Moving Average Process

$$\tilde{Z}_t = a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q}$$

$\uparrow$ MA(q)

(i) It is clearly stationary.

For invertibility, it can be rewritten as

$$\tilde{Z}_t = (1 - \theta_1 B - \cdots - \theta_q B^q) a_t$$

$$= \theta(B) a_t$$

$$\Rightarrow \quad \theta^{-1}(B) \tilde{Z}_t = a_t \text{ (formal expression)}$$

We need to check that it is well-defined

Suppose $\theta(B) = \prod_{1}^{q} (1 - H_i B)$. Then

$$\theta^{-1}(B)\hat{Z}_t = a_t \Rightarrow \frac{1}{\prod (1 - H_i B)} \hat{Z}_t = a_t$$

$$\underset{\text{Partial Fraction}}{\Rightarrow} \sum \frac{M_i}{1 - H_i B} \hat{Z}_t = a_t$$

Let $(LHS) = \hat{Z}_t + \pi_1 \hat{Z}_{t-1} + \pi_2 \hat{Z}_{t-2} + \cdots$

We have $\pi_j = \sum M_i H_i^j$ and

$$\sum |\pi_j| = \sum_j \sum_i |M_i H_i^j| < \infty \text{ if}$$

$|H_i| < 1$.

<Remark> $\theta^{-1}(B) = \dfrac{1}{\prod (1 - H_i B)}$ can be regarded as a function of B in $|B| \leq 1$ (from stationarity).

In order to be well-defined,

the denominator can never be zero, which implies that $\frac{1}{H_i} \notin [-1, 1]$ and hence $|H_i| < 1$.

Proposition: MAs are invertible if all roots of $\underbrace{\Theta(B) = 0}$ lie outside the unit circle.

↳ Characteristic equation.

(ii) ACF: $\gamma_k = E(\tilde{Z}_t \tilde{Z}_{t-k})$

$= -\theta_k E a_{t-k}^2 + \theta_1 \theta_{k+1} E a_{t-k-1}^2 + \cdots + \theta_{q-k} \theta_q E a_{t-q}^2$

$$= \begin{cases} (1 + \theta_1^2 + \cdots + \theta_q^2)\sigma_a^2 & (k=0) \\ (-\theta_k + \theta_1\theta_{k+1} + \cdots + \theta_{q-k}\theta_q)\sigma_a^2 & (0<k\leq q) \end{cases}$$

$$\rho_k = \begin{cases} 1 & (k=0) \\ \dfrac{-\theta_k + \theta_1\theta_{k+1} + \cdots + \theta_{q-k}\theta_q}{1+\theta_1^2+\cdots+\theta_q^2} & (0<k\le q) \end{cases}$$

Example 1 : MA(1) $\Rightarrow \tilde{Z}_t = a_t - \theta_1 a_{t-1}$.

(i) Invertibility : $\theta(B) = 1-\theta_1 B \overset{\text{set}}{=} 0$

$$\Rightarrow \frac{1}{|\theta_1|} > 1 \Rightarrow |\theta_1| < 1$$

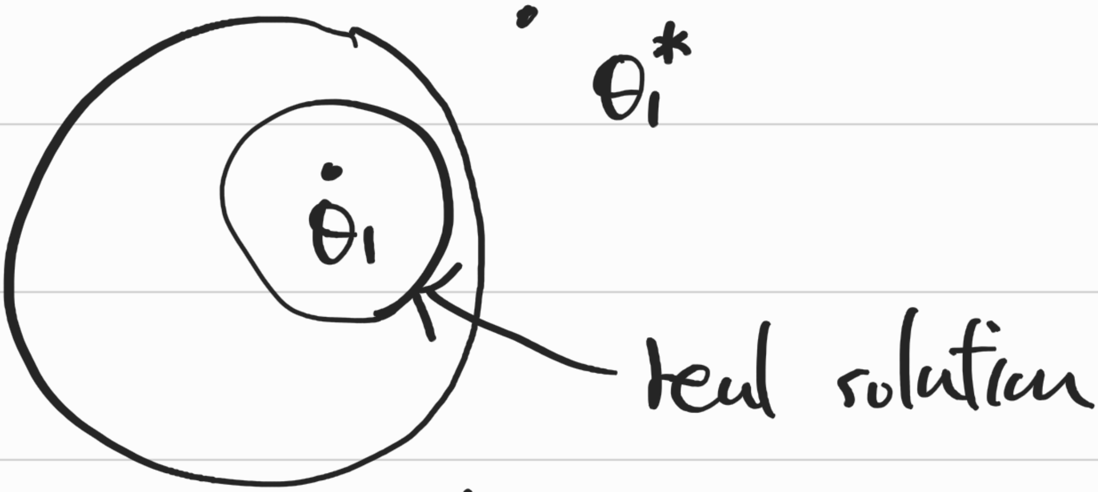
(ii) $\gamma_0 = (1+\theta_1^2)\sigma_a^2$

(iii) $$\rho_k = \begin{cases} 1 & k=0 \\ \dfrac{-\theta_1}{1+\theta_1^2} & \\ 0 & k>1 \end{cases}$$

<Note> If we know ρ_1, then

$$\rho_1 = \frac{-\theta_1}{1+\theta_1^2} \Rightarrow \rho_1\theta_1^2 + \theta_1 + \rho_1 = 0$$

$$\Rightarrow \theta_1 = \frac{-1 \pm \sqrt{1-4\rho_1^2}}{2\rho_1}$$



by $|\theta_1 \theta_1^*| = 1$!

<Note 2> Using Yule-Walker equation, we get PACF. In this case,

$$\phi_{kk} = \frac{-\theta_1^k(1-\theta_1^2)}{1-\theta_1^{2(k+1)}}$$

$\le |\theta_1|^k$ ~ exponentially decayed