#4.6.

궁금증: Table 4.8을 보고 overdispersion은 어떻게 확인할까?

만약 overdispersion이 확인되면 해결책

① Random effects Poisson/logistic regression.

② Seek other model (Poisson → Negative binomial)

③ Quasi-likelihood estimation. ④ Mixture model.

* Natural exponential family ⇝ $Var(Y) = v(\mu)$ — v 에 의해 characterized !

* QLE : $\sum \frac{(y_i - \mu_i)x_{ij}}{v(\mu_i)} \cdot \frac{\partial \mu_i}{\partial \eta_i} = 0$.

  (i) $v(\mu_i) = \mu_i$ → poisson dist^{n}.

  (ii) $v(\mu_i) = \frac{\mu_i(1-\mu_i)}{n_i}$ → binomial dist^{n}

* Overdispersion for Poisson GLM.

→ $v(\mu_i) = \phi(\mu_i)$. , $\phi > 1$.

→ QL eq. = ML eq. $Cov(\hat{\beta}_{QL}) = \phi \, Cov(\hat{\beta}_{ML})$

How can we estimate ϕ ??

$X^2/\phi \sim \chi^2_{N-p}$. ⇒ $E(X^2/\phi) \approx N-p$ ⇒ $\hat{\phi} = X^2/N-p$.

Multiply SE → $\hat{\phi}^{1/2} SE$.

$$\hat{\phi} = \frac{535.8957}{171} = 3.1339 \rightarrow \hat{\phi}^{1/2} = 1.77.$$

$$SE(\hat{\beta}) \rightarrow 1.77 \times 0.065 = 0.115$$

#4.9

Negative Binomial : Poisson 보다 Variance가 더 큰 distribution $\in$ exponential family

$k \to \infty \Rightarrow Var(Y) \searrow \mu$. , NB $\to$ Poisson

$\therefore$ glm.nb 를 사용하면 glm(poisson) 보다 $\hat{\beta}$의 SE가 더 크다.

$\Rightarrow$ overdispersion을 해결해주므로 더 좋다.

#4.12

??

#4.15.

binomial : $\pi_i \in (0,1)$, $\alpha + \beta x \in \mathbb{R}$

Poisson : $\lambda \in \mathbb{R}_+$, $\alpha + \beta x \in \mathbb{R}$.

#4.18.

$$\mathcal{L}(\pi) = \prod_{i=1}^{N} \binom{n_i}{y_i} \pi^{y_i} (1-\pi)^{n_i - y_i} \rightarrow l(\pi) = k + \Sigma y_i \log \pi + (\Sigma n_i - \Sigma y_i) \log(1-\pi).$$

$$= \prod_{i=1}^{N} \binom{n_i}{y_i} \cdot \pi^{\Sigma y_i} (1-\pi)^{\Sigma(n_i - y_i)}$$

$$\frac{\partial l}{\partial \pi} = \frac{\Sigma y_i}{\pi} - \frac{\Sigma n_i - \Sigma y_i}{1-\pi} = 0 \rightarrow (1-\pi)\Sigma y_i - (\Sigma n_i - \Sigma y_i)\pi = 0$$

$$\Rightarrow \Sigma y_i - \Sigma n_i \pi = 0$$

$$\Rightarrow \hat{\pi} = \frac{\Sigma y_i}{\Sigma n_i}$$

when $n_i = 1$ $\forall i = 1, \cdots, N$.

$\hat{\pi} = \frac{1}{N} \Sigma y_i$.

$$\Rightarrow X^2 = \sum_{i=1}^{N} \frac{(y_i - \hat{\mu}_i)^2}{v(\mu_i)} = \sum_{i=1}^{N} \frac{(y_i - \hat{\pi})^2}{\hat{\pi}(1-\hat{\pi})} = \cancel{(N - \Sigma y_i)}^{N} \frac{\hat{\pi}}{\cancel{1-\hat{\pi}}} + \cancel{\Sigma y_i}^{N} \cdot \frac{1-\hat{\pi}}{\cancel{\hat{\pi}}}$$

$= N\hat{\pi} + N - N\hat{\pi} = N \rightarrow$ Not depend on observed data

$\rightarrow$ uninformative

#4.21

$\pi_i = \Phi(\Sigma_j \beta_j x_{ij})$ $\quad n_i Y_i \sim bin(n_i, \pi_i)$

$$\frac{\frac{e^y}{1+e^y}}{1 - \frac{e^y}{1+e^y}}$$

$w_i = \left(\frac{\partial \mu_i}{\partial \eta_i}\right)^2 / Var(Y_i)$.

$$\log \frac{\pi}{1-\pi} = y \qquad \pi = \frac{e^y}{1+e^y}$$

$$\frac{\pi}{1-\pi} = e^y$$

$$Var(Y_i) = \frac{1}{n_i^2} Var(n_i Y_i) = \frac{\pi_i(1-\pi_i)}{n_i}.$$

$$\mu_i = \Phi(\eta_i) \Rightarrow \frac{\partial \mu_i}{\partial \eta_i} = \phi(\eta_i) \Rightarrow w_i = \phi(\eta_i)^2 / \frac{\pi_i(1-\pi_i)}{n_i} = \frac{\phi(\eta_i)^2 n_i}{\pi_i(1-\pi_i)}.$$

$$\Rightarrow J = X^T W X, \; J^{-1} = Cov(\hat{\beta})$$

<Logistic regression>

$$\Rightarrow \Phi(\eta) = \frac{e^\eta}{1+e^\eta} \Rightarrow \phi(\eta) = \frac{e^\eta(1+e^\eta) - e^\eta \cdot e^\eta}{(1+e^\eta)^2} = \frac{e^\eta}{(1+e^\eta)^2} = \Phi(\eta)(1-\Phi(\eta))$$

$$= \pi_i(1-\pi_i).$$

$\therefore \; w_i = n_i \pi_i (1-\pi_i)$.

#4.27

$$f(y;k,\mu) = \frac{\Gamma(y+k)}{\Gamma(k)\,\Gamma(y+1)} \left(\frac{k}{\mu+k}\right)^{k} \left(1-\frac{k}{\mu+k}\right)^{y}$$

, k is known.

$$= \frac{1}{\Gamma(k)} \cdot \left(\frac{k}{\mu+k}\right)^{k} \cdot \frac{\Gamma(y+k)}{\Gamma(y+1)} \cdot \exp\left(y \log \frac{\mu}{\mu+k}\right)$$

natural parameter.

#4.30

$$0 = \ell'(\beta^{(0)}) + (\beta^{(1)} - \beta^{(0)})\, \ell''(\beta^{(0)})$$

$$\Rightarrow \quad \beta^{(1)} = \beta^{(0)} - \frac{\ell'(\beta^{(0)})}{\ell''(\beta^{(0)})}$$

#4.33

observed information matrix : $-H^{-1}$

expected information matrix : $E\left(-\frac{\partial^2 L}{\partial \beta^T \partial \beta}\right)^{-1} = E\left(\frac{\partial L}{\partial \beta}\right)^T \left(\frac{\partial L}{\partial \beta}\right)$

$$= \sum \frac{x_{ij}\, x_{ih}}{Var(y_i)} \left(\frac{\partial \mu_i}{\partial \eta_i}\right)^2$$

$$= \sum \frac{x_{ij}\, x_{ih}}{Var(y_i)}\, \phi(\eta_i)^2$$

$$H = \frac{\partial}{\partial \beta_h}\left(\frac{\partial L}{\partial \beta_j}\right)$$

$$\frac{\partial L}{\partial \beta_j} = \sum_{i=1}^{N} \frac{(y_i - \mu_i)\, x_{ij}}{Var(y_i)} \cdot \frac{\partial \mu_i}{\partial \eta_i}.$$

depend on data!!

$$\frac{\partial^2 L}{\partial \beta_j \partial \beta_h} = \sum_{i=1}^{N} \left[\frac{-x_{ih}\, x_{ij}}{Var(y_i)} \cdot \underbrace{\frac{\partial \mu_i}{\partial \eta_i}}_{\phi(\eta_i)} + \frac{(y_i - \mu_i)\, x_{ij}}{Var(y_i)} \cdot \underbrace{\frac{\partial^2 \mu_i}{\partial \eta_i^2}}_{\phi''(\eta_i)} \cdot x_{ih} \right].$$