#12 on page 244

12. Two components of a computer have the following joint pdf for their useful lifetimes X and Y:

$$f(x,y) = \begin{cases} xe^{-x(1+y)} & x \ge 0 \text{ and } y \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

a. What is the probability that the lifetime X of the first component exceeds 3?

b. What are the marginal pdf's of X and Y? Are the two lifetimes independent? Explain.

c. What is the probability that the lifetime of at least one component exceeds 3?

(a) $P(X > 3)$

$= P\{(x,y) \mid x>3,\ y \ge 0\}$

$= \int_3^\infty \int_0^\infty x e^{-x(1+y)}\, dy\, dx$

c 10

$\int e^x dx = e^x + c$

$(e^x)' = e^x$

$(e^{cx})' = e^{cx}(cx)' = e^{cx}\cdot c = c\cdot e^{cx}$

chain rule

$x e^{-x(1+y)}\, d(y+1)$

constant

$= e^{-x(1+y)}\, d[x(y+1)]$

$= \int -d\{e^{-x(1+y)}\}$

$= -e^{-x(1+y)}$

$= \int_3^\infty \left[-e^{-x(1+y)}\right]_0^\infty dx$

$y=0$

$= \int_3^\infty \left[\lim_{y\to\infty}\left(-e^{-x(1+y)}\right) + e^{-x}\right] dx$

$\dfrac{d\{-e^{-x(1+y)}\}}{dy} = -\dfrac{d\{e^{-x(1+y)}\}}{dy}$

$= -$

$\int [-x\cdot e^{-x(1+y)}]\, dy$

$= \int d\{-e^{-x(1+y)}\}$

$= -e^{-x(1+y)} + c$

e^u where $u = -x(1+y)$

$\dfrac{de^u}{dy} = \dfrac{de^u}{du}\cdot\dfrac{du}{dy}$

$= e^u \cdot \dfrac{d(-x(1+y))}{dy}$

$= e^u \cdot -x\, \dfrac{d(1+y)}{dy}$

$= e^u\cdot(-x)$

$= -x\cdot e^{-x(1+y)}$

$\Rightarrow d\{-e^{-x(1+y)}\} = -x\cdot e^{-x(1+y)}\, dy$

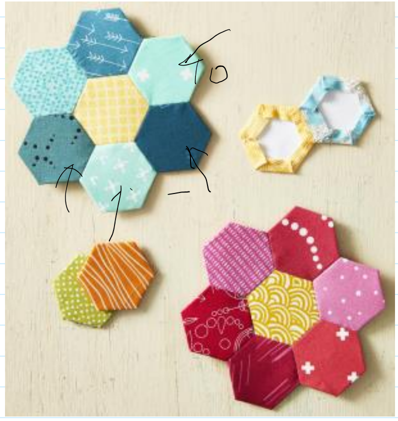
Calculus Integrals Reference Sheet

<https://www.eeweb.com/tools/calculus-integrals-sheet/>

Techniques of Integration

knowledge of derivatives

<https://www.whitman.edu/mathematics/calculus_online/chapter08.html>



"derivative piecing"

$\int \bigcirc\, dx =$

$= \int \{\ \}\, dx = \square$

— joint distribution for continuous r.v. (X, Y)

· joint pdf

Suppose $(X,Y) \sim$ joint pdf $f(x,y)$ where $(x,y) \in$ 2-D set S

· prob.

For any 2-D set A

✓ $P\{(X,Y)\in A\} = \iint_A f(x,y)\,dx\,dy$

* exp. $A = \{(x,y) \mid a \le x \le b,\ c \le y \le d\}$

$P\{(X,Y)\in A\} = \int_a^b \int_c^d f(x,y)\,dy\,dx$

· marginal pdfs

marginal pdf of X, $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

= only x

$= \int f(x,y)\,dy$

Y. $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$

$= \int f(x,y)\,dx$

y only

· independence

"X and Y are independent" $\Leftrightarrow f(x,y) = f_X(x)\, f_Y(y)$

$$= -e^{-x(1+y)} + C$$

$$\lim_{y\to\infty} -e^{-x(1+y)} = \lim_{y\to\infty} -\frac{1}{\boxed{e^{x(1+y)}}} = 0$$

$$= \int_3^\infty e^{-x}\,dx$$

$$= -e^{-x}\Big|_3^\infty = \lim_{x\to\infty}(-e^{-x}) - (-e^{-x})\Big|_{x=3}$$

$$= 0 + e^{-3} = e^{-3}$$

(b) $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

$$= \int_0^\infty x e^{-x(1+y)}\,dy$$

$$= -e^{-x(1+y)}\Big|_0^\infty = \lim_{y\to\infty}\underbrace{(-e^{-x(1+y)})}_{=?\,0} + e^{-x} = e^{-x}, \quad x \geq 0$$

$(e^{-x})' = e^{-x}\cdot(-1) = -e^{-x}$

$\boxed{\int u\,dv = uv - \int v\,du}$

$$f_Y(y) = \int_0^\infty x\,\boxed{e^{-x(1+y)}\,dx} \qquad d(?) \qquad d(e^{-x(1+y)}) \quad x$$

$$= \int_0^\infty x\cdot\left(-\frac{1}{1+y}\right) d(e^{-x(1+y)}) \qquad = -(1+y)\cdot\boxed{e^{-x(1+y)}\,dx}$$

$$= -\frac{1}{1+y}\int_0^\infty x\,d(e^{-x(1+y)})$$

$$= -\frac{1}{1+y}\left[\underbrace{x e^{-x(1+y)}\Big|_0^\infty}_{0-0} - \int_0^\infty e^{-x(1+y)}\,dx\right]$$

$$\lim_{x\to\infty} x\cdot e^{-x(1+y)} = \lim_{x\to\infty}\frac{x}{e^{x(1+y)}} \quad \frac{\infty}{\infty} = \lim_{x\to\infty}\frac{1}{e^{x(1+y)}\cdot(1+y)} = 0$$

$$= \left(\frac{1}{1+y}\right)\int_0^\infty e^{-x(1+y)}\,dx$$

$$= \left(\frac{1}{1+y}\right)\cdot -e^{-x(1+y)}\cdot\left(-\frac{1}{1+y}\right)\Big|_0^\infty$$

$$= -\left(\frac{1}{1+y}\right)^2\cdot e^{-x(1+y)}\Big|_0^\infty$$

$$= -\left(\frac{1}{1+y}\right)^2\left(\lim_{x\to\infty}\boxed{e^{-x(1+y)}} - \underbrace{e^{-x(1+y)}\Big|_{x=0}}_{e^0}\right)$$

$$= -\left(\frac{1}{1+y}\right)^2(0-1)$$

$$= \left(\frac{1}{1+y}\right)^2$$

$$f_Y(y) = \frac{1}{(1+y)^2} \quad , \quad y \geq 0$$