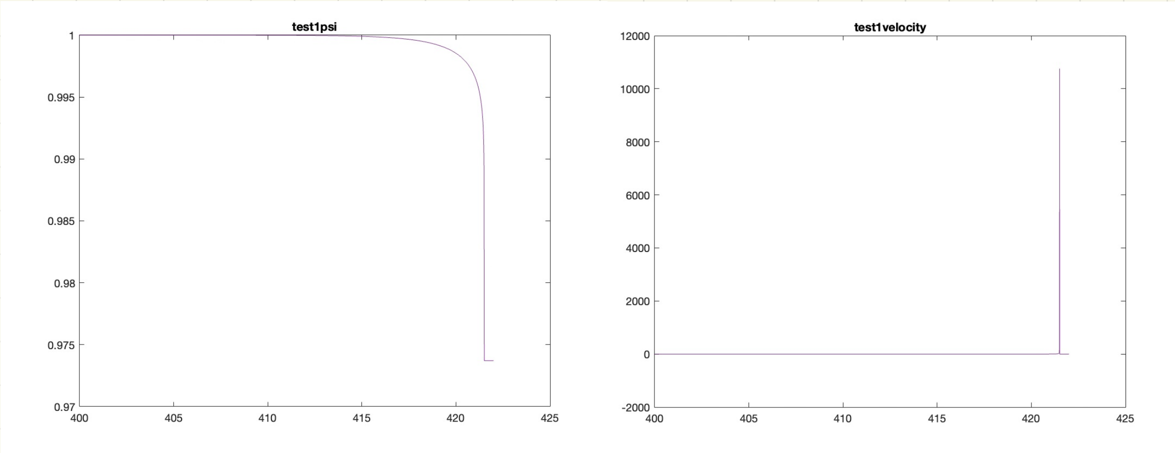
θ & p coupling (psi or coupling term)

① psi

- Simulation:



The slope of ψ is related to velocity (as shown in the eq. also), the "stair-like" pattern actually relates to reaccumulate of velocity.

increase velocity ⟶ increase slope of psi

velocity drops to zero ⟶ slope of psi drops to zero

- Analytical:

$$\psi' + \frac{1}{c}\left(\bar{v} + \frac{V_p}{v_0}\right)\hat{\psi} - \frac{1}{c}\left(\bar{v} + \frac{V_p}{v_0}\right)\frac{\bar{\sigma}_n'}{\bar{\sigma}_n} = 0$$

$\bar{v}(t)$ — function of $\bar{v}$ (Not easy to derive)

② Coupling term

$$\frac{\partial \theta}{\partial t} = 1 - \frac{\theta v}{L} - \frac{\alpha \theta}{b \sigma_n'} \frac{d\sigma_n'}{dt}$$

$$\frac{\theta_0 v_0}{L} \frac{\partial \hat{\theta}}{\partial \hat{t}} = 1 - \frac{\theta \bar{v} + V_p}{L} + \frac{\alpha \theta}{b \sigma_n'} \frac{v_0 \sigma_n}{L} \frac{d\hat{P}}{d\hat{t}}$$

$$\begin{cases} \theta = \theta_0 \hat{\theta} \\ v = \bar{v} + V_p \\ \sigma_n' = \sigma_n - P \\ P = \sigma_n \hat{P} \\ \hat{v} = v_0 \bar{v} \end{cases}$$

$$\frac{\partial \hat{\theta}}{\partial \hat{t}} = \frac{L}{\theta_0 v_0} - \left(\hat{v} + \frac{V_p}{v_0}\right)\hat{\theta} + \frac{\alpha \hat{\theta} \sigma_n}{b \sigma_n'} \frac{d\hat{P}}{d\hat{t}}$$ ← analytical solution (Pressure BC w. Tanh Approximation)

↑ approximate using first order finite difference. (Rate BC)

$$\frac{d\hat{P}}{d\hat{t}} = \frac{\hat{P}_n - \hat{P}_{n-1}}{\Delta t}$$

- Governing Equations: (Implementation level)

$$\frac{d\hat{\theta}}{d\hat{t}} = \frac{L}{\theta_o v_o} - abs\left(\hat{v} + \frac{V_p}{v_o}\right)\hat{\theta} + \frac{\alpha \hat{\theta} \sigma_n}{b \sigma_n'} \frac{\hat{P}_n - \hat{P}_{n-1}}{\Delta t}$$

$$\frac{d\hat{u}}{d\hat{t}} = \hat{v}$$

$$\frac{d\hat{v}}{d\hat{t}} = \frac{kL^2}{Mv_o^2}\left(-\hat{u} - sign(\hat{v} + \frac{V_p}{v_o})\frac{\sigma_n'}{kL}\left(f_o + a\ln\left(\overset{abs}{\left(\hat{v} + \frac{V_p}{v_o}\right)}\right) + b\ln\left(\overset{abs}{(\hat{\theta})}\right)\right)\right)$$

variable [θ u v]

```
function [f2] = sbm_sys_test1(y,coeff1,coeff2,coeff3,coeff4,sigma_initial,f_o,a,b,t_step,alpha)
%unknown [theta,u,v,psi]
global p_cur p_pre
f2 = [ coeff4 -  abs(y(3)+coeff3) * y(1) + ( alpha * sigma_initial * y(1) ) / ( b * (sigma_initial - p_cur * sigma_initial) ) * ( p_cur - p_pre) / t_step;
       y(3);
       coeff1 * (-y(2) - sign(y(3)+coeff3) * ( coeff2 * (sigma_initial - p_cur * sigma_initial) ) * ( f_o + a * log(abs(y(3)+coeff3)) + b * log(abs(y(1)))));
       %-0.1 * abs(y(3)+coeff3) * ( y(4) - (sigma_initial - p_cur) / (sigma_initial) )];
end
```

```
%Nondimensionlization
coeff1 = k_s * L ^ 2 / ( M * v_o ^ 2);
coeff2 = 1 / ( k_s * L );
coeff3 = V_p / v_o;
coeff4 = L / ( theta_o * v_o);
```