $< i$  when column  $= j$ .

$$a_{i,j} = \frac{1}{a_{i,i}} \left( m_{i,j} - \sum_{s=0}^{i-1} \overline{a_{s,i}} a_{s,j} \right).$$

If  $k \nmid (i - j)$ , then  $m_{i,j} = 0$ . For each  $s \in \{0, \dots, i-1\}$ :

- If  $k \nmid (s - i)$ , then  $a_{s,i} = 0$  (by induction on column  $i < j$ ).
- If  $k \nmid (s - j)$ , then  $a_{s,j} = 0$  (by induction on row  $s < i$ ).
- If both nonzero, then  $k \mid (s - i)$  and  $k \mid (s - j)$ , implying  $k \mid (i - j)$  (contradiction).

Thus  $\overline{a_{s,i}} a_{s,j} = 0$ , so  $a_{i,j} = 0$ . Diagonal entries  $a_{j,j}$  have  $i - j = 0$  (always divisible by  $k$ ).  $\square$

**Lemma 3.6.** *For any unitary curve  $v : S \rightarrow \mathbb{P}^1$ , let us define a subspace*

$$V_v = \{A \in \text{Mat}_{n+1}(\mathbb{C}) \mid AU = UA, \forall U \in \sigma(G_v)\}$$

*of the complex vector space  $\text{Mat}_{n+1}(\mathbb{C})$  formed by all  $n+1$ -order matrices. Then we have  $\dim_{\mathbb{R}} (\Delta_v) = \dim_{\mathbb{C}} (V_v) - 1$ .*

**Proof.** We divide the proof into the following three steps.

- Let  $\text{Herm}_{n+1}(1)$  denote the set of  $(n+1)$  by  $(n+1)$  Hermitian matrices with determinant 1. Since the map  $\Delta_{n+1} \rightarrow \text{Herm}_{n+1}^+(1)$ ,  $\delta \mapsto \delta^*\delta$  is a bijection (Lemma 3.3), it induces a bijection  $\Delta_v \rightarrow \text{Herm}_{n+1}^+(1) \cap C(\sigma(G_v))$  by restricting the domain to  $\Delta_v$ . Therefore,  $\dim_{\mathbb{R}}(\Delta_v) = \dim_{\mathbb{R}}(\text{Herm}_{n+1}^+(1) \cap C(\sigma(G_v)))$ . Since  $\text{Herm}_{n+1}^+(1)$  is an open subset of  $\text{Herm}_{n+1}(1)$  and  $\text{Herm}_{n+1}^+(1) \cap C(\sigma(G)) \neq \emptyset$ , we conclude that

$$\dim_{\mathbb{R}}(\Delta_v) = \dim_{\mathbb{R}}(\text{Herm}_{n+1}^+(1) \cap C(\sigma(G_v))) = \dim_{\mathbb{R}}(\text{Herm}_{n+1}(1) \cap C(\sigma(G_v))).$$

- Let  $\text{Herm}_{n+1}$  denote the set of  $(n+1)$  by  $(n+1)$  Hermitian matrices, and let  $\mathcal{H}_{n+1}$  be its projection in  $\mathbb{P}(\text{Mat}_{n+1}(\mathbb{C}))$ . Since  $\text{Herm}_{n+1}(1)$  is an open dense subset of  $\mathcal{H}_{n+1}$ , and  $\text{PSL}(n+1, \mathbb{C})$  is an open dense subset of  $\mathbb{P}(\text{Mat}_{n+1}(\mathbb{C}))$ , it follows that  $\text{Herm}_{n+1}(1) \cap C(\sigma(G_v)) = \text{Herm}_{n+1}(1) \cap (\text{PSL}(n+1, \mathbb{C}) \cap \mathbb{P}(V_v))$  is also a non-empty open subset of  $\mathcal{H}_{n+1} \cap \mathbb{P}(V_v)$ . Therefore, we conclude that

$$\dim_{\mathbb{R}}(\Delta_v) = \dim_{\mathbb{R}}(\text{Herm}_{n+1}(1) \cap C(\sigma(G_v))) = \dim_{\mathbb{R}}(\mathcal{H}_{n+1} \cap \mathbb{P}(V_v)).$$

- Since  $\text{Mat}_{n+1}(\mathbb{C}) = \text{Herm}_{n+1} \otimes_{\mathbb{R}} \mathbb{C}$ , any matrix  $A \in \text{Mat}_{n+1}(\mathbb{C})$  can be expressed uniquely as  $A = H_1 + \sqrt{-1}H_2$ , where  $H_1, H_2 \in \text{Herm}_{n+1}$ . For any  $U \in \text{SU}(n+1)$ , the matrix  $U^*HU$  remains Hermitian for any  $H \in \text{Herm}_{n+1}$ . Hence,  $A \in V_v$  if and only if  $H_1, H_2 \in \text{Herm}_{n+1} \cap V_v$ . Thus, we can write  $V_v = (\text{Herm}_{n+1} \cap V_v) \otimes_{\mathbb{R}} \mathbb{C}$ , which implies  $\dim_{\mathbb{R}}(\text{Herm}_{n+1} \cap V_v) = \dim_{\mathbb{C}}(V_v)$ . Since  $\mathcal{H}_{n+1} \cap \mathbb{P}(V_v)$  is the projection of  $\text{Herm}_{n+1} \cap V_v$  in  $\mathbb{P}(\text{Mat}_{n+1}(\mathbb{C}))$ , we conclude:

$$\dim_{\mathbb{R}}(\Delta_v) = \dim_{\mathbb{R}}(\mathcal{H}_{n+1} \cap \mathbb{P}(V_v)) = \dim_{\mathbb{R}}(\text{Herm}_{n+1} \cap V_v) - 1 = \dim_{\mathbb{C}}(V_v) - 1. \quad \square$$

**Remark 3.7.** Recall that  $p : \text{SU}(2) \rightarrow \text{PSU}(2)$  is the projection and  $V$  is the natural representation of  $\text{SU}(2)$ . Because  $p^{-1}(G_v)$  is a subgroup of  $\text{SU}(V)$ , we could see  $\text{Sym}^n(V)$  as a representation space of  $p^{-1}(G_v)$ . Then  $V_v$  is just the space  $\text{End}_{p^{-1}(G_v)}(\text{Sym}^n(V))$  of module homomorphisms.

Let us recall some results of the representation theory.

**Lemma 3.8.** [3] Let  $G$  be a group. If  $V = V_1^{\oplus a_1} \oplus \cdots \oplus V_n^{\oplus a_n}$  is a complex representation of  $G$ , where all  $V_i, i = 1, \dots, n$  are distinct irreducible representation spaces, then  $\dim_{\mathbb{C}} \text{End}_G(V) = a_1^2 + \cdots + a_n^2$ . In particular, when  $G$  is a finite group,  $\dim_{\mathbb{C}} \text{End}_G(V) = \frac{1}{|G|} \sum_{g \in G} |\chi_V(g)|^2$ , where  $\chi_V : G \rightarrow \mathbb{C}$  is the character of  $V$ .

**Proof of Theorem 1.3.** (1)  $G_v = \text{PSU}(2)$ .

Notice  $\sigma : \text{PSU}(2) = \text{PSU}(V) \rightarrow \text{PSU}(n+1) = \text{PSU}(\text{Sym}^n(V))$  (Lemma 2.12) is a irreducible representation of  $\text{PSU}(2)$ . It is irreducible because symmetric product representation  $\text{Sym}^n(V)$  is an irreducible representation of  $\text{SL}(V)$  [3, Section 11.1].

Thus  $C(\text{Im } \sigma) = \{I\}$  by Schur's Lemma [3, Lemma 1.7]. Therefore,  $\delta \in \Delta_v$  if and only if  $\delta^* \delta = I_{n+1}$ , which means  $\delta = I_{n+1}$ . Consequently, we conclude that  $\Delta_v = \{I_{n+1}\}$ .

(2)  $G_v = \text{PU}(1)$ .

We have  $\sigma(G_v) = \{\text{diag}(c^n, c^{n-2}, \dots, c^{-n}) \mid |c| = 1\}$ . Then  $C(\sigma(\text{PU}(1))) = \{\text{all diagonal matrices}\}$ . Thus,  $\delta \in \Delta_v$  if and only if  $\delta^* \delta$  is diagonal. Since  $\delta$  is induced by an upper triangular matrix with positive diagonal entries, it must be diagonal with positive entries, which implies

$$\Delta_v = \{\text{diag}(a_0, \dots, a_n) \in \Delta_{n+1}\}.$$

It is obviously  $\dim_{\mathbb{R}} \Delta_v = n$ .

(3)  $G_v = \text{PO}(2)$

Let  $g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ . Since  $C(\sigma(g)) \cap C(\sigma(\text{PU}(1))) = \{\text{diag}(\lambda_1, \dots, \lambda_{n+1}) \mid \lambda_i = \lambda_{n+1-i}\}$ , it follows that  $C(\sigma(G_v)) = \{\text{diag}(\lambda_1, \dots, \lambda_{n+1}) \mid \lambda_i = \lambda_{n+1-i}\}$ . Then  $\delta \in \Delta_v$  if and only if  $\delta^* \delta \in C(\sigma(G_v))$ . Since  $\delta$  is an upper triangular matrix with positive diagonal entries, it implies

$$\Delta_v = \{\text{diag}(a_0, \dots, a_n) \in \Delta_{n+1} \mid a_i = a_{n-i}\}.$$

It is obviously  $\dim_{\mathbb{R}} \Delta_v = \lfloor \frac{n}{2} \rfloor$ .

(4)  $G_v = p(C_k)$ .

The group  $\sigma(G)$  is generated by  $\text{diag}(\xi_k^n, \xi_k^{n-2}, \dots, \xi_k^{-n})$ , where  $\xi_k$  is a primitive  $k$ -th root of unity. Then, the centralizer of  $\sigma(G_v)$  in  $\text{PSL}(n+1, \mathbb{C})$  is  $C(\sigma(G_v)) = \{(z_{ij})_{0 \leq i, j \leq n} \mid z_{ij} = 0 \text{ if } k \nmid 2(i-j)\} = \{(z_{ij})_{0 \leq i, j \leq n} \mid z_{ij} = 0 \text{ if } \frac{k}{\gcd(k, 2)} \nmid (i-j)\}$ . Then,  $\delta \in \Delta_v$  if and only if  $\delta^* \delta \in C(\sigma(G_v))$ . Hence, by Lemma 3.5

$$\Delta_v = \{(a_{ij})_{0 \leq i, j \leq n} \in \Delta_{n+1} \mid a_{i,j} = 0 \text{ if } \frac{k}{\gcd(k, 2)} \nmid (i-j)\}.$$

Thus, the number of independent equations given by  $C(\sigma(G_v))$  are  $|\{(i, j) \mid 0 \leq i, j \leq n, \frac{k}{\gcd(k, 2)} \nmid (i-j)\}|$ . It means  $\dim_{\mathbb{R}} \Delta_v = |\{(i, j) \mid 0 \leq i, j \leq n, \frac{k}{\gcd(k, 2)} \mid (i-j)\}| - 1$ . The number can be a sum by row:

$$\begin{aligned} \dim_{\mathbb{R}} \Delta_v &= 2 \sum_{i=0}^n \lfloor \frac{i \gcd(k, 2)}{k} \rfloor + (n+1) - 1 \\ &= -\frac{k}{\gcd(k, 2)} \lfloor \frac{n \gcd(k, 2)}{k} \rfloor^2 + (2n+2 - \frac{k}{\gcd(k, 2)}) \lfloor \frac{n \gcd(k, 2)}{k} \rfloor + n \end{aligned}$$

(5)  $G_v = p(D_k)$ .

In this case, the centralizer  $C(\sigma(G_v))$  can be expressed as  $C(\sigma(G_v)) = \{(z_{ij})_{0 \leq i, j \leq n} \in C(\sigma(p(C_{2k}))) \mid z_{i,j} = (-\sqrt{-1})^{i-j} z_{n-i, n-j}\}$ . Then  $\delta \in \Delta_v$  if and only if  $\delta^* \delta \in C(\sigma(G_v))$ . Hence, by Lemma 3.5,

$$\Delta_v = \left\{ (a_{i,j})_{0 \leq i,j \leq n} \in \Delta_{n+1} \middle| \begin{array}{l} a_{i,j} = 0 \text{ if } k \nmid i-j \\ \sum_{l=0}^n \bar{a}_{l,i} a_{l,j} = (-\sqrt{-1})^{i-j} \sum_{l=0}^n \bar{a}_{l,n-i} a_{l,n-j} \end{array} \right\}.$$

The number of independent equations given by  $C(\sigma(G_v))$  are  $|\{(i, j) | 0 \leq i, j \leq n, k \nmid i - j\}| + |\{(i, j) | 0 \leq i, j \leq n, k \mid i - j\}|/2$ . Thus

$$\begin{aligned} \dim_{\mathbb{R}} \Delta_v &= n^2 + 2n - |\{(i, j) | 0 \leq i, j \leq n, k \nmid i - j\}| \\ &\quad - |\{(i, j) | 0 \leq i, j \leq n, k \mid i - j\}|/2 \\ &= |\{(i, j) | k \mid i - j\}|/2 \\ &= -\frac{k}{2} \lfloor \frac{n}{k} \rfloor^2 + (n+1-\frac{k}{2}) \lfloor \frac{n}{k} \rfloor + \lfloor \frac{n}{2} \rfloor \end{aligned}.$$

(6)  $G_v = p(E_6)$ .

Denote  $g_1 = \begin{pmatrix} 0 & -\sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix}$ ,  $g_2 = \begin{pmatrix} \frac{1+\sqrt{-1}}{2} & \frac{1+\sqrt{-1}}{2} \\ \frac{\sqrt{-1}-1}{2} & \frac{1-\sqrt{-1}}{2} \end{pmatrix}$ . Let  $V$  be the natural representation of  $SU(2)$  and  $\rho : E_6 \rightarrow SU(V)$  be the given embedding. Then  $V_v = \text{End}_{E_6}(\text{Sym}^n(V))$ . Since  $E_6$  is a finite group, direct computation shows that the conjugacy classes of  $E_6$  are listed in the following table (which is also a well-known result of tetrahedral group):

conjugacy classes	$I$	$-I$	$g_1$	$g_2$	$g_2^2$	$g_2^4$	$g_2^5$
their cardinality	1	1	6	4	4	4	4

Notice that  $\chi_{\text{Sym}^n(V)}(g) = \sum_{i=0}^n \lambda_1^i \lambda_2^{n-i}$ , where  $\lambda_1, \lambda_2$  are two eigenvalues of  $\rho(g)$ . The character of  $\text{Sym}^n(V)$  is as the following table:

$\text{Sym}^n(V)$	$I$	$-I$	$g_1$	$g_2$	$g_2^2$	$g_2^4$	$g_2^5$
$n \equiv 1 \pmod{2}$	$n+1$	$-n-1$	0	c	-c	-c	c
$n \equiv 0 \pmod{2}$	$n+1$	$n+1$	$(-1)^{n/2}$	c	c	c	c

where  $c = \cos \frac{n\pi}{3} + \frac{\sqrt{3}}{3} \sin \frac{n\pi}{3}$ . Thus,

- i. When  $n$  is odd, by Lemma 3.8, we have  $\dim_{\mathbb{C}} V_v = \frac{n^2}{12} + \frac{2c^2}{3} + \frac{n}{6} + \frac{1}{12}$ . Thus,  $\dim_{\mathbb{R}} \Delta_v = \frac{n^2}{12} + \frac{2c^2}{3} + \frac{n}{6} - \frac{11}{12}$ .
- ii. When  $n$  is even, by Lemma 3.8, we have  $\dim_{\mathbb{C}} V_v = \frac{n^2}{12} + \frac{n}{6} + \frac{2c^2}{3} + \frac{1}{3}$ . Thus,  $\dim_{\mathbb{R}} \Delta_v = \frac{n^2}{12} + \frac{n}{6} + \frac{2c^2}{3} - \frac{2}{3}$ .

(7)  $G_v = p(E_7)$ . Let  $g_1 = \begin{pmatrix} 0 & \frac{\sqrt{2}(1+\sqrt{-1})}{2} \\ \frac{\sqrt{2}(\sqrt{-1}-1)}{2} & 0 \end{pmatrix}$  and  $g_2 = \begin{pmatrix} \frac{1+\sqrt{-1}}{2} & \frac{1+\sqrt{-1}}{2} \\ \frac{\sqrt{-1}-1}{2} & \frac{1-\sqrt{-1}}{2} \end{pmatrix}$ . Let  $V$  be the natural representation of  $SU(2)$  and  $\rho :$

$E_7 \rightarrow \mathrm{SU}(V)$  be the given embedding. Then  $V_v = \mathrm{End}_{E_7}(\mathrm{Sym}^n(V))$ . The conjugacy classes of  $E_7$  are listed as (which is also a well-known result of octahedral group)

conjugacy classes	$I_2$	$-I_2$	$(g_1 g_2)^2$	$g_2$	$g_2^2$	$g_1 g_2$	$(g_1 g_2)^3$	$g_1$
their cardinality	1	1	6	8	8	6	6	12

Similar to  $E_6$ , the character of  $\mathrm{Sym}^n(V)$  of  $E_7$  is expressed in the following table:

$\mathrm{Sym}^n(V)$	$I_2$	$-I_2$	$(g_1 g_2)^2$	$g_2$	$g_2^2$	$g_1 g_2$	$(g_1 g_2)^3$	$g_1$
$n \equiv 1 \pmod{2}$	$n+1$	$-n-1$	0	$c_1$	$-c_1$	$-c_2$	$c_2$	0
$n \equiv 0 \pmod{2}$	$n+1$	$n+1$	$(-1)^{n/2}$	$c_1$	$c_1$	$c_2$	$c_2$	$(-1)^{n/2}$

where  $c_1 = \cos \frac{n\pi}{3} + \frac{\sqrt{3}}{3} \sin \frac{n\pi}{3}$  and  $c_2 = \cos \frac{n\pi}{4} + \sin \frac{n\pi}{4}$ . Thus,

- i. When  $n$  is odd, by Lemma 3.8, we have  $\dim_{\mathbb{C}} V_v = \frac{n^2}{24} + \frac{n}{12} + \frac{c_1^2}{3} + \frac{c_2^2}{4} + \frac{1}{24}$ . Thus,  $\dim_{\mathbb{R}} \Delta_v = \frac{n^2}{24} + \frac{n}{12} + \frac{c_1^2}{3} + \frac{c_2^2}{4} - \frac{23}{24}$ .
- ii. When  $n$  is even, by Lemma 3.8, we have  $\dim_{\mathbb{C}} V_v = \frac{n^2}{24} + \frac{n}{12} + \frac{c_1^2}{3} + \frac{c_2^2}{4} + \frac{5}{12}$ . Thus,  $\dim_{\mathbb{R}} \Delta_v = \frac{n^2}{24} + \frac{n}{12} + \frac{c_1^2}{3} + \frac{c_2^2}{4} - \frac{7}{12}$ .

(8)  $G_v = p(E_8)$ . Let

$$g_1 = \begin{pmatrix} 0 & -\sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix} \quad \text{and} \quad g_2 = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{5}-1}{4} + \frac{\sqrt{5}+1}{4}\sqrt{-1} \\ \frac{\sqrt{5}-1}{4} + \frac{\sqrt{5}+1}{4}\sqrt{-1} & \frac{1}{2} \end{pmatrix}.$$

Let  $V$  be the natural representation of  $\mathrm{SU}(2)$  and  $\rho : E_8 \rightarrow \mathrm{SU}(V)$  be the given embedding. Then  $V_v = \mathrm{End}_{E_8}(\mathrm{Sym}^n(V))$ . The conjugacy classes of  $E_8$  are listed as (which is also a well-known result of icosahedral group):

conjugacy classes	$I$	$-I$	$g_2^2$	$g_1$	$(g_1 g_2)^2$	$(g_1 g_2)^4$	$g_2$	$g_1 g_2$	$(g_1 g_2)^3$
their cardinality	1	1	20	30	12	12	20	12	12

Similar to  $E_6$ , the character of  $\mathrm{Sym}^n(V)$  is expressed in the following table:

$\mathrm{Sym}^n(V)$	$I$	$-I$	$g_2^2$	$g_1$	$(g_1 g_2)^2$	$(g_1 g_2)^4$	$g_2$	$g_1 g_2$	$(g_1 g_2)^3$
$n \equiv 1 \pmod{1}$	$n+1$	$-n-1$	$-c_1$	0	$c_3$	$-c_2$	$c_1$	$c_2$	$-c_3$
$n \equiv 0 \pmod{2}$	$n+1$	$n+1$	$c_1$	$(-1)^{n/2}$	$c_3$	$c_2$	$c_1$	$c_2$	$c_3$

where  $c_1 = \cos \frac{n\pi}{3} + \frac{\sqrt{3}}{3} \sin \frac{n\pi}{3}$ ,  $c_2 = \sqrt{1 + \frac{2}{\sqrt{5}}} \sin \frac{n\pi}{5} + \cos \frac{n\pi}{5}$  and  $c_3 = \sqrt{1 - \frac{2}{\sqrt{5}}} \sin \frac{2n\pi}{5} + \cos \frac{2n\pi}{5}$ . Thus,

- i. When  $n$  is odd, by Lemma 3.8, we have  $\dim_{\mathbb{C}} V_v = \frac{n^2}{60} + \frac{n}{30} + \frac{c_1^2}{3} + \frac{c_2^2}{5} + \frac{c_3^2}{5} + \frac{1}{60}$ .  
 Thus,  $\dim_{\mathbb{R}} \Delta_v = \frac{n^2}{60} + \frac{n}{30} + \frac{c_1^2}{3} + \frac{c_2^2}{5} + \frac{c_3^2}{5} - \frac{59}{60}$ .
- ii. When  $n$  is even, by Lemma 3.8, we have  $\dim_{\mathbb{C}} V_v = \frac{n^2}{60} + \frac{n}{30} + \frac{c_1^2}{3} + \frac{c_2^2}{5} + \frac{c_3^2}{5} + \frac{2}{15}$ .  
 Thus,  $\dim_{\mathbb{R}} \Delta_v = \frac{n^2}{60} + \frac{n}{30} + \frac{c_1^2}{3} + \frac{c_2^2}{5} + \frac{c_3^2}{5} - \frac{13}{15}$ .  $\square$

#### 4. Examples

Let us recall the result of the spherical metric.

**Theorem 4.1.** [10, Theorem A] Let  $g > 0$  be an integer. Assume  $\beta_1, \dots, \beta_m > 0$  satisfy

$$\beta_1 + \dots + \beta_m > 2g - 2 + m,$$

then there exists a compact orientable Riemann surface  $X$  of genus  $g$  with a spherical metric  $\omega$  on  $X$  that represents  $D = \sum_{j=1}^m (\beta_j - 1)[P_j]$  for some distinct points  $P_1, \dots, P_m \in X$ .

Then there will be a natural corollary.

**Corollary 4.2.** Let  $X$ ,  $\omega$  and  $D$  be the same as above. Then, for each positive integer  $n > 1$ , the  $SU(n+1)$  Toda system on  $X$  with cone singularities

$$\underbrace{(D, D, \dots, D)}_{n \text{ divisors}}$$

has a family of reduced solutions, including  $(i(n+1-i)\omega)_{i=1}^n$  and is characterized in Theorem 1.3.

**Remark 4.3.** Consider the  $SU(n+1)$ -Toda system with cone singularities

$$\text{Ric}(\vec{\omega}) = 2\vec{\omega} C_n + (\delta_{P_1}, \dots, \delta_{P_m})\Gamma,$$

where  $\delta_P$  denotes the Dirac measure at  $P$  and  $\Gamma = (\gamma_{j,i})_{m \times n}$  is a real matrix with  $\gamma_{j,i} > -1$ . The solution  $\vec{\omega}$  represents an  $n$ -tuple of divisors  $(D_i = \sum_{j=1}^m \gamma_{j,i}[P_j])_{i=1}^n$ . The readers may find the detail of this framework of Toda system with cone singularities in [11, Section 1]. When  $\gamma_{j,i} = \beta_j - 1$  for all  $i$  and  $j$ , this corollary shows that the system with cone singularities is solvable. It should be noted that Lin, Yang and Zhong [9, Theorem 1.9] provide a sufficient condition for the solvability of the Toda system with cone singularities. Our corollary, however, offers a different sufficient condition. These conditions are not equivalent. For example, in the case  $n > 1, \beta_i \in \mathbb{Z}_{>1}, g > 0$ , which does not satisfy the condition in [9, Theorem 1.9], our corollary demonstrates that the system is solvable.

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## Data availability

No data was used for the research described in the article.

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