Introduction) Assume $\theta \in \mathbb{R}$.

$$\log p_{\theta_0+h} = \log p_{\theta_0} + h \cdot \dot{l}_\theta + \frac{1}{2} h^2 \ddot{l}_\theta$$

$$\Rightarrow \log \frac{p_{\theta_0+h/\sqrt{n}}}{p_{\theta_0}} = \frac{h}{\sqrt{n}} \dot{l}_\theta + \frac{h^2}{2n} \ddot{l}_\theta$$

$$\Rightarrow \log \frac{p^n_{\theta_0+h/\sqrt{n}}}{p^n_{\theta_0}} = \frac{h}{\sqrt{n}} \Sigma \dot{l}_\theta + \frac{h^2}{2n} \Sigma \ddot{l}_\theta$$

Note) ① $E\dot{l}_\theta = \int \frac{\dot{p}_\theta}{p_\theta} p_\theta \overset{?}{=} \frac{d}{d\theta} \int p_\theta = 0$

② $E\ddot{l}_\theta = \int \frac{\ddot{p}_\theta p_\theta - (\dot{p}_\theta)^2}{p_\theta^2} \cdot p_\theta$

$$= \int \ddot{p}_\theta - \int \frac{(\dot{p}_\theta)^2}{p_\theta} \overset{?}{=} 0 - \int \frac{(\dot{p}_\theta)^2}{p_\theta^2} p_\theta$$

$$= - E(\dot{l}_\theta)^2 \overset{\text{def}}{=} -I_\theta$$

③ $E(\dot{l}_\theta)^2 = I_\theta$

$$\log \frac{P^n_{\theta_0+h/\sqrt{n}}}{P^n_{\theta_0}} = \frac{h}{\sqrt{n}} \Sigma \dot{l}_\theta - \frac{h^2}{2n} \Sigma \ddot{l}_\theta$$

$$\rightsquigarrow h \cdot \boxed{N(0, I_\theta)} - \frac{h^2}{2} I_\theta = N\left(-\frac{h^2}{2} I_\theta, h^2 I_\theta\right)$$

①, ②, ③ by CLT and LLN

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On the other hand, $\log \frac{dN(h, I_\theta^{-1})}{dN(0, I_\theta^{-1})}$

$$= \log \exp \frac{I_\theta}{2}\left[x^2 - (x-h)^2\right]$$

$$= h \boxed{I_\theta X} - \frac{h^2}{2} I_\theta$$

prop) Assume $\theta \in \mathbb{R}$.

WTS $\displaystyle\int \left[ \frac{s_{\theta+h} - s_\theta}{h} - \frac{1}{2} \dot{l}_\theta \sqrt{p_\theta} \right]^2$

$\to 0$ as $h \to 0$

Let $\dot{l}_\theta = \dfrac{\dot{p}_\theta}{p_\theta}$. Then we have

$\dot{s}_\theta = (\dot{\sqrt{p_\theta}}) = \dfrac{\dot{p}_\theta}{2\sqrt{p_\theta}}$ and hence

$\frac{1}{2} \dot{l}_\theta \sqrt{p_\theta} = \dot{s}_\theta$. That is,

ETS $\displaystyle\int \left( \frac{s_{\theta+h} - s_\theta}{h} - \dot{s}_\theta \right)^2 \to 0$

as $h \to 0$

ETS $\int |f_n|^2 \leq \int f^2 < \infty$

$$S_{\theta+h} - S_\theta \underset{\uparrow}{=} h \int_0^1 \dot{S}_{\theta+uh}\, du$$

MVT for integrals

$$\Rightarrow (S_{\theta+h} - S_\theta)^2 \leq h^2 \cdot \left(\int_0^1 \dot{S}_{\theta+uh}\, du\right)^2$$

$$\underset{\uparrow}{\leq} h^2 \int_0^1 (\dot{S}_{\theta+uh})^2\, du$$

Cauchy-Schwartz

$$\Rightarrow \int \left(\frac{S_{\theta+h} - S_\theta}{h}\right)^2 \leq \int \int_0^1 \dot{S}_{\theta+uh}\, du$$

$$\underset{\uparrow}{=} \int_0^1 \int \dot{S}_{\theta+uh}\, du = \frac{1}{4} \int_0^1 \int \frac{\dot{P}^2_{\theta+uh}}{P^2_{\theta+uh}} P_{\theta+uh}$$

Fubini's

$$= \frac{1}{4}\int_0^1 I_{\theta+uh}\, du \underset{\substack{\uparrow \\ \text{MVT for integrals}}}{=} \frac{1}{4} I_{\theta+\tilde{u}h}$$

$$\longrightarrow \frac{1}{4} I_\theta \quad \text{as } h \to 0$$

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Example 1) Let $\theta \in \{\theta \mid \int h \exp Qt \, d\mu < \infty\}$.

$\Rightarrow$ ① $S_\theta = \sqrt{P_\theta} = \sqrt{h \exp(Q(\theta) t)}$

② $I_\theta = Q'_\theta \, \mathrm{Cov}\, t(X)\, {Q'_\theta}^T$

If $Q(\theta)$ is continuously differentiable,

then $\mathcal{P}$ is QMD.

Example 2: ① $\sqrt{f}$ is continuously diff

③ $I_\theta = \int \left(\frac{f'(x-\theta)}{f(x-\theta)}\right)^2 f(x-\theta)$

$= \int \left(\frac{f'}{f}\right)^2 f = I_f$ (constant)

$\Rightarrow \{f(x-\theta)\}$ is QMD.

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$\dot{l}_\theta = \left(\log \frac{1}{2} \exp - |x-\theta|\right)'$ ?

it is not diff at $\theta = x$, but

$= - \operatorname{sign}(x-\theta)$ a.e.

Example 3) $\int \left(S_{\theta+h} - S_\theta - \frac{1}{2} h \dot{l}_\theta \sqrt{p_\theta}\right)^2$

$$\geq \int_\theta^{\theta+h} \quad '' \quad = \int_\theta^{\theta+h} (S_{\theta+h})^2$$

$$= \int_\theta^{\theta+h} \frac{1}{\theta+h} = \frac{h}{\theta+h} = O(h)$$

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$$\begin{array}{ccccc}
 & \xrightarrow{h_1} T_n & \overset{h_1}{\rightsquigarrow} & L_{h_1} & \\
X_1, \dots, X_n & T_n & \overset{h_2}{\rightsquigarrow} & L_{h_2} & \left.\right\} \; T_n \rightsquigarrow X \\
P^n_{\theta+h/\sqrt{n}} & \xrightarrow{h_2} & \vdots & & \\
 & \xrightarrow{h_m} T_n & \overset{h_m}{\rightsquigarrow} & L_{h_m} & N(h, I_\theta^{-1})
\end{array}$$

Thm 7.10) WTS ① $\exists$ randomized $T$

② $T_n \overset{h}{\rightsquigarrow} T$

Assume $\left(T_n, \frac{dP_{n,h}}{dP_{n,0}} \frac{P_n}{Q_n}\right) \overset{0}{\rightsquigarrow} {}_{Q_n} \left(S, \underbrace{\exp N\left(-\frac{h^2}{2}J, h^2 J\right)}_{V}\right)$

$\Longrightarrow$ $dP_{n,0} \triangleleft dP_{n,h}$

Example 6.5 $\boxed{dP_{n,h} \triangleleft dP_{n,0}}$ iff

$-\frac{h^2}{2}J = -\frac{1}{2}\cdot\sigma^2$

$\Longrightarrow$ $L_h(B) = E\,1_B(S)V$

Thm 6.6 and $T_n \overset{h}{\rightsquigarrow} L_h \overset{?}{=} T$

$\Longrightarrow$ $\exists$ randomized statistic $T = T(\Delta, U)$

Lemma s.t. $(T, \Delta) \overset{D}{=} (S, \Delta)$

($\Rightarrow$ given $\Delta$, $T = S$)

Let $X \sim N(h, J^{-1})$. Then

$$P_h(T(JX, U) \in B)$$

$$= \int_{\{x \mid T(Jx, U) \in B\}} \sqrt{\frac{J}{2\pi}}\, e^{-\frac{J}{2}(x-h)^2}$$

$$= \int_{\text{''}} \underbrace{\sqrt{\frac{J}{2\pi}}\, e^{-\frac{J}{2}x^2}}_{N(0,J)} \cdot e^{Jhx - \frac{h^2}{2}J}$$

$$= E_{h=0}\, I\left(T(JX, U) \in B\right) e^{JhX - \frac{h^2}{2}J}$$

Note) $h = 0 \Rightarrow JX \sim N(0, J)$

$\Rightarrow JX \overset{D}{=} \Delta \Rightarrow T(JX, U) = T(\Delta, U)$

$\Rightarrow T = S$, as desired.

for each $x$, $Jx$ should be fixed.

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The remaining is to verify that

$$\left(T_n, \frac{dP_{n,h}}{dP_{n,0}}\right) \rightsquigarrow \left(S, e^{N\left(-\frac{h^2}{2}J,\, h^2 J\right)}\right)$$

$$T_n \overset{o}{\rightsquigarrow} S \quad \Rightarrow \text{They are uniformly}$$
$$\Delta_n \overset{o}{\rightsquigarrow} \Delta \qquad\qquad \text{tight}$$

$\Rightarrow (T_n, \Delta_n)$ is also uniformly tight

$\Rightarrow \exists (S, \Delta)$ s.t $(T_n, \Delta_n) \rightsquigarrow (S, \Delta)$

$\Rightarrow$ Apply Thm 9.2 with this fact!

(??)