1. From Fluid Continuity [1] [Note]

Sum of mass fluxes = rate of loss of mass in the volume

$$\frac{\Delta m_f}{\Delta t} = -\frac{\partial \rho_f q_x}{\partial x}\Delta x(\Delta y \Delta z) - \frac{\partial \rho_f q_y}{\partial y}\Delta y(\Delta x \Delta y) - \frac{\partial \rho_f q_z}{\partial z}\Delta z(\Delta x \Delta y) + \tilde{S}$$

$$= -\nabla\cdot(\rho_f \vec{q})\,\Delta x \Delta y \Delta z = -\nabla\cdot(\rho_f \vec{q})\,V$$

$$m_f = \rho_f V_p = \rho_f \phi V$$

Ignore change of ρ_f in space: $\nabla\cdot(\rho_f \vec{q}) = \rho_f \nabla\cdot\vec{q}$ $\quad(\rho = \rho_f)$

$$[1]\quad \boxed{\frac{\partial(\rho\phi)}{\partial t} = -\rho\,\nabla\cdot\vec{q} + \tilde{m}} \qquad \left(\tilde{m} = \frac{\tilde{S}}{V}\right)$$

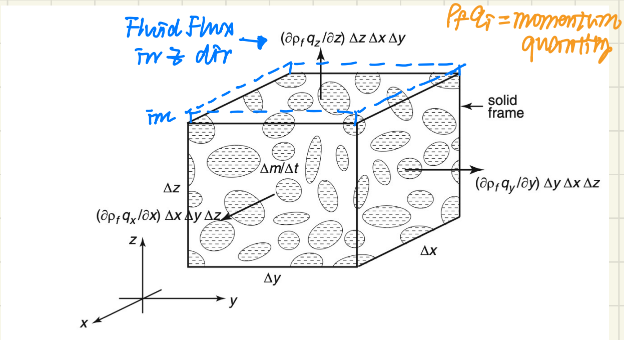

Fig. 6.3 Fluid mass conservation

ρ_f: fluid density $\left[\frac{M}{L^2}\right]$

q_i: fluid flux in i direction $\left[\frac{L}{T}\right]$

$\tilde{S}$: source term rate $\left[\frac{M}{T}\right]$

p: pressure $\left[\frac{M}{LT^2}\right]$

k: permeability

Substitute Darcy's Law [2] in [1]:

$$\vec{q} = -\frac{k}{\mu}\nabla p$$

$$\boxed{\frac{\partial(\rho\phi)}{\partial t} = \frac{k}{\mu}\nabla(\rho\nabla p) + \tilde{m}}$$

Consider 1D case, homogeneous and isotropic permeability

$$\frac{\partial(\rho\phi)}{\partial t} = \frac{k}{\mu}\frac{\partial}{\partial x}\left(\rho\frac{\partial}{\partial x}p\right) + \tilde{m}$$

Introduce formation volume factor B_w ($B_w \sim 1.0$ for water)

$$B_w = \frac{V_{Rc}}{V_{sc}} = \frac{\rho_{sc}}{\rho_{Rc}}$$

sc : surface condition
Rc : reservoir condition.

replace ρ by $\frac{\rho_{sc}}{B_w}$:

$$\frac{\partial}{\partial t}\left(\frac{\rho_{sc}}{B_w}\phi\right) = \frac{k}{\mu}\frac{\partial}{\partial x}\left(\frac{\rho_{sc}}{B_w}\frac{\partial p}{\partial x}\right) + \tilde{m}$$

$$\frac{\partial}{\partial t}\left(\frac{1}{B_w}\phi\right) = \frac{k}{\mu}\frac{\partial}{\partial x}\left(\frac{1}{B_w}\frac{\partial p}{\partial x}\right) + \frac{\tilde{m}}{\rho_{sc}}$$

Expand chain and product rule on both sides

(chain rule)

$$\frac{\partial p}{\partial t}\left(\frac{\partial}{\partial p}\left(\frac{1}{B_w}\right)\phi + \frac{1}{B_w}\frac{\partial \phi}{\partial p}\right) = \frac{k}{\mu}\left(\frac{\partial}{\partial p}\left(\frac{1}{B_w}\right)\left(\frac{\partial p}{\partial x}\right)^2 + \frac{1}{B_w}\frac{\partial^2 p}{\partial x^2}\right) + \frac{\tilde{m}}{\rho_{sc}}$$

Define

By definition

$$C_r = \boxed{\frac{1}{V_p}\frac{\partial V_p}{\partial p}} = \frac{1}{\phi}\frac{\partial \phi}{\partial p} \quad \text{(rock compressibility)}$$

$$C_f = -\frac{1}{V}\frac{\partial V}{\partial p}\bigg|_T = \frac{1}{\rho}\frac{\partial \rho}{\partial p}\bigg|_T = \frac{B_w}{\rho_{sc}}\frac{\partial}{\partial p}\left(\frac{\rho_{sc}}{B_w}\right) = B_w\frac{\partial}{\partial p}\left(\frac{1}{B_w}\right)$$

(fluid compressibility)

$C_t = C_r + C_f$ (total compressibility)

$$\frac{\partial p}{\partial t}\frac{\phi}{B_w}\underbrace{\left(\underbrace{\frac{1}{B_w}\frac{\partial}{\partial p}\left(\frac{1}{B_w}\right)}_{C_f} + \underbrace{\frac{1}{\phi}\frac{\partial \phi}{\partial p}}_{C_r}\right)}_{C_t} = \frac{k}{\mu}\left(\frac{1}{B_w}\frac{\partial^2 p}{\partial x^2} + \cancel{\frac{C_f}{B_w}\left(\frac{\partial p}{\partial x}\right)^2}\right) + \frac{\tilde{m}}{\rho_{sc}}$$

can ignore for small fluid compressibility ($<10^{-5}$)

Take $B_w = 1.0$, then

$$\boxed{\frac{\partial p}{\partial t} = \frac{k}{\mu \phi C_t}\frac{\partial^2 p}{\partial^2 x} + \frac{\tilde{m}}{\rho_{sc}\phi C_t}}$$

p: pressure
t: time
k: permeability
ϕ: porosity
c_t: total compressibility
$\tilde{m}$: source rate (per unit volume)
ρ_{sc}: surface fluid density

Nondimensionize the system:

$$\left.\begin{aligned} p &= \sigma_n \hat{p} \\ t &= \frac{L}{v_0}\hat{t} \\ x &= L_x \hat{x} \\ \tilde{m} &= m_0 \hat{\tilde{m}} \end{aligned}\right\} \boxed{\frac{\partial \hat{p}}{\partial \hat{t}} = \frac{kL}{\mu \phi c_t L_x^2 v_0}\frac{\partial^2 \hat{p}}{\partial \hat{x}^2} + \frac{m_0 L}{\rho \phi c_t \sigma_n v_0}\hat{\tilde{m}}}$$

2. Show analytical solution of peak value of diffusion equation (in terms of time)

Start from fundamental solution of diffusion equation:

$$G(x,t) = \frac{1}{\sqrt{4\pi D t}} e^{-\frac{x^2}{4Dt}}$$

$$@\, x=0 \quad G = \frac{1}{\sqrt{4\pi D t}} \propto \boxed{\frac{1}{\sqrt{t}}}$$

We could view constant injection rate as continuous point source and use Duhamel integral.

$$p|_{x=0} = \int_0^t \frac{1}{\sqrt{4\pi D t}} = \sqrt{\frac{t}{\pi D}} \propto \boxed{\sqrt{t}}$$