Analytical Expressions for Combined Eigenstrain - Pure Shear and Volumetric

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This report involves the implementation of the Eshelby analytical solution for a circular inclusion with pure shear and volumetric Eigenstrain. For the case of pure shear, all formulae are simplified from Eshelby's general solution given in the book by Mura [1] and for the case of volumetric, formulae are derived by solving the 2D plain strain axisymmetric elasticity problem (one can also derive the same formulae from the Eshelby's general solution). Having said this, the aim here is to have closed-form analytical solutions that can reduce the computation cost one needs to compute Eshelby's general solution using numerics.

1 Coordinate systems and corresponding components of Eigenstrain tensor

Fig. 1 shows a schematic of the involved coordinate systems of the problem. (x, y) is a lab coordinate system, and (x', y') is a local coordinate system along θ_s direction (this direction can be maximum shear or maximum shear +/- friction angle). The following decomposed (in pure shear and volumetric parts) Eigenstrain tensor is introduced into the local frame (x', y').

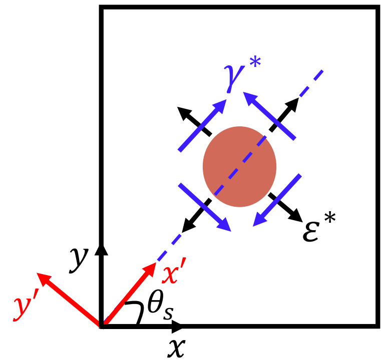

Figure 1: Schematic of two coordinate systems: Laboratory coordinate system (x, y) (*Black*) and local coordinate system (x', y') (*Red*)
.

$$\epsilon^*_{i'j'} = \begin{bmatrix} 0 & \gamma^* \\ \gamma^* & 0 \end{bmatrix} + \begin{bmatrix} \epsilon^* & 0 \\ 0 & \epsilon^* \end{bmatrix} \; ; \; i', j' = x', y' \tag{1}$$

Its corresponding components into the lab frame are then given by

$$\epsilon^*_{ij} = \gamma^* \begin{bmatrix} -\sin 2\theta_s & \cos 2\theta_s \\ \cos 2\theta_s & \sin 2\theta_s \end{bmatrix} + \epsilon^* \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \; ; \; i, j = x, y \tag{2}$$

2 Analytical Expression of elastic fields in the lab frame

This section describes the analytical expressions of elastic fields which get generated at a point p located at (x, y) (equivalently at $(x'y')$ into the local frame) due to a STZ located at $(y_{\text{STZ}}, y_{\text{STZ}})$ (See schematic in Fig. 2). The distance between the point and STZ is r and the angle is θ concerning the lab coordinate. The components of the Eigenstrain tensor are given by Eq. 1 in the local coordinate system and by Eq. 2 in the lab coordinate system.

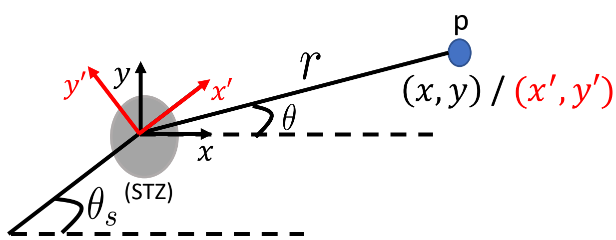

Figure 2: Schematic showing the location of STZ, maximum shear direction, and point location.

2.1 Displacement and strain fields

2.1.1 Pure shear

Components in the local frame (x',y')

Inside the inclusion

Displacements:

$$u^{\mathrm{s}}_{x'} = \frac{(3-4\nu)\gamma^*}{4(1-\nu)} y' \ ; \ u^{\mathrm{s}}_{y'} = \frac{(3-4\nu)\gamma^*}{4(1-\nu)} x'. \tag{3}$$

Strains:

$$\epsilon^{\mathrm{s}}_{x'x'} = \epsilon^{\mathrm{s}}_{y'y'} = 0 \ ; \ \epsilon^{\mathrm{s}}_{x'y'} = \frac{(3-4\nu)\gamma^*}{4(1-\nu)}. \tag{4}$$

Where, superscript 's' stands for mechanical fields corresponding to the pure shear.

Outside the inclusion

Displacements:

$$\begin{aligned} u^{\mathrm{s}}_{x'} &= \frac{a^2\gamma^*}{8(1-\nu)r'^6}\left[(4-8\nu)y'^5 + (12-8\nu)x'^4 y + 16(1-\nu)x'^2 y'^3 + 2a^2 y'^3 - 6a^2 x'^2 y'\right] \\ u^{\mathrm{s}}_{y'} &= \frac{a^2\gamma^*}{8(1-\nu)r'^6}\left[(4-8\nu)x'^5 + (12-8\nu)y'^4 x + 16(1-\nu)y'^2 x'^3 + 2a^2 x'^3 - 6a^2 y'^2 x'\right] \end{aligned} \tag{5}$$

Here, $r'^2 = x'^2 + y'^2$. For making use of some simple tricks at a later stage, let me factorize the above equation and put them in the following form

$$\begin{aligned} u^{\mathrm{s}}_{x'} &= \frac{a^2\gamma^*}{4(1-\nu)r'^6}\ y'\left\{2(x'^2+y'^2)\left[(1-2\nu)(x'^2+y'^2)+2x'^2\right] + a^2\left[(x'^2+y'^2)-4x'^2\right]\right\} \\ u^{\mathrm{s}}_{y'} &= \frac{a^2\gamma^*}{4(1-\nu)r'^6}\ x'\left\{2(x'^2+y'^2)\left[(1-2\nu)(x'^2+y'^2)+2y'^2\right] + a^2\left[(x'^2+y'^2)-4y'^2\right]\right\} \end{aligned} \tag{6}$$

Strains:

$$\begin{aligned} \epsilon^{\mathrm{s}}_{x'x'} &= \frac{a^2\gamma^*}{2(1-\nu)r'^4}\left[r'^2(2\nu-1) + (3a^2-2r'^2)\frac{x'^2-y'^2}{r'^2}\right]\frac{2x'y'}{r'^2} \\ \epsilon^{\mathrm{s}}_{y'y'} &= \frac{a^2\gamma^*}{2(1-\nu)r'^4}\left[r'^2(2\nu-1) - (3a^2-2r'^2)\frac{x'^2-y'^2}{r'^2}\right]\frac{2x'y'}{r'^2} \\ \epsilon^{\mathrm{s}}_{x'y'} &= -\frac{a^2\gamma^*}{4(1-\nu)r'^4}\left(3a^2-2r'^2\right)\left[\frac{(x'^2-y'^2)^2}{r'^4} - \frac{4x'^2y'^2}{r'^4}\right] \end{aligned} \tag{7}$$

Components in the local frame but in terms of lab coordinates (x, y)

Before transforming components of displacements and strain into the lab frame, let me rewrite them in terms of lab coordinates (x, y), using coordinate transformation (x, y) to (x', y'), *i.e.*

$$\left\{ \begin{array}{c} x' \\ y' \end{array} \right\} = \left[\begin{array}{cc} \cos\theta_s & \sin\theta_s \\ -\sin\theta_s & \cos\theta_s \end{array} \right] \left\{ \begin{array}{c} x \\ y \end{array} \right\} \tag{8}$$

and replacing $x = r\cos\theta$ and $y = r\sin\theta$, we get

Inside the inclusion

Displacements:

$$\begin{aligned} u^{\mathrm{s}}_{x'} &= \frac{3-4\nu}{4(1-\nu)} r\left(\sin\theta\cos\theta_s - \cos\theta\sin\theta_s\right)\gamma^* \\ u^{\mathrm{s}}_{y'} &= \frac{3-4\nu}{4(1-\nu)} r\left(\sin\theta\sin\theta_s + \cos\theta\cos\theta_s\right)\gamma^* \end{aligned} \tag{9}$$

Strains:

$$\epsilon^{\mathrm{s}}_{x'x'} = \epsilon^{\mathrm{s}}_{y'y'} = 0 \ ; \ \epsilon^{\mathrm{s}}_{x'y'} = \frac{(3-4\nu)\gamma^*}{4(1-\nu)}. \tag{10}$$

Outside the inclusion

Displacements:

$$\begin{aligned} u^{\mathrm{s}}_{x'} &= \frac{a^2\gamma^*(\sin\theta\cos\theta_s - \cos\theta\sin\theta_s)}{4(1-\nu)r}\left\{\left[4(1-\nu) - \left(\frac{a}{r}\right)^2\right] + 2\left[1 - \left(\frac{a}{r}\right)^2\right](\cos 2\theta\cos 2\theta_s + \sin 2\theta\sin 2\theta_s)\right\} \\ u^{\mathrm{s}}_{y'} &= \frac{a^2\gamma^*(\cos\theta\cos\theta_s + \sin\theta\sin\theta_s)}{4(1-\nu)r}\left\{\left[4(1-\nu) - \left(\frac{a}{r}\right)^2\right] - 2\left[1 - \left(\frac{a}{r}\right)^2\right](\cos 2\theta\cos 2\theta_s + \sin 2\theta\sin 2\theta_s)\right\} \end{aligned} \tag{11}$$

Where I have made the use of identity, $x'^2 + y'^2 = x^2 + y^2 = r'^2 = r^2$. Similarly,
Strains:

$$\begin{aligned} \epsilon^{\mathrm{s}}_{x'x'} &= \frac{a^2\gamma^*}{2(1-\nu)r^4}\left[r^2(2\nu-1) + (3a^2-2r^2)\left(\cos 2\theta\cos 2\theta_s + \sin 2\theta\sin 2\theta_s\right)\right](\sin 2\theta\cos 2\theta_s - \cos 2\theta\sin 2\theta_s) \\ \epsilon^{\mathrm{s}}_{y'y'} &= \frac{a^2\gamma^*}{2(1-\nu)r^4}\left[r^2(2\nu-1) - (3a^2-2r^2)\left(\cos 2\theta\cos 2\theta_s + \sin 2\theta\sin 2\theta_s\right)\right](\sin 2\theta\cos 2\theta_s - \cos 2\theta\sin 2\theta_s) \\ \epsilon^{\mathrm{s}}_{x'y'} &= -\frac{a^2\gamma^*}{4(1-\nu)r^4}\left(3a^2-2r^2\right)(\cos 4\theta\cos 4\theta_s + \sin 4\theta\sin 4\theta_s) \end{aligned} \tag{12}$$

Components in the lab frame

Now by rotating the local frame by an angle $-\theta_s$, we get these fields in the lab frame in terms of components of ϵ^*_{ij}.

Inside the inclusion

Displacements:

$$\begin{aligned} u^{\mathrm{s}}_x &= \frac{3-4\nu}{4(1-\nu)} r \left(\epsilon^*_{xx}\cos\theta + \epsilon^*_{xy}\sin\theta\right) \\ u^{\mathrm{s}}_y &= \frac{3-4\nu}{4(1-\nu)} r \left(\epsilon^*_{xy}\cos\theta + \epsilon^*_{yy}\sin\theta\right) \end{aligned} \tag{13}$$

Strains:

$$\begin{aligned} \epsilon^{\mathrm{s}}_{xx} &= \frac{3-4\nu}{4(1-\nu)}\epsilon^*_{xx} \\ \epsilon^{\mathrm{s}}_{yy} &= \frac{3-4\nu}{4(1-\nu)}\epsilon^*_{yy} \\ \epsilon^{\mathrm{s}}_{xy} &= \frac{3-4\nu}{4(1-\nu)}\epsilon^*_{xy} \end{aligned} \tag{14}$$

Outside the inclusion

Displacements:

$$\begin{aligned} u^{\mathrm{s}}_x &= \frac{a^2}{4(1-\nu)r}\left\{\left[4(1-\nu) - \left(\frac{a}{r}\right)^2\right]\left(\epsilon^*_{xx}\cos\theta + \epsilon^*_{xy}\sin\theta\right) + 2\sin\theta\left[1-\left(\frac{a}{r}\right)^2\right](\epsilon^*_{xy}\cos 2\theta - \epsilon^*_{xx}\sin 2\theta)\right\} \\ u^{\mathrm{s}}_y &= \frac{a^2}{4(1-\nu)r}\left\{\left[4(1-\nu) - \left(\frac{a}{r}\right)^2\right]\left(\epsilon^*_{yy}\sin\theta + \epsilon^*_{xy}\cos\theta\right) - 2\cos\theta\left[1-\left(\frac{a}{r}\right)^2\right](\epsilon^*_{xy}\cos 2\theta + \epsilon^*_{yy}\sin 2\theta)\right\} \end{aligned} \tag{15}$$

Strains:

$$\epsilon^{\mathrm{s}}_{xx} = \frac{1}{4(1-\nu)}\left(\frac{a}{r}\right)^2\Bigg\{2\left[2\nu-1\right]\left[\epsilon^*_{xy}\sin 2\theta + \epsilon^*_{xx}\cos 2\theta\right] + \left[3\left(\frac{a}{r}\right)^2 - 2\right]\Bigg[\epsilon^*_{xy}\left(\sin 4\theta\cos 4\theta_s - \cos 4\theta \sin 4\theta_s\right)$$
$$+ \epsilon^*_{yy}\left(\cos 4\theta\cos 4\theta_s + \sin 4\theta\sin 4\theta_s\right)\Bigg]\Bigg\}$$
$$\epsilon^{\mathrm{s}}_{yy} = \frac{1}{4(1-\nu)}\left(\frac{a}{r}\right)^2\Bigg\{2\left[2\nu-1\right]\left[\epsilon^*_{xy}\sin 2\theta + \epsilon^*_{xx}\cos 2\theta\right] - \left[3\left(\frac{a}{r}\right)^2 - 2\right]\Bigg[\epsilon^*_{xy}\left(\sin 4\theta\cos 4\theta_s - \cos 4\theta \sin 4\theta_s\right)$$
$$+ \epsilon^*_{yy}\left(\cos 4\theta\cos 4\theta_s + \sin 4\theta\sin 4\theta_s\right)\Bigg]\Bigg\}$$
$$\epsilon^{\mathrm{s}}_{xy} = \frac{1}{4(1-\nu)}\left(\frac{a}{r}\right)^2\left[3\left(\frac{a}{r}\right)^2 - 2\right]\left[\epsilon^*_{yy}\left(\sin 4\theta\cos 4\theta_s - \cos 4\theta\sin 4\theta_s\right) - \epsilon^*_{xy}\left(\cos 4\theta\cos 4\theta_s + \sin 4\theta\sin 4\theta_s\right)\right] \tag{16}$$

Eq. 2 is used to compute terms $\cos 4\theta_s$ and $\sin 4\theta_s$, *i.e.*

$$\begin{aligned} \sin 4\theta_s &= 2\sin 2\theta_s\cos 2\theta_s = \frac{2\epsilon^*_{xy}\epsilon^*_{yy}}{{\gamma^*}^2} \\ \cos 4\theta_s &= \cos^2 2\theta_s - \sin^2 2\theta_s = \frac{{\epsilon^*_{xy}}^2 - {\epsilon^*_{yy}}^2}{{\gamma^*}^2} \end{aligned} \tag{17}$$

Inserting the above expressions in Eq. 16, we get a strain field outside the inclusion in terms of everything in the lab coordinate $(r, \theta, \epsilon^*_{ij})$

$$\begin{aligned} \epsilon^{\mathrm{s}}_{xx} &= A\left\{2\left[2\nu-1\right]\left[\epsilon^*_{xy}\sin 2\theta + \epsilon^*_{xx}\cos 2\theta\right] + \left[3\left(\frac{a}{r}\right)^2 - 2\right]\left(\frac{{\epsilon^*_{xy}}^2 + {\epsilon^*_{yy}}^2}{{\gamma^*}^2}\right)\left(\epsilon^*_{xy}\sin 4\theta - \epsilon^*_{yy}\cos 4\theta\right)\right\} \\ \epsilon^{\mathrm{s}}_{yy} &= A\left\{2\left[2\nu-1\right]\left[\epsilon^*_{xy}\sin 2\theta + \epsilon^*_{xx}\cos 2\theta\right] - \left[3\left(\frac{a}{r}\right)^2 - 2\right]\left(\frac{{\epsilon^*_{xy}}^2 + {\epsilon^*_{yy}}^2}{{\gamma^*}^2}\right)\left(\epsilon^*_{xy}\sin 4\theta - \epsilon^*_{yy}\cos 4\theta\right)\right\} \\ \epsilon^{\mathrm{s}}_{xy} &= -A\left[3\left(\frac{a}{r}\right)^2 - 2\right]\left(\frac{{\epsilon^*_{xy}}^2 + {\epsilon^*_{yy}}^2}{{\gamma^*}^2}\right)\left(\epsilon^*_{xy}\cos 4\theta + \epsilon^*_{yy}\sin 4\theta\right) \end{aligned} \tag{18}$$

where $A = \frac{1}{4(1-\nu)}\left(\frac{a}{r}\right)^2$.

2.1.2 Volumetric strain

Components in the local frame (x',y')

Inside the inclusion

Displacements:

$$u^{\mathrm{v}}_{x'} = \frac{\epsilon^*}{2(1-\nu)} x' \;;\; u^{\mathrm{v}}_{y'} = \frac{\epsilon^*}{2(1-\nu)} y'. \tag{19}$$

Where superscript 'v' stands for the volumetric part. *Strains:*

$$\epsilon^{\mathrm{v}}_{x'x'} = \epsilon^{\mathrm{v}}_{y'y'} = \frac{\epsilon^*}{2(1-\nu)} \;;\; \epsilon^{\mathrm{v}}_{x'y'} = 0. \tag{20}$$

Outside the inclusion

Displacements:

$$\begin{aligned} u^{\mathrm{v}}_{x'} &= \frac{\epsilon^*}{2(1-\nu)} \left(\frac{a}{r'^2}\right)^2 x' \\ u^{\mathrm{v}}_{y'} &= \frac{\epsilon^*}{2(1-\nu)} \left(\frac{a}{r'^2}\right)^2 y' \end{aligned} \tag{21}$$

Strains:

$$\begin{aligned} \epsilon^{\mathrm{v}}_{x'x'} &= -\frac{\epsilon^*}{2(1-\nu)} \left(\frac{a}{r'^2}\right)^2 \left(\frac{x'^2-y'^2}{r'^2}\right) \\ \epsilon^{\mathrm{v}}_{y'y'} &= \frac{\epsilon^*}{2(1-\nu)} \left(\frac{a}{r'^2}\right)^2 \left(\frac{x'^2-y'^2}{r'^2}\right) \\ \epsilon^{\mathrm{v}}_{x'y'} &= -\frac{\epsilon^*}{(1-\nu)} \left(\frac{a}{r'^2}\right)^2 \left(\frac{x'y'}{r'^2}\right) \end{aligned} \tag{22}$$

Components in the local frame but in terms of lab coordinates (x, y)

Inside the inclusion

Displacements:

$$\begin{aligned} u^{\mathrm{v}}_{x'} &= \frac{\epsilon^*}{2(1-\nu)} r\left(\cos\theta\cos\theta_s + \sin\theta\sin\theta_s\right) \\ u^{\mathrm{v}}_{y'} &= \frac{\epsilon^*}{2(1-\nu)} r\left(\sin\theta\cos\theta_s - \cos\theta\sin\theta_s\right) \end{aligned} \tag{23}$$

Strains:

$$\epsilon^{\mathrm{v}}_{x'x'} = \epsilon^{\mathrm{v}}_{y'y'} = \frac{\epsilon^*}{2(1-\nu)} \;;\; \epsilon^{\mathrm{v}}_{x'y'} = 0. \tag{24}$$

Outside the inclusion

Displacements:

$$
\begin{aligned}
u^{\mathrm{v}}_{x'} &= \frac{\epsilon^*}{2(1-\nu)}\frac{a^2}{r}\left(\cos\theta\cos\theta_s + \sin\theta\sin\theta_s\right) \\
u^{\mathrm{v}}_{y'} &= \frac{\epsilon^*}{2(1-\nu)}\frac{a^2}{r}\left(\sin\theta\cos\theta_s - \cos\theta\sin\theta_s\right)
\end{aligned} \tag{25}
$$

Again, identity $x'^2 + y'^2 = x^2 + y^2 = r'^2 = r^2$ has been used.
Strains:

$$
\begin{aligned}
\epsilon^{\mathrm{v}}_{x'x'} &= -\frac{\epsilon^*}{2(1-\nu)}\left(\frac{a}{r}\right)^2\left(\cos 2\theta\cos 2\theta_s + \sin 2\theta\sin 2\theta_s\right) \\
\epsilon^{\mathrm{v}}_{y'y'} &= \frac{\epsilon^*}{2(1-\nu)}\left(\frac{a}{r}\right)^2\left(\cos 2\theta\cos 2\theta_s + \sin 2\theta\sin 2\theta_s\right) \\
\epsilon^{\mathrm{v}}_{x'y'} &= -\frac{\epsilon^*}{2(1-\nu)}\left(\frac{a}{r}\right)^2\left(\sin 2\theta\cos 2\theta_s - \cos 2\theta\sin 2\theta_s\right)
\end{aligned} \tag{26}
$$

Components fields in the lab frame

By rotating the local frame by an angle $-\theta_s$, we get these fields in the lab frame in terms of components of ϵ^*_{ij}.

Inside the inclusion

Displacements:

$$
\begin{aligned}
u^{\mathrm{v}}_{x} &= \frac{\epsilon^*}{2(1-\nu)}x \\
u^{\mathrm{v}}_{y} &= \frac{\epsilon^*}{2(1-\nu)}y
\end{aligned} \tag{27}
$$

Strains:

$$
\epsilon^{\mathrm{v}}_{xx} = \epsilon^{\mathrm{v}}_{yy} = \frac{\epsilon^*}{2(1-\nu)};\ \epsilon^{\mathrm{v}}_{xy} = 0. \tag{28}
$$

Outside the inclusion

Displacements:

$$
\begin{aligned}
u^{\mathrm{v}}_{x} &= \frac{\epsilon^*}{2(1-\nu)}\left(\frac{a}{r}\right)^2 x \\
u^{\mathrm{v}}_{y} &= \frac{\epsilon^*}{2(1-\nu)}\left(\frac{a}{r}\right)^2 y
\end{aligned} \tag{29}
$$

Strains:

$$
\begin{aligned}
\epsilon^{\mathrm{v}}_{xx} &= -\frac{\epsilon^*}{2(1-\nu)}\left(\frac{a}{r}\right)^2\cos 2\theta \\
\epsilon^{\mathrm{v}}_{yy} &= \frac{\epsilon^*}{2(1-\nu)}\left(\frac{a}{r}\right)^2\cos 2\theta \\
\epsilon^{\mathrm{v}}_{xy} &= -\frac{\epsilon^*}{2(1-\nu)}\left(\frac{a}{r}\right)^2\sin 2\theta
\end{aligned}
\tag{30}
$$

2.2 Stress fields

Corresponding stress components in the lab frame are given.
Inside the inclusion:

$$
\left\{\begin{array}{c}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{array}\right\} = \frac{E}{(1+\nu)(1-2\nu)}\left[\begin{array}{ccc}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & 1-2\nu\end{array}\right]\left\{\begin{array}{c}\epsilon_{xx}-\epsilon^*_{xx}\\ \epsilon_{yy}-\epsilon^*_{yy}\\ \epsilon_{xy}-\epsilon^*_{xy}\end{array}\right\}
\tag{31}
$$

Outside the inclusion:

$$
\left\{\begin{array}{c}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{array}\right\} = \frac{E}{(1+\nu)(1-2\nu)}\left[\begin{array}{ccc}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & 1-2\nu\end{array}\right]\left\{\begin{array}{c}\epsilon_{xx}\\ \epsilon_{yy}\\ \epsilon_{xy}\end{array}\right\}
\tag{32}
$$

Total fields

As Eshelby solution is an elastic solution, I just superposed both solutions due to pure shear and volumetric transformation, *i.e.*

$$
\begin{aligned}
u_i &= u^{\mathrm{s}}_i + u^{\mathrm{v}}_i \\
u_y &= u^{\mathrm{s}}_y + u^{\mathrm{v}}_y \\
\epsilon_{ij} &= \epsilon^{\mathrm{s}}_{ij} + \epsilon^{\mathrm{v}}_{ij} \\
\sigma_{ij} &= \sigma^{\mathrm{s}}_{ij} + \sigma^{\mathrm{v}}_{ij}
\end{aligned}
\tag{33}
$$

3 Validation

This section is about validating the above analytical expressions with a simple configuration as shown below figure (Fig. 3). A STZ with radius 0.05mm and located at the center, *i.e.* $(x_{STZ}, y_{STZ}) = (0, 0)$ is considered. The elastic properties of matrix are $E = 7Gpa$ and $\nu = 0.4$. For testing purposes, I have taken a decomposed (pure shear + volumetric part) Eigenstrain tensor in the local frame along the direction of the 45^o angle. *I.e.*

$$\epsilon_{i'j'} = \begin{bmatrix} 0 & 0.2 \\ 0.2 & 0 \end{bmatrix} + \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} \tag{34}$$

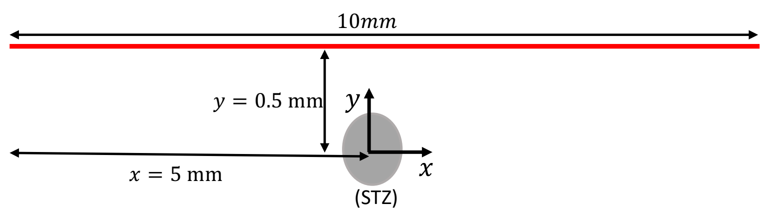

Figure 3: Schematic of the test configuration with a single STZ located at the center along the direction of 45^o and the aim is to compute stress field along the line $y = 0.5$mm (*red solid line*).

I then computer stress along a horizontal line located at $y = 0.5$mm (as shown as a red-solid line in Fig. 3). I compute it using analytical with closed-form expression as given above and numerically calculating general solution as given in [1]. The below figures plot the results from these two methods for three cases: (1) pure shear eigenstrain, (2) volumetric eigenstrain, and (3) combining both pure shear and volumetric eigenstrain.

Pure shear eigenstrain

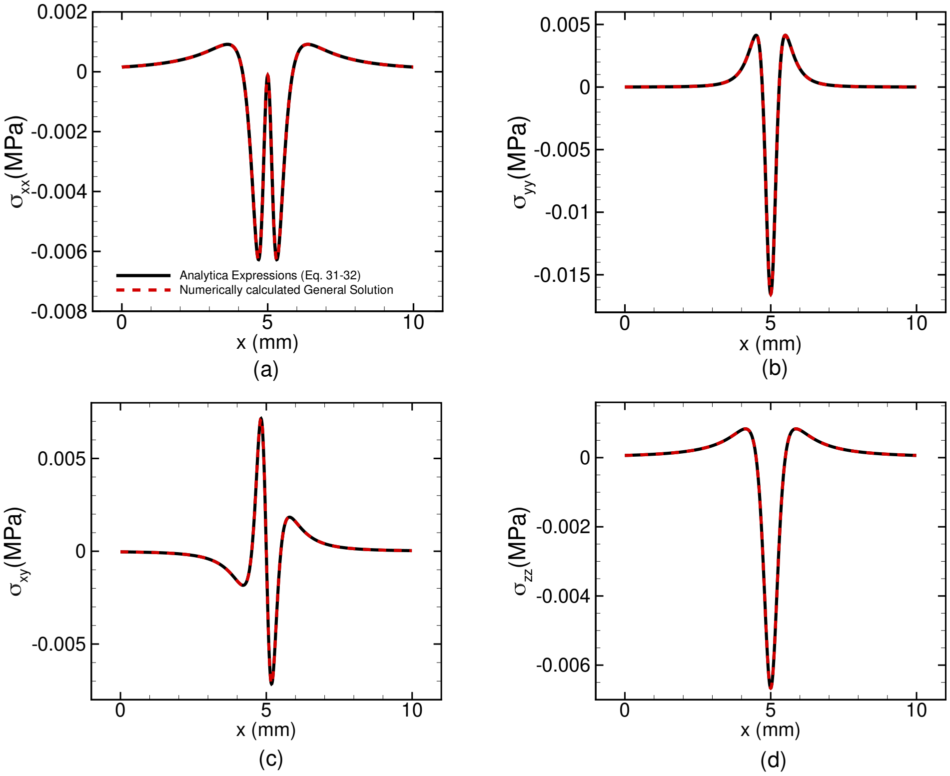

Figure 4: **Pure shear eigenstrain:** All plane-strain stress components due to presence of STZ along the $y = 0.5$ mm: (a) σ_{xx}, (b) σ_{yy}, (c) σ_{xy}, (d) σ_{zz}.

Volumetric eigenstrain

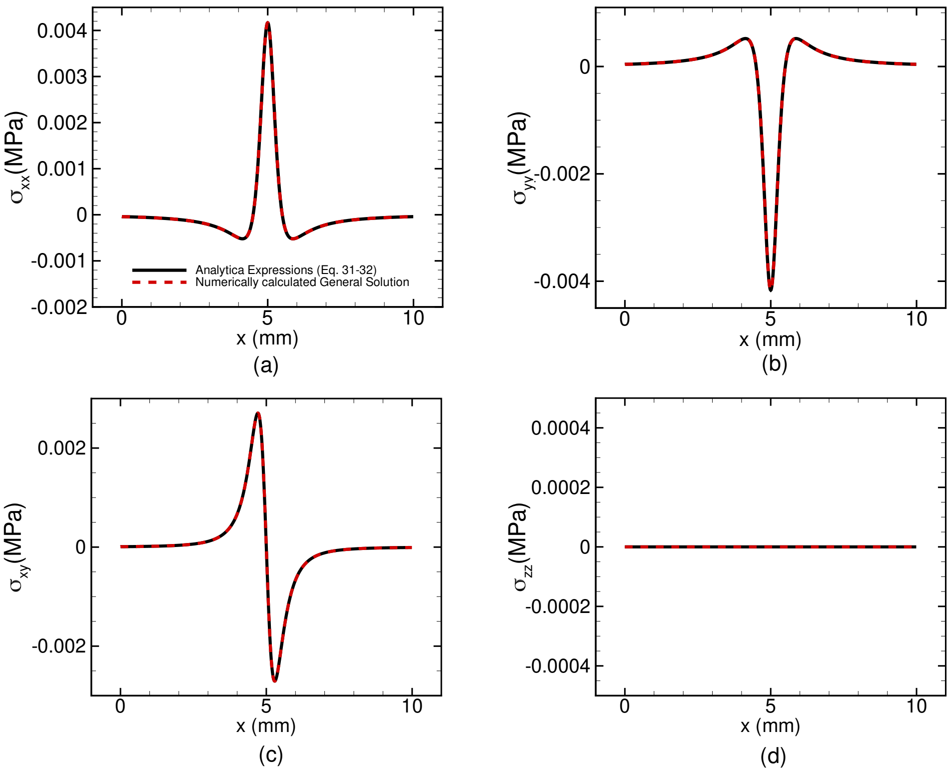

Figure 5: **Volumetric eigenstrain:** All plane-strain stress components due to presence of STZ along the $y = 0.5$ mm: (a) $\sigma_{\rm xx}$, (b) $\sigma_{\rm yy}$, (c) $\sigma_{\rm xy}$, (d) $\sigma_{\rm zz}$.

Combined both pure shear and volumetric eigenstrain

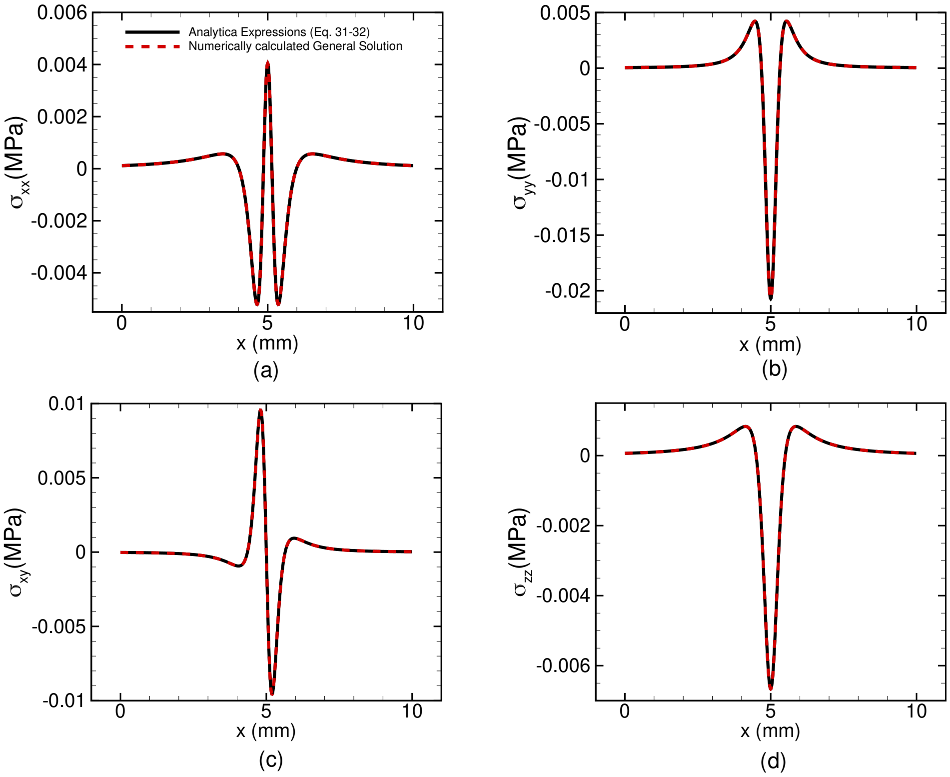

Figure 6: **Combined pure shear and volumetric eigenstrain:**All plane-strain stress components due to presence of STZ along the $y = 0.5$ mm: (a) σ_{xx}, (b) σ_{yy}, (c) σ_{xy}, (d) σ_{zz}.

In all three cases, both ways are producing the exact same results. In the current code version, I have kept both subroutines - *stzfield* and *stzfiledan* -' an' stands for analytical. But I am only calling *stzfiledan* subroutine of the analytical/closed form expression for fast calculations.

References

[1] T. Mura, Micromechanics of Defects in Solids, 2^{nd} Edt. (Mechanics of elastic and inelastic solids), ISSN 0924-2163 (1987).