北京航空航天大學
BEIHANG UNIVERSITY

计算流体力学

第三章：有限差分方法

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本章主要内容

- 有限差分的基本概念
- 有限差分格式的构造

- 编程作业：迁移方程的离散及求解

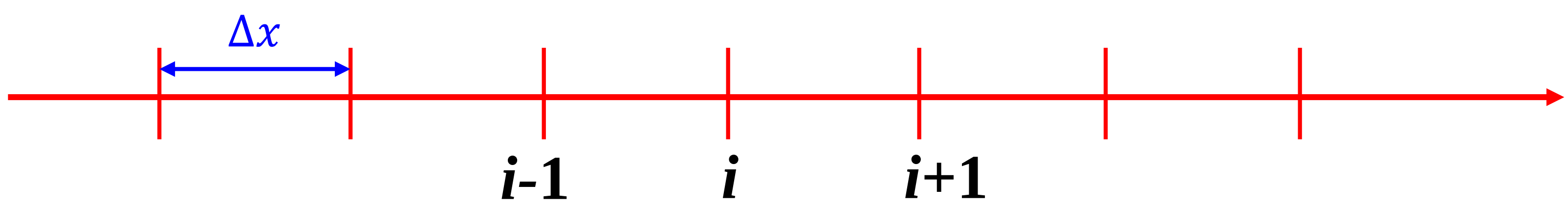
3.1 有限差分的基本概念

- 有限差分：**Finite difference**

$$u_x = \frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{u(x+\Delta x) - u(x)}{\Delta x} \qquad \frac{\partial u}{\partial x} \approx \frac{u(x+\Delta x) - u(x)}{\Delta x}$$

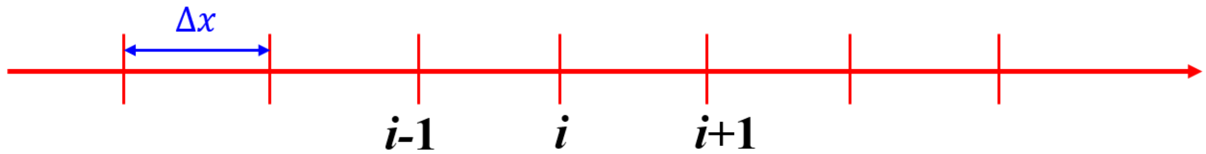
$$u_x = \frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{u(x) - u(x-\Delta x)}{\Delta x} \qquad \frac{\partial u}{\partial x} \approx \frac{u(x) - u(x-\Delta x)}{\Delta x}$$

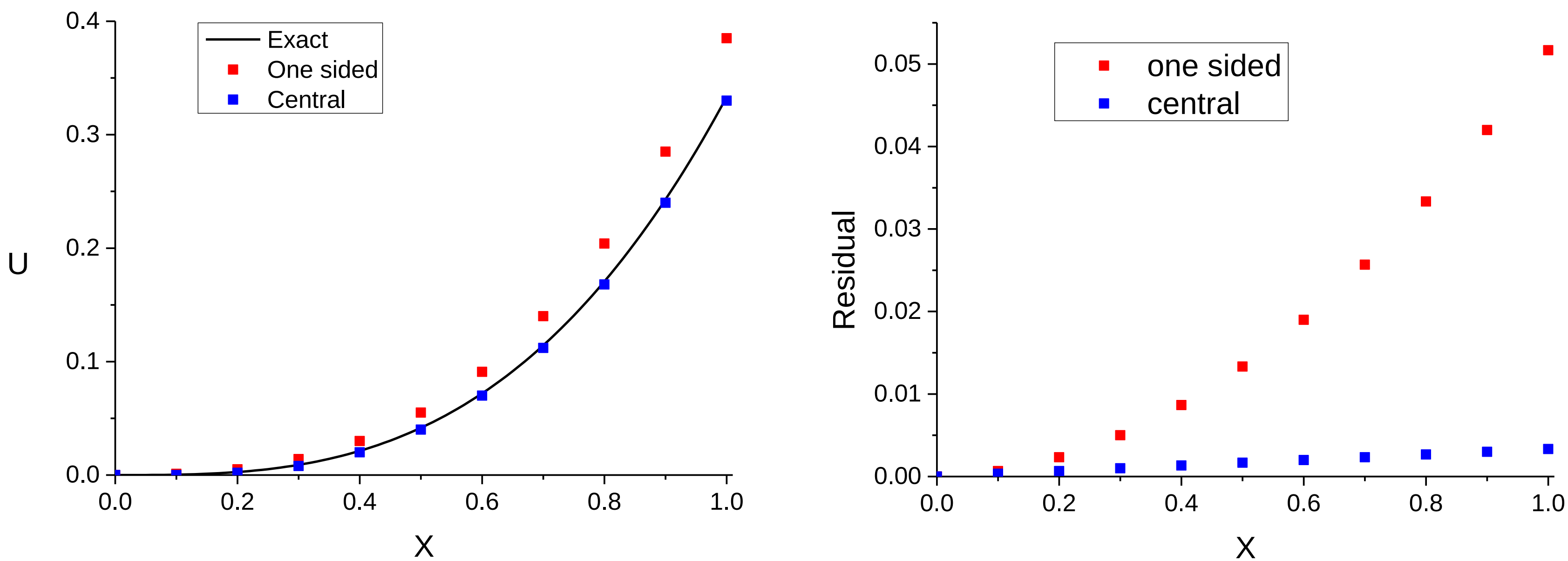
$$u_x = \frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{u(x+\Delta x) - u(x-\Delta x)}{2\Delta x} \qquad \frac{\partial u}{\partial x} \approx \frac{u(x+\Delta x) - u(x-\Delta x)}{2\Delta x}$$



3.1 有限差分的基本概念

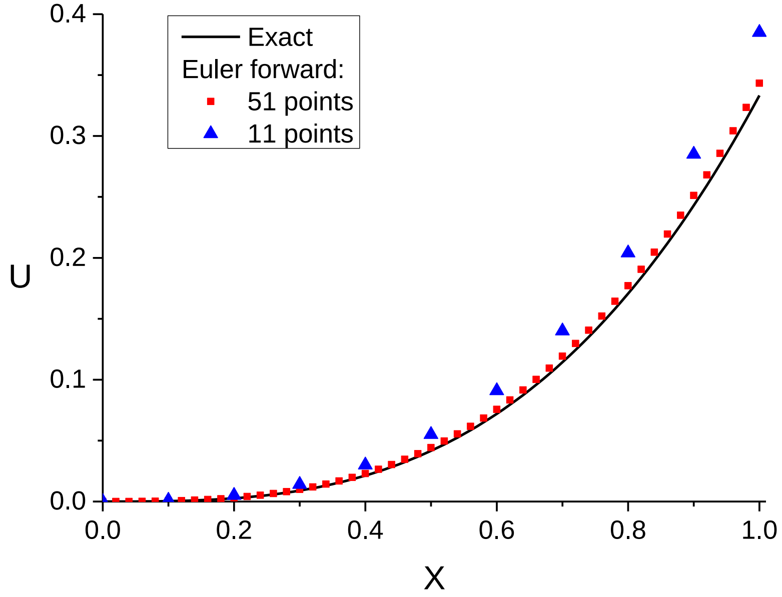
$$\frac{du}{dx} = x^2, \qquad u(0) = 0$$

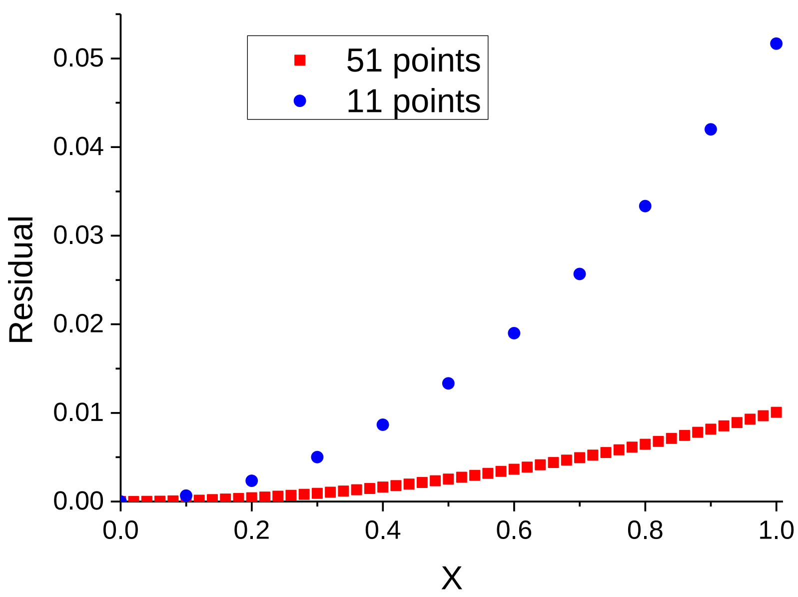




不同格式间的比较

3.1 有限差分的基本概念





单侧差分、不同离散点数的比较

不同差分格式、离散点 Δx 对精度都有影响，如何评价它们？

3.1 有限差分的基本概念

$$u_x = \frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{u(x+\Delta x) - u(x)}{\Delta x} \qquad u_x = \frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{u(x+\Delta x) - u(x-\Delta x)}{2\Delta x}$$

$$u(x+\Delta x) = u(x) + \Delta x \frac{\partial u}{\partial x} + \frac{(\Delta x)^2}{2} \frac{\partial^2 u}{\partial x^2} + \frac{(\Delta x)^3}{6} \frac{\partial^3 u}{\partial x^3} + \cdots$$

$$u(x-\Delta x) = u(x) - \Delta x \frac{\partial u}{\partial x} + \frac{(\Delta x)^2}{2} \frac{\partial^2 u}{\partial x^2} - \frac{(\Delta x)^3}{6} \frac{\partial^3 u}{\partial x^3} + \cdots$$

$$\frac{\partial u}{\partial x} = \frac{u(x+\Delta x) - u(x)}{\Delta x} \quad - \frac{\Delta x}{2} \frac{\partial^2 u}{\partial x^2} \quad - \frac{(\Delta x)^2}{6} \frac{\partial^3 u}{\partial x^3} - \cdots$$

$$\frac{\partial u}{\partial x} = \frac{u(x+\Delta x) - u(x-\Delta x)}{2\Delta x} \quad - 0 \quad - \frac{(\Delta x)^2}{6} \frac{\partial^3 u}{\partial x^3} - \cdots$$

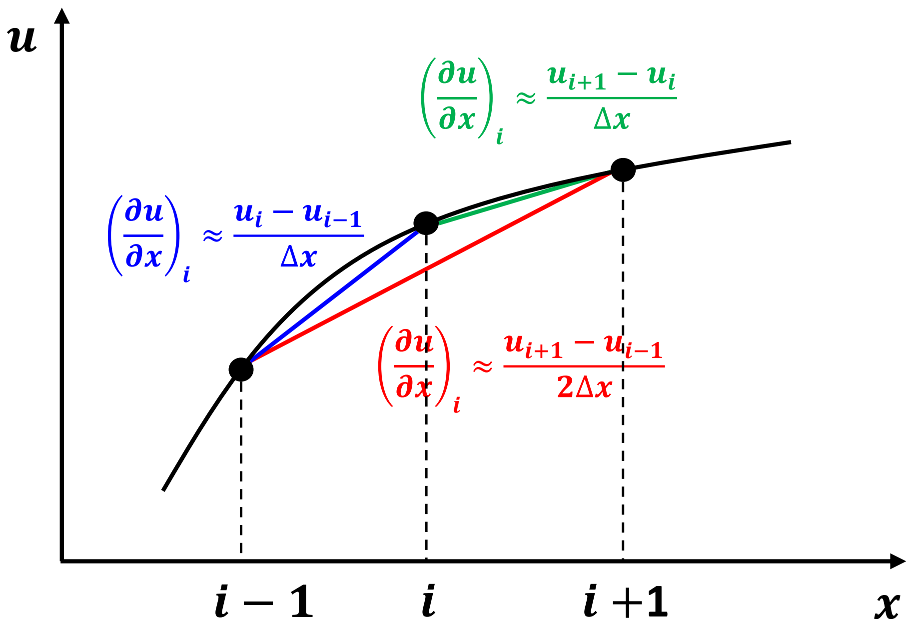
截断误差

3.1 有限差分的基本概念

$$\frac{\partial u}{\partial x}=\frac{u(x+\Delta x)-u(x)}{\Delta x}-\frac{\Delta x}{2}\frac{\partial^2 u}{\partial x^2}-\frac{(\Delta x)^2}{6}\frac{\partial^3 u}{\partial x^3}-\cdots=\frac{u(x+\Delta x)-u(x)}{\Delta x}+O(\Delta x)$$

$$\frac{\partial u}{\partial x}=\frac{u(x+\Delta x)-u(x-\Delta x)}{2\Delta x}-0-\frac{(\Delta x)^2}{6}\frac{\partial^3 u}{\partial x^3}-\cdots=\frac{u(x+\Delta x)-u(x-\Delta x)}{2\Delta x}+O((\Delta x)^2)$$

当$\Delta x \to 0$时，能更快收敛至$\frac{\partial u}{\partial x}$



- 截断误差$O((\Delta x^n))$，n称为差分格式的精度阶数，n越大，则差分格式的精度越高

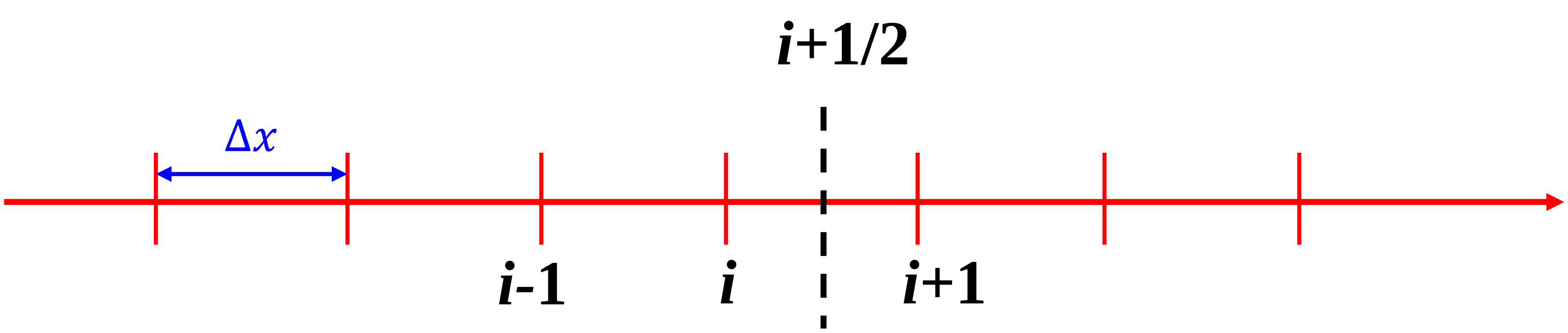
3.2 有限差分格式的构造

- 一维一阶导数的差分离散

$$\left(\frac{\partial u}{\partial x}\right)_i = \frac{u_{i+1} - u_i}{\Delta x} + O(\Delta x)$$ Euler前差

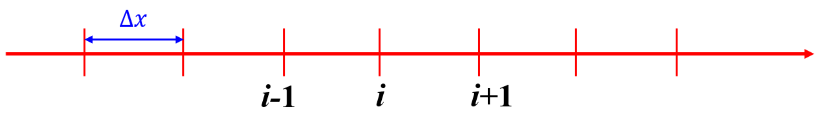
$$\left(\frac{\partial u}{\partial x}\right)_i = \frac{u_i - u_{i-1}}{\Delta x} + O(\Delta x)$$ Euler后差

$$\left(\frac{\partial u}{\partial x}\right)_i = \frac{u_{i+1} - u_{i-1}}{2\Delta x} + O((\Delta x)^2)$$ 中心差分 $$\left(\frac{\partial u}{\partial x}\right)_{i+1/2} = \frac{u_{i+1} - u_i}{\Delta x} + O((\Delta x)^2)$$



3.2 有限差分格式的构造

- 一维一阶导数的差分离散



待定系数法：

$$\left(\frac{\partial u}{\partial x}\right)_{i+1} \approx (au_{i+1} + bu_i + cu_{i-1})/\Delta x$$

$$u_i = u_{i+1} - \Delta x\left(\frac{\partial u}{\partial x}\right)_{i+1} + \frac{(\Delta x)^2}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)_{i+1} - \frac{(\Delta x)^3}{6}\left(\frac{\partial^3 u}{\partial x^3}\right)_{i+1} + \cdots$$

$$u_{i-1} = u_{i+1} - 2\Delta x\left(\frac{\partial u}{\partial x}\right)_{i+1} + \frac{(2\Delta x)^2}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)_{i+1} - \frac{(2\Delta x)^3}{6}\left(\frac{\partial^3 u}{\partial x^3}\right)_{i+1} + \cdots$$

$$\left.\begin{array}{l} a + b + c = 0 \\ -b - 2c = 1 \\ \frac{1}{2}b + 2c = 0 \end{array}\right\} \quad a = \frac{3}{2} \qquad b = -2 \qquad c = \frac{1}{2}$$

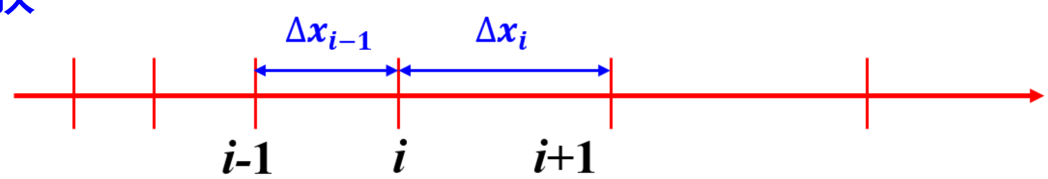
$$\left(\frac{\partial u}{\partial x}\right)_{i+1} = \frac{3u_{i+1} - 4u_i + u_{i-1}}{2\Delta x} + O((\Delta x)^2)$$

三点Euler后差

3.2 有限差分格式的构造

- **一维一阶导数的差分离散**

非均匀网格点分布：



$$\left(\frac{\partial u}{\partial x}\right)_i \approx (au_{i+1} + bu_i + cu_{i-1})/(\Delta x_{i-1} + \Delta x_i)$$

$$u_{i+1} = u_i + \Delta x_i \left(\frac{\partial u}{\partial x}\right)_i + \frac{(\Delta x_i)^2}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)_i + \frac{(\Delta x_i)^3}{6}\left(\frac{\partial^3 u}{\partial x^3}\right)_i + \cdots$$

$$u_{i-1} = u_i - \Delta x_{i-1}\left(\frac{\partial u}{\partial x}\right)_i + \frac{(\Delta x_{i-1})^2}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)_i - \frac{(\Delta x_{i-1})^3}{6}\left(\frac{\partial^3 u}{\partial x^3}\right)_i + \cdots$$

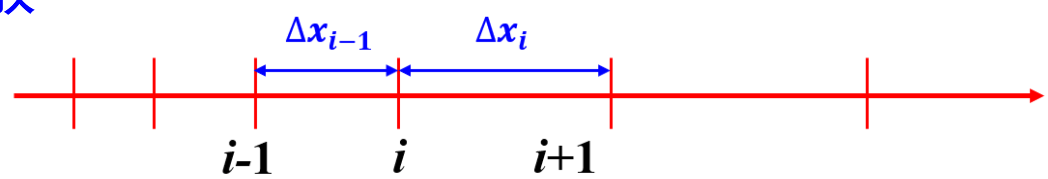
$$a = \frac{\Delta x_{i-1}}{\Delta x_i} \qquad b = \frac{\Delta x_i}{\Delta x_{i-1}} - \frac{\Delta x_{i-1}}{\Delta x_i} \qquad c = -\frac{\Delta x_i}{\Delta x_{i-1}} \qquad r = \frac{\Delta x_i}{\Delta x_{i-1}}$$

$$\left(\frac{\partial u}{\partial x}\right)_i = \frac{\frac{1}{r}u_{i+1} + (r - \frac{1}{r})u_i - ru_{i-1}}{\Delta x_{i-1} + \Delta x_i} + O((\Delta x)^2)$$

3.2 有限差分格式的构造

- 一维一阶导数的差分离散

非均匀网格点分布：



$$\left(\frac{\partial u}{\partial x}\right)_i=\frac{\frac{1}{r}u_{i+1}+(r-\frac{1}{r})u_i-ru_{i-1}}{\Delta x_{i-1}+\Delta x_i}+O((\Delta x)^2)$$

如果忽略网格点间距的变化：

$r=1$

$$\left(\frac{\partial u}{\partial x}\right)_i=\frac{u_{i+1}-u_{i-1}}{\Delta x_{i-1}+\Delta x_i}+O((\Delta x)^?)$$

$$u_{i+1}=u_i+\Delta x_i\left(\frac{\partial u}{\partial x}\right)_i+\frac{(\Delta x_i)^2}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)_i+\cdots$$

$$u_{i-1}=u_i-\Delta x_{i-1}\left(\frac{\partial u}{\partial x}\right)_i+\frac{(\Delta x_{i-1})^2}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)_i+\cdots$$

⇩

$$\left(\frac{\partial u}{\partial x}\right)_i=\frac{u_{i+1}-u_{i-1}}{\Delta x_{i-1}+\Delta x_i}-\frac{\Delta x_i-\Delta x_{i-1}}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)_i+\cdots$$

- 如果$(\Delta x_i-\Delta x_{i-1})\sim\Delta x$，则格式只有一阶精度
- 当$(\Delta x_i-\Delta x_{i-1})\sim O(\Delta x^2)$时，格式才有二阶精度

3.2 有限差分格式的构造

- **一维一阶导数的差分离散**

多项式法：

Step1：给定N阶多项式

$$u(x)=a_0+a_1(x-x_i)+a_2(x-x_i)^2+a_3(x-x_i)^3+\cdots$$

Step2：N阶多项式有N+1个待定系数，用多项式拟合N+1个离散点处u的值，可确定多项式系数

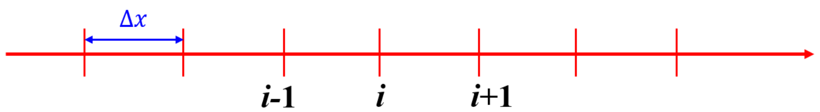
$$\left.\begin{aligned}x&=x_{i-k},x_{i-k+1},\cdots,x_i,\cdots,x_{i+m-1},x_{i+m}\\u&=u_{i-k},u_{i-k+1},\cdots,u_i,\cdots,u_{i+m-1},u_{i+m}\end{aligned}\right\}\text{代入多项式}$$

Step3：在i点的N阶精度的一阶差分为 $u'(x_i)=a_1$

在i点的N-1阶精度的二阶差分为 $u''(x_i)=2a_2$

3.2 有限差分格式的构造

- 紧致差分离散



$$\left(\frac{\partial u}{\partial x}\right)_i = \frac{u_{i+1}-u_{i-1}}{2\Delta x} + O((\Delta x)^2) \qquad \frac{u_{i+1}-u_{i-1}}{2\Delta x} = \left(\frac{\partial u}{\partial x}\right)_i + O((\Delta x)^2)$$

$$\frac{u_{i+1}-u_{i-1}}{2\Delta x} = a(u_x)_{i+1} + b(u_x)_i + c(u_x)_{i-1} + O((\Delta x)^n)$$

$$\frac{u_{i+1}-u_{i-1}}{2\Delta x} = (u_x)_i + \cancel{\frac{\Delta x^2}{3!}(u_{xxx})_i} + \frac{\Delta x^4}{5!}(u_{5x})_i + O((\Delta x)^6)$$

$$(u_x)_{i+1} = (u_x)_i + \cancel{\Delta x(u_{xx})_i} + \cancel{\frac{\Delta x^2}{2}(u_{xxx})_i} + \cancel{\frac{\Delta x^3}{3!}(u_{4x})_i} + \frac{\Delta x^4}{4!}(u_{5x})_i + O((\Delta x)^5)$$

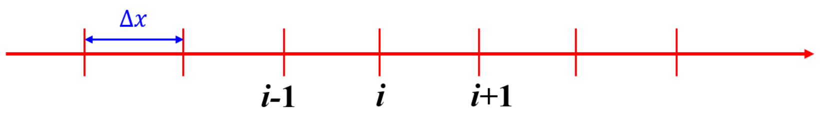
$$(u_x)_{i-1} = (u_x)_i - \cancel{\Delta x(u_{xx})_i} + \cancel{\frac{\Delta x^2}{2}(u_{xxx})_i} - \cancel{\frac{\Delta x^3}{3!}(u_{4x})_i} + \frac{\Delta x^4}{4!}(u_{5x})_i + O((\Delta x)^5)$$

$$a+b+c=1 \quad a=c \quad a+c=1/3$$

$$\frac{1}{6}[(u_x)_{i+1} + 4(u_x)_i + (u_x)_{i-1}] = \frac{u_{i+1}-u_{i-1}}{2\Delta x} + O((\Delta x)^4)$$

3.2 有限差分格式的构造

- 紧致差分离散



$$u(x)=a_0+a_1(x-x_i)+a_2(x-x_i)^2+a_3(x-x_i)^3+a_4(x-x_i)^4$$

$$u'(x)=a_1+2a_2(x-x_i)+3a_3(x-x_i)^2+4a_4(x-x_i)^3$$

$$\left.\begin{aligned}&x=x_{i-1},\ x_i,\ x_{i+1}:\quad u=u_{i-1},\ u_i,\ u_{i+1}\\&x=x_{i-1},\ x_{i+1}:\quad u'=u'_{i-1},\ u'_{i+1}\end{aligned}\right\}\ \text{代入以上两式}$$

$$\left(\frac{\partial u}{\partial x}\right)_i=a_1=-\frac{1}{4}\left(\frac{\partial u}{\partial x}\right)_{i-1}-\frac{1}{4}\left(\frac{\partial u}{\partial x}\right)_{i+1}+\frac{3}{4}\frac{u_{i+1}-u_{i-1}}{\Delta x}$$

$$\frac{1}{6}\left[(u_x)_{i+1}+4(u_x)_i+(u_x)_{i-1}\right]=\frac{u_{i+1}-u_{i-1}}{2\Delta x}+O((\Delta x)^4)$$

具有4阶精度，一样的结果

3.2 有限差分格式的构造

- 紧致差分离散

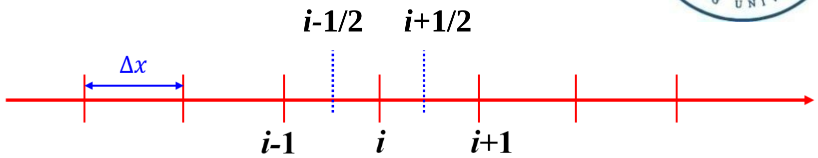
$$\frac{1}{6}[(u_x)_{i+1}+4(u_x)_i+(u_x)_{i-1}]=\frac{u_{i+1}-u_{i-1}}{2\Delta x}+O((\Delta x)^4)$$

$$\begin{bmatrix} \ddots & & & & \\ 1 & 4 & 1 & & \\ & 1 & 4 & 1 & \\ & & 1 & 4 & 1 \\ & & & & \ddots \end{bmatrix}\cdot\begin{bmatrix} \vdots \\ (u_x)_{i-1} \\ (u_x)_i \\ (u_x)_{i+1} \\ \vdots \end{bmatrix}=\frac{3}{\Delta x}\begin{bmatrix} \vdots \\ u_i-u_{i-2} \\ u_{i+1}-u_{i-1} \\ u_{i+2}-u_i \\ \vdots \end{bmatrix}$$

- 采用紧致格式需要隐式求解
- 任意一处离散点的u值的变化会影响所有的导数值
- 紧致格式可以用更少的点获得高阶精度

3.2 有限差分格式的构造

- 一维高阶导数的差分离散



$$\left(\frac{\partial^2 u}{\partial x^2}\right)_i = \frac{(u_x)_{i+1/2}-(u_x)_{i-1/2}}{\Delta x} = \frac{u_{i+1}-2u_i+u_{i-1}}{\Delta x^2} + O((\Delta x)^2)$$

$$\left(\frac{\partial^3 u}{\partial x^3}\right)_i = \frac{(u_{xx})_{i+1}-(u_{xx})_{i-1}}{2\Delta x} = \frac{u_{i+2}-2u_{i+1}+2u_{i-1}-u_{i-2}}{2\Delta x^3} + O((\Delta x)^2)$$

$$\left(\frac{\partial^3 u}{\partial x^3}\right)_{i+1/2} = \frac{(u_{xx})_{i+1}-(u_{xx})_{i}}{\Delta x} = \frac{u_{i+2}-3u_{i+1}+3u_{i}-u_{i-1}}{\Delta x^3} + O((\Delta x)^2)$$

$$\left(\frac{\partial^4 u}{\partial x^4}\right)_i = \frac{(u_{xxx})_{i+1/2}-(u_{xxx})_{i-1/2}}{\Delta x} = \frac{u_{i+2}-4u_{i+1}+6u_{i}-4u_{i-1}+u_{i-2}}{\Delta x^4} + O((\Delta x)^2)$$

3.2 有限差分格式的构造

• 迁移方程的有限差分格式

$$\frac{\partial u}{\partial t}+a\frac{\partial u}{\partial x}=0$$

计算域：$x_B \le x \le x_T$

初始条件：$t=0$时，$u(x,0)=u^0(x)$

边界条件：$x=x_U$时，$u(x_U,t)=g(t)$ $\begin{cases} a>0: & x_U=x_B \\ a<0: & x_U=x_T \end{cases}$

空间离散：

$$(u_t)_i=\boxed{-a\frac{u_{i+1}-u_{i-1}}{2\Delta x}}$$ 中心差分

$$(u_t)_i=\boxed{-a\frac{u_i-u_{i-1}}{\Delta x}}$$ Euler后差

$$(u_t)_i=\boxed{-a\frac{3u_i-4u_{i-1}+u_{i-2}}{2\Delta x}}$$ 三点Euler后差

$$R_i$$

3.2 有限差分格式的构造

- 迁移方程的有限差分格式

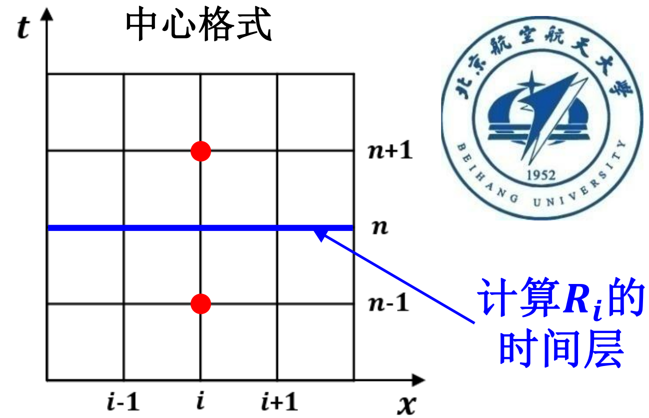
空间离散： $(u_t)_i = R_i$

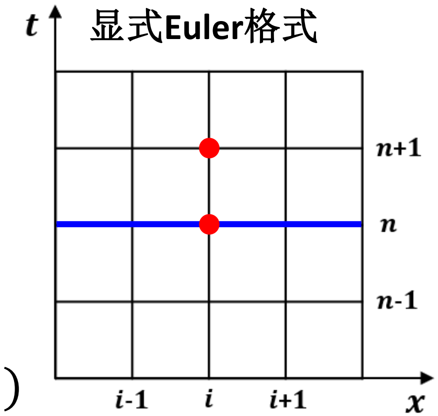
时间离散： 将时刻表示为：$t = n\Delta t$

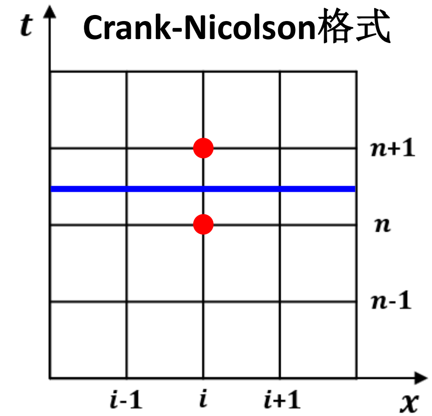
$$\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} = R_i^n$$

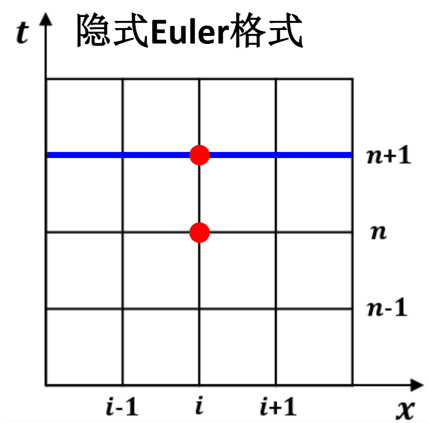
$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = R_i^n \,, R_i^{n+1}, or\ \frac{1}{2}(R_i^n + R_i^{n+1})$$

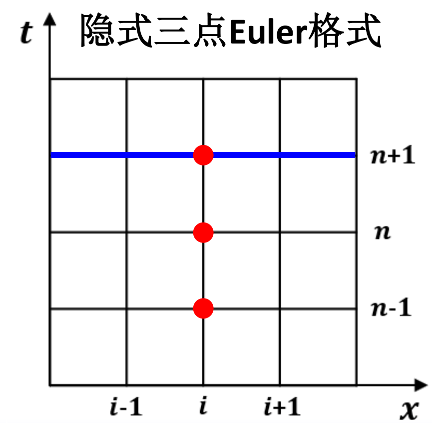
$$\frac{3u_i^{n+1} - 4u_i^n + u_i^{n-1}}{2\Delta t} = R_i^n \quad or \quad R_i^{n+1}$$









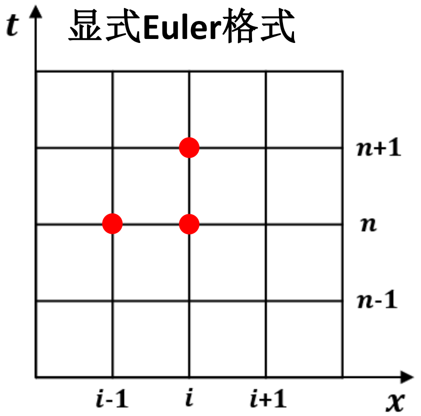


3.2 有限差分格式的构造

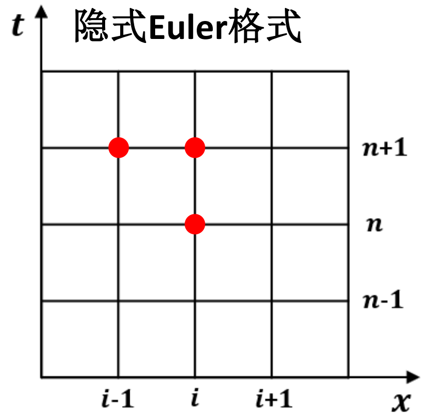
- 迁移方程的有限差分格式

更多离散格式：

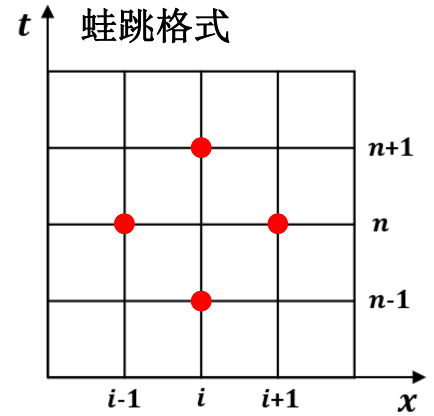
显式Euler格式



隐式Euler格式



蛙跳格式



$$\frac{u_i^{n+1}-u_i^n}{\Delta t}=-a\frac{u_i^n-u_{i-1}^n}{\Delta x}$$

$$u_i^{n+1}=u_i^n-a\frac{\Delta t}{\Delta x}(u_i^n-u_{i-1}^n)$$

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}=-a\frac{u_i^{n+1}-u_{i-1}^{n+1}}{\Delta x}$$

$$u_i^{n+1}=u_i^n-a\frac{\Delta t}{\Delta x}(u_i^{n+1}-u_{i-1}^{n+1})$$

$$\frac{u_i^{n+1}-u_i^{n-1}}{2\Delta t}=-a\frac{u_{i+1}^n-u_{i-1}^n}{2\Delta x}$$

$$u_i^{n+1}=u_i^{n-1}-a\frac{\Delta t}{\Delta x}(u_{i+1}^n-u_{i-1}^n)$$

列出的这些格式不一定都能给出正确的结果！

3.2 有限差分格式的构造

- 迁移方程的有限差分格式

Lax-Wendroff格式:

$$u_i^{n+1} = u_i^n + \Delta t\left(\frac{\partial u}{\partial t}\right)_i^n + \frac{\Delta t^2}{2}\left(\frac{\partial^2 u}{\partial t^2}\right)_i^n + O(\Delta t^3)$$

$$\left(\frac{\partial u}{\partial t}\right)_i = -a\frac{u_{i+1} - u_{i-1}}{2\Delta x} + O((\Delta x)^2)$$

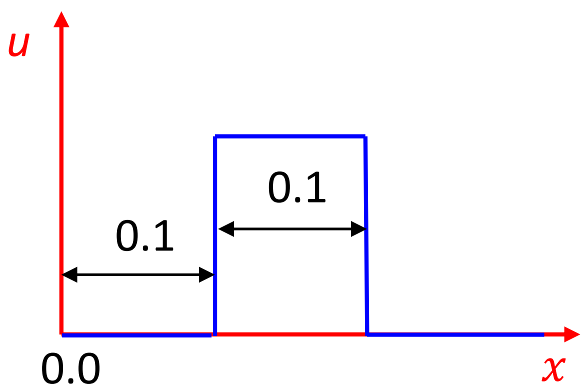
$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = -a\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right) = a^2\frac{\partial^2 u}{\partial x^2} = a^2\frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} + O((\Delta x)^2)$$

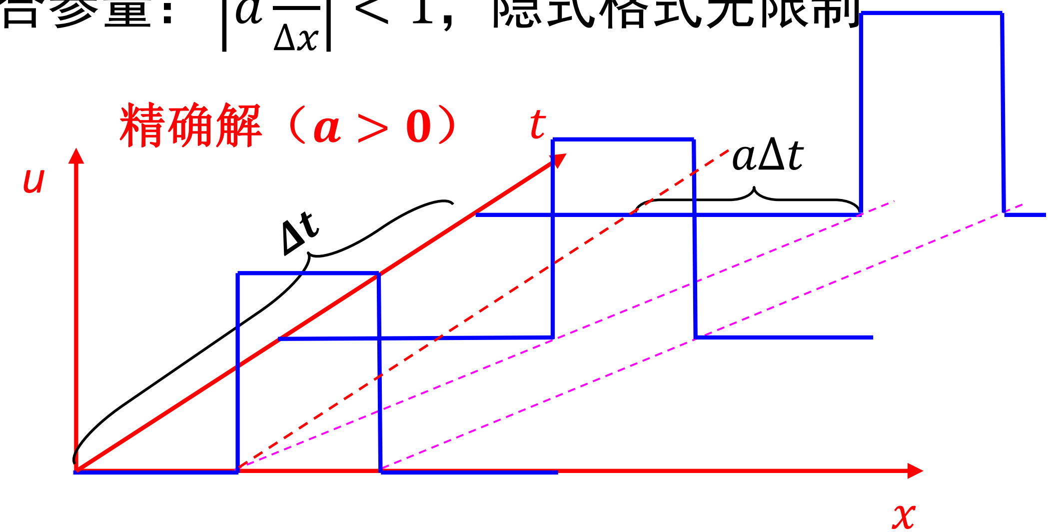
$$u_i^{n+1} = u_i^n - \frac{1}{2}\frac{a\Delta t}{\Delta x}(u_{i+1}^n - u_{i-1}^n) + \frac{1}{2}\left(\frac{a\Delta t}{\Delta x}\right)^2(u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$

$$u_i^{n+1} = u_i^n - \frac{\sigma}{2}(u_{i+1}^n - u_{i-1}^n) + \frac{\sigma^2}{2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n) \qquad \sigma = \frac{a\Delta t}{\Delta x}$$

编程作业一：迁移方程的离散及求解

- **算例具体要求：**
  - 空间计算域：[0.0−1.0]，用至少100个点离散
  - 对于显式格式，建议组合参量：$\left|a\dfrac{\Delta t}{\Delta x}\right| < 1$，隐式格式无限制
  - 初场可以如下：





  - 沿时间推进求解至时刻$t = 0.7/a$，$(a > 0$时$)$
  - 完成不少于3种迁移方程离散格式的推导；
  - 对每种离散格式，进行编程求解；
  - 分析每种格式的结果、收敛性、精度等，完成分析报告，报告后附源程序。

3.2 有限差分格式的构造

- 高维空间导数的差分离散

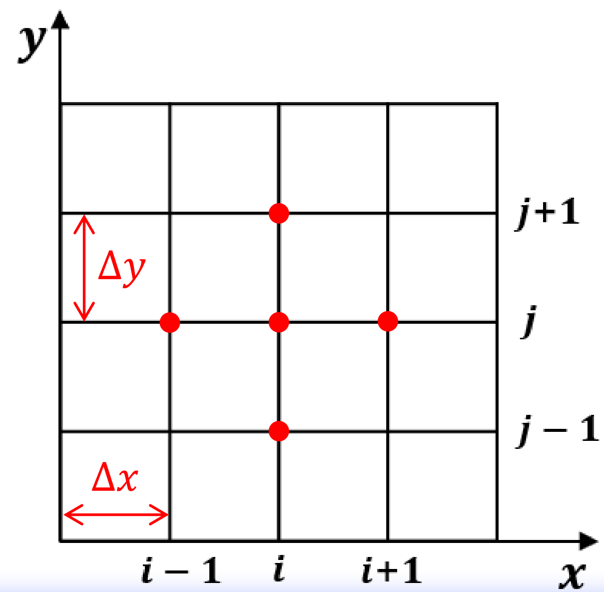
$$u(x+\Delta x, y+\Delta y)=u(x,y)+\left(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right)u+\frac{1}{2}\left(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right)^2 u+\frac{1}{3!}\left(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right)^3 u+\cdots$$

$$\left(\frac{\partial u}{\partial x}\right)_{i,j}=\frac{u_{i+1,j}-u_{i,j}}{\Delta x}+O(\Delta x)\,;\quad \frac{u_{i,j}-u_{i-1,j}}{\Delta x}+O(\Delta x);\quad \frac{u_{i+1,j}-u_{i-1,j}}{2\Delta x}+O\left((\Delta x)^2\right)$$

$$\left(\frac{\partial u}{\partial y}\right)_{i,j}=\frac{u_{i,j+1}-u_{i,j}}{\Delta y}+O(\Delta y)\,;\quad \frac{u_{i,j}-u_{i,j-1}}{\Delta y}+O(\Delta y);\quad \frac{u_{i,j+1}-u_{i,j-1}}{2\Delta y}+O\left((\Delta y)^2\right)$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j}=\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{\Delta x^2}+O\left((\Delta x)^2\right)$$

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{i,j}=\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{\Delta y^2}+O\left((\Delta y)^2\right)$$

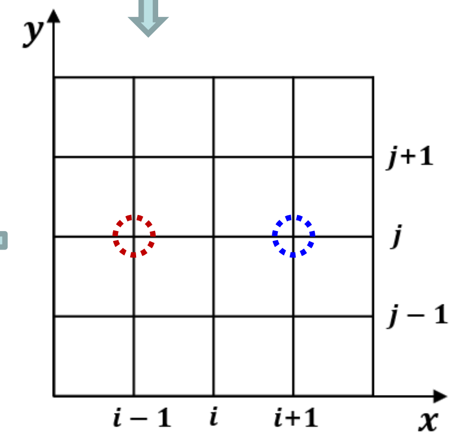


在笛卡尔网格下，函数的空间导数可视为沿各自坐标方向的一维导数，所以可用一维差分格式进行离散

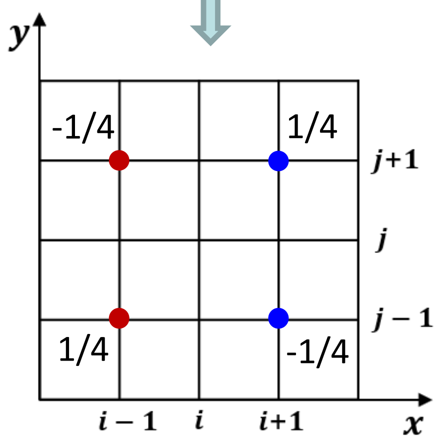
3.2 有限差分格式的构造

- 二维空间导数的差分离散

$$\left(\frac{\partial^2 u}{\partial x\partial y}\right)_{i,j} = \left[\frac{\partial}{\partial x}(u_y)\right]_{i,j} = \left[\frac{\partial}{\partial y}(u_x)\right]_{i,j} = \quad \text{e.g.} \quad \frac{(u_y)_{i+1,j}-(u_y)_{i-1,j}}{2\Delta x}+O((\Delta x)^2)$$



$$\frac{\dfrac{u_{i+1,j+1}-u_{i+1,j-1}}{2\Delta y}-\dfrac{u_{i-1,j+1}-u_{i-1,j-1}}{2\Delta y}}{2\Delta x}+O((\Delta x)^2,(\Delta y)^2)$$

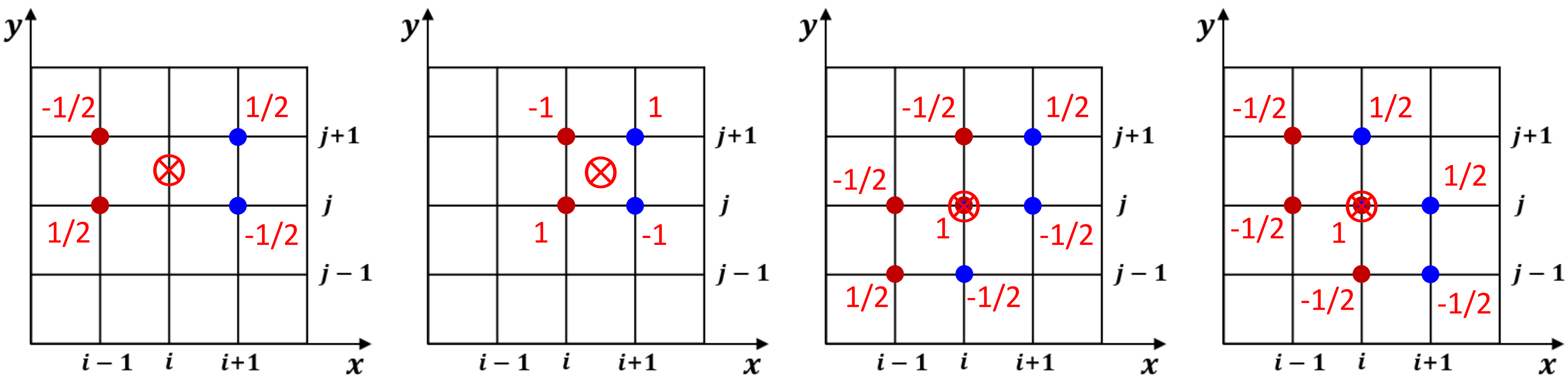


$$\frac{u_{i+1,j+1}-u_{i+1,j-1}-u_{i-1,j+1}+u_{i-1,j-1}}{4\Delta x\Delta y}+O((\Delta x)^2,(\Delta y)^2)$$

3.2 有限差分格式的构造

- 二维空间导数的差分离散

$$\left(\frac{\partial^2 u}{\partial x \partial y}\right)_{i,j} = \left[\frac{\partial}{\partial x}(u_y)\right]_{i,j} = \left[\frac{\partial}{\partial y}(u_x)\right]_{i,j}$$



$O(\Delta x, \Delta y)$ $O(\Delta x, \Delta y)$ $O((\Delta x)^2, (\Delta y)^2)$ $O((\Delta x)^2, (\Delta y)^2)$

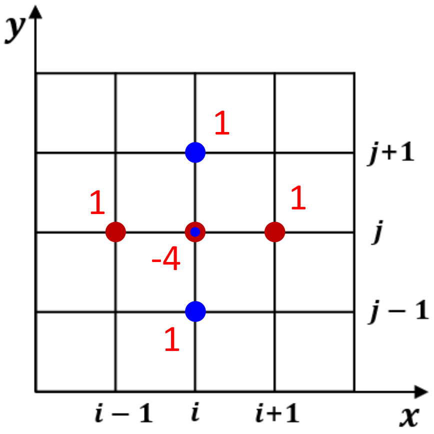
3.2 有限差分格式的构造

- **Laplace算子的差分离散**

$$\nabla^2 u = \vec{\nabla} \cdot \left(\vec{\nabla} u\right) = u_{xx} + u_{yy}$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} + O\left((\Delta x)^2\right)$$

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{i,j} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2} + O\left((\Delta y)^2\right)$$



$$\left(\nabla^2 u\right)_{i,j} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2} + O\left((\Delta x)^2, (\Delta y)^2\right)$$

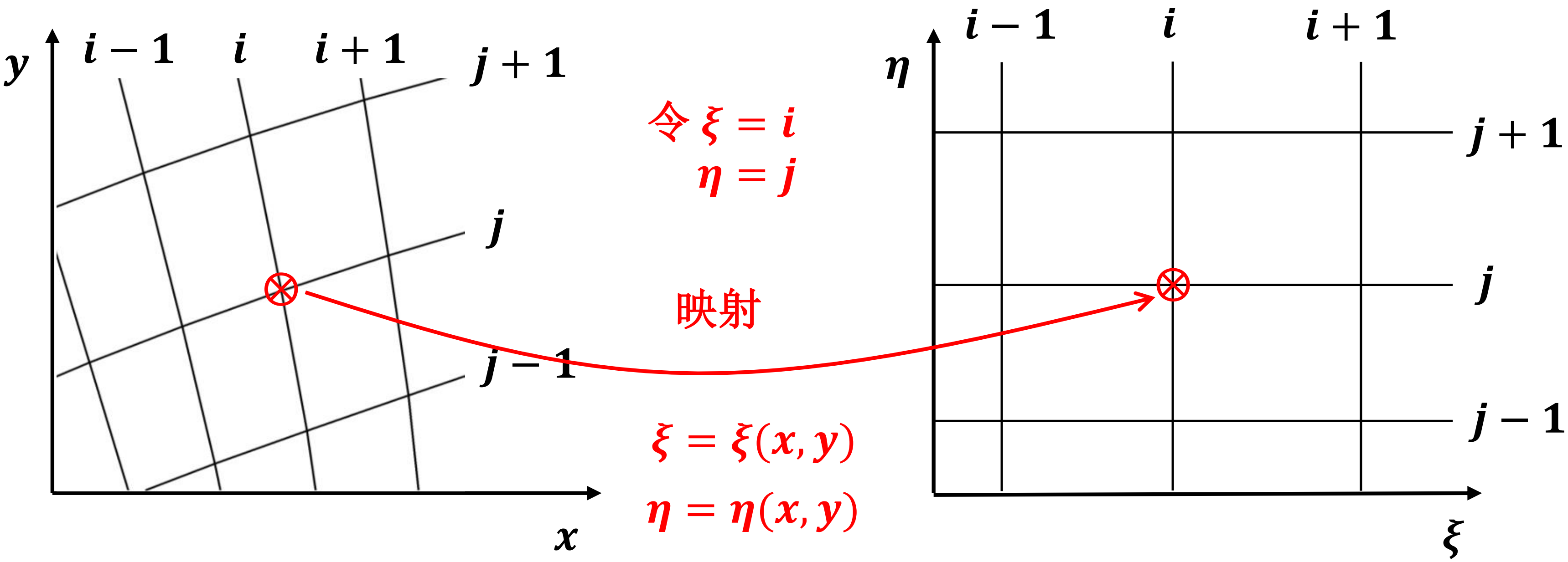
非线性扩散算子：

$$\vec{\nabla}(\mu \vec{\nabla} u)_{i,j} = \left[\frac{\partial}{\partial x}\left(\mu \frac{\partial u}{\partial x}\right)\right]_{i,j} + \left[\frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right)\right]_{i,j}$$

$$\mu = \mu(x, y)$$

3.2 有限差分格式的构造

- 曲线网格系统下的差分离散



$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}$$

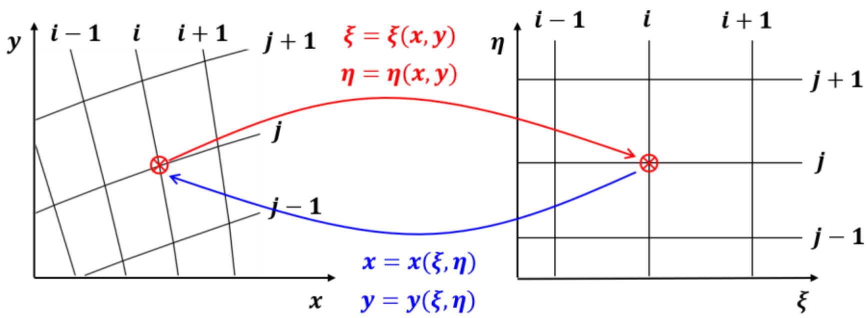
以中心差分为例:

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{1}{2}(u_{i+1,j} - u_{i-1,j})(\xi_x)_{i,j} + \frac{1}{2}(u_{i,j+1} - u_{i,j-1})(\eta_x)_{i,j}$$

同样:

$$\left(\frac{\partial u}{\partial y}\right)_{i,j} = \frac{1}{2}(u_{i+1,j} - u_{i-1,j})(\xi_y)_{i,j} + \frac{1}{2}(u_{i,j+1} - u_{i,j-1})(\eta_y)_{i,j}$$

3.2 有限差分格式的构造



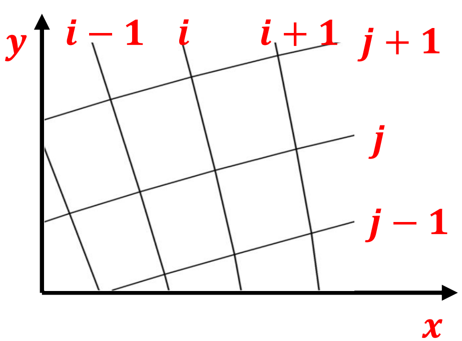
- 曲线网格系统下的差分离散

$$\begin{cases}\xi=\xi(x,y)\\ \eta=\eta(x,y)\end{cases} \xrightarrow{\text{求全微分}} \begin{cases}d\xi=\xi_x dx+\xi_y dy\\ d\eta=\eta_x dx+\eta_y dy\end{cases} \Longrightarrow \begin{bmatrix}d\xi\\ d\eta\end{bmatrix}=\begin{bmatrix}\xi_x & \xi_y\\ \eta_x & \eta_y\end{bmatrix}\begin{bmatrix}dx\\ dy\end{bmatrix}$$

$$\begin{cases}x=x(\xi,\eta)\\ y=y(\xi,\eta)\end{cases} \qquad \begin{cases}dx=x_\xi d\xi+x_\eta d\eta\\ dy=y_\xi d\xi+y_\eta d\eta\end{cases} \Longrightarrow \begin{bmatrix}dx\\ dy\end{bmatrix}=\begin{bmatrix}x_\xi & x_\eta\\ y_\xi & y_\eta\end{bmatrix}\begin{bmatrix}d\xi\\ d\eta\end{bmatrix}$$

$$\begin{bmatrix}\xi_x & \xi_y\\ \eta_x & \eta_y\end{bmatrix}=\begin{bmatrix}x_\xi & x_\eta\\ y_\xi & y_\eta\end{bmatrix}^{-1}$$

$$\begin{bmatrix}\xi_x & \xi_y\\ \eta_x & \eta_y\end{bmatrix}=\begin{bmatrix}y_\eta & -x_\eta\\ -y_\xi & x_\xi\end{bmatrix}\Big/(x_\xi y_\eta - x_\eta y_\xi)$$



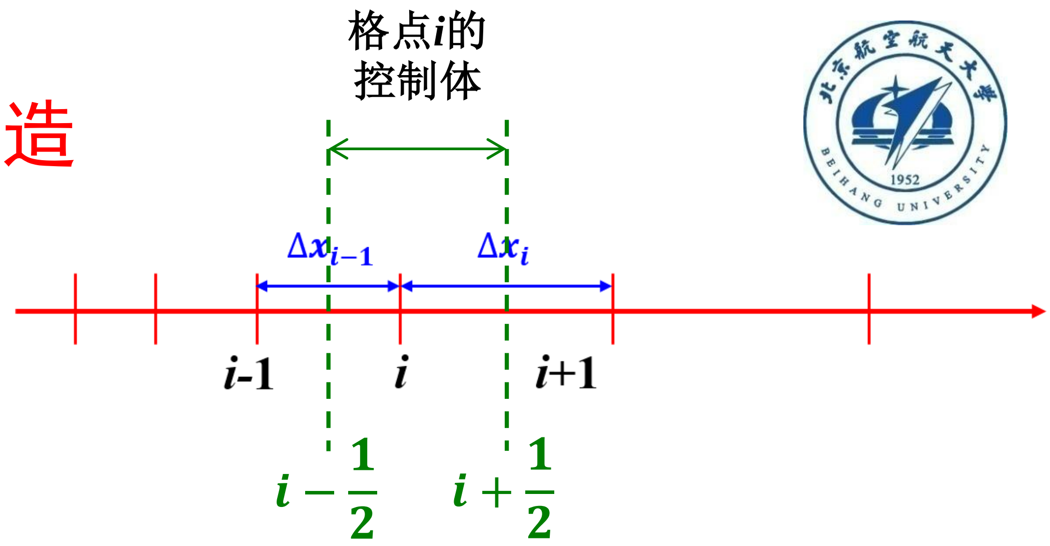
$$(x_\xi)_{i,j}=\frac{1}{2}(x_{i+1,j}-x_{i-1,j})$$

$$(x_\eta)_{i,j}=\frac{1}{2}(x_{i,j+1}-x_{i,j-1})$$

$$(x_\xi)_{i+\frac{1}{2},j}=\cdots \qquad (x_\xi)_{i,j+\frac{1}{2}}=\cdots$$

3.2 有限差分格式的构造

- 差分格式的守恒性概念



考虑如下守恒律：

$$\frac{\partial u}{\partial t}+\frac{\partial f}{\partial x}=0$$

对空间导数作如下离散：

$$\left(\frac{\partial f}{\partial x}\right)_i=\frac{f_{i+\frac{1}{2}}-f_{i-\frac{1}{2}}}{x_{i+\frac{1}{2}}-x_{i-\frac{1}{2}}}$$

称之为守恒差分格式。**Why？**

控制方程的守恒性

⇓

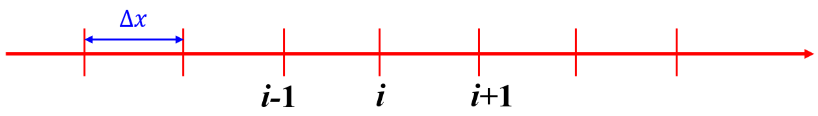
离散方程的守恒性

⇓

数值解的守恒性

3.2 有限差分格式的构造

- 一维Euler方程的差分离散



$$\frac{\partial U}{\partial t}+\frac{\partial F}{\partial x}=0 \qquad U=\begin{bmatrix}\rho\\ \rho u\\ E\end{bmatrix} \qquad F=\begin{bmatrix}\rho u\\ \rho u^2+p\\ \rho u H\end{bmatrix}$$

守恒差分格式：

$$\left(\frac{\partial U}{\partial t}\right)_i+\frac{\widehat{\widehat{F}}_{i+\frac{1}{2}}-\widehat{\widehat{F}}_{i-\frac{1}{2}}}{\Delta x}=0$$

上式不能视为对空间导数的中心差分，实际上依据Lagrange中值定理，$(\widehat{\widehat{F}}_{i+\frac{1}{2}}-\widehat{\widehat{F}}_{i-\frac{1}{2}})/\Delta x = or \neq \frac{\partial F}{\partial x}$

仅与 $\widehat{\widehat{F}}_{\frac{1}{2}}$ 的通量计算格式有关！

Lagrange中值定理：

$f(x)$在$[x_B,x_T]$上连续，在(x_B,x_T)内可导，则在开区间(x_B,x_T)内至少有一点x_0，使下式成立：

$$f'(x_0)=\frac{f(x_T)-f(x_B)}{x_T-x_B}$$

3.2 有限差分格式的构造

- 二维Euler方程的差分离散

$$\frac{\partial U}{\partial t}+\frac{\partial F}{\partial x}+\frac{\partial G}{\partial y}=0 \qquad U=\begin{bmatrix}\rho\\ \rho u\\ \rho v\\ E\end{bmatrix} \qquad F=\begin{bmatrix}\rho u\\ \rho u^2+p\\ \rho uv\\ \rho uH\end{bmatrix} \qquad G=\begin{bmatrix}\rho v\\ \rho uv\\ \rho v^2+p\\ \rho vH\end{bmatrix}$$

$$\frac{\partial U}{\partial t}+\frac{\partial F}{\partial \xi}\xi_x+\frac{\partial F}{\partial \eta}\eta_x+\frac{\partial G}{\partial \xi}\xi_y+\frac{\partial G}{\partial \eta}\eta_y=0 \qquad \frac{1}{J}\frac{\partial U}{\partial t}+\frac{\partial}{\partial \xi}\frac{1}{J}\left(F\xi_x+G\xi_y\right)+\frac{\partial}{\partial \eta}\frac{1}{J}\left(F\eta_x+G\eta_y\right)=0$$

$$\hat{F}=\frac{1}{J}\left(F\xi_x+G\xi_y\right)=\frac{1}{J}\begin{bmatrix}\rho U_\xi\\ \rho U_\xi u+p\xi_x\\ \rho U_\xi v+p\xi_y\\ \rho U_\xi H\end{bmatrix} \qquad U_\xi=u\xi_x+v\xi_y$$

$$\frac{1}{J}\frac{\partial U}{\partial t}+\frac{\partial \hat{F}}{\partial \xi}+\frac{\partial \hat{G}}{\partial \eta}=0$$

（未考虑变形网格）

$$\hat{G}=\frac{1}{J}\left(F\eta_x+G\eta_y\right)=\frac{1}{J}\begin{bmatrix}\rho U_\eta\\ \rho U_\eta u+p\eta_x\\ \rho U_\eta v+p\eta_y\\ \rho U_\eta H\end{bmatrix} \qquad U_\eta=u\eta_x+v\eta_y$$

守恒差分格式?